1990

Density Continuity Versus Continuity

Krzysztof Ciesielski

West Virginia University, krzysztof.ciesielski@mail.wvu.edu

Follow this and additional works at: https://researchrepository.wvu.edu/faculty_publications

Part of the [Mathematics Commons](#)

Digital Commons Citation

Ciesielski, Krzysztof, "Density Continuity Versus Continuity" (1990). *Faculty Scholarship*. 820.
https://researchrepository.wvu.edu/faculty_publications/820

This Article is brought to you for free and open access by The Research Repository @ WVU. It has been accepted for inclusion in Faculty Scholarship by an authorized administrator of The Research Repository @ WVU. For more information, please contact ian.harmon@mail.wvu.edu.

Density Continuity Versus Continuity

Krzysztof Ciesielski Lee Larson Krzysztof Ostaszewski *

Abstract

Real-valued functions of a real variable which are continuous with respect to the density topology on both the domain and the range are called density continuous. A typical continuous function is nowhere density continuous. The same is true of a typical homeomorphism of the real line. A subset of the real line is the set of points of discontinuity of a density continuous function if and only if it is a nowhere dense F_σ set. The corresponding characterization for the approximately continuous functions is a F_σ set. An alternative proof of that result is given. Density continuous functions belong to the class Baire*1, unlike the approximately continuous functions.

1 Introduction

The density topology is a completely regular refinement of the natural topology on the real line. It consists of all measurable subsets A of $\mathbf{R}$ such that, for every $x \in A$, x is a density point of A . Ostaszewski [6,7] studied the class of functions $f: \mathbf{R} \rightarrow \mathbf{R}$ which are continuous with respect to the density topology on the domain and the range. These are termed *density continuous*. Bijections of the real line whose inverses are density continuous were investigated by Bruckner [2] and Niewiarowski [4]. Ostaszewski [8] considered the class as a semigroup with composition as the operation, and showed that the semigroup, and three of its subsemigroups, have the inner automorphism property. Ciesielski and Larson [3] showed that real-analytic functions are density continuous, and that the class of density continuous functions is not a linear space. Furthermore, there exist C^∞ functions which are not density continuous.

*This author was partially supported by a University of Louisville research grant.

In this work we are concerned with the relationship between the classes of continuous and density continuous functions.

We will use the following notation:

$\mathbf{R}$ $\pm$ the set of real numbers;

$\mathbf{N}$ $\pm$ the set of natural numbers;

$\mathcal{C}$ $\pm$ the space of continuous functions $f: [0, 1] \rightarrow \mathbf{R}$;

$\|f\|$ $\pm$ the norm of an $f \in \mathcal{C}$, $\|f\| = \sup_{x \in [0, 1]} |f(x)|$;

$C(f)$ $\pm$ the set of points at which f is continuous;

$Z(f)$ $\pm$ the set of points at which f is not continuous;

$\omega(f, x)$ $\pm$ the oscillation of f at x ;

$\text{supp}(f) = \{x : f(x) \neq 0\}$ $\pm$ the support of f ;

$\mathcal{H}$ $\pm$ the space of all automorphisms of $[0, 1]$ equipped with the metric

$$\sigma(g, h) = \|g - h\| + \|g^{-1} - h^{-1}\|$$

for $g, h \in \mathcal{H}$;

$|A|$ $\pm$ the Lebesgue measure of a measurable set $A \subset \mathbf{R}$;

A^c $\pm$ the complement of the set A ;

$\text{int}(A)$ $\pm$ the interior of the set A ;

$\overline{d}(A, x)$, $\underline{d}(A, x)$, $d(A, x)$ $\pm$ the upper, lower, and ordinary (respectively) densities of a set $A \subset \mathbf{R}$ at a point $x \in \mathbf{R}$.

2 Typical Continuous Functions

In this section we prove that a typical continuous function is nowhere density continuous. The same is true of a typical homeomorphism of the real line. To do this, some preliminary definitions and lemmas must be presented.

Let $\{J_n\}$ be a sequence of intervals and let $\{I_n\}$ be a sequence of closed intervals such that I_n and J_n have the same center and $I_n \subset J_n$, for each n . We say that the sequence J_n *captures* the sequence I_n . This relationship between the sequences is denoted $I_n \triangleleft J_n$.

