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West Virginia University

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Results on hyperspaces

Jorge M. Martínez-Montejano

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in
Mathematics

Sam B. Nadler, Jr., Ph.D., Chair

Krzysztof Ciesielski, Ph.D.

Jack T. Goodykoontz, Jr., Ph.D.

Andrzej Karwowski, Ph.D.

Leonardo Golubovic, Ph.D.

Department of Mathematics

Morgantown, West Virginia

2004

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ABSTRACT

Results in hyperspaces

Jorge M. Martínez-Montejano

A characterization of $C_p(X)$, the family of subcontinua of X containing a fixed point of X , when X is an atriodic continuum is given as follows. Assume Z is a continuum and consider the following three conditions: (1) Z is a planar absolute retract; (2) cut points of Z have component number two; (3) any true cyclic element of Z contains at most two cut points of Z . If X is an atriodic continuum and $p \in X$, then $C_p(X)$ satisfies (1)-(3) and, conversely, if Z satisfies (1)-(3), then there exist an arc-like continuum (hence, atriodic) X and a point $p \in X$ such that $C_p(X)$ is homeomorphic to Z . For $n \geq 3$, it is shown that the n^{th} symmetric product of nondegenerate continua is mutually aposyndetic, and that the natural map of the Cartesian product onto the n^{th} symmetric product of nondegenerate continua is not k -confluent for any positive integer k . It is also shown that for every nondegenerate continuum X there is a non- k -confluent map of some continuum onto $F_2(X)$ for any positive integer k . Answers are provided to questions of S. Macías and S. B. Nadler, Jr., about when the space of singletons is a Z -set in the hyperspace $C(X)$. In answering one of these questions it is shown in general that $C(X)$ being contractible is sufficient, but not necessary, for X to be a $C(X)$ -coselection space.

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I dedicate this dissertation to Manuel and Martha.

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Introduction

Hyperspace theory has been largely investigated through the years with special interest in the hyperspace 2^X of all compact subsets of X and the hyperspace $C(X)$ of all subcontinua of X , where X is a continuum. Recently, there has been increasing interest in investigating the properties of other hyperspaces: for example, $F_n(X)$, the n th symmetric product of X , and $C_p(X)$, the containment hyperspace for $\{p\}$ in $C(X)$. This work deals mainly with results about symmetric products and containment hyperspaces.

This dissertation consists of five chapters. In Chapter 1, we give the preliminaries that we will need for the development of our work.

In Chapter 2, answering a question of Patricia Pellicer-Covarrubias, we construct all possible models for $C_p(X)$ when X is an atriodic continuum.

In Chapter 3 we show that for $n \geq 3$, the n^{th} symmetric product of X is mutually aposyndetic. This answers a question of Alejandro Illanes and is analogous to a known result about Cartesian Products.

In Chapter 4 we show that the natural map from the Cartesian product onto the symmetric product is not as “nice” as one would initially think.

Finally, in Chapter 5, we answer some questions of Sergio Macías and Sam B. Nadler, Jr., concerning when the hyperspace of singletons is a Z -set in the hyperspace of subcontinua.

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CHAPTER 1

Preliminaries

In this chapter we introduce general concepts that we need for our work.

NOTATION 1.1. The following symbols will be used throughout this work.

- (1) The symbol $\mathbb{N}$ stands for the set of natural numbers $\{1, 2, 3, \dots\}$.
- (2) The symbol I denotes the unit interval $[0, 1]$.
- (3) The symbol $\overline{A}$ stands for the topological closure of A .
- (4) The symbol $\text{int}A$ denotes the topological interior of A .
- (5) The symbol BdA stands for the topological boundary of A .
- (6) The symbol $A - B$ denotes the set theoretical difference of A minus B ; that is, $A - B = \{x \in A: x \notin B\}$.
- (7) The symbol $\prod_{i \in S} X_i$ stands for the Cartesian product of the collection $\{X_i: i \in S\}$; in the case that $X_i = X$ for all $i \in S$, we will write X^S instead of $\prod_{i \in S} X_i$. If S is countable and each X_i is a bounded metric space, then we assume that the metric d_∞ for $\prod_{i \in S} X_i$ is given by

$$d_\infty \left((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty \right) = \sum_{i \in S} \frac{d(x_i, y_i)}{2^i}.$$

DEFINITION 1.2. A continuum is a compact connected metric space.

DEFINITION 1.3. Given a continuum X , we define the hyperspace 2^X , called the *hyperspace of compact subsets of X* , the hyperspace $C(X)$, called the *hyperspace of subcontinua of X* , and, for each $n \in \mathbb{N}$, the hyperspace $F_n(X)$, called the n^{th} *symmetric product of X* , as follows:

$$2^X = \{A \subset X: A \text{ is nonempty and compact}\},$$

$$C(X) = \{A \in 2^X: A \text{ is connected}\},$$

and

$$F_n(X) = \{A \in 2^X: A \text{ has at most } n \text{ points}\}.$$

The topology of these hyperspaces is obtained from the Hausdorff metric induced by the metric on X . We define the *Hausdorff metric* as follows: Let d denote the metric for X . If $\varepsilon > 0$ and $A \in 2^X$, then

$$N_d(\varepsilon, A) = \{x \in X: d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

Now, for $A, B \in 2^X$, we define the Hausdorff distance between A and B by

$$H_d(A, B) = \inf\{\varepsilon > 0: A \subset N_d(\varepsilon, B) \text{ and } B \subset N_d(\varepsilon, A)\}.$$

The Hausdorff metric for $C(X)$ and $F_n(X)$ is the metric that they inherit as subspaces of 2^X .

