2002

Whitney preserving maps

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Dissertation submited to the

Eberly College of Arts and Sciences

at West Virginia University

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

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2002

Keywords: arc component, arcwise connected continuum, continuous decomposition, continuum, hereditarily irreducible map, hereditarily weakly confluent map, hereditarily Whitney preserving map, hyperspace, induced map, order arc, strictly Whitney preserving map, weakly confluent map, Whitney level, Whitney map, Whitney preserving map

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ABSTRACT

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We connect Whitney levels and continuous functions between continua to obtain the new notion of Whitney preserving maps. We introduce the basic properties of Whitney preserving maps. We give conditions on a continuum X in order that a Whitney preserving map f from X to the unit interval is a homeomorphism, and we give examples to show the conditions are necessary and that the result is false when the range of the map is the unit circle. Concerning the structure of any continuum X, we show that if $\mathcal{A}$ is a continuous decomposition of X into nondegenerate terminal continua, then there is a Whitney preserving map from X to the quotient space $X/\mathcal{A}$. We also show that if $f : X \longrightarrow Y$ is a Whitney preserving map, then the highest Whitney level of $C(X)$ mapping to the zero Whitney level of $C(Y)$ is a continuous decomposition of X into terminal continua. We introduce the notion of a strictly Whitney preserving map; we show that being strictly Whitney preserving is equivalent to being hereditarily irreducible when the map is weakly confluent.

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ACKNOWLEDGMENTS

I thank the members of my committee: Professor Krzysztof Ciesielski, Professor Jack Goodykoontz, Professor Jerzy Wojciechowski, Professor Wathiq Abdul-Razzaq for all their comments and suggestions which made the dissertation readable. I also thank them for taking the time to read the dissertation, despite they got a first draft late March.

I specially thank Prof. Nadler for all the help and knowledge he provided me during the realization of this dissertation. I also thank him for all the time he spent with me discussing the work in the dissertation.

I thank all my friends and my family in Mexico who have always supported me. I also thank my officemates: Jorge Martinez and Likin Simon.

Finally, I thank CONACyT for the economic support provided during my studies.

I dedicate this dissertation to Larissa.

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Contents

1 Introduction **1**

2 Preliminaries **5**

3 Whitney Preserving Maps **15**

4 Maps onto the interval **27**

5 Maps onto S^1 **41**

6 Maps onto the interval, revisited **51**

7 Questions **57**

Chapter 1

Introduction

Whitney maps and their levels sets, called Whitney levels, are perhaps the most important tools in the study of the structure of hyperspaces. Intuitively, a Whitney map assigns to each subcontinuum of a given continuum its size relative to other subcontinua. The set of subcontinua which have the same "size" (under a Whitney map) are the Whitney levels.

Given a continuous function between two continua X and Y, there exists a natural map from the hyperspace $C(X)$ to the hyperspace $C(Y)$ called the induced map. By connecting continuous functions with Whitney levels, one can formulate some interesting questions: What do the images of the Whitney levels in $C(X)$ look like in $C(Y)$ under the induced map? Or, must a map be a homeomorphism whenever the induced map maps Whitney levels of $C(X)$ onto Whitney levels of $C(Y)$?

In line with the second question, we have developed the new notion of Whitney preserving maps. In general terms, a Whitney preserving map is a map such that the induced map maps Whitney levels onto Whitney levels. Since some maps are not Whitney preserving (Example 3.3), the class of Whitney preserving maps is a distinguished class of maps.

Whitney preserving maps can be used to study the structure of the Whitney levels for some Whitney map. For example, assume that $\mathcal{P}$ is a property (preserved under continuous functions) such that all the positive Whitney levels of a given continuum X with respect to a given Whitney map μ have property $\mathcal{P}$; then every Whitney level of every image of X under a map that is Whitney preserving with respect to μ will have property $\mathcal{P}$. This dissertation is concerned with the problem of when there is a Whitney preserving map of a continuum X to a continuum Y.

This dissertation is in six basic parts. In the first part (Chapter 2), we give the basic theory of hyperspaces that we will need to develop our theory.

In the second part (Chapter 3), we introduce the notion of Whitney preserving maps, we establish their basic properties, and we show that for weakly confluent maps, being hereditarily irreducible is equivalent to being strictly Whitney preserving (Theorem 3.19). This implies that the induced map of a strictly Whitney preserving map is light. We also show that if a continuum X is connected im kleinen at some point or if X is hereditarily continuumwise accessible, then only the Whitney level of singletons maps onto the Whitney level of singletons under a Whitney preserving map (Theorem 3.10). This result implies the following: assume that Y is a continuum of dimension 1 and that either X is a continuum which is connected im kleinen at some point or it is hereditarily continuumwise accessible. If there is a Whitney preserving map of X onto Y, then the dimension of X is 1.

In the next part of the dissertation (Chapter 4), we consider Whitney preserving maps whose range is the unit interval I and we prove the following important result (Theorem 4.10): Let X be a continuum such that X contains a dense arc component. If $f : X \longrightarrow I$ is Whitney preserving, then f is a homeomorphism.

In the fourth part (Chapter 5), we give a brief discussion about Whitney preserving maps onto the unit circle; in particular, we show that a generalization of Theorem 4.10 is not possible.

In the fifth part (Chapter 6) we again consider Whitney preserving maps onto the unit interval and we give an example that shows the necessity of the hypothesis in Theorem 4.10. In contrast to Theorem 3.10 and in connection to the structure of any continuum X, we show in Theorem 6.4 that the highest Whitney level that is mapped to the singletons is a continuous decomposition of X into terminal continua.

Finally, we raise questions related to our work (Chapter 7).

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Chapter 2

Preliminaries

In this chapter we introduce the general properties of continua and their hyperspaces. The properties we present are needed to understand the notion of a Whitney preserving map. Some of the propositions and theorems in this chapter will be stated without proof; the proofs can be found in [7], [13], or [14].

We use standard terminology nevertheless, we mention that by a neigborhood of a point x in a space X we mean a set with no empty interior containing x in its interior.

Notation. The following symbols will be used throughout this work.

1. The symbol $\mathbb{N}$ stands for the set of natural numbers $\{1, 2, 3, \ldots\}$.

2. The symbol I denotes the unit interval $[0, 1]$.

3. The symbol S^1 denotes the unit circle in the real plane $\mathbb{R}^2$ given by $S^1 = \{x \in \mathbb{R}^2 : ||\mathrm{x}|| = 1\}$.

Let A and B be sets.

3. $A \backslash B = \{a \in A : a \notin B\}$.

4. The symbol $\overline{A}$ stands for the topological closure of A.

5. The symbol $\mathrm{int}(A)$ denotes the topological interior of A.

Let $f : X \longrightarrow Y$ be a function, and let $A \subset X$.

6. The symbol $f_{|A}$ will denote the restriction of f to the set A, *i.e.*, the function $f_{|A} : A \longrightarrow Y$.

Definition 2.1. A metric space X is said to be a *continuum* provided X is nonempty, compact, and connected. The capital letters X and Y always denote continua unless otherwise stipulated.

Given a continuum X, we define the *hyperspace* 2^X, also known as the hyperspace of compact subsets of X, and the *hyperspace* $C(X)$, also known as the hyperspace of subcontinua of X, as follows:

$$2^X = \{A \subset X \ : \ A \text{ is nonempty and compact}\}$$

and

$$C(X) = \{A \in 2^X \ : \ A \text{ is connected}\}.$$

The topology of these hyperspaces is obtained from the *Hausdorff metric* induced by the metric on X. We define the Hausdorff metric as follows:

Let d denote the metric for X. If $\varepsilon > 0$ and $A \in 2^X$, then

$$N_d(\varepsilon, A) = \{x \in X \ : \ d(x,a) < \varepsilon \ \text{ for some } \ a \in A\}.$$

Now, for $A, B \in 2^X$, we define the *Hausdorff distance* between A and B by

$$H_d(A, B) = \inf\{\varepsilon > 0 \ : \ A \subset N_d(\varepsilon, B) \ \text{ and } \ B \subset N_d(\varepsilon, A)\}.$$

The Hausdorff metric for $C(X)$ is the metric that $C(X)$ inherits as a subspace of 2^X.

For some continua is possible to construct a geometric representation of $C(X)$. For example, if X is the unit interval $[0, 1]$ and let A be the triangular region in the plane given by

$$A = \{(a, b) \in \mathbb{R}^2 : 0 \leq \mathrm{a} \leq \mathrm{b} \leq 1\},$$

then the function $\phi : C(X) \longrightarrow A$, given by $\phi([a, b]) = (a, b)$, is a homeomorphism. Figure 2 shows the set A; notice that the line $y = x$ corresponds under ϕ to the subcontinua of X that consist of only one point, and the point $(0, 1)$ corresponds to the point X of $C(X)$.

Definition 2.2. Let $f : X \longrightarrow Y$ be a continuous function between continua. We define the *induced map* $\hat{f} : C(X) \longrightarrow C(Y)$ by letting, for each A in $C(X)$,

$$\hat{f}(A) = \{f(a) : a \in A\}.$$

It follows from the continuity of f that $\hat{f}$ is continuous.

In the following definition we introduce three kinds of maps.

Definition 2.3. Let X and Y be continua and let $f : X \longrightarrow Y$ be an onto continuous function.

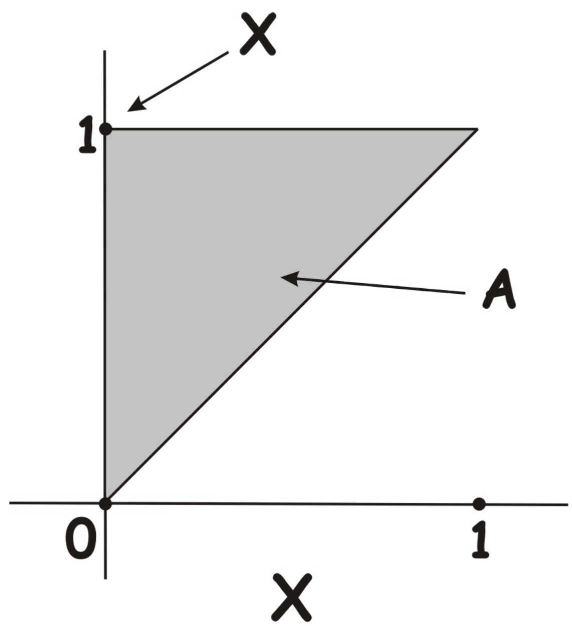

Figure 2.1: Geometric representation of $C(X)$ when $X = [0,1]$

(1) f is *weakly confluent* provided that for each subcontinuum B of Y there exists a component A of $f^{-1}(B)$ such that $f(A) = B$; f is *hereditarily weakly confluent* if $f_{|K} : K \longrightarrow f(K)$ is weakly confluent for every subcontinuum K of X.

(2) f is *open* provided that $f(U)$ is an open subset of Y for each open subset U of X.

(3) f is *confluent* provided that for each subcontinuum K of Y and each component A of $f^{-1}(K)$, we have $f(A) = K$

Weakly confluent maps play a natural role in the theory of hyperspaces because of the following easy-to-prove result: If $f : X \longrightarrow Y$ is a continuous function between continua, then $\hat{f} : C(X) \longrightarrow C(Y)$ is a surjection if and only if f is weakly confluent (see [13, 0.49.1]). Weakly confluent maps are important for this work since, as we show in Theorem 3.4, Whitney preserving maps are weakly confluent.

Definition 2.4. Let X be a continuum. We say that a continuous function $\mu : C(X) \longrightarrow \mathbb{R}$ is called a *Whitney map* if satisfies the following two conditions:

(i) if $A, B \in C(X)$ such that $A \subset B$ and $A \neq B$, then $\mu(A) < \mu(B)$;

(ii) $\mu(\{x\}) = 0$ for all $x \in X$.

It is not difficult to prove that for the hyperspace of subintervals of the unit interval, the assignment of the length of an interval to the interval is a Whitney function. It is a well-known fact that for any continuum X, there exists a Whitney map for $C(X)$ (see 0.50 of [13, p. 25]).