If $I_n \triangleleft J_n$, as above, we define

$$J_x = \bigcup_{\{n: x \notin J_n\}} I_n.$$

The properties of captured sequences which are useful in what follows are contained in the following two propositions.

Lemma 1 *If $\{I_n\}$ and $\{J_n\}$ are sequences of intervals such that $I_n \triangleleft J_n$ and*

$$\sum_{n \in \mathbf{N}} \frac{|I_n|}{|J_n|} < \infty,$$

then $d(J_x, x) = 0$, $\forall x \in \mathbf{R}$.

Proof. Without loss of generality, we may assume that $x \notin \bigcup_{n \in \mathbf{N}} J_n$. Let $\varepsilon \in (0, 1)$ and choose n_0 and $\delta_0 > 0$ such that

$$\sum_{n \geq n_0} \frac{|I_n|}{|J_n|} < \varepsilon/3 \text{ and } (x - \delta_0, x + \delta_0) \cap I_n = \emptyset, \forall n \leq n_0. \quad (1)$$

Observe that the choice of $\varepsilon \in (0, 1)$ and (1) guarantees that for all $n \geq n_0$, it is true that if $(x - \delta, x + \delta) \cap I_n \neq \emptyset$, then $J_n \subset (x - 3\delta, x + 3\delta)$.

Let

$$S_\delta = \{n : (x - \delta, x + \delta) \cap I_n \neq \emptyset\} \text{ and } M_\delta = \sup_{n \in S_\delta} |J_n|.$$

If $\delta \in (0, \delta_0)$, the observation and (1) show

$$\begin{aligned}
\frac{|(x - \delta, x + \delta) \cap J_x|}{2\delta} &\leq \frac{|\bigcup_{n \in S_\delta} I_n|}{2\delta} \\
&\leq 3 \frac{\sum_{n \in S_\delta} |I_n|}{|(x - 3\delta, x + 3\delta)|} \\
&\leq 3 \frac{\sum_{n \in S_\delta} |J_n| |I_n| / |J_n|}{\bigcup_{n \in S_\delta} |J_n|} \\
&\leq 3 \frac{M_\delta \sum_{n \in S_\delta} |I_n| / |J_n|}{M_\delta} \\
&< \varepsilon.
\end{aligned}$$

From this, Lemma 1 follows at once.

It is interesting to note that the following can be proved in much the same way as Lemma 1.

Corollary 1 *If I_n and J_n are sequences of intervals such that $I_n \triangleleft J_n$, $J_i \cap J_j = \emptyset$ when $i \neq j$ and $|I_n|/|J_n| \rightarrow 0$, then $\bigcup_{n=1}^\infty I_n$ is density closed.*

Lemma 2 *Let $x \in \mathbf{R}$ and let $\{L_n\}$ be a sequence of intervals such that $x \in \bigcap_{n \geq 1} L_n$ and $\lim_{n \rightarrow \infty} |L_n| = 0$. If K_n is a subinterval of L_n for every n and*

$$\limsup_{n \rightarrow \infty} |K_n|/|L_n| > 0,$$

then $\bar{d}(\bigcup_{n \geq 1} K_n, x) > 0$.

Proof. Let

$$\limsup_{n \rightarrow \infty} |K_n|/|L_n| = a > 0.$$

It will be shown that

$$\bar{d}\left(\bigcup_{n \in \mathbf{N}} K_n, x\right) = \limsup_{\delta \rightarrow 0^+} \frac{|(x - \delta, x + \delta) \cap \bigcup_{n \in \mathbf{N}} K_n|}{2\delta} \geq a/4.$$

To do this, it is enough to show that for every $\delta_0 > 0$ there is a $\delta \in (0, \delta_0]$ and a number $n \geq 1$ such that

$$|K_n \cap (x - \delta, x + \delta)|/2\delta \geq a/4.$$

Let n be such that $|L_n| < \delta_0$ and $|K_n|/|L_n| > a/2$. Set $\delta = \inf\{t : L_n \subset (x-t, x+t)\}$. Then $0 < \delta < \delta_0$, $L_n \subset [x-\delta, x+\delta]$ and $|L_n|/2\delta \geq 1/2$. Hence,

$$\frac{|K_n \cap (x-\delta, x+\delta)|}{2\delta} = \frac{|K_n|}{2\delta} = \frac{|K_n|}{|L_n|} \frac{|L_n|}{2\delta} \geq \frac{a}{2} \frac{1}{2} = \frac{a}{4}$$

and the lemma is proved.