DEFINITION 1.4. A *retraction* is a continuous function r from a space, Y , into Y such that r is the identity map on its range, that is, $r(r(y)) = r(y)$ for each $y \in Y$. A subset Z of Y is said to be a *retract* of Y provided there is a retraction of Y onto Z . A continuum X is called an *absolute retract* (written *AR*) provided whenever X is embedded in a metric space Y , the embedded copy of X is a retract of Y .

DEFINITION 1.5. An *arc* is a space homeomorphic to I .

DEFINITION 1.6. If S is a connected space and $s \in S$, then the *component number of s (in S)* is the cardinality of the set of all components of $S - \{s\}$.

DEFINITION 1.7. Let S be a connected topological space and let $s \in S$. If $S - \{s\}$ is connected, then s is called a *non-cut point of S* . If $S - \{s\}$ is not connected, then s is called a *cut point of S* .

DEFINITION 1.8. A *half ray* is a space homeomorphic to $[0, \infty)$.

DEFINITION 1.9. A continuum is said to be *decomposable* provided that it is the union of two proper subcontinua. A continuum that it is not decomposable is said to be *indecomposable*.

DEFINITION 1.10. Let (X, T) be a topological space, and let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of X . We define the *limit inferior of $\{A_i\}_{i=1}^{\infty}$* , denoted by $\liminf A_i$, and the *limit superior of $\{A_i\}_{i=1}^{\infty}$* , denoted by $\limsup A_i$, as follows:

$\liminf A_i = \{x \in X: \text{for any } U \in T \text{ such that } x \in U, U \cap A_i \neq \emptyset \text{ for all but finitely many } i\};$
 $\limsup A_i = \{x \in X: \text{for any } U \in T \text{ such that } x \in U, U \cap A_i \neq \emptyset \text{ for infinitely many } i\}.$

DEFINITION 1.11. A *map* is a continuous function.

DEFINITION 1.12. An *n-od* ($2 \leq n < \infty$) is a continuum, X , for which there is a subcontinuum, Z , such that $X - Z = \bigcup_{i=1}^n U_i$, $U_i \neq \emptyset$ for each $i \in \{1, \dots, n\}$, and $\overline{U_i} \cap U_j = \emptyset$ whenever $i \neq j$. A 3-od is also called a *trioid*.

DEFINITION 1.13. A continuum X is *atriodic* provided that it contains no triods.

DEFINITION 1.14. Let X be a continuum. A subset A of X is called *nowhere dense* in X if its closure has empty interior.

DEFINITION 1.15. Let X be a continuum and let $\mathcal{H} \subset 2^X$. An *order arc* in $\mathcal{H}$ is an arc α in $\mathcal{H}$ such that if $A, B \in \alpha$, then $A \subset B$ or $B \subset A$.

DEFINITION 1.16. A continuum X is said to be *semi-locally-connected at a point* x of X provided that for any $\varepsilon > 0$ there exist a neighborhood V of x in X of diameter less than ε such that $X - V$ has only a finite number of components. If X is semi-locally-connected at each of its points, it is said to be *semi-locally-connected*.

DEFINITION 1.17. Let S be a connected topological space. An *E_0 -set in S* is a nondegenerate connected subset of S which has no cut point and is maximal with respect to the property of being a connected subset of S without cut points.

DEFINITION 1.18. Let X be a semi-locally-connected continuum. By a *cyclic element of X* is meant a cut point of X , an end point of X or an E_0 -set in X . The cut points and the end points of X are called *degenerate cyclic elements of X* and the E_0 -sets are called *true cyclic elements of X* .

DEFINITION 1.19. For each $n \in \mathbb{N}$, an *n-cell* is a space homeomorphic to I^n .

DEFINITION 1.20. If A_1 , A_2 , and B are mutually disjoint subsets of a space X , we say that A_1 and A_2 are *separated* in X by B if $X - B$ can be written as the union of two disjoint open sets in $X - B$ containing A_1 and A_2 , respectively.

DEFINITION 1.21. Let X be a continuum. A subcontinuum A of X is *terminal* provided that if $B \in C(X)$ and $A \cap B \neq \emptyset$, then $A \subset B$ or $B \subset A$.

DEFINITION 1.22. Let X be a continuum and let $\mathcal{H} \subset 2^X$. A *Whitney map* for $\mathcal{H}$ is a continuous function $\mu: \mathcal{H} \longrightarrow [0, \infty)$ that satisfies the following two conditions:

- (1) for any $A, B \in \mathcal{H}$ such that $A \subsetneq B$, $\mu(A) < \mu(B)$;
- (2) $\mu(A) = 0$ if and only if $A \in \mathcal{H} \cap F_1(X)$.

It is a well known fact that for any continuum X , there is a Whitney map for any hyperspace of X ([8], p. 107).

CHAPTER 2

Models for $C_p(X)$ for atriodic continua

DEFINITION 2.1. Let X be a continuum. For $K \in C(X)$ define the *containment hyperspaces* as follows:

$$2_K^X = \{A \in 2^X: K \subset A\} \text{ and } C(K, X) = \{A \in C(X): K \subset A\}.$$

More precisely, 2_K^X is the *containment hyperspace for K in 2^X* and $C(K, X)$ is the *containment hyperspace for K in $C(X)$* .