Notation. From now on, if $\mu : C(X) \longrightarrow \mathbb{R}$ is a Whitney map and Z is a proper subcontinuum of X, μ_Z will denote the restriction of μ to $C(Z)$ (*i.e.*, $\mu_{|C(Z)}$).

Definition 2.5. Let X be a continuum and let μ be a Whitney map for $C(X)$. For every $0 \leq s < \mu(X)$, $\mu^{-1}(s)$ is called a *Whitney level of* $C(X)$. We will say that $\mu^{-1}(s)$ is a *positive Whitney level* if $s > 0$.

Assume μ is a Whitney map for some $C(X)$. Then, since every $s \in (0, \mu(X))$ is a cut point of $[0, \mu(X)]$, every positive Whitney level $\mu^{-1}(s)$

separates $C(X)$. Furthermore, for each $s \in [0, \mu(X)]$, $\mu^{-1}(s)$ is a continuum (see 19.9 of [7, p. 160]).

Intuitively, a Whitney function can be thought as a "measure" of the relative sizes of subcontinua of X. A Whitney level is then the set of all subcontinua with the same "measure". Figure 2 shows the Whitney level $\mu^{-1}(\frac{1}{2})$ in $C(X)$ in the case when X is the unit interval and μ is the Whitney function that assigns to each subcontinuum of X its length.

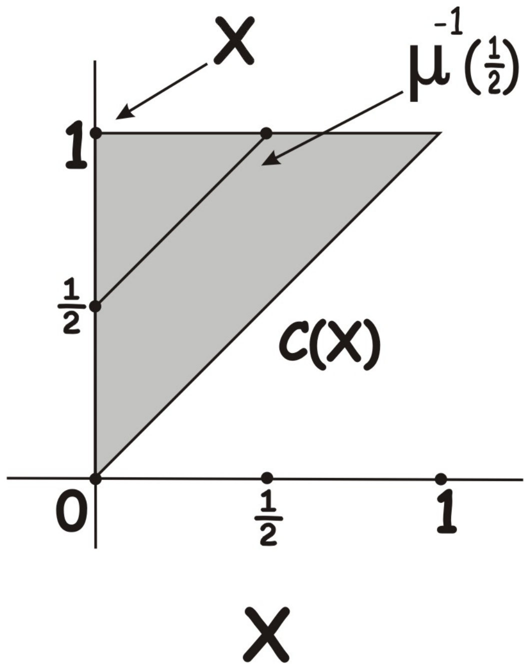

Figure 2.2: $\mu^{-1}(\frac{1}{2})$

Definition 2.6. By and *arc* we will mean a homeomorphism h of the interval $[0, 1]$, or we will mean the range $h([0, 1])$ of such a homeomorphism. When it is necessary to know which is meant, the context will make it clear.

An *order arc in* $C(X)$ is an arc α in $C(X)$ such that if $A, B \in \alpha$, then $A \subset B$ or $B \subset A$.

It is an immediate consequence of [13, (1.8)] that for any two subcontinua A and B of a continuum X, there is an order arc from A to B if and only if $A \subset B$. This gives us that $C(X)$ is arcwise connected.

We will use the next lemma several times; basically, the lemma says that if α is an order arc from A to B, then α intersects every Whitney level between the Whitney levels that contain A and B.

Lemma 2.7. *Let X be a continuum, let $\mu : C(X) \longrightarrow \mathbb{R}$ be a Whitney map for $C(X)$, and let A and B be two proper subcontinua of X such that $\mu(A) < \mu(B)$. If α is an order arc from A to B, then $\alpha \cap \mu^{-1}(s) \neq \emptyset$ for all $s \in (\mu(A), \mu(B))$.*

Proof. Let $s \in (\mu(A), \mu(B))$. Then s is a cut point of $[0, \mu(X)]$, since s is a cut point of $[\mu(A), \mu(B)]$. Therefore, $\mu^{-1}([0, s))$ and $\mu^{-1}((s, \mu(X)])$ are disjoint. Furthermore, $C(X)\setminus\mu^{-1}(s) = \mu^{-1}([0, s)) \cup \mu^{-1}((s, \mu(X)])$. Now, if $\mu(A) < s < \mu(B)$, then $A \in \mu^{-1}([0, s))$ and $B \in \mu^{-1}((s, \mu(X)])$. So, since α is connected and $A \in \alpha$ and $B \in \alpha$ we have that $\alpha \cap \mu^{-1}(s) \neq \emptyset$. □

Corollary 2.8. *Let X be a continuum, let $\mu : C(X) \longrightarrow \mathbb{R}$ be a Whitney map for $C(X)$, and let $s \in (0, \mu(X)]$. If A is a subcontinuum of X such that either $\mu(A) < s$ or $s < \mu(A)$, then there exists a subcontinuum C of X such that either $A \subset C$ or $C \subset A$, respectively, and $\mu(C) = s$.*

Proof. Let $x \in A$ and let α be an order arc from $\{x\}$ to X passing through A. We can find such an α by taking first an order arc from $\{x\}$ to A and then joining it to an order arc from A to X (see comment following Definition 2.6). Then $\alpha \cap \mu^{-1}(s) \neq \emptyset$ by Lemma 2.7. Hence, $A \subset C$ (or $C \subset A$ depending on the case) for $C \in \alpha \cap \mu^{-1}(s)$. □

Proposition 2.9. *Let X be a nondegenerate continuum and let $\mu : C(X) \longrightarrow \mathbb{R}$ be a Whitney map for $C(X)$. Then $\mu^{-1}(s)$ is a nondegenerate continuum for every $s \in [0, \mu(X))$.*

Proof. Let X be a continuum, let $\mu : C(X) \longrightarrow \mathbb{R}$ be a Whitney map for $C(X)$, and let $\mu^{-1}(s)$ be a Whitney level with $s < \mu(X)$. By 19.9 of [7, p. 160], $\mu^{-1}(s)$ is a continuum. So, we need only to show that $\mu^{-1}(s)$ contains more than one point. Let $A \in \mu^{-1}(s)$. Then A is a proper subcontinuum of X, since $\mu(A) < \mu(X)$. Let $x \in X \setminus A$ and let α be an order arc from $\{x\}$ to X. Then, by Lemma 2.7, there exists a $B \in C(X)$ such that $B \in \alpha \cap \mu^{-1}(s)$. Note that B is different from A since $x \in B$ and $x \notin A$. Hence, $\mu^{-1}(s)$ is nondegenerate. □

The *cone over* X is the decomposition space of the upper semi-continuous decomposition $(X \times [0,1])/(X \times \{1\})$ of $X \times [0,1]$ obtained by shrinking $X \times \{1\}$ to a point.

It is customary to think of the hyperspace $C(X)$ as a cone over X, where the base represents the set of all one-point subcontinua of X and the vertex represents the point X. However, the hyperspaces of most continua are not homeomorphic to cones; thus, thinking of hyperspaces as cones is merely intuitive. Nevertheless, it provides insightful intuition.

Definition 2.10. Let X be a metric space. A collection $\mathcal{A}$ of nonempty, compact subsets of X is called a *set theoretic decomposition of* X provided the elements of $\mathcal{A}$ are mutually disjoint and $\bigcup \mathcal{A} = X$.

Definition 2.11. Let X be a continuum. A nondegenerate subcontinuum K of X is called a *convergence continuum* (of X), provided that there is a sequence $\{K_i\}_{i \in \mathbb{N}}$ of subcontinua of X such that

$$K = \lim K_i \text{ and } K \cap K_i = \emptyset \text{ for every } i.$$

It is not difficult to prove that if a continuum X does not contain a convergence continuum, then X is locally connected .

Definition 2.12. Let X be a continuum and let p be any point of X. The set

$$\kappa(p) = \{x \in X : \text{ there is a proper subcontinuum } K \text{ of } X \text{ such that } p, x \in K\}$$

is called a *composant* of X.

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Chapter 3

Whitney Preserving Maps

We introduce the notion of Whitney preserving maps and some stronger notions. We give some basic properties for these notions. In particular, hereditarily Whitney preserving maps play an important role in the next chapter.

Definition 3.1. Let $f : X \longrightarrow Y$ be a continuous function between continua. Then f is said to be *Whitney preserving* if there are Whitney maps $\mu : C(X) \longrightarrow \mathbb{R}$ and $\nu : C(Y) \longrightarrow \mathbb{R}$ such that for every real number $s \in [0, \mu(X)]$, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(t)$ for some $t \in [0, \nu(Y)]$. When we want to be more specific, we will say that f is μ, ν-Whitney preserving.

In order to simplify notation, from now on whenever we say that $f : X \longrightarrow Y$ is a Whitney preserving map we will assume that f is μ, ν-Whitney preserving, where μ and ν are the Whitney maps for which f is Whitney preserving (unless otherwise stipulated).

Lemma 3.2. *Let $f : X \longrightarrow Y$ be a Whitney preserving map. Then f is surjective and $\hat{f}(\mu^{-1}(0)) = \nu^{-1}(0)$.*

Proof. Let $f : X \longrightarrow Y$ be a Whitney preserving map. Let $y \in Y$. For each $x \in X$, $\hat{f}(\{x\}) = \{f(x)\} \in \nu^{-1}(0)$. Hence, $\hat{f}(\mu^{-1}(0)) \subset \nu^{-1}(0)$. By definition, there exists $t \in [0, \nu(Y)]$ such that $\hat{f}(\mu^{-1}(0)) = \nu^{-1}(t) \subset \nu^{-1}(0)$. It follows that $t = 0$ and $\hat{f}(\mu^{-1}(0)) = \nu^{-1}(0)$. Therefore, there exists $x \in X$ such that $f(x) = y$. □

The following example shows that the converse of Lemma 3.2 is false.

Example 3.3. Let $X = [0, \pi]$, $Y = S^1$ and let $f : X \longrightarrow Y$ be given by $f(t) = e^{2ti}$. Then f is an onto continuous function which is not Whitney preserving (by the next theorem).

Theorem 3.4. *If $f : X \longrightarrow Y$ is Whitney preserving, then f is weakly confluent.*

Proof. Since f is surjective, by Lemma 3.2, $\hat{f}(X) = Y$ and $\hat{f}(\mu^{-1}(0)) = \nu^{-1}(0)$. Hence, by connectness, $\hat{f}(C(X))$ intersects every Whitney level $\nu^{-1}(t)$ (see comment following Definition 2.5). Thus, since f is Whitney preserving, it follows that every Whitney level $\nu^{-1}(t)$ is the image under $\hat{f}$ of a Whitney level for $C(X)$. Therefore, $\hat{f}(C(X)) = C(Y)$, which shows f is weakly confluent (see comment following Definition 2.2). □

Remark. In Proposition 6.5 we will show that the converse of Theorem 3.4 is false.

Corollary 3.5. *Let Y be a nondegenerate continuum. If $f : I \longrightarrow Y$ is a Whitney preserving map, then Y is an arc or a simple closed curve.*

Proof. By Theorem 3.4 f is weakly confluent. Then, by II.3 of [4], Y is either an arc or a simple closed curve. □

Lemma 3.6. *Let X be a continuum and let $\mu : C(X) \longrightarrow \mathbb{R}$ be a Whitney map for $C(X)$. If $\mu^{-1}(s)$ is a Whitney level that is a set theoretic decomposition of X and $A \in \mu^{-1}(s)$, then $int(A) = \emptyset$.*

Proof. Suppose the Lemma is false. Then there exists an $A \in \mu^{-1}(s)$ such that $\text{int}(A) \neq \emptyset$. Let $x_0 \in \text{int}(A)$ and let $\epsilon > 0$ such that $N_d(\varepsilon, \{x_0\}) \subset \text{int}(A)$. By Proposition 2.9, there exists a $B \in \mu^{-1}(s)$ such that $H_d(A, B) < \varepsilon$ and $B \neq A$. Then, by the definition of the Hausdorff metric, $A \subset N_d(\varepsilon, B)$; this means, in particular, that there exists $y \in B$ such that $d(x_0, y) < \varepsilon$. Hence, $y \in N_d(\varepsilon, \{x_0\})$. Thus, $y \in A$. Therefore, $A \cap B \neq \emptyset$. However, this contradicts the fact that $\mu^{-1}(s)$ is a set theoretic decomposition of X. □

Definition 3.7. A topological space X is said to be *connected im kleinen at x*, written *cik at x*, provided that every neighborhood of x contains a connected neighborhood of x (Recall from Chapter 2 that a neigborhood is not necessarily open). Note that the definition of local connectness differs from the definition of connected im kleinen since for local connectness the neigborhoods are asked to be open.