Here is the main result of this section.

Theorem 1 *If $\mathcal{C}_{\mathcal{D}}$ denotes the subset of $\mathcal{C}$ consisting of all functions which have at least one point of density continuity, then $\mathcal{C}_{\mathcal{D}}$ is a $\mathcal{A}$ -category subset of $\mathcal{C}$.*

Proof. We will show that there exists a dense G_δ subset E of $\mathcal{C}$ such that every $f \in E$ is nowhere density continuous.

For every $n \in \mathbf{N}$ denote by D_n the set of all $f \in \mathcal{C}$ such that for every $i = 1, 2, \dots, 2^n$, f is linear and nonconstant on each interval $[(i-1)2^{-n}, i2^{-n}]$. Notice that $D_{n+1} \subset D_n$ for every $n \in \mathbf{N}$ and $D = \bigcup_{n \in \mathbf{N}} D_n$ is a dense subset of $\mathcal{C}$.

For $f \in \mathcal{C}$ define

$$\|f\|_n = \max_{i=1,2,\dots,2^n} |f(i2^{-n}) - f((i-1)2^{-n})|. \quad (2)$$

We claim that for each open set U in $\mathcal{C}$, there exists an $n \in \mathbf{N}$ and a function $f \in D_n$ such that the ball in $\mathcal{C}$ centered at f of radius $\|f\|_n$ is entirely contained in U . To see this, choose $m \in \mathbf{N}$ and an $f \in D_m$ such that $f \in U$. Since U is open, there is a $\delta > 0$ such that the open ball of radius δ centered at f is contained in U . Using the uniform continuity of f , we can find $n > m$ such that whenever $|x - y| < 2^{-n}$, then $|f(x) - f(y)| < \delta$. From this it is clear that $f \in D_n$ and $\|f\|_n < \delta$. The claim is evident.

We will now start the construction of the promised G_δ set E as an intersection of dense open sets, W_k .

Let $k \geq 1$ and U be a nonempty open subset of $\mathcal{C}$, and choose f and n as above. For $j = 0, 1, 2, \dots, 2^{n+1}$, define

$$g\left(\frac{j}{2^{n+1}}\right) = f\left(\frac{j}{2^{n+1}}\right).$$

If $i2^{-n} \leq j2^{-n-1} < (j+1)2^{-n-1} \leq (i+1)2^{-n}$, where $i \in \{0, 1, 2, \dots, 2^n - 1\}$, put $L_i = (i2^{-n}, (i+1)2^{-n})$, $M_j = (j2^{-n-1}, (j+1)2^{-n-1})$ and let $K_j = [a_j, b_j]$ be centered in M_j such that

$$\frac{|K_j|}{|M_j|} = 1 - \frac{1}{2^n} = \frac{2|K_j|}{|L_i|}.$$

Figure 1

Let us choose $I_j^0 = [c_j, d_j]$ centered in the interval $f(M_j)$ and such that

$$\frac{|I_j^0|}{|f(M_j)|} = \frac{1}{2^n}.$$

Define g to be linear on each of the intervals

$$[j2^{-n-1}, a_j], [a_j, b_j] \text{ and } [b_j, (j+1)2^{-n-1}],$$

such that $g([a_j, b_j]) = [c_j, d_j] = I_j^0$. Thus, if $J_j = f(M_j) = g(M_j)$, then

$$\frac{|g(K_j)|}{|g(M_j)|} = \frac{|I_j^0|}{|J_j|} = \frac{1}{2^n}$$

and

$$\frac{|g^{-1}(I_j^0)|}{|g^{-1}(J_j)|} = \frac{|K_j|}{|M_j|} = 1 - \frac{1}{2^n}.$$

Notice that g is contained in the open ball centered at f of radius $\|f\|_n$. Thus, $g \in U$.

Let W_U^k be the open ball centered at g of radius

$$\varepsilon_k = 2^{-n-1} \min_{i=1,2,\dots,2^n} \left| f\left(\frac{i}{2^n}\right) - f\left(\frac{i-1}{2^n}\right) \right| > 0. \quad (3)$$

Obviously $W_k = \bigcup \{W_U^k : U \text{ is open and nonempty in } \mathcal{C}\}$ is open and dense in $\mathcal{C}$, so that $E = \bigcap_{k \in \mathbf{N}} W_k$ is a residual set in $\mathcal{C}$. We will show that if $h \in E$ then h is nowhere density continuous.