NOTATION 2.2. For convenience, we denote $2_{\{p\}}^X$ simply by 2_p^X and we denote $C(\{p\}, X)$ simply by $C_p(X)$.

The hyperspaces $C_p(X)$ have not been largely investigated. Nevertheless, there are some known results about them; the following two are among the most important for us:

THEOREM 2.3. ([3], p. 220) *Let X be a continuum and let $p \in X$. Then $C_p(X)$ is an AR.*

Recently, in [21], Patricia Pellicer-Covarrubias proved the following theorem:

THEOREM 2.4. *A continuum X is atriodic if and only if for each $p \in X$ there exists a map $g_p: I \longrightarrow I$ such that $g_p(0) = 0 = g_p(1)$ and $C_p(X)$ is homeomorphic to*

$$\{(r, s) \in I \times I: 0 \leq s \leq g_p(r)\}.$$

Also in [21], Patricia Pellicer-Covarrubias asked the following question:

QUESTION 2.5. *Let $g: I \longrightarrow I$ be a map such that $g(0) = 0 = g(1)$ and let*

$$K = \{(r, s) \in I \times I: 0 \leq s \leq g(r)\}.$$

Does there exist a continuum X and a point $p \in X$ such that $C_p(X)$ is homeomorphic to K ?

In this chapter, we answer this question affirmatively, giving a characterization of $C_p(X)$ when X is an atriodic continuum. We begin with some general properties of $C_p(X)$; then we give several examples, and after that we construct all possible models for $C_p(X)$ when X is an atriodic continuum. At the end of the chapter we investigate when the planarity of $C_p(X)$ implies the planarity of X .

It is surprising to note that the models we obtain for $C_p(X)$ when X is an atriodic continuum are exactly the same as those that Sam B. Nadler, Jr. and Thelma West obtained for the size levels of an arc [20] – see Theorem 2.33.

General results

In this section we prove some general properties of $C_p(X)$. We use the properties in our characterization in the next section.

LEMMA 2.6. *Let X be a continuum. If X contains an n -od Y , then there is a $p \in Y$ such that $C_p(X)$ contains an n -cell.*

PROOF. Since X contains an n -od Y , we have that there is a subcontinuum Z of Y such that $Y - Z = \bigcup_{i=1}^n U_i$, $U_i \neq \emptyset$ for each $i \in \{1, \dots, n\}$, and $\overline{U_i} \cap U_j = \emptyset$ whenever $i \neq j$. For each $i \in \{1, \dots, n\}$, let V_i be a component of U_i . We know from 5.9 of [18] that $Z \cup V_i$ is a continuum for each $i \in \{1, \dots, n\}$. For each $i \in \{1, \dots, n\}$, let $\alpha_i: I \rightarrow C(X)$ be an order arc from Z to $Z \cup V_i$. Define $f: I^n \rightarrow C(X)$ by $f(x_1, \dots, x_n) = \bigcup_{i=1}^n \alpha_i(x_i)$. By 1.23 of [8], f is continuous. It is easy to see that f is a one-to-one function from an n -cell into $C(X)$. Hence, $f(I^n)$ is an n -cell in $C(X)$. Note that for every $A \in f(I^n)$, $Z \subset A$. Therefore, for each $p \in Z$, $f(I^n) \subset C_p(X)$. $\square$

DEFINITION 2.7. An *inverse sequence* is a double sequence $\{X_i, f_i\}_{i=1}^\infty$ of continua X_i , called *coordinate spaces*, and maps $f_i: X_{i+1} \rightarrow X_i$ called *bounding maps*. If $\{X_i, f_i\}_{i=1}^\infty$ is an inverse sequence, then the *inverse limit* of $\{X_i, f_i\}_{i=1}^\infty$, denoted by $\varprojlim \{X_i, f_i\}_{i=1}^\infty$, is the subcontinuum of the Cartesian product space $\prod_{i \in \mathbb{N}} X_i$ defined by

$$\varprojlim \{X_i, f_i\}_{i=1}^\infty = \left\{ (x_i)_{i=1}^\infty \in \prod_{i \in \mathbb{N}} X_i : f_i(x_{i+1}) = x_i \text{ for all } i \right\}.$$

NOTATION 2.8. Let $\{X_i, f_i\}_{i=1}^\infty$ be an inverse sequence. For each $n \in \mathbb{N}$, let

$$\pi_n: \varprojlim \{X_i, f_i\}_{i=1}^\infty \rightarrow X_n$$

denote the restriction to $\varprojlim \{X_i, f_i\}_{i=1}^\infty$ of the n^{th} projection map of $\prod_{i \in \mathbb{N}} X_i$ to X_n , i.e., $\pi_n((x_i)_{i=1}^\infty) = x_n$.