Definition 3.8. A continuum X is said to be *hereditarily continuumwise accessible* if and only if for every proper subcontinuum A of X there exists a nondegenerate subcontinuum B of X and a point $x \in A$ such that $A \cap B = \{x\}$.

Proposition 3.9. *Let X be a continuum. If X is cik at some point or X is hereditarily continuumwise accessible, then no positive Whitney level of $C(X)$ is a set theoretic decomposition.*

Proof. Let μ denote a Whitney map for $C(X)$. Fix s such that $0 < s < \mu(X)$.

Now, assume that X is cik at some point x. Then there is a connected neighborhood N of x such that $\mu(N) < s$. Hence, by Corollary 2.8, there exists $A \in C(X)$ such that $N \subset A$ and $\mu(A) = s$. Since $N \subset A$, $\text{int}(A) \neq \emptyset$. Therefore, by Lemma 3.6, $\mu^{-1}(s)$ is not a set theoretic decomposition of X.

Next, assume that X is hereditarily continuumwise accessible.

Let $A \in \mu^{-1}(s)$. Then there exists a continuum B and a point $x \in A$ such that $A \cap B = \{x\}$. Hence, $A \cup B$ is a continuum such that $\mu(A \cup B) > s$. Thus, taking an order arc from $\{x\}$ to $A \cup B$ through B, we can find a subcontinuum C of X in $\mu^{-1}(s)$ such that $C \neq A$ and $C \cap A \neq \emptyset$. Therefore, $\mu^{-1}(s)$ is not a set theoretic decomposition. □

We will use the following theorem in the next chapter to prove that Whitney preserving maps of arcwise connected continua onto $[0, 1]$ are homeomorphisms (Theorem 4.3).

Theorem 3.10. *Let X and Y be continua. If X is cik at some point or X is hereditarily continuumwise accessible and $f : X \longrightarrow Y$ is Whitney preserving, then $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$ if and only if $s = 0$.*

Proof. By Lemma 3.2, $\hat{f}(\mu^{-1}(0)) = \nu^{-1}(0)$. Thus, we need only to prove that $s = 0$ if $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$.

Suppose there exists an $s > 0$ such that $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$. Let $s_0 = \sup\{s : \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Note from the continuity of $\hat{f}$ that

$$\hat{f}(\mu^{-1}(s_0)) = \nu^{-1}(0). \tag{1}$$

Also, note that $s_0 \neq \mu(X)$ since $\hat{f}(X) = Y$ by Lemma 3.2. By Proposition 3.9, there exist A,$B \in \mu^{-1}(s_0)$ such that $A \neq B$ and $A \cap B \neq \emptyset$. Then $A \cup B$ is a continuum such that

$$r > s_0, \text{ where } r = \mu(A \cup B). \tag{2}$$

By (1), $\hat{f}(A), \hat{f}(B) \in \nu^{-1}(0)$; in other words, $f(A) = \{p\}$ and $f(B) = \{q\}$ for $p, q \in Y$. Hence,

$$f(A \cup B) = f(A) \cup f(B) = \{p, q\}.$$

Thus, since $f(A \cup B)$ is a continuum, we have that $p = q$. Hence, $f(A \cup B) \in \nu^{-1}(0)$. Thus, $\hat{f}(A \cup B) \in \nu^{-1}(0)$. Then, since f is Whitney preserving and $A \cup B \in \mu^{-1}(r)$, $\hat{f}(\mu^{-1}(r)) = \nu^{-1}(0)$. Therefore, by (2), we have a contradiction to the way we defined s_0. □

Note that Theorem 3.10 says that the only subcontinua of X whose Whitney preserving images are degenerate are precisely the degenerate subcontinua of X.

Corollary 3.11. *Let X be a continuum and let Y be a continuum of dimension 1. Assume X is cik at some point or that X is hereditarily continu-*

umwise accessible. If $f : X \longrightarrow Y$ is a Whitney preserving map, then the dimension of X is 1.

Proof. By Theorem 3.10, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$ if and only if $s = 0$. Then $f^{-1}(y)$ is totally disconnected for every $y \in Y$. Hence, by Theorem VI 7 of [6, p. 91], X has dimension 1. □

Lemma 3.12. *Let X be a continuum. If X is arcwise connected, then X is hereditarily continuumwise accessible.*

Proof. Let A be a proper subcontinua of X. Let $x \in X \setminus A$ and let $y \in A$. Then, since X is arcwise connected, there is an arc α connecting x and y. Hence, there is a subarc β of α such that $\beta \cap A = \{p\}$. Hence, X is hereditarily continuumwise accessible. □

Corollary 3.13. *Let X and Y be continua. If X is arcwise connected and $f : X \longrightarrow Y$ is Whitney preserving, then $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$ if and only if $s = 0$.*

Proof. By Lemma 3.12, X is hereditarily continuumwise accessible. Hence, the corollary follows from Theorem 3.10. □

Definition 3.14. A Whitney preserving map $f : X \longrightarrow Y$, between continua, is said to be *strictly Whitney preserving* if for any two different Whitney levels $\mu^{-1}(s)$ and $\mu^{-1}(r)$ of $C(X)$ we have that $\hat{f}(\mu^{-1}(s)) \cap \hat{f}(\mu^{-1}(r)) = \emptyset$. In other words, the images of two different Whitney levels under $\hat{f}$ are different Whitney levels.

Definition 3.15. A map $f : X \longrightarrow Y$ is said to be *hereditarily irreducible* provided that whenever A and B are subcontinua of X such that $A \subset B$ and $A \neq B$, then $f(A) \neq f(B)$.

We show next that strictly Whitney preserving maps are related to hereditarily irreducible maps. Our main result is Theorem 3.19.

Theorem 3.16. *If $f : X \longrightarrow Y$ is strictly Whitney preserving, then f is hereditarily irreducible.*

Proof. Assume $f : X \longrightarrow Y$ is a strictly Whitney preserving map. Let A and B be subcontinua of X such that $A \subset B$ and $A \neq B$. Then $s < r$ where $s = \mu(A)$ and $r = \mu(B)$. Thus, since f is strictly Whitney preserving, $\hat{f}(\mu^{-1}(s)) \cap \hat{f}(\mu^{-1}(r)) = \emptyset$. Therefore, $f(A) \neq f(B)$. □

The following lemma is a combination of 1.212.3 and 1.213.6 of [13, p. 204, 206]. We include the proof of the lemma for completeness since the results just mentioned are not proved in [13].

Lemma 3.17. *Let X and Y be continua, let $f : X \longrightarrow Y$ be a hereditarily irreducible map and let $\nu : C(Y) \longrightarrow \mathbb{R}$ be a Whitney map for $C(Y)$. Then the map $\mu = \nu \circ \hat{f} : C(X) \longrightarrow \mathbb{R}$ is a Whitney map for $C(X)$.*

Proof. Let A and B be two subcontinua of X such that $A \subset B$ and $A \neq B$. We prove that $\mu(A) < \mu(B)$. Clearly, $f(A) \subset f(B)$ and $f(A) \neq f(B)$ since f is a hereditarily irreducible map. So, since ν is a Whitney map, we have $\nu(f(A)) < \nu(f(B))$. Therefore, $\mu(A) < \mu(B)$.

Next, we see that for any $x \in X$, $\mu(\{x\}) = 0$ since $\mu(\{x\}) = \nu(\{f(x)\})$. Finally, μ is continuous because it is a composition of two continuous functions. □

Proposition 3.18. *Let $f : X \longrightarrow Y$ be hereditarily irreducible. If f is Whitney preserving, then f is strictly Whitney preserving.*

Proof. Assume that f is Whitney preserving. Assume that $0 \leq s_1 < s_2 \leq \mu(X)$. Let $A_1 \in \mu^{-1}(s_1)$. Then, by Corollary 2.8, there exists $A_2 \in \mu^{-1}(s_2)$ such that $A_1 \subset A_2$; note that $A_1 \neq A_2$ since $s_1 \neq s_2$. Thus, since f is hereditarily irreducible, $f(A_1) \neq f(A_2)$. Hence, since $f(A_1) \subset f(A_2)$, we have that $\nu(f(A_1)) < \nu(f(A_2))$. Therefore, $f(A_1) \in \hat{f}(\mu^{-1}(s_1))$ and $f(A_2) \in \hat{f}(\mu^{-1}(s_2))$ and since f is Whitney preserving, $\hat{f}(\mu^{-1}(s_1)) \cap \hat{f}(\mu^{-1}(s_2)) = \emptyset$. This proves that f is strictly Whitney preserving. □

Theorem 3.19. *Let $f : X \longrightarrow Y$ be a weakly confluent map. Then* (1) *and* (2) *are equivalent.*

(1) *f is strictly Whitney preserving.*

(2) *f is hereditarily irreducible.*

Proof. By Theorem 3.16, (1) implies (2). We prove that (2) implies (1).

Assume (2) holds. Let ν be a Whitney map for $C(Y)$. Then, by (2) and Lemma 3.17, $\nu \circ \hat{f} = \mu$ is a Whitney map for $C(X)$. We prove that f is strictly μ, ν-Whitney preserving. By Proposition 3.18, it suffices to prove that f is μ, ν-Whitney preserving.

Let $s \in [0, \mu(X)]$. We prove that $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(s)$.

It is easy to see that $\hat{f}(\mu^{-1}(s)) \subset \nu^{-1}(s)$: For any $K \in \mu^{-1}(s)$, $\mu(K) = s$; thus, since $\mu = \nu \circ \hat{f}$, we have that $\hat{f}(K) \in \nu^{-1}(s)$.

Next, we prove that $\nu^{-1}(s) \subset \hat{f}(\mu^{-1}(s))$. Let $B \in \nu^{-1}(s)$. Then, by our assumption that f is weakly confluent, there exists $C \in C(X)$ such that $f(C) = B$. Hence, using order arcs (by Corollary 2.8), there exists an $L \in \mu^{-1}(s)$ such that $C \subset L$ or $L \subset C$. Thus, having already proved that $\hat{f}(\mu^{-1}(s)) \subset \nu^{-1}(s)$, we have that $f(L) \in \nu^{-1}(s)$; furthermore, since $f(C) = B$, we know that $B \subset f(L)$ or $f(L) \subset B$. Thus, since $B, f(L) \in \nu^{-1}(s)$, we conclude that $f(L) = B$. Therefore, since $L \in \mu^{-1}(s)$, we have proved that $\nu^{-1}(s) \subset \hat{f}(\mu^{-1}(s))$.

This completes the proof. □

Definition 3.20. A continuum X is said to be in $Class(W)$ provided that every map of any continuum onto X is weakly confluent.

It is known that every hereditarily indecomposable continuum is in $Class(W)$ as so is every arc-like continuum. See [7, p. 198, 253] for a complete reference about $Class(W)$. Theorem 3.19 shows, in particular, that if Y is in $Class(W)$ then every hereditarily irreducible map from any continuum onto Y is a Whitney preserving map. In connection with Theorem 3.19 and $Class(W)$ we have the following question.

Question. Is $Class(W)$ characterized as follows: Y is in $Class(W)$ if and only if for every continuum X and every onto map $f : X \longrightarrow Y$ such that f is hereditarily irreducible, f is strictly Whitney preserving?