Now let x be an arbitrary point of $[0, 1]$. We will choose intervals $I_m, m \in \mathbf{N}$ such that

$$d\left(\bigcup_{m \in \mathbf{N}} I_m, h(x)\right) = 0,$$

and

$$\bar{d}\left(h^{-1}\left(\bigcup_{m \in \mathbf{N}} I_m\right), x\right) > 0.$$

This will prove that h is not density continuous at x .

Let $m \in \mathbf{N}$. We have $h \in W_m$ so there exists a set U , open in $\mathcal{C}$, such that $h \in W_U^m$. Let g be the center of W_U^m . Let $n \geq m$ be the number given in the construction of W_U^m . Let $i \in \{0, 1, 2, \dots, 2^n - 1\}$ be such that $x \in [i2^{-n}, (i+1)2^{-n}]$.

Put $L_m = [i2^{-n}, (i+1)2^{-n}]$. Let $M^1 = (2i2^{-n-1}, (2i+1)2^{-n-1})$, $M^2 = ((2i+1)2^{-n-1}, 2(i+1)2^{-n-1})$ and $M_m \in \{M^1, M^2\}$ such that $h(x) \notin g(M_m)$.

Put $J_m = g(M_m)$ and let $I_m^0 = [c_j, d_j]$ and $K_m = [a_j, b_j]$ be as in the construction of g . Thus

$$\frac{|I_m^0|}{|J_m|} = 1/2^n \leq 1/2^m \text{ and } \frac{|K_m|}{|M_m|} = 1 - 1/2^n \geq 1 - 1/2^m.$$

Put $I_m = [c_j - \varepsilon_m, d_j + \varepsilon_m]$. As $h(x) \notin J_m$ for every m and

$$\sum_{m=1}^{\infty} \frac{|I_m|}{|J_m|} \leq \sum_{m=1}^{\infty} \frac{3|I_m^0|}{|J_m|} \leq 3 \sum_{m=1}^{\infty} \frac{1}{2^m} = 3,$$

Lemma 1 yields $d(\bigcup_{m=1}^{\infty} I_m, h(x)) = 0$.

On the other hand, by the choice of ε_m , $K_m \subset h^{-1}(I_m)$. Thus, by Lemma 2, the fact that $x \in L_m$ for every m and using

$$\lim_{m \rightarrow \infty} \frac{|K_m|}{|L_m|} = \lim_{m \rightarrow \infty} \frac{|K_m|}{2|M_m|} = 1/2 > 0$$

we have

$$\bar{d}(h^{-1}(\bigcup_{m=1}^{\infty} I_m), x) \geq \bar{d}(\bigcup_{m=1}^{\infty} K_m, x) > 0.$$

Therefore, h is not density continuous at x .

Theorem 2 *If $\mathcal{H}_{\mathcal{D}}$ denotes the class of all elements of $\mathcal{H}$ which have at least one point of density continuity, then $\mathcal{H}_{\mathcal{D}}$ is a $\mathcal{AE}$ category subset of $\mathcal{H}$.*

Proof. As discussed in [9, page 50], $\mathcal{H}$ is a G_{δ} subset of $\mathcal{C}$. It is actually complete with the metric σ defined in the introduction of this work.

Let W be the dense G_{δ} subset of $\mathcal{C}$ constructed in the proof of Theorem 1. It is obvious that $W \cap \mathcal{H}$ is a G_{δ} subset of $\mathcal{H}$. Thus, it would be sufficient to show that $W \cap \mathcal{H}$ is dense in $\mathcal{H}$ in order to prove Theorem 2.

Unfortunately, in general, this is not the case. However, the set $D \cap \mathcal{H}$ is dense in $\mathcal{H}$. Thus, if in the choice of f in the proof of Theorem 1, we assume additionally that for nonempty $U \cap \mathcal{H}$ we choose $f \in U \cap \mathcal{H} \cap D$, then the corresponding function g will be also in $\mathcal{H}$. $\mathcal{H} \cap W$ will be dense in W . This proves Theorem 2.