NOTATION 2.9. Let X be a continuum, let K be a subcontinuum of X , and assume that $X = \varprojlim \{X_i, f_i\}_{i=1}^\infty$. For each $i \in \mathbb{N}$, let $K_i = \pi_i(K)$. Also, for each $i \in \mathbb{N}$, let

$$f_i^*: 2_{K_{i+1}}^{X_{i+1}} \longrightarrow 2_{K_i}^{X_i}$$

be given by $f_i^*(A) = f_i(A)$ for each $A \in 2_{K_{i+1}}^{X_{i+1}}$. Note that for each $i \in \mathbb{N}$, the restriction of f_i^* to $C(K_{i+1}, X_{i+1})$ is a map, which we call $\widehat{f}_i$, from $C(K_{i+1}, X_{i+1})$ onto $C(K_i, X_i)$. Also, note that $\{2_{K_i}^{X_i}, f_i^*\}_{i=1}^\infty$ and $\{C(K_i, X_i), \widehat{f}_i\}_{i=1}^\infty$ are inverse sequences. Denote $\varprojlim \{2_{K_i}^{X_i}, f_i^*\}_{i=1}^\infty$ by $(2_K^X)^\infty$ and denote $\varprojlim \{C(K_i, X_i), \widehat{f}_i\}_{i=1}^\infty$ by $C^\infty(K, X)$. Thus, by definition

$$(2_K^X)^\infty = \varprojlim \{2_{K_i}^{X_i}, f_i^*\}_{i=1}^\infty \text{ and } C^\infty(K, X) = \varprojlim \{C(K_i, X_i), \widehat{f}_i\}_{i=1}^\infty.$$

We will consider $C^\infty(K, X)$ as being contained in $(2_K^X)^\infty$ by inclusion. Finally, for convenience, we denote $(2_{\{p\}}^X)^\infty$ simply by $(2_p^X)^\infty$ and we denote $C^\infty(\{p\}, X)$ simply by $C_p^\infty(X)$.

THEOREM 2.10. Let X be a continuum, let K be a subcontinuum of X , and assume that $X = \varprojlim \{X_i, f_i\}_{i=1}^\infty$. For each $i \in \mathbb{N}$, let $K_i = \pi_i(K)$. Let $(2_K^X)^\infty$ and $C^\infty(K, X)$ be as in Notation 2.9. Then, $(2_K^X)^\infty$ is homeomorphic to 2_K^X and $C^\infty(K, X)$ is homeomorphic to $C(K, X)$. Furthermore, there is a homeomorphism of $(2_K^X)^\infty$ onto 2_K^X such that $h(C^\infty(K, X)) = C(K, X)$.

PROOF. Let $A = (A_i)_{i=1}^\infty \in (2_K^X)^\infty$. Then $f_i^*(A_{i+1}) = A_i$ for each $i \in \mathbb{N}$; hence, $f_i(A_{i+1}) = A_i$ for each $i \in \mathbb{N}$. Therefore, $\{A_i, f_i \mid A_{i+1}\}_{i=1}^\infty$ is an inverse sequence. Also, since $A \in (2_K^X)^\infty$, $A_i \in 2_{K_i}^{X_i}$. Since $X = \varprojlim \{X_i, f_i\}_{i=1}^\infty$, it follows that $(\varprojlim \{A_i, f_i \mid A_{i+1}\}_{i=1}^\infty) \in 2_K^X$. Define $h: (2_K^X)^\infty \longrightarrow 2_K^X$ by

$$h(A) = \varprojlim \{A_i, f_i \mid A_{i+1}\}_{i=1}^\infty.$$

We have that h is equal to the restriction to $(2_K^X)^\infty$ of the function that Sam B. Nadler, Jr., define in Theorem 1.169 of [17]; which he proves is one-to-one and continuous (p. 172-174). Using similar arguments as in (d) of Nadler's proof (p. 174) we can show that $h(C^\infty(K, X)) = C(K, X)$. Therefore, h is a homeomorphism between $(2_K^X)^\infty$ and 2_K^X that sends $C^\infty(K, X)$ to $C(K, X)$. $\square$

LEMMA 2.11. *If $X = \varprojlim \{X_i, f_i\}_{i=1}^\infty$ and for each $i \in \mathbb{N}$ there is a homeomorphism $h_i: X_i \longrightarrow Y_i$, then X is homeomorphic to $\varprojlim \{Y_i, \psi_i\}_{i=1}^\infty$ where $\psi_i = h_{i+1}^{-1} \circ f_i \circ h_i$.*

In [2] (p.244), C. E. Capel proved the following fact about inverse limit of arcs.

LEMMA 2.12. *If $X = \varprojlim \{A_i, f_i\}_{i=1}^\infty$ where each A_i is an arc and each f_i is monotone, then X is arc.*

In [4] (p. 257), M. K. Fort, Jr. and Jack Segal proved the following analogue of Lemma 2.12 for 2-cells.

LEMMA 2.13. *If $X = \varprojlim \{B_i, f_i\}_{i=1}^\infty$ is locally connected, each B_i is a 2-cell with boundary J_i and $f_{i+1}(J_{i+1}) = J_i$, then X is a 2-cell with boundary $J = \varprojlim \{J_i, f_i \mid J_{i+1}\}_{i=1}^\infty$.*

In our journey to the characterization of $C_p(X)$ we discovered the following fact, which helps us realize that a continuum X for which $C_p(X)$ has the properties described in Theorem 2.33 must be indecomposable. We do not use the proposition in the chapter; we include it because we think it is of interesting in itself.