Next, we examine what happens when we require the restrictions of Whitney preserving maps to be Whitney preserving.

Definition 3.21. A continuous function, $f : X \longrightarrow Y$, between continua is said to be *hereditarily Whitney preserving* if for every subcontinuum Z of X, $f_{|Z} : Z \longrightarrow f(Z)$ is Whitney preserving.

Example 3.22. Let $X = [0, \pi]$, $Y = S^1$, and let $f : X \longrightarrow Y$ be given by $f(t) = e^{4ti}$. If $\mu : C(X) \longrightarrow \mathbb{R}$ is the diameter map and $\nu : C(Y) \longrightarrow \mathbb{R}$ is the arc-length map, then f is μ, ν-Whitney preserving but f is not hereditarily Whitney preserving since $f_{|[0,\frac{\pi}{2}]}$ is not Whitney preserving by Theorem 3.4.

Observe that for a map f to be hereditarily Whitney preserving, there must exist two Whitney maps for every subcontinuum Z, one Whitney map for $C(Z)$ and another for $C(f(Z))$, such that the restriction of f takes Whitney levels onto Whitney levels with respect to these Whitney maps. The following proposition tells us that hereditarily Whitney preserving maps are, in fact, hereditarily Whitney preserving with respect to the restrictions of the given Whitney maps.

Proposition 3.23. *Let $f : X \longrightarrow Y$ be a hereditarily μ, ν-Whitney preserving map. Then for every subcontinuum Z of X, $f_{|Z}$ is a $\mu_Z, \nu_{f(Z)}$-Whitney preserving map.*

Proof. Let Z be a subcontinuum of X. Any Whitney level of $C(Z)$ with respect to μ_Z is of the form $\mu^{-1}(s) \cap C(Z)$ where $\mu^{-1}(s)$ is a Whitney level of

X with respect to μ (see [13, p. 411–412]). Denote by $(\mu^{-1}(s))^Z$ the Whitney levels of $C(Z)$ with respect to μ_Z and denote by $(\nu^{-1}(t))^{f(Z)}$ the Whitney levels of $f(Z)$ with respect to $\nu_{f(Z)}$.

We want to prove that for every $s \in [0, \mu(Z)]$, $\hat{f}((\mu^{-1}(s))^Z) = (\nu^{-1}(t))^{f(Z)}$ for some $t \in [0, \nu(f(Z))]$.

Let $(\mu^{-1}(s))^Z$ be any Whitney level of C(Z). Since f is μ, ν-Whitney preserving, we have that $\hat{f}((\mu^{-1}(s))^Z) \subset (\nu^{-1}(t))^{f(Z)}$. Hence, we need only to prove that the reverse inclusion holds.

Let $B \in (\nu^{-1}(t))^{f(Z)}$; by hypothesis there exist two Whitney maps $\mu^* : C(Z) \longrightarrow \mathbb{R}$ and $\nu^* : C(f(Z)) \longrightarrow \mathbb{R}$ such that $f_{|Z}$ is μ^*, ν^*-Whitney preserving. Hence, $f_{|Z}$ is weakly confluent by Theorem 3.4. Thus, there is $A \in C(Z)$ such that $\hat{f}_{|Z}(A) = B$. Let $\mu(A) = s_1$. We have two cases: $s_1 \leq s$ or $s_1 \geq s$. Assume $s_1 \leq s$. Since f is Whitney preserving, we have that

$$\hat{f}(\mu^{-1}(s_1)) = \hat{f}(\mu^{-1}(s)) = \nu^{-1}(t). \tag{1}$$

Now, by Corollary 2.8, there exists $D \in C(X)$ such that $A \subset D$ and $\mu(D) = s$. Then $f(A) \subset f(D)$, and, by (1), $\nu(f(A)) = \nu(f(D))$. Therefore, $f(A) = f(D)$. Hence, $f(D) = B$. Therefore, since $D \in (\mu^{-1}(s))^Z$, $(\nu^{-1}(t))^{f(Z)} \subset \hat{f}((\mu^{-1}(s))^Z)$.

The case when $s_1 \geq s$ can be proved with a similar argument. □

Proposition 3.24. *Let X and Y be continua. If $f : X \longrightarrow Y$ is Whitney preserving and $f_{|Z}$ is weakly confluent for some subcontinuum Z of X, then $f_{|Z}$ is Whitney preserving.*

Proof. Let Z be a subcontinuum of X such that $f_{|Z}$ is weakly confluent.

If $f(Z)$ is a singleton then $f_{|Z} : Z \longrightarrow f(Z)$ is a Whitney preserving map.

Assume $f(Z)$ is nondegenerate. Let $(\mu^{-1}(s))^Z$ be a Whitney level of $C(Z)$. It is easy to see that $\hat{f}((\mu^{-1}(s))^Z) \subset (\nu^{-1}(t))^{f(Z)}$ where $(\nu^{-1}(t))^{f(Z)}$ is a Whitney level of $f(Z)$. We need only prove the reverse containment.

To prove that $(\nu^{-1}(t))^{f(Z)} \subset \hat{f}((\mu^{-1}(s))^Z)$, let $D \in (\nu^{-1}(t))^{f(Z)}$. Since $f_{|Z}$ is weakly confluent, there exists a subcontinuum C of Z such that $f(C) = D$.

If $\mu(C) = s$, then $D \in \hat{f}((\mu^{-1}(s))^Z)$. So, assume that $\mu(C) < s$ or $\mu(C) > s$. Then, by Corollary 2.8, there exists $E \in (\mu^{-1}(s))^Z$ such that $C \subset E$ or $E \subset C$. Hence, $f(C) \subset f(E)$ or $f(E) \subset f(C)$. Now, since f is Whitney preserving, we have in either case, that $f(E) \in \nu^{-1}(t)$. Hence, $\nu(f(C)) = \nu(f(E))$; therefore, $f(C) = f(E)$. Hence, $f(E) = D$. Therefore, we have proved that $(\nu^{-1}(t))^{f(Z)} \subset \hat{f}((\mu^{-1}(s))^Z)$. □

Corollary 3.25. *Let X and Y be continua. If $f : X \longrightarrow Y$ is Whitney preserving and hereditarily weakly confluent, then f is hereditarily Whitney preserving.*

Proof. The proof is an immediate consequence of Proposition 3.24 since for any subcontinuum Z of X, the map $f_{|Z}$ is weakly confluent. □

Chapter 4

Maps onto the interval

In this chapter we take advantage of the fact that the unit interval is in $Class(W)$ (see the definition following Theorem 3.19). We give conditions on a continuum X in order that a Whitney preserving map f from X to the unit interval is a homeomorphism (e.g., Theorem 4.10). From now on, we will denote the unit interval by I. Our first result is a consequence of Corollary 3.25:

Corollary 4.1. *Let X be a continuum. If $f : X \longrightarrow I$ is a Whitney preserving map, then f is hereditarily Whitney preserving.*

Proof. Since I is in $Class(W)$, we have that any continuous function onto I is hereditarily weakly confluent. Then it follows from Corollary 3.25 that every Whitney preserving map onto I is hereditarily Whitney preserving. □

The following lemma will be used to prove a more general result (Theorem 4.3).

Lemma 4.2. *If $f : I \longrightarrow I$ is Whitney preserving, then f is a homeomorphism.*

Proof. Assume that f is μ, ν-Whitney preserving. It is enough to prove that f is injective, since f is surjective by Lemma 3.2. To do this suppose there exist two points $x, y \in I$, $x < y$, such that $f(x) = r = f(y)$.

By Theorem 3.10, $f([x, y])$ is nondegenerate. Hence, there exists $d \in [x, y]$ such that $f(d) \neq r$. Assume without loss of generality that $f(d) > r$. Let $c \in [x, y]$ such that $f(c)$ is a maximum of f on $[x, y]$ and let

$$x_1 = \max\{z \in [x, c] : f(z) = r\}$$

$$y_1 = \min\{z \in [c, y] : f(z) = r\}$$

By construction:

(1) $f(z) > r$ for all $z \in (x_1, y_1)$ and $f([x_1, c]) = f([c, y_1]) = [r, f(c)]$.

Now, by (1) we have that

(2) $f(D) \subset [r, f(c)] = f([x_1, c])$ and $f(D) \neq [r, f(c)]$ for each subcontinuum D of (x_1, y_1), *i.e.*, $f(D)$ and $[r, f(c)]$ are in different Whitney levels.

Let $\rho = \min\{\mu([x_1, c]), \mu([c, y_1])\}$. Notice that $\rho < \mu([x_1, y_1])$. Then, since μ is a Whitney map, there exists a natural number N such that $\rho < \mu([x_1 + \frac{1}{N}, y_1 - \frac{1}{N}]) < \mu([x_1, y_1])$

Hence, by Corollary 2.8, there exists a subcontinuum D of $[x_1 + \frac{1}{N}, y_1 - \frac{1}{N}]$ such that $\mu(D) = \rho$ and such that $c \in D$. Note that since $x_1, y_1 \notin D$, D is different from $[x_1, c]$ and from $[c, y_1]$. By the choice of D, D is in the same Whitney level as, either $[x_1, c]$ or $[c, y_1]$ (see the definition of ρ). This and (2) contradict the fact that f is Whitney preserving. Hence, f is injective. □

The following theorem plays an important role in Lemma 4.9 and in Theorem 4.10. Our main result in this chapter is Theorem 4.10. We develop the necessary tools to prove Theorem 4.10.

Theorem 4.3. *Let X be an arcwise connected continuum. If $f : X \longrightarrow I$ is Whitney preserving, then f is a homeomorphism.*

Proof. Assume f is Whitney preserving. Since every Whitney preserving map is onto (by Lemma 3.2), it is enough to prove that f is injective.

Let p and q be two different points of X, and let A be an arc from p to q. Then, by Theorem 3.10 and Corollary 4.1, $f(A)$ is nondegenerate and $f_{|A} : A \longrightarrow f(A)$ is Whitney preserving. Hence, by Lemma 4.2, $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism. Therefore, $f(p) \neq f(q)$, which proves that f is injective.

□

Corollary 4.4. *Let Y be a nondegenerate continuum. If $f : S^1 \longrightarrow Y$ is a Whitney preserving map, then Y is a simple closed curve.*

Proof. By Theorem 3.4, f is weakly confluent. Then, by II.3 of [4], Y is either an arc or a simple closed curve. By Theorem 4.3 and since the domain of f is S^1, Y can not be an arc. Therefore Y is a simple closed curve. □

Definition 4.5. A *simple triod* is a continuum which is the union of two arcs such that their intersection is one and only one point, that point being a noncut point of one of the arcs and a cut point of the other; we will call this point the *vertex* of the simple triod. A *half-ray triod* is a continuum which is the union of a half-ray H and an arc A such that $A \cap H = \emptyset$ and $\overline{H} \setminus H$ is a subarc or a point of A which contains neither noncut point of A.

Note that a simple triod is also a half-ray triod.

Definition 4.6. Let X be a metric space and let $\alpha \subset X$ be an arc. The arc α is said to be *free in* X if α without its end points is an open subset of X. When the context is clear, we will only say that α is a free arc.

Clearly, from the definition of half-ray triod, every half-ray triod contains a nonfree arc. The following remark will be useful in the proof of Lemma 4.7.

Remark. A *circle-like continuum* is an inverse limit (see Definition 5.5) in which all the coordinates spaces are S^1. Theorem 6 of [10] shows that if X is an arcwise connected circle-like continuum, then either X is a simple closed curve or X can be written in the form $A \cup C$ where A is an arc, C is a chainable continuum with exactly two arc components, and $A \cap C$ is exactly the two noncut points of A. Regarding the structure of C, Theorem 1 of [9] shows that one of the arc components of C is an arc, say D, and the other component is homeomorphic to $[0, \infty)$, say E. Hence, since C is a continuum, $\overline{E} \cap D \neq \emptyset$. Note that $\overline{E} \cap D$ contains a point different from the end points of D. In other words, C has a nonfree arc.