Let us note that the fact that a typical homeomorphism is not density continuous is mentioned in [6], but without a detailed proof.

3 Continuity of Density Continuous Functions

In this section, the set on which a density continuous function can be continuous is characterized as any nowhere dense F_σ set.

A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is in the class Baire*1 if for each perfect set P , there is a portion Q of P such that $f|_Q$ is continuous. In other words, f is continuous on a relative subinterval of each closed set. This class was introduced by Richard O'Malley [5], who studied the Baire*1 functions having the Darboux property.

Theorem 3 *If f is a density continuous function, then f is in Baire*1.*

Proof. We assume the theorem is not true. Then there is a nonempty perfect set P such that

$$Z = \{x \in P : f|_P \text{ is not continuous at } x\}$$

is dense in P . We will show that this assumption assures that there is an $x \in P$ such that f is not density continuous at x . The proof uses induction to find a sequence $x_n \in P$ a sequence of open intervals (a_n, b_n) and two sequences of compact intervals, I_n and J_n such that $x_n \in I_n \subset J_n$, $I_n \triangleleft J_n$ and $x_n \rightarrow x$.

To start, let $x_0 \in Z$, $J_0 = I_0 = \emptyset$ and $(a_0, b_0) = (x_0 - 1, x_0 + 1)$. Assume that x_i , closed intervals J_i and I_i , and an open interval (a_i, b_i) have been chosen for $1 \leq i \leq n$ to satisfy the following properties:

- (a) $f(x_i) \in I_i \subset J_i$;
- (b) $J_{i-1} \cap J_i = \emptyset$;
- (c) $0 < |I_i| \leq |J_i|/2^i$ and $|J_i| < \omega(f|_P, x_i)$;
- (d) $x_i \in (a_i, b_i) \cap Z \subset [a_i, b_i] \subset (a_{i-1}, b_{i-1})$;
- (e) $b_i - a_i < 1/2^i$; and,
- (f) $|f^{-1}(I_i) \cap (a_i, b_i)| > (1 - 2^{-i})(b_i - a_i)$.

To continue with the inductive step, we note that from (c), we are able to choose

$$y \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n).$$

If $y \in Z$, then let $x_{n+1} = y$. Otherwise, $f|_P$ is continuous at y . In this case, the fact that Z is dense in P guarantees the existence of

$$x_{n+1} \in P \cap f^{-1}(J_n^c) \cap (a_n, b_n) \cap Z.$$

Because J_n is closed and $x_{n+1} \in Z$, there is a closed interval J_{n+1} centered at $f(x_{n+1})$ such that $J_{n+1} \cap J_n = \emptyset$ and $0 < |J_{n+1}| < \omega(f|_P, x_{n+1})$. Setting I_{n+1} to be the closed interval centered at $f(x_{n+1})$ with length $|J_{n+1}|/2^{n+1}$, it follows that (a), (b) and (c) are true with $i = n+1$. Next, use the approximate continuity of f at x_{n+1} to find an interval $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$ containing x_{n+1} such that (d), (e) and (f) are satisfied. The induction is complete.

From (d) and (e) we see that there is an $x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \cap P$. We claim that there is a subsequence J_{n_m} of J_n such that $f(x) \notin J_{n_m}$ for every m . Otherwise, $f(x)$ is contained in all but a finite number of the J_n , which is easily seen to violate (b). From (c) and the construction, it follows that

$$\sum_{m=1}^{\infty} \frac{|I_{n_m}|}{|J_{n_m}|} < \infty \text{ and } I_{n_m} \triangleleft J_{n_m},$$

so Lemma 1 implies that $d(\bigcup_{m=1}^{\infty} I_{n_m}, f(x)) = 0$. The density continuity of f now implies that

$$d(f^{-1}(\bigcup_{m=1}^{\infty} I_{n_m}), x) = 0. \quad (4)$$

On the other hand, $x \in (a_{n_m}, b_{n_m})$ for all m , so (f) implies

$$\bar{d}(f^{-1}(\bigcup_{m=1}^{\infty} J_{n_m}), x) \geq \lim_{m \rightarrow \infty} \frac{|f^{-1}(I_{n_m}) \cap (a_{n_m}, b_{n_m})|}{|(a_{n_m}, b_{n_m})|} = 1. \quad (5)$$

But, (4) and (5) contradict each other, so we are forced to conclude that Z cannot be dense in P , which finishes the proof.