PROPOSITION 2.14. *Let X be a continuum. If $\{K_i\}_{i=1}^\infty$ is a sequence of terminal subcontinua of X such that, for each $i \in \mathbb{N}$, $K_i \subsetneq K_{i+1}$ and $\bigcup_{i=1}^\infty K_i = X$, then X is indecomposable.*

PROOF. Define $K = \bigcup_{i=1}^\infty K_i$. Since each K_i is nowhere dense in X , by Baire's Category Theorem ([9], p. 414), there exist $p \in X - K$. Let A be a proper subcontinuum of X such that $p \in A$. Assume $A \cup K_n \neq \emptyset$ for some $n \in \mathbb{N}$. Then, since $p \notin K$, $K_n \subset A$. Hence, $K_i \subset A$ for all $i \in \mathbb{N}$. Thus, $K \subset A$; therefore, we have that $X = A$, which is a contradiction. Therefore, every proper subcontinuum A that contains p is nowhere dense in X (since $A \subset X - K$). Therefore, X is indecomposable. $\square$

Examples

In this section we construct some models for $C_p(X)$. Even though they are models for simple continua, they give an insight into how we construct the models in our characterization of $C_p(X)$ when X is an atriodic continuum.

EXAMPLE 2.15. *$C_0(I)$ is an arc and, for each p different from 0 and 1, $C_p(I)$ is a 2-cell.*

PROOF. Clearly, the subcontinua of I that contain 0 are of the form $[0, a]$ with $a \in I$; hence, they are completely determined by the end point different from 0 (or by 0 if the subcontinuum is $\{0\}$). Therefore, $C_0(I)$ is an arc by the homeomorphism $h(t) = [0, t]$.

When $0 < p < 1$, the subcontinua of I that contain p are of the form $[a, b]$; so, they are completely determined by their end points. Therefore, $C_p(I)$ is a 2-cell, as is seen from the homeomorphism $h: [0, p] \times [p, 1] \longrightarrow C_p(I)$ given by $h(s, t) = [s, t]$. $\square$

EXAMPLE 2.16. *Let $X = (\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) : x \in (0, 1]\}$. Then $C_{(0,0)}(X)$ is the join of a 2-cell and an arc at a boundary point of each.*

PROOF. There are two types of subcontinua of X containing $(0, 0)$: the subcontinua that are contained in $\{0\} \times [-1, 1]$ and the subcontinua that properly contain $\{0\} \times [-1, 1]$. The subcontinua of X containing $(0, 0)$ that are contained in $\{0\} \times [-1, 1]$ are of the form $\{0\} \times [a, b]$ with $0 \in [a, b]$; so, they are completely determined by the end points of the interval; hence, they form a 2-cell in $C_{(0,0)}(X)$. Next, the subcontinua of X containing $\{0\} \times [-1, 1]$ are of the form $(\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) : x \in (0, c]\}$ with $c \in (0, 1]$; so, they are completely determined by c ; hence, they form an arc in $C_{(0,0)}(X)$. Therefore, $C_{(0,0)}(X)$ is the join of a 2-cell and an arc at a boundary point of each. $\square$

EXAMPLE 2.17. *Let $X = (\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) : 0 < |x| \leq 1\}$. Then $C_{(0,1)}(X)$ is the join of a 2-cell and an arc at a boundary point of each; $C_{(0,0)}(X)$ is the join of two disjoint 2-cells at a boundary point of each.*

PROOF. As in Example 2.16, there are two types of subcontinua of X containing $(0, 1)$: the subcontinua that are contained in $\{0\} \times [-1, 1]$ and the subcontinua that properly contain $\{0\} \times [-1, 1]$. The subcontinua of X containing $(0, 1)$ that are contained in $\{0\} \times [-1, 1]$ are of the form $\{0\} \times [a, 1]$ with $a \in [-1, 1]$; so, they are completely determined by the end point of the interval different from 1 (or by 1 if the subcontinuum is $\{(0, 1)\}$); hence, they form an arc in $C_{(0,1)}(X)$ with end points $\{(0, 1)\}$ and $\{0\} \times [-1, 1]$. Next, the subcontinua of X containing $\{0\} \times [-1, 1]$ are of the form $(\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) : x \in [b, 0) \cup (0, c]\}$ with $b \in [-1, 0)$ and $c \in (0, 1]$; so, they are completely determined by b and c ; hence, they form a 2-cell in $C_{(0,1)}(X)$. Therefore, $C_{(0,1)}(X)$ is the join of a 2-cell and an arc at the point $\{0\} \times [-1, 1]$, which is a boundary point of each.

There are two types of subcontinua of X containing $(0, 0)$, the ones that are contained in $\{0\} \times [-1, 1]$ and the ones that contain $\{0\} \times [-1, 1]$. The subcontinua of X containing

$(0, 0)$ that are contained in $\{0\} \times [-1, 1]$ are of the form $\{0\} \times [a, b]$ with $0 \in [a, b]$; so, they are completely determined by the end points of the interval $[a, b]$; hence, they form a 2-cell in $C_{(0,0)}(X)$. Next, the subcontinua of X containing $\{0\} \times [-1, 1]$ are of the form $(\{0\} \times [-1, 1]) \cup \{(x, \sin \frac{1}{x}) : x \in [c, 0) \cup (0, d]\}$ with $c \in [-1, 0)$ and $d \in (0, 1]$; so, they are completely determined by c and d ; hence, they form a 2-cell in $C_{(0,0)}(X)$. Therefore, $C_{(0,0)}(X)$ is the join of two disjoint 2-cells at the point $\{0\} \times [-1, 1]$, which is a boundary point of each. $\square$

Arc-like continua

To answer Question 2.5 we will prove the following theorem:

THEOREM 2.18. *For every map $g: I \longrightarrow I$ such that $g(0) = 0 = g(1)$ there exists an arc-like continuum (hence, atriodic) X and a point $p \in X$ such that $C_p(X)$ is homeomorphic to $\{(r, s) \in I \times I : 0 \leq s \leq g(r)\}$.*

It is important to note that to prove Theorem 2.18 we can work with closed subsets of I instead of maps from I into I . To explain this we need the following definitions, which we will use later in the section.