Lemma 4.7. *Let Z be a separable metric, arcwise connected, locally compact space such that every arc is free. Then Z is homeomorphic to either*

(1) $(0,1)$

(2) $[0,1)$

(3) $[0,1]$

(4) S^1.

Proof. Implicitly, the proof of this lemma is divided into two cases; first, we consider the case when Z is compact and we prove that Z is either homeomorphic to [0,1] or homeomorphic to S^1. Then we consider the case when Z is not compact, and we prove that the one point compactification of Z is either homeomorphic to [0,1] or homeomorphic to S^1.

Since every arc in Z is free, Z does not contain half-ray triods.

Assume first that Z is compact. Then, by Theorem 2 of [12], Z is homeomorphic to either an interval or to an arcwise connected circle-like continuum.

Suppose Z is not an arc. Since every arc in Z is free, we see from the remark preceding our lemma that Z is homeomorphic to S^1.

Now, assume Z is not compact. Denote by Z^* the one point compactification of Z and let v be the point of compactification. We show that Z^* is a locally connected continuum containing no simple triods, implying that Z^* must be either homeomorphic to [0,1] or homeomorphic to S^1.

First we will show that Z^* does not properly contain a copy of S^1. Suppose Z^* contains a copy of S^1, call it S. Since Z is arcwise connected and does not contain simple triods, Z does not properly contain a copy of S^1. Hence, the point of compactification v is in S. If there is a point $p \in (Z^* \setminus S)$, then let γ be an arc in Z with p and some point of $S \setminus \{v\}$ as its two noncut points. Clearly $(S \setminus \{v\}) \cup \gamma$ contains a simple triod, which is a contradiction to the fact that Z does not contain simple triods. Thus $S = Z^*$. Therefore Z^* does not properly contain a copy of S^1. In other words, if Z^* contains a copy of S^1, then Z^* is homeomorphic to S^1.

Note that Z^* does not contain a half-ray triod that is not a simple triod, otherwise, by definition of half-ray triod, Z will contain a nowhere dense arc, *i.e.*, a nonfree arc, which is impossible by hypothesis. Now, we prove that Z^* does not contain simple triods. For this purpose suppose Z^* has a simple triod, call it T. Since Z does not contain simple triods, we have that the vertex of T must be the point of compactification v. Let C_1, C_2 and C_3 be the components of $T \setminus \{v\}$. For each $i \in \{1, 2, 3\}$, let $a_i \in C_i$. Then, since Z is arcwise connected, there exist arcs α_1^2, from a_1 to a_2, and α_2^3, from a_2 to a_3, in Z^* such that $v \notin \alpha_1^2$ and $v \notin \alpha_2^3$. Then, because Z does not contain simple triods, $\alpha_1^2 \cup \alpha_2^3$ is an arc. Assume without loss of generality that a_2 is a cut point of $\alpha_1^2 \cup \alpha_2^3$ and let D be the arc in C_2 from v to a_2. Then $(\alpha_1^2 \cup \alpha_2^3) \cup D$ is a simple triod with a_2 as its vertex. Therefore, we have a simple triod contained in Z, which is a contradiction. Thus, Z^* does not contain simple triods.

In order to show that Z^* is locally connected, we show, in Claim 2, that Z^* does not contain a convergence continuum (see Definition 2.11).

Claim 1. Let A be a convergence continuum of Z^*. If $a \in A$ and γ is an arc in Z such that $a \in \gamma$, then a is an end point of γ.

Proof. Let $a \in A$ and let γ be an arc in Z such that $a \in \gamma$. Since A is a convergence continuum and every arc in Z is free, A does not contain any subarc of γ. Hence, there exists a sequence $\{y_m\} \subset A \setminus \gamma$ such that $y_m \to a$. Thus, since γ is free in Z, a must be an end point of γ. This proves Claim 1.

Claim 2. Z^* does not contain a convergence continuum.

Proof. Suppose Z^* has a convergence continuum, say A.

Let $a, b \in A \setminus \{v\}$ be two different points. Denote by β an arc between a and b. Then, by Claim 1, a and b are end points of β. Now, since A does not contain arcs, there is a point $c \in A \setminus (\beta \cup \{v\})$. Let γ be an arc from c to a. Then, by Claim 1, a and c are end points of γ. Now, since Z contains no simple triods and contains no simple closed curves, $\beta \cup \gamma$ is an arc. In either case, one of the points a or b is not an end point of some arc, which contradicts Claim 1. This proves Claim 2.

By Claim 2, Z^* is locally connected. Hence, Z^* is a locally connected continuum containing no simple triods. Hence, Z^* is either homeomorphic to S^1 or homeomorphic to $[0, 1]$. Therefore, since Z^* is a one point compactification of Z, Z is either homeomorphic to $[0, 1)$ or homeomorphic to $(0, 1)$. □

Definition 4.8. Let X be a continuum. Then $A \subset X$ is said to be an *arc component of* X if A is a maximal arcwise connected subset of X. If A is also dense in X, then we say that A is a *dense arc component of* X.

The next lemma is similar to Theorem 3.10: The lemma shows that only the degenerate subcontinua of X are mapped to degenerate subcontinua of I when X has a dense arc component and f is a Whitney preserving map.

Lemma 4.9. *Let X be a continuum that contains a dense arc component. If $f : X \longrightarrow I$ is Whitney preserving, then $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$ if and only if $s = 0$.*

Proof. Since f is a Whitney preserving map, by Lemma 3.2, it is enough to prove that if $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$, then $s = 0$.

Let A be a dense arc component of X and let $B \in C(A)$ such that $\mu(B) > 0$.

Since A is dense in X, for every $\varepsilon > 0$ there exists and an arcwise connected subcontinuum C of A such that $H_d(X, C) < \varepsilon$. Hence, by the continuity of ν and f, and since $\hat{f}(X) = I$, we can find an arcwise connected subcontinuum C of A such that

$$\nu(f(C)) > 0. \tag{1}$$

Let $b \in B$ and $c \in C$ and denote by β_b^c an arc from b to c. Then $B \cup \beta_b^c \cup C$ is an arcwise connected continuum. Denote by Y this new continuum.

By (1), $f(Y)$ is nondegenerate. Hence, by Corollary 4.1 and Theorem 4.3, $f_{|Y}$ is a homeomorphism. Therefore, for any subcontinuum B of A, $f(B)$ is nondegenerate, *i.e.*, $\nu(f(B)) > 0$.

Now, let D be any subcontinuum of X such that $\mu(D) > 0$. Then, since A is dense, we can use order arcs to construct a subcontinuum E of A such that $\mu(E) = \mu(D)$. Then, by the previous argument and since f

is Whitney preserving, $\nu(f(E)) > 0$ and, since f is Whitney preserving, $\nu(f(D)) > 0$. □

The following proposition appeared in the proceedings of the First International Conference on Continuum Theory [5]. This proposition has two proofs. One is my original proof and the other one was suggested by the referee of the paper as a shorter alternative to my proof. I decided to include both proofs here. The first is my proof and the second is the referee's proof.

Theorem 4.10. *Let X be a continuum such that X contains a dense arc component. If $f : X \longrightarrow I$ is Whitney preserving, then f is a homeomorphism.*

Proof 1. Let X be a continuum and let A be a dense arc component of X. Assume there is a Whitney preserving map $f : X \longrightarrow I$.

By Lemma 4.9, $f(C)$ is nondegenerate for every nondegenerate subcontinuum C of A. Hence, A does not contain a copy of S^1 and A does not contain simple triods otherwise, the restriction of the map to either a simple closed curve or a simple triod would be a homeomorphism by Theorem 4.3. Therefore, A is uniquely arcwise connected. By Theorem 4.3, we can assume that A is not closed in X.

Claim 1. Every arc in A is free in X.

Proof. Suppose A contains an arc that is not free, say α, and assume that a and b are the noncut points of α. Since α is not free there exist $c \in \alpha$ and a sequence $\{x_n\}$ such that for all n, $x_n \notin \alpha$ and $x_n \to c$. Without loss of generality, since A is dense, we can assume that $\{x_n\} \subset A$.

For each n let D_n be the arc in A from c to x_n. Since A is uniquely arcwise connected either $a \in D_n$ or $b \in D_n$, so there exists an infinite subsequence $\{D_{n_k}\}$ such that $a \in D_{n_k}$ or $b \in D_{n_k}$. Assume without loss of generality that $a \in D_{n_k}$. For simplicity, assume that $D_{n_k} \subset D_{n_{k+1}}$ for all k.

Now, by Lemma 4.9, $f(D_{n_k})$ is nondegenerate for all k. By Corollary 4.1 and Theorem 4.3:

(i) $f_{|D_{n_k}}$ is a homeomorphism for every k.

Thus,

(ii) $f(D_{n_k}) \subset f(D_{n_{k+1}})$ for every k.

Hence, $f(c) \neq f(x_{n_1})$. Assume without loss of generality that $f(c) < f(x_{n_1})$. Then, by (i) and (ii), $f(x_{n_k}) < f(x_{n_{k+1}})$. Therefore, $f(c) < \lim f(x_{n_k})$. On the other hand, since $x_{n_k} \to c$ and f is continuous, $f(c) = \lim f(x_{n_k})$, so we have a contradiction. This proves Claim 1.

Claim 2. Let $c \in A$. If A is not locally compact at c, then c is an end point of every arc α in A such that $c \in \alpha$.

Proof. Let $c \in A$ and assume A is not locally compact at c. Let α be an arc in A such that $c \in A$. Then, since every arc in A is free in X, we have that c cannot be in the interior of α. Thus, c must be an end point of α.

Claim 3. A has at most one point of nonlocal compactness.

Proof. Suppose the claim is not true. Let c and d be two different points of A such that A is not locally compact at c and at d.

Let β be the arc from c to d in A.

By Claim 2, and since A is arcwise connected and does not contain neither simple closed curves nor simple triods, we have that $A \subset \beta$. Hence, A is an arc, which implies that A is closed in X. This contradicts our assumption of A being a dense proper subset of X. This proves Claim 3.

Since A is not closed in X and is properly contained in X, we have that if A is locally compact, then, by Lemma 4.7, A is homeomorphic to either $(0, 1)$ or to $[0, 1)$.

Now, assume that A is not locally compact at some point c. Then, by Claim 2, $A\setminus\{c\}$ is arcwise connected and, by Claim 3, $A\setminus\{c\}$ is locally compact. Therefore, by Lemma 4.7, $A\setminus\{c\}$ is homeomorphic to either $(0, 1)$ or to $[0, 1)$. Clearly, $A\setminus\{c\}$ cannot be homeomorphic to $[0, 1)$, because this would imply that A is homeomorphic to $[0, 1]$, and then, closed in X.

So far, we have that A is either homeomorphic to $[0, 1)$ or to $(0, 1)$. We show next that $\overline{A}$ is homeomorphic to $[0, 1]$.

Assume first that A is homeomorphic to $[0, 1)$ and let $h : [0, 1) \longrightarrow A$ be a surjective homeomorphism. Let $a_0 = h(0)$. For a and b in A, we say that $a < b$ if and only if $h^{-1}(a) < h^{-1}(b)$. Also, we denote the arc in A between a and b as $[a, b]$.

Now, let $x \in X\setminus A$. Since A is dense, there is a sequence $\{a_n\} \subset A$ such that $a_n \to x$. Let $\{x_n\} \subset [0, 1)$ be such that $h(x_n) = a_n$. Because $a_n \to x$ and $x \notin A$, we have that $x_n \to 1$. Therefore,

(iii) for all $x \in X\setminus A$, there exists a sequence $\{x_n\} \subset [0, 1)$ such that $x_n \to 1$ and $x = \lim h(x_n)$.

Let $\{z_n\}$ be an increasing sequence in $[0,1)$ such that $z_n \to 1$. Then, by (iii),

(iv) $X \setminus A \subset \bigcap_{n \in \mathbb{N}} \overline{h([z_n, 1))}$.