It is evident from the definition of Baire*1 that if f is in Baire*1, then $C(f)$ contains a dense open set. Because $C(f)$ is a G_{δ} set, we have proved that a density continuous function can be discontinuous on at most a nowhere dense F_{σ} set. The converse to this statement is also true.

Theorem 4 *If $\mathcal{Z} = \{Z(f) : f \text{ is density continuous}\}$, then*

$$\mathcal{Z} = \{F : F \text{ is a nowhere dense } F_{\sigma} \text{ set}\}.$$

In order to prove this theorem, it suffices to show that given an arbitrary nowhere dense F_σ set F , a density continuous function f can be constructed such that $Z(f) = F$. In order to do this, two lemmas are needed.

Lemma 3 *If F is a nowhere dense F_σ set, then there exist sequences of pairwise disjoint compact intervals, I_n and J_n with $I_n \triangleleft J_n$ such that*

$$F \subset \overline{\bigcup_{n \in \mathbf{N}} I_n} \setminus \bigcup_{n \in \mathbf{N}} J_n.$$

Moreover, if $F = \bigcup_{n \in \mathbf{N}} F_n$, where F_n is closed for each n , then there are disjoint subsequences m_k^n from $\mathbf{N}$ such that

$$F_n = \overline{\bigcup_{k \in \mathbf{N}} I_{m_k^n}} \setminus \bigcup_{k \in \mathbf{N}} J_{m_k^n}.$$

Proof. Let $\{(a_n, b_n) : n \in \mathbf{N}\}$ be the components of $\overline{F}^c$. For each n , choose a decreasing sequence $\{x_i^n\} \subset (a_n, b_n)$ such that $\lim_{i \rightarrow \infty} x_i^n = a_n$. The set $\{x_i^n : i, n \in \mathbf{N}\}$ is discrete, so it can be enumerated as a sequence y_i . Let J_i be a sequence of pairwise disjoint closed intervals such that J_i is centered at y_i and let I_i be a closed interval centered in J_i such that $|I_i|/|J_i| = 2^{-i}$. Then $I_n \triangleleft J_n$ and

$$F \subset \overline{F} = \overline{\{a_i : i \in \mathbf{N}\}} = \overline{\{y_i : i \in \mathbf{N}\}} \setminus \{y_i : i \in \mathbf{N}\} = \overline{\bigcup_{i \in \mathbf{N}} I_i} \setminus \bigcup_{i \in \mathbf{N}} J_i.$$

The second part of the lemma follows easily by choosing appropriate subsequences of y_i .

Lemma 4 *Let F be a closed nowhere dense set, $\lambda > 0$ and suppose that I_n and J_n are sequences of compact intervals such that $I_n \triangleleft J_n$ and the J_n are pairwise disjoint. If $F = \overline{\bigcup_{n \in \mathbf{N}} I_n} \setminus \bigcup_{n \in \mathbf{N}} J_n$, then there exists a density continuous function $f : \mathbf{R} \rightarrow [0, \lambda]$ such that*

- (a) $Z(f) = F$,
- (b) $\omega(f, x) = \lambda, \forall x \in F$, and
- (c) $f^{-1}((0, \lambda]) = \bigcup_{n \in \mathbf{N}} \text{int}(I_n)$.

Proof. Let

$$f_n(x) = \begin{cases} 0 & x \notin I_n \\ 2\lambda \text{dist}(x, I_n^c)/|I_n| & x \in I_n \end{cases}$$

and

$$f(x) = \sum_{n \in \mathbf{N}} f_n.$$

The disjointness of the J_n and the fact that $I_n \subset \text{int}(J_n)$ for all n guarantees that (a), (b) and (c) are true. To see that f is density continuous, there are two cases to consider. First, suppose that $x \in I_n$, for some n . In this case, the definitions of f_n and the fact that the J_n are pairwise disjoint guarantee that f is piecewise linear on some neighborhood of x . So, f is density continuous at x . Second, if x is in no I_n , then (c) implies that $f(x) = 0$. Using (c) again, along with Corollary 1, it follows that $f = 0$ on a density open neighborhood of x . This implies that f is density continuous at x .

We now proceed with the proof of Theorem 4.