DEFINITION 2.19. If $g: I \longrightarrow I$ is a map, then define

$$K(g) = \{(r, s) \in I \times I : 0 \leq s \leq g(r)\}.$$

DEFINITION 2.20. If Z is a closed subset of I , then define

$$K(Z) = \{(r, s) \in I \times I : 0 \leq s \leq d(r, Z)\}.$$

PROPOSITION 2.21. *If $g: I \longrightarrow I$ is a map and we let $Z = g^{-1}(0)$, then $K(g)$ is homeomorphic to $K(Z)$.*

PROOF. The homeomorphism $h: K(Z) \longrightarrow K(g)$ is given by $h(x, y) = (x, f_x(y))$ where $f_x(t) = 0$ for $x \in Z$ and $f_x(t) = \frac{g(x)}{d(x, Z)} t$ otherwise. $\square$

Before we can prove Theorem 2.18 we have to build the tools that we are going to need in the proof.

DEFINITION 2.22. Let X be a continuum. A *chain in X* is a nonempty, finite, indexed collection $\mathcal{C} = \{U_1, \dots, U_n\}$ of open subsets U_i of X such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. If $\mathcal{C} = \{U_1, \dots, U_n\}$ is a chain in X , then the members of $\mathcal{C}$ are called *links of $\mathcal{C}$* , U_i being called the i^{th} link of $\mathcal{C}$; the *mesh of $\mathcal{C}$* , written $\text{mesh}(\mathcal{C})$, is defined by $\text{mesh}(\mathcal{C}) = \max\{\text{diameter}(U_i) : 1 \leq i \leq n\}$. If $\mathcal{C}$ is a chain in X and $\text{mesh}(\mathcal{C}) < \varepsilon$, then we say that $\mathcal{C}$ is an ε -chain in X . We say that X is *chainable* provided that X is nondegenerate and for each $\varepsilon > 0$ there is an ε -chain in X covering X .

DEFINITION 2.23. Let X and Y be metric spaces and let $f: X \rightarrow Y$. Then f is called an ε -map provided that f is continuous and the diameter of $f^{-1}(f(x))$ is less than ε for all $x \in X$.

DEFINITION 2.24. Let X be a compact metric space and let $\mathcal{P}$ be a given collection of compact metric spaces. Then X is said to be $\mathcal{P}$ -like provided that for each $\varepsilon > 0$ there is an ε -map from X onto some member Y_ε of $\mathcal{P}$.

It is known that a continuum is arc-like if and only if it is chainable ([18], p. 235).

The following is a well known fact ([18], p. 24-25). We include the proof for completeness.

LEMMA 2.25. Let $\{X_i, f_i\}_{i=1}^\infty$ be an inverse sequence with surjective bonding maps such that each X_i is $\mathcal{P}$ -like. Then $X = \varprojlim \{X_i, f_i\}_{i=1}^\infty$ is $\mathcal{P}$ -like.

PROOF. Let $\varepsilon > 0$. Then there is an $n \in \mathbb{N}$ such that the natural projection $\pi_n: X \rightarrow X_n$ is an ε -map. Let d denote the metric for X_n and d_∞ the metric for X . Since π_n is an ε -map, there is $\delta > 0$ such that if $x, y \in X_n$ and $d(x, y) < \delta$, then $d_\infty(\pi_n^{-1}(x), \pi_n^{-1}(y)) < \frac{\varepsilon}{2}$. Since X_n is $\mathcal{P}$ -like, there is a δ -map f from X_n onto some member Y_δ of $\mathcal{P}$. Define $g = f \circ \pi_n$. If $t \in Y_\delta$, then the diameter of $f^{-1}(t) < \delta$; this implies that the diameter of $\pi_n^{-1}(f^{-1}(t)) < \varepsilon$. Hence, g is an ε -map from X onto Y_δ . Therefore, X is $\mathcal{P}$ -like. $\square$

DEFINITION 2.26. We say that a chain $\{U_1, \dots, U_n\}$ is a *chain from the point a to the point b* provided that $U_1 - U_2$ contains a and $U_n - U_{n-1}$ contains b . The points a and b of the arc-like continuum X are called *opposite end points of X* if for each $\varepsilon > 0$ there is an ε -chain from a to b covering X .

The idea for the construction in the proof of the following lemma comes from the proof of Lemma 2 in [1].

LEMMA 2.27. *Let X be an arc-like continuum with opposite end points a and b or let X consist of a single point. Then there is an arc-like subcontinuum $M = X \cup A \cup B$ of $X \times [-1, 1]$ such that $X \cup A$ and $X \cup B$ are subcontinua of M , X is nowhere dense in $X \cup A$ and in $X \cup B$, and the Cartesian product projections from $X \cup A$ onto X , from $X \cup B$ onto X and from M onto X are retractions.*

PROOF. If $X = \{p\}$, then let $M = \{p\} \times [-1, 1]$. Identify X with $X \times \{0\}$ and define $A = \{p\} \times (0, 1]$ and $B = \{p\} \times [-1, 0)$. Clearly, M , $X \cup A$ and $X \cup B$ are arcs that have the required properties.