We show the reverse inclusion holds. Let $x \in A$ and let $y = h^{-1}(x)$. Since $\{z_n\}$ is an increasing sequence, there exists a natural number N such that $y < z_n$ for all $n \geq N$. Suppose $x \in \overline{h([z_N, 1))}$. Then there exists a sequence $\{a_k\}$ in $h([z_n, 1))$ such that $a_k \to x$. Note that for all n, there is k such that $h(z_n) < a_k$; otherwise $\{a_k\} \subset h([z_n, z_{n_0}])$ for some n_0. Hence, $\lim h^{-1}(a_k) = 1$. Assume, without loss of generality, that $\{h^{-1}(a_k)\}$ is an increasing sequence. Then for all k, $[x, a_k] \subset [x, a_{k+1}]$; also for all k, $f_{|[x,a_k]}$ is a homeomorphism. Assume that $f(x) < f(a_1)$. Then $\{f(a_k)\}$ is an increasing sequence. Therefore, $f(x) < \lim f(a_k)$. On the other hand, since $a_k \to x$ and f is continuous, we have that $f(x) = \lim f(a_k)$, which is a contradiction. This contradiction arose because we supposed $x \in \overline{h([z_N, 1))}$. So, we have proved that

(v) $\bigcap_{n \in \mathbb{N}} \overline{h([z_n, 1))} \subset X \setminus A$.

Therefore, by (iv) and (v), we have that $X \setminus A$ is a subcontinuum of X.

Let $x, y \in X \setminus A$, and let $\{a_n\}$ and $\{b_m\}$ be sequences in A such that $a_n \to x$ and $b_m \to y$; in addition, assume for simplicity that $a_n < a_{n+1}$ and $b_m < b_{m+1}$ for all n and all m. Then, by Lemma 4.9 and Theorem 4.3, $f_{|[a_0,a_n]}$ and $f_{|[a_0,b_m]}$ are homeomorphisms for all n and all m. Assume $f(a_0) < f(a_1)$. Then $f(a_0) < f(a_n)$ and $f(a_0) < f(b_m)$ for every n and m. It now follows from (iii) that $f(x) = f(y)$. Hence, $f(X\backslash A)$ is degenerate. Then, by Lemma 4.9, $X\backslash A$ is a point. Therefore, $\overline{A}$ is homeomorphic to $[0, 1]$.

The case when A is homeomorphic to $(0, 1)$ follows from the case when A is homeomorphic with $[0, 1)$ since we can write $(0, 1)$ as the union of $(0, \frac{1}{2}]$ and $[\frac{1}{2}, 1)$. □

Proof 2. Let A be a dense arc component of X. Given a subarc L of A, $f_{|L} : L \longrightarrow f(L)$ is Whitney preserving by Corollary 4.1; also, $f(L)$ is nondegenerate by Lemma 4.9, and $f_{|L} : L \longrightarrow f(L)$ is a homeomorphism by Lemma 4.2.

We want to prove that, for each $t \in I$, $f^{-1}(t)$ is degenerate. We consider two cases.

Case 1. $0 < t < 1$.

Let points $p, q \in A$ be such that $f(p) < t < f(q)$. Let $L \subset A$ be an arc joining p and q. Then $f_{|L} : L \longrightarrow [f(p), f(q)]$ is a homeomorphism. Suppose that $f^{-1}(t)$ is nondegenerate. Then there exists $y \in X\backslash L$ such that $f(y) = t$. By the density of A, there exists an element $r \in A\backslash L$ such that $f(p) < f(r) < f(q)$. Let $J \subset A$ be an arc joining r and a point $u \in L$ such that $J \cap L = \{u\}$. Let pu and qu be the subcontinua of L that are irreducible about $\{p, u\}$ and $\{q, u\}$, respectively. Since $f(r) \in [f(p), f(u)]$ or $f(r) \in [f(u), f(q)]$, one of the maps $f_{|J\cup pu}$ or $f_{|J\cup qu}$ is not one-to-one. This is a contradiction, which proves that $f^{-1}(t)$ is degenerate.

Case 2. $t = 0$ or $t = 1$.

For each integer $n \geq 2$, let $p_n \in X$ be such that $f(p_n) = \frac{1}{n}$. By Case 1, $f^{-1}(\frac{1}{n}) = \{p_n\}$, so $X\backslash\{p_n\} = f^{-1}([0, \frac{1}{n})) \cup f^{-1}((\frac{1}{n}, 1])$ is a separation of

$X \setminus \{p_n\}$. This implies that $f^{-1}([0, \frac{1}{n}])$ is a subcontinuum of X. Therefore, $f^{-1}(0) = \bigcap\{f^{-1}([0, \frac{1}{n}]) : n \geq 2\}$ is a subcontinuum of X. Similarly, $f^{-1}(1)$ is a subcontinuum of X. Thus, f is monotone.

Now, by Lemma 4.9, $f^{-1}(0)$ and $f^{-1}(1)$ are degenerate subcontinua of X. Therefore, f is a homeomorphism. □

Example 6.1 will show the necessity of the hypothesis in Theorem 4.10 of X having a dense arc component.

Chapter 5

Maps onto $\mathbf{S^1}$

In this chapter we give two examples of Whitney preserving maps onto S^1. In the first example (Example 5.8) we show the existence of a Whitney preserving map from the solenoid to S^1. This example is important since it shows that a result similar to Theorem 4.10, having S^1 as the range of a Whitney preserving map, is not possible (since every composant of the solenoid is a dense arc component of the solenoid). The second example (Example 5.16) shows a Whitney preserving map from a continuum having no arc components to S^1. It is from this example that we obtain an example that shows that the assumption in Theorem 4.10 is necessary. Example 5.16 was suggested by Professor Sergio Macías when he was a visiting professor at West Virgnia University. A modification of this example was also suggested later in the referee report of [5].

Definition 5.1. A space X is said to be *semi-locally-connected at p*, written *slc at p*, provided that each neighborhood of p contains a neighborhood V of

p such that $X \backslash V$ has only finitely many components. A space X is said to be *semi-locally-connected*, written *slc*, provided that X is slc at every point.

It is not difficult to prove that every locally connected continuum is slc (see 8.44 of [14, p. 136]).

Proposition 5.2. *Let X be an arcwise connected slc continuum. If $f : X \longrightarrow S^1$ is a hereditarily weakly confluent, Whitney preserving map, then X is homeomorphic to S^1.*

Proof. By Corollary 8.5 of [3], X contains a simple closed curve S.

We show $X \subset S$. Assume, by way of contradiction, that X is not contain in S. Let $x \in X \backslash S$. Then, since X is arcwise connected, there exists an arc α such that $x \in \alpha$ and $\alpha \cap S = \{p\}$. Let β be a proper arc of S such that $p \in \beta$ and such that p is neither of the end points of β. Then $\alpha \cup \beta$ is a simple triod. By Corollary 3.13, $f(\alpha \cup \beta)$ is nondegenerate. Hence, there is a simple triod $T \subset \alpha \cup \beta$ such that $f(T)$ is a proper arc of S^1. Now, $f_{|T} : T \longrightarrow f(T)$ is Whitney preserving by Corollary 3.25. Hence, $f_{|T}$ is a homeomorphism by Theorem 4.3. This is a contradiction to T being a simple triod and $f(T)$ being an arc. Therefore, $X \subset S$. □

Proposition 5.3. *Let X be a locally connected continuum and let $f : X \longrightarrow S^1$ be a Whitney preserving map. Then X is homeomorphic to either I or S^1. Furthermore, if f is hereditarily weakly confluent, then X is homeomorphic to S^1.*

Proof. Since the only locally connected continua that do not contain simple triods are the interval and the unit circle, we will prove that X does not contain a simple triod.

Assume, by way of contradiction, that X contains a simple triod T.

By Theorem 3.10, the image of T is nondegenerate. Using order arcs, we can find a simple triod $T_2 \subset T$ such that $f(T_2) \subset S^1$ and $f(T_2) \neq S^1$. Then $f_{|T_2}$ is Whitney preserving by Proposition 3.24 and since $f(T_2)$ is an arc. Hence, $f_{|T_2}$ is a homeomorphism by Theorem 4.3, which is a clear contradiction with T_2 being a simple triod. Therefore, X does not contain a simple triod.

Now, assume f is hereditarily weakly confluent. Then, since X is locally connected, X is slc. Hence, by Proposition 5.2, X is homeomorphic to a simple closed curve. □

Theorem 5.4. *Let X be an arcwise connected slc continuum and let $f : X \longrightarrow Y$ be a map. If f is hereditarily weakly confluent and Whitney preserving, then f is a homeomorphism.*

Proof. Observe that from Theorem 3.10, $\hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)$ iff $s = 0$. Now,

Let x and y be two different points of X and let α be an arc from x to y. Then, by Corollary 3.13, $f(\alpha)$ is nondegenerate. Then, by Corollary 3.25, $f_{|\alpha}$ is Whitney preserving. Therefore, $f_{|\alpha}$ is hereditarily weakly confluent and Whitney preserving. By 13.70 of [14, p. 310], since f is hereditarily weakly confluent and α is an arc, $f(\alpha)$ is either an arc or a simple closed curve. But, by Proposition 5.2, we have that $f(\alpha)$ is not a simple close curve. Hence, $f_{|\alpha} : \alpha \longrightarrow f(\alpha)$ is a Whitney preserving map between arcs. Then, by Lemma

4.2, $f_{|\alpha}$ is a homeomorphism. Therefore $f(x) \neq f(y)$; this shows that f is one to one. Thus, f is a homeomorphism. □

Next, we give an example of a Whitney preserving map from a continuum having a dense arc component to S^1. This example shows that Theorem 4.10 can not be generalized to the case when the range of the map is S^1. In order to give this example, we need first, a pair of definitions and a proposition.

Definition 5.5. An *inverse sequence* is a sequence $\{X_i, f_i\}_{i\in\mathbb{N}}$ of spaces X_i, called *coordinate spaces*, and continuous functions $f_i : X_{i+1} \longrightarrow X_i$ called *bonding maps.* If $\{X_i, f_i\}_{i\in\mathbb{N}}$ is an inverse sequence, then the *inverse limit* of $\{X_i, f_i\}_{i\in\mathbb{N}}$, denoted by $\varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$, is the subspace of the cartesian product space $\prod_{i\in\mathbb{N}} X_i$ defined by

$$\varprojlim \{X_i, f_i\}_{i\in\mathbb{N}} = \{(x_i) \in \prod_{i\in\mathbb{N}} X_i : f_i(x_{i+1}) = x_i \text{ for all } i\}.$$

It is known that an inverse limit of continua is a continuum (see 2.4 of [14, p. 19]).

Definition 5.6. Let $\{X_i, f_i\}_{i\in\mathbb{N}}$ be an inverse sequence. We will say that $\{X_i, f_i\}_{i\in\mathbb{N}}$ is a *Whitney inverse sequence* if for every $i \in \mathbb{N}$, there exists a Whitney map $\mu_i : C(X_i) \longrightarrow \mathbb{R}$ such that f_i is μ_{i+1}, μ_i-Whitney preserving. The inverse limit, $X = \varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$, will be called an *inverse Whitney limit.*

Proposition 5.7. *Let $X = \varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$ be an inverse Whitney limit. Then for every $j \in \mathbb{N}$, the projection $P_j : X \longrightarrow X_j$ is a μ, μ_j-Whitney* *preserving map; where $\mu : C(X) \longrightarrow \mathbb{R}$ is given by $\mu(A) = \sum_{i\in\mathbb{N}} \frac{\mu_i(P_i(A))}{2^i}$ for all $A \in C(X)$.*

Proof. It is easy to see that μ is a Whitney map for $C(X)$.