Let F be as in the statement of the theorem. Suppose $F = \bigcup_{n \in \mathbf{N}} F_n$, where F_n is closed and $F_n \subset F_{n+1}$ for $n \geq 1$. Let $I_n \triangleleft J_n$ and the sequences m_k^n be as in Lemma 3. For each n , use Lemma 4 with $\lambda = 3^{-n}$ and the pair of intervals $I_{m_k^n} \triangleleft J_{m_k^n}$ to construct a function f_n . Define

$$f = \sum_{n \in \mathbf{N}} f_n. \quad (6)$$

We see that (6) converges uniformly. Because of this, part (a) of Lemma 4 yields $Z(f) \subset F$. On the other hand, if $x_0 \in F$, then $f(x_0) = 0$ and $x \in F_n$ for some n . It follows that

$$\limsup_{x \rightarrow x_0} f(x) \geq \limsup_{x \rightarrow x_0} f_n(x) = 3^{-n} > f(x), \quad (7)$$

so $F = Z(f)$.

Since $f = 0$ on $(\bigcup_{n \in \mathbf{N}} I_n)^c$, Corollary 1 implies that f is density continuous on that set. If $x \in I_n$ for some n , then the fact that $\text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset$ whenever $m \neq n$ shows that there is a neighborhood G of x such that $f = f_n$ on G . The density continuity of f_n at x implies the density continuity of f at x . Therefore, f is a density continuous function.

The structure of $Z(f)$ for an approximately continuous function f is well-known. (See, e.g. Bruckner [2, page 48].) But, the proof of Theorem 4 can be used to give an alternative proof of this characterization.

Theorem 5 *If $Z = \{Z(f) : f \text{ is approximately continuous}\}$ then $Z = \{F : F \text{ is } F_\sigma \text{ and } \mathcal{A}\text{-category}\}$.*

Proof. Since approximately continuous functions are continuous on a dense G_δ set, we see

$$Z \subset \{F : F \text{ is } F_\sigma \text{ and } \mathcal{A}\text{-category}\}.$$

Let F be a $\mathcal{A}$ -category F_σ set and suppose $F = \bigcup_{n \in \mathbf{N}} F_n$, where each F_n is closed and nowhere dense with $F_n \subset F_{n+1}$, $\forall n \in \mathbf{N}$. The functions f_n and f can be defined in the proof of Theorem 4. Density continuous functions are approximately continuous and the uniform limit of approximately continuous functions is approximately continuous. Therefore, f is approximately continuous.

As before, it is clear that $Z(f) \subset F$. To establish the opposite containment, we note that if $x \in F_{n+1} \setminus F_n$, then

$$\omega(f_{n+1}, x) = \omega\left(\sum_{i \leq n+1} f_i, x\right) = 1/3^{n+1} > \sum_{i > n+1} f_i,$$

so $x \in Z(f)$ and the theorem follows.

References

- [1] Bruckner, A.M., Density-preserving homeomorphism and the theorem of Maximoff, *Quart. J. Math. Oxford*, (2)**21** (1970), 337-347.
- [2] Bruckner, A.M., *Differentiation of Real Functions*, Lecture Notes in Mathematics, 659, Springer-Verlag, 1978.
- [3] Ciesielski, K., and Larson, L., The space of density continuous functions, submitted.
- [4] Niewiarowski, J., Density-preserving homeomorphisms, *Fund. Math.*, **106** (1980), 77-87.
- [5] O'Malley, R. J., Baire*1Darboux functions, *Proc. Amer. Math. Soc.*, **60** (1976), 187-192.
- [6] Ostaszewski, K., Continuity in the density topology, *Real Anal. Exchange*, **7**(2) (1982), 259-270.
- [7] Ostaszewski, K., Continuity in the density topology II, *Rend. Circ. Mat. Palermo*, (2)**32** (1983), 398-414.
- [8] Ostaszewski, K., Semigroups of density-continuous functions, *Real Anal. Exch.*, to appear.
- [9] Oxtoby, J.C., *Measure and Category*, Springer, 1971.

Krzysztof Ciesielski, Department of Mathematics, Methodist College, Fayetteville, NC 28301.

Lee Larson, Department of Mathematics, University of Louisville, Louisville, KY 40292

Krzysztof Ostaszewski, Department of Mathematics, University of Louisville, Louisville, KY 40292