Now, assume that X is an arc-like continuum with opposite end points a and b . In the Cartesian product $X \times [-1, 1]$, identify X with $X \times \{0\}$ and consider the following subcontinua: For each $i \in \mathbb{Z} - \{0\}$, let

$$X_i = X \times \left\{ \frac{1}{i} \right\}$$

and, for each $i \in \mathbb{N}$,

$$\text{let } K_i = \{a\} \times \left[\frac{1}{i+1}, \frac{1}{i} \right] \text{ and } L_i = \{a\} \times \left[\frac{-1}{i}, \frac{-1}{i+1} \right] \text{ if } i \text{ is odd}$$

and

$$\text{let } K_i = \{b\} \times \left[\frac{1}{i+1}, \frac{1}{i} \right] \text{ and } L_i = \{b\} \times \left[\frac{-1}{i}, \frac{-1}{i+1} \right] \text{ if } i \text{ is even.}$$

Set

$$A = \bigcup_{i \in \mathbb{N}} (X_i \cup K_i) \text{ and } B = \bigcup_{i \in \mathbb{N}} (X_{-i} \cup L_i).$$

Let $M = X \cup A \cup B$ (see Figure 1 top of the next page).

Clearly, M , $X \cup A$ and $X \cup B$ are continua, X is nowhere dense in $X \cup A$ and in $X \cup B$ and the Cartesian product projections from $X \cup A$ to X , from $X \cup B$ to X and from M to X are retractions.

Now, to prove that M is arc-like, take $\varepsilon > 0$. Since X is arc-like and a and b are opposite end points of X , there is an $\frac{\varepsilon}{2}$ -chain, $\mathcal{U} = \{U_1, \dots, U_n\}$, in X covering X such that $a \in U_1$ and $b \in U_n$. Let $m \in \mathbb{N}$ be such that $\frac{1}{m} < \frac{\varepsilon}{2}$. Suppose without loss of generality that m is odd. For each $i \in \{1, \dots, n\}$, let $V_i = U_i \times \left(\frac{-(2m+3)}{2(m+1)(m+2)}, \frac{2m+1}{2m(m+1)} \right)$. Consider the chain $\mathcal{V}$ in M given by $\mathcal{V} = \{V_1, \dots, V_n\}$. Since K_m and L_{m+1} are arcs, there are chains, $\mathcal{O} = \{O_1, \dots, O_j\}$ and $\mathcal{P} = \{P_1, \dots, P_k\}$, in M covering K_m and L_{m+1} , respectively, such that their mesh is less than $\frac{1}{(m+2)(m+3)}$, $(a, \frac{1}{m}) \in O_1$, $(a, \frac{1}{m+1}) \in O_j$, $(b, \frac{-1}{m+2}) \in P_1$ and $(b, \frac{-1}{m+1}) \in P_k$. Let r be

FIGURE 1

the last index in $\{1, \dots, j\}$ such that O_r is not contained in V_1 and let s be the first index in $\{1, \dots, k\}$ such that P_s is not contained in V_n . Let $\mathcal{H} = \{H_1, \dots, H_p\}$ and $\mathcal{G} = \{G_1, \dots, G_q\}$ be chains, in M covering X_m and $X_{-(m+1)}$ respectively, of mesh less than $\frac{1}{(m+2)(m+3)}$ such that $(b, \frac{1}{m}) \in H_1$, $(a, \frac{1}{m}) \in H_p$, $(b, \frac{-1}{m+1}) \in G_1$ and $(a, \frac{-1}{m+1}) \in G_q$. Let e be the last index in $\{1, \dots, p\}$ such that H_e is not contained in O_1 and let f be the first index in $\{1, \dots, q\}$ such that G_f is not contained in P_k . Note that

$$\mathcal{W} = \{H_1, \dots, H_e, O_1, \dots, O_r, V_1, \dots, V_n, P_s, \dots, P_k, G_f, \dots, G_q\}$$

is an ε -chain in M covering $X \cup \left(\bigcup_{i=m}^{\infty} (X_m \cup K_m) \right) \cup \left(\bigcup_{i=m+1}^{\infty} (X_{-i} \cup L_i) \right)$. Continuing in this fashion we can construct an ε -chain in M covering M ; hence, M is an arc-like continuum. $\square$

LEMMA 2.28. *Let X be an arc-like continuum with opposite end points a and b or let X consist of a single point. Then there is an arc-like subcontinuum $M = X \cup A$ of $X \times I$ such that X is nowhere dense in M and the Cartesian product projection from M onto X is a retraction.*

PROOF. If $X = \{p\}$, then let $M = X \times I$. Identify X with $X \times \{0\}$ and define $A = \{p\} \times (0, 1]$. Clearly, M is an arc with the required properties.