Let P_j be the j-th projection of X to X_j, let $\mu^{-1}(s)$ be a Whitney level of $C(X)$ and let $A, B \in \mu^{-1}(s)$. Because each f_i is μ_{i+1}, μ_i-Whitney preserving, we have that $\mu_j(P_j(A)) = \mu_j(P_j(B))$. Otherwise, if we assume, for example, that $\mu_j(P_j(A)) < \mu_j(P_j(B))$, we will have that for every $i > j$, $\mu_i(P_i(A)) < \mu_j(P_i(B))$, so

$$\mu(A) = \sum_{i\in\mathbb{N}} \frac{\mu_i(P_i(A))}{2^i} < \mu(B) = \sum_{i\in\mathbb{N}} \frac{\mu_i(P_i(B))}{2^i},$$

which would be a contradiction. This shows that $\hat{P}_j(\mu^{-1}(s)) \subset \mu_j^{-1}(t)$ for some $t \in [0, \mu_j(X_j)]$.

To show the other inclusion take $C \in \mu_j^{-1}(t)$. For every $i > j$, let $D_i = f_i^{-1}(f_{i-1}^{-1}(\cdots(f_{j+1}^{-1}(f_j^{-1}(C)))\cdots))$ and let $D_0 = C$. Let $D = \varprojlim \{D_i, f_{|D_i}\}_{i\in\mathbb{N}}$.

By construction, $P_j(D) = C$. We have two cases, $\mu(D) \geq s$ and $\mu(D) \leq s$. Assume $\mu(D) \geq s$. Then, using an order arc, there exists a continuum $E \subset D$ such that $\mu(E) = s$. Hence, $P_j(E) \subset P_j(D)$, but we know that $\hat{P}_j(\mu^{-1}(s)) \subset \nu^{-1}(t)$. Therefore, $\mu_j(P_j(E)) = t$. This, together with $\mu_j(C) = t$, and $P_j(E) \subset P_j(D)$ imply that $P_j(E) = C$. Hence, $\nu^{-1}(t) \subset \hat{P}_j(\mu^{-1}(s))$. The case when $\mu(D) \leq s$ can be done in a similar way. □

The following example is a consequence of the preceding proposition.

Example 5.8. Let $\Sigma_n = \varprojlim \{X_i, f_i\}_{i\in\mathbb{N}}$ where $X_i = S^1$ and $f_i = z^n$ for all i. Then $P_1 : \Sigma_n \longrightarrow S^1$ given by $P_1(x_1, x_2, x_3, \ldots) = x_1$ is a Whitney preserving map.

The continuum Σ_n is known as the *n-adic solenoid*. It is not difficult to prove that Σ_n is an indecomposable continuum and that all nondegenerate proper subcontinua of Σ_n are arcs (see 2.16 of [14, p. 25]). Therefore, every composant of Σ_n is a dense arc component of Σ_n. Hence, Example 5.8 shows that Theorem 4.10 is false if we consider S^1 instead of I as the range of f.

Next, we develop the necessary tools to construct a Whitney map from a continuum X having no arc dense components to S^1. We will use a modification of this example to show the necessity of X having a dense arc component in Theorem 4.10.

Proposition 5.9. *Let X be a continuum. Assume there exists a Whitney map $\mu : C(X) \longrightarrow \mathbb{R}$ for which there is a Whitney level $\mu^{-1}(s)$ such that $\mu^{-1}(s)$ is a set theoretic decomposition of X. Then every subcontinuum K with $\mu(K) > s$ contains every subcontinuum A intersecting K such that $\mu(A) \leq s$.*

Proof. Let K be a subcontinuum of X such that $\mu(K) > s$ and let A be any subcontinuum of X such that $A \cap K \neq \emptyset$, and $\mu(A) \leq s$.

Assume, first, that $\mu(A) = s$. Let $c \in A \cap K$. By Corollary 2.8, there exists a subcontinuum B of K such that $c \in B$ and $\mu(B) = s$. Since $c \in B$, we have that $A \cap B \neq \emptyset$. Hence, since $\mu^{-1}(s)$ is a decomposition of X, $A = B$. Therefore, $A \subset K$.

If $\mu(A) < s$, then, by Corollary 2.8, we can find a continuum C such that $A \subset C$ and $\mu(C) = s$. Hence, this case is reduced to the previous case. □

Definition 5.10. A subcontinuum A of a continuum X is said to be a *terminal continuum*, if whenever B is a subcontinuum of X intersecting A, then either $A \subset B$ or $B \subset A$.

Proposition 5.11. *Let X be a continuum. If there is a continuous decomposition $\mathcal{A}$ of X such that every element of $\mathcal{A}$ is a nondegenerate terminal continuum, then there exists a Whitney preserving map $f : X \longrightarrow Y$ where $Y = X/\mathcal{A}$.*

Proof. Let $f : X \longrightarrow Y$ be the natural quotient map. We prove that f is a Whitney preserving map.

Since $\mathcal{A}$ is a continuous decomposition, we have that f is an open map (see 13.11 of [14, p. 283]) and f is confluent (by 13.14 of [14, p. 285]).

Let c be a positive real number and let $\mu' : \mathcal{A} \longrightarrow \mathbb{R}$ be given by $\mu'(A) = c$ for all $A \in \mathcal{A}$. Now, since $\mathcal{A}$ is a continuous decomposition of X, $\mathcal{A}$ is closed in $C(X)$. Then, by 16.10 of [7, p. 132], μ' can be extended to a Whitney map $\mu : C(X) \longrightarrow \mathbb{R}$. By construction, $\mathcal{A} \subset \mu^{-1}(c)$ and, since $\mathcal{A}$ is a continuous decomposition into terminal continua, $\mu^{-1}(c) \subset \mathcal{A}$. Hence, $\mathcal{A}$ is a Whitney level of $C(X)$.

Since $\mathcal{A}$ is a continuous decomposition of X and all of its elements are terminal continua, we have that if B is a nondegenerate subcontinuum of Y such that $f^{-1}(B)$ intersects an element A of $\mathcal{A}$, then $A \subset f^{-1}(B)$. Therefore, since f is the natural projection, the map $\nu : C(Y) \longrightarrow \mathbb{R}$, defined as $\nu(B) = \mu(f^{-1}(B)) - c$ for all $B \in C(Y)$, is continuous. In order to prove that ν is a Whitney map, we need to show that $\mu(\{y\}) = 0$ for all $y \in Y$, and that $\mu(A) < \mu(B)$ if $A \subset B$ and $A \neq B$. Let $y \in Y$. Then $f^{-1}(\{y\}) \in \mathcal{A}$. Hence, $\nu(\{y\}) = \mu(f^{-1}(y)) - c = c - c = 0$. Now, let A and B be two subcontinua of Y such that $A \subset B$ and $A \neq B$. Since f is the natural projection, $f^{-1}(A) \subset f^{-1}(B)$ and $f^{-1}(A) \neq f^{-1}(B)$. Hence, since μ is a Whitney map, $\nu(A) = \mu(f^{-1}(A)) - c < \nu(B) = \mu(f^{-1}(B)) - c$. This shows that ν is a Whitney map.

By the construction of ν and because f is the natural projection, f is a μ, ν-Whitney preserving map. □

The following example shows that we can not weaken the hypothesis of $\mathcal{A}$ being a continuous decomposition in Proposition 5.11. In other words, not every set theoretic decomposition of a continuum X, into nondegenerate terminal continua, is a continuous decomposition. In order to give this example, we need two definitions and a lemma.

Definition 5.12. A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. A continuum X is said to be *indecomposable* if is not decomposable. A continuum X is said to be *hereditarily indecomposable* if all its subcontinua are indecomposable. Note that, if X is a hereditarily indecomposable continuum, and A and B are two proper subcontinua of X that intersect, then either $A \subset B$ or $B \subset A$. In other words, any subcontinuum of a hereditarily indecomposable continuum is a terminal continuum.

Lemma 5.13. *Let X be a continuum. If A is a terminal subcontinuum of X, then int$(A) = \emptyset$.*

Proof. By contrapositive, let A be a subcontinuum of X and assume $\text{int}(A) \neq \emptyset$. Hence, there is an open set O such that $\overline{O} \subset A$. Let $x \in X \setminus A$, and let K be the component of $X \setminus O$ containing x. Then, by the Boundary Bumping Theorem (see 5.4 of [14, p. 73]), $K \cap \overline{O} \neq \emptyset$. Hence, K is a proper subcontinuum of X intersecting A such that neither $A \subset K$ nor $K \subset A$. Therefore, A is not a terminal continuum. □

Theorem 1 of [1] shows that the *pseudo-arc* is the only hereditarily indecomposable arc-like continuum. Example 5.14 shows that we can not replace the assumption that the decomposition be continuous in Proposition 5.11 with the condition that the decomposition be upper semi-continuous.

Example 5.14. Let P be the pseudo-arc, and let μ be any Whitney map for $C(P)$. Let A be a proper subcontinuum of P with $\mu(A) > 0$, and let $\mathcal{B} = \{B \in C(P) : \mu(B) = \frac{1}{2}\mu(A)$ and such that $B \cap A = \emptyset\}$. (Note that $\mathcal{B}$ is contained in a Whitney level). Then the set $\mathcal{A} = \mathcal{B} \cup A$ is an upper semicontinuous decomposition of P which is not a continuous decomposition. We prove this as follows:

Let $x \in P$ and assume $x \notin B$ for all $B \in \mathcal{B}$. Then, by Corollary 2.8, there is a subcontinuum C of P with $x \in C$ and $\mu(C) = \frac{1}{2}\mu(A)$. By our assumption, $C \notin \mathcal{B}$. Hence, $C \cap A \neq \emptyset$. Therefore, since P is hereditarily indecomposable and $\mu(C) < \mu(A)$, $C \subset A$. Hence, $x \in A$. This shows that $P = \bigcup \mathcal{A}$.

Given any two different elements A and B of $\mathcal{A}$, we have, by the hereditary indecomposability of P and the choice of the elements in $\mathcal{B}$, that $A \cap B = \emptyset$. This proves that $\mathcal{A}$ is a set theoretic decomposition of P.

Now, we prove $\mathcal{A}$ is not a continuous decomposition.

Let $x \in A$. Then, by Lemma 5.13, $\text{int}(A) = \emptyset$. Hence, there is a sequence $\{x_n\}$ in X such that $x_n \to x$ and $x_n \notin A$ for all n. This implies that there is a sequence $\{B_n\}$ in S converging to a proper subcontinuum of A. Therefore, $\mathcal{A}$ is not a continuous decomposition. However, since every sequence in $\mathcal{A}$ converges either to an element of S or to a proper subcontinuum of A, $\mathcal{A}$ is an upper semicontinuous decomposition.

The following example, which is a direct application of Proposition 5.11, is not only an example of a Whitney preserving map from the circle of pseudo-arcs to S^1, but it also leads to an example (in the next chapter) of a Whitney preserving map from a continuum having no arc dense components to the unit interval. Thus, the hypothesis that X has a dense arc component in Theorem 4.10 is a necessary condition.

Definition 5.15. A *circle of pseudo-arcs* is a continuum X such that there is a continuous decomposition of X into pseudo-arcs such that the decomposition space is a simple closed curve S. Hence, the preimage of every point of S is a pseudo-arc. If X is a circle of pseudo-arcs, then, it follows from Theorem 5 of [2], that every maximal pseudo-arc is a terminal continuum in X.

Example 5.16. Let X be the circle of pseudoarcs. Let $f : X \longrightarrow S^1$ be the quotient map that shrinks each maximal pseudoarc of X to a point of S^1. Then f is Whitney preserving by Proposition 5.11.

Chapter 6

Maps onto the interval, revisited

We begin this chapter with an example. This example shows the necessity of having a dense arc component in the hypothesis of Theorem 4.10. Example 6.1 was not included together with Theorem 4.10. As a matter of coincidence, namely, we obtain the example from Example 5.16.

Example 6.1. Let X and $f : X \longrightarrow S^1$ be as in Example 5.16 and let A be a proper subcontinuum of X intersecting at least two pseudoarcs. By Proposition 3.24 and since $f(A)$ is an arc, $f_{|A} : A \longrightarrow f(A)$ is a Whitney preserving map. Note that A does not contain a dense arc component.

Proposition 5.11 shows that if $\mathcal{A}$ is a continuous decomposition of X into terminal continua, then the natural projection $\pi : X \longrightarrow X/\mathcal{A}$ is a Whitney preserving map. The following question is natural: If $\mathcal{A}$ is a set theoretic upper semi-continuous decomposition of X, then is there a Whitney preserving map from X onto $X/\mathcal{A}$?. In Proposition 6.5 we give an example of a continuum Γ and a set theoretic upper semi-continuous decomposition $\mathcal{A}$ of X for which the natural projection of X onto $X/\mathcal{A}$ is not Whitney preserving. In fact, we show that there does not exist a Whitney preserving map from X onto $X/\mathcal{A}$. Proposition 6.5 also shows that the converse of Theorem 3.4 is false. In the following definition, we construct the continuum known as the $V - \Lambda$ continuum.

Definition 6.2. Let $\mathcal{C}$ denote the Cantor ternary set. Let $\mathcal{C}_0$ be the set $\mathcal{C} \times \{0\}$ and let $\mathcal{C}_1$ be the set $\mathcal{C} \times \{1\}$. Join every point $(x, 0)$ of $\mathcal{C}_0$ to the corresponding point $(x, 1)$ of $\mathcal{C}_1$ with a straight line (see Figure 6.1).

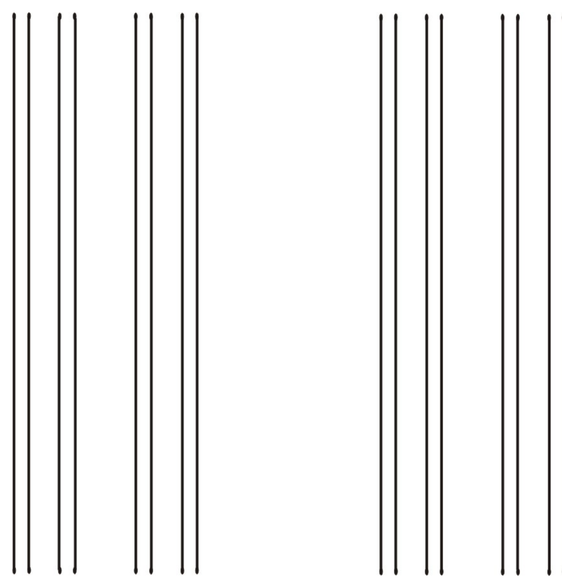

Figure 6.1:

Next, add the contiguous intervals to $\mathcal{C}_0$ with lengths $\frac{1}{3}, \frac{1}{3^3}, \ldots$, and add the contiguous intervals to $\mathcal{C}_1$ with lengths $\frac{1}{3^2}, \frac{1}{3^4}, \ldots$ (see Figure 6.2).

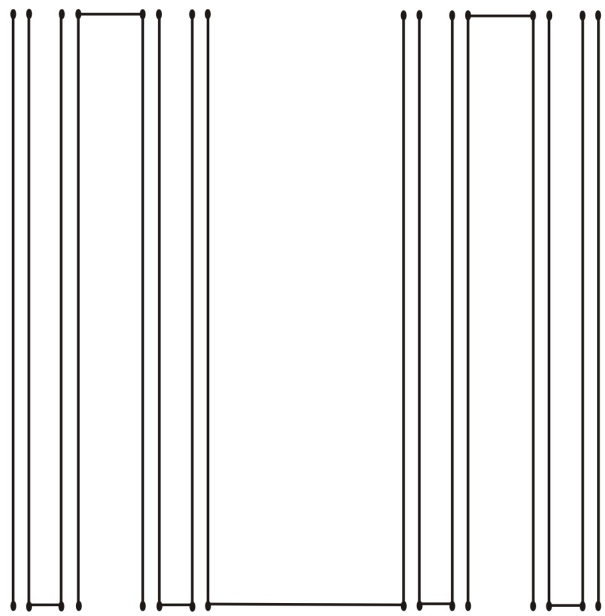

Figure 6.2:

Finally, identify each of these intervals into a different point (see Figure 6.3).

The continuum obtained is called the $V - \Lambda$ *continuum*, which we will denote by Γ. The subcontinua of Γ, which are maximal with respect to being an arc, are called traunches. Note that there are three diferent types of traunches. The traunches having form of V will be called *V-traunches*, the traunches with form Λ will be called *Λ-traunches*, and the traunches which are straight lines will be called *I-traunches*.

The following lemma shows how a Whitney preserving map from Γ onto I acts on the traunches of Γ.

Lemma 6.3. *Let $f : \Gamma \longrightarrow I$ be a Whitney preserving map. Then the image under f of every traunch of Γ is a point.*

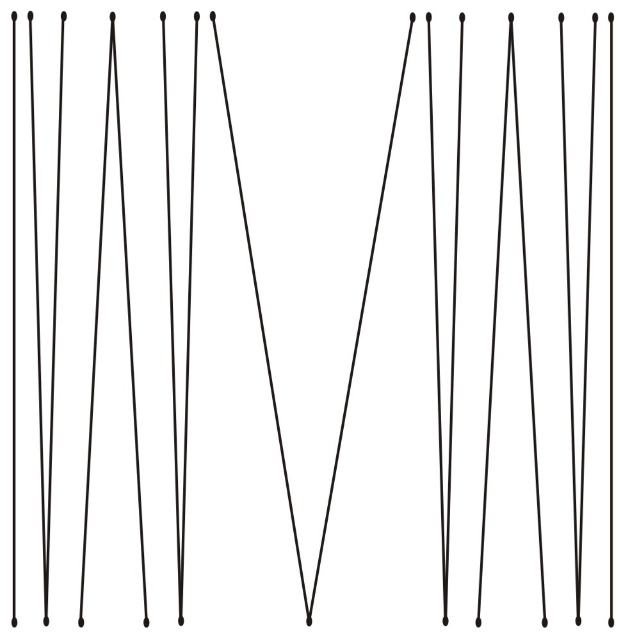

Figure 6.3:

Proof. We first show that f is constant for all I-traunches. Let $A = [a, b]$ be an I-traunch. Then there exists a sequence of Λ-traunches $\{A_n\}$ such that $A_n \to A$. For every n, denote each Λ-traunch A_n by $[a_n, b_n, c_n]$, where a_n and c_n are the end points of the traunch, and b_n is the bending point of the traunch.

Because $A_n \to A$, we have that

$$a_n \to a, c_n \to a \text{ and } b_n \to b. \tag{1}$$

We can assume that there is a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $f_{|A_{n_k}}$ is nondegenerate for all n_k, otherwise, by continuity, $f(A)$ would be a point. Then $f_{|A_{n_k}}$ is a homeomorphism by Corollary 4.1 and Lemma 4.2. Hence,

$f(A_{n_k})$ is an arc for every n; furthermore, the images of the end points a_{n_k} and c_{n_k} are end points of $f(A_{n_k})$. This, together with continuity, and (1), imply $f(a_{n_k}) \to f(a)$ and that $f(c_{n_k}) \to f(a)$. Therefore, $f(b_{n_k}) \to f(a)$, *i.e.*, $f(A)$ is a point. This argument shows that the image of any I-traunch is a point.

Now, let B be any Λ-traunch, denote by B_1 the "right" part of B and by B_2 the "left" part of B. So, B_1 and B_2 are limits of I-traunches. Therefore, by continuity, $f(B_1)$ and $f(B_2)$ are points. Hence, since $B = B_1 \cup B_2$ and $B_1 \cap B_2 \neq \emptyset$, $f(B)$ is a point. This shows that the image of any Λ-traunch is a point. A similar argument, shows that the image of any V-traunch is a point. □

Theorem 6.4. *Let $f : X \longrightarrow Y$ be a Whitney preserving map and let $s_0 = \max\{s \in [0, \mu(X)] : \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. If $s_0 > 0$, then $\mu^{-1}(s_0)$ is a continuous decomposition of X and A is terminal for every $A \in \mu^{-1}(s_0)$.*

Proof. Let $s_0 = \max\{s \in [0, \mu(X)] : \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Then $\bigcup \mu^{-1}(s_0) = X$ since $\mu^{-1}(s_0)$ is a Whitney level and by 1.213.1 of [13, p. 205]. Now, let $A, B \in \mu^{-1}(s_0)$ and assume that $A \cap B \neq \emptyset$. Then $A \cup B$ is a continuum with $A \subset A \cup B$, and $f(A) \subset f(A \cup B) = f(A) \cup f(B)$. Then, since $f(A)$ and $f(B)$ are points, and $s_0 = \max\{s \in [0, \mu(X)] : \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$, we have that $A = A \cup B$. Therefore, $A = B$. This shows that $\mu^{-1}(s_0)$ is a set theoretic decomposition of X and, since $\mu^{-1}(s_0)$ is compact in $C(X)$, that any convergent sequence in $\mu^{-1}(s_0)$ converges to an element of $\mu^{-1}(s_0)$. Thus, $\mu^{-1}(s_0)$ is a continuous decomposition.

Now, let $A \in \mu^{-1}(s_0)$ and let K be a subcontinuum of X such that $A \cap K \neq \emptyset$. If $\mu(K) \geq s_0$, then, by Proposition 5.9, $A \subset K$, and if $\mu(K) < s_0$, then $K \subset A$; otherwise, if $x \in K \setminus A$, we can take an order arc α from x to A containing K, and find a subcontinuum B in α such that $\mu(B) = s_0$ and $K \subset B$, which would contradict the fact that $\mu^{-1}(s_0)$ is a continuous decomposition. Therefore, A is a terminal continuum. □

Proposition 6.5. *There is no Whitney preserving map from* Γ *to the unit interval.*

Proof. Suppose, by way of contradiction, that there is a Whitney preserving map $f : \Gamma \longrightarrow I$. Let $s_0 = \max\{s \in [0, \mu(X)] : \hat{f}(\mu^{-1}(s)) = \nu^{-1}(0)\}$. Then, by Lemma 6.3, $s_0 > 0$ and by Theorem 6.4, $\mu^{-1}(s_0)$ is a continuous decomposition of Γ into terminal continua. On the other hand, the only nondegenerate terminal subcontinua of Γ are the I-traunches and, since the images of the V-traunches are points, $\mu^{-1}(s_0)$ has elements which are not terminal continua; this is a contradiction. Hence, f can not exist. □

Chapter 7

Questions

Question 1. For which continua X, is there a Whitney preserving map $f : X \longrightarrow I$?

The main question, still remaining, is Question 1. By Theorem 4.10, the question is reduced to finding the nonlocally connected continua which do not contain a dense arc component. Proposition 6.5 shows a non locally connected continuum for which there is not a Whitney preserving map onto the unit interval. However, Example 6.1 shows a non locally connected continuum for which there is one. Proposition 5.11 shows that for the continua that have a continuous decomposition into terminal continua, for which its quotient space is the unit interval, there is a Whitney preserving map. If the answer to the following question is affirmative, we can show that the continua that have a continuous decomposition into terminal continua, for which its quotient space is the unit interval are the only ones for which there is a monotone Whitney preserving map to the unit interval. Note that the

$V - \Lambda$ continuum has a partition whose quotient space is the unit interval; however, this partition is not a partition into terminal continua.

Question 2. Let $f : X \longrightarrow I$ be a Whitney preserving map. Is it true that if f is not one-to-one, then there exists a positive Whitney level that is mapped onto the singletons?

Regarding Question 1, we have a more general question:

Question 3. Let Y be a continuum. What are the continua X, for which there exists a Whitney preserving map $f : X \longrightarrow Y$?

Example 5.14 shows that not every decomposition of a space into terminal continua is a continuous decomposition. So, the natural question to ask is the following one:

Question 4. For which continua X, is it true that every decomposition of X into terminal continua is a continuous decomposition?

Question 5. What continua have the property that hereditarily irreducible surjective maps are strictly Whitney preserving? (See Proposition 3.18).

Question 6. What continua X have the property that every map of any continuum onto X is Whitney preserving?

Question 7. See question above Definition 3.21.

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