Now, assume that X is an arc like continuum with opposite end points a and b . Let A be as in the proof of Lemma 2.27, and take $M = X \cup A$. The verifications of the properties of M are similar to the verifications in Lemma 2.27. $\square$

LEMMA 2.29. *For every function h from a finite nonempty linearly ordered set $S = \{s_1, \dots, s_n\}$ into $\{0, 1\}$ there exist an arc-like continuum $X_h \subset [-1, 1]^S$ and a point $p_h \in X_h$ such that $C_{p_h}(X)$ is homeomorphic to $K(g_h^{-1}(0))$, where g_h is the function from I to $\{0, 1\}$ defined by $g_h(t) = h(s_i)$ if $t \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$ and $g_h(t) = 0$ if $t \in \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$. Thus, $C_{p_h}(X_h)$ is a system of continua C_i , where C_i is an $(2 - h(s_i))$ -cell and two consecutive continua are joined at a boundary point of each.*

Moreover, if, for every $j \in \{1, \dots, n\}$, λ_j is the Cartesian product projection from $[-1, 1]^S$ onto $[-1, 1]^{S-\{s_j\}} \times \{0\}^{\{s_j\}}$, then $X_{h|S-\{s_j\}}$ is homeomorphic to $\lambda_j(X_h)$ and $C_{p_h|S-\{s_j\}}(X_{h|S-\{s_j\}})$ is homeomorphic to $K(g_{h|S-\{s_j\}}^{-1}(0))$.

PROOF. Assume without loss of generality that $s_1 < \dots < s_n$. Let h be a function from S into $\{0, 1\}$. Define $M_0 = \{p_0\}$. Take $k \in \{1, \dots, n\}$. Suppose that $M_0, \dots, M_{k-1}$ have already been defined. We have that $h(s_k)$ is equal to either 0 or 1. If $h(s_k) = 0$, then define M_k to be the arc-like continuum that we obtain when we apply Lemma 2.27 to M_{k-1} and let $p_k = (p_{k-1}, 0)$. If $h(s_k) = 1$, then define M_k to be the arc-like continuum that we obtain when we apply Lemma 2.28 to M_{k-1} and let $p_k = (p_{k-1}, 0)$.

So, we have constructed arc-like continua $M_1 \subset \dots \subset M_n$ with the following properties:

(1) For each $i \in \{1, \dots, n\}$, M_i is a subcontinuum of $M_{i-1} \times [-1, 1]$; hence, M_i is a subcontinuum of $[-1, 1]^i$, and

(2) $M_{i-1} \times \{0\}$ is nowhere dense in $M_i \cap (M_{i-1} \times I)$ and, if M_i was constructed by applying Lemma 2.27 to M_{i-1} , then $M_i \times \{0\}$ is nowhere dense in $M_i \cap (M_{i-1} \times [-1, 0])$.

Define $X_h = M_n$ and $p_h = p_n$. From the ideas of Examples 2.15, 2.16 and 2.17, it is easy to see that $C_p(X)$ is homeomorphic to $K(g_h^{-1}(0))$.

Now, take $j \in \{1, \dots, n\}$ and construct $X_{h|S-\{s_j\}}$ and $p_{h|S-\{s_j\}}$ in the same way we constructed X_h and p_h . Again, following the ideas of Examples 2.15, 2.16 and 2.17, it is not difficult to see that $C_{p_{h|S-\{s_j\}}}(X_{h|S-\{s_j\}})$ is homeomorphic to $K(g_{h|S-\{s_j\}}^{-1}(0))$. Define $\lambda_j: [-1, 1]^S \longrightarrow [-1, 1]^{S-\{s_j\}} \times \{0\}^{\{s_j\}}$ by $\lambda_j((t_{s_i})_{s_i \in S}) = (r_{s_i})_{s_i \in S}$ where $r_{s_i} = t_{s_i}$ if $i \neq j$ and $r_{s_j} = 0$. Also, define $\varphi: X_{h|S-\{s_j\}} \longrightarrow \lambda_j(X_h)$ by $\varphi((t_{s_i})_{s_i \in S-\{s_j\}}) = (r_{s_i})_{s_i \in S}$ where

$r_{s_i} = t_{s_i}$ if $i \neq j$ and $r_{s_j} = 0$. Clearly, the function φ is a continuous bijection; hence, φ is a homeomorphism. $\square$

NOTATION 2.30. If Z is a closed subset of I , then

- let $D_0(Z)$ be the set of all components of the interior of $I - Z$ and
- let $D_1(Z)$ be the set of all components of the interior of Z .
- Define $D(Z) = D_0(Z) \cup D_1(Z)$.

Note that $D(Z)$ is nonempty, linearly ordered (with the order that $D(Z)$ implicitly inherits from I) and has at most countably many elements.

Now, we are ready to prove Theorem 2.18.

PROOF OF THEOREM 2.18. Let $Z = g^{-1}(0)$. Define $h: D(Z) \longrightarrow \{0, 1\}$ by $h(J) = 0$ if and only if $J \in D_0(Z)$.

If $D(Z)$ is finite, then let X_h and p_h be the arc-like continuum and the point, respectively, that we obtain when we apply Lemma 2.29 to h . Then $X = X_h$ and $p = p_h$ have the desired properties.

If $D(Z)$ is infinite, then let $\{s_i: i \in \mathbb{N}\}$ be an enumeration of $D(Z)$. Define

$$S_n = \{s_i: i \leq n\} \text{ and } h_n = h \mid S_n.$$

Let X_n and p_n be the arc-like continuum and the point, respectively, that we obtain when we apply Lemma 2.29 to h_n . Also, for each $n \in \mathbb{N}$, we have a map $f_n = \varphi_n^{-1} \circ (\lambda_n \mid X_{n+1})$ from X_{n+1} onto X_n (where λ_n and φ_n are defined as in the proof of Lemma 2.29). We have that $\{X_i, f_i\}_{i=1}^\infty$ is an inverse sequence (see Figures 2 and 3 on the next page) and that $C_{p_i}(X_i)$ is homeomorphic to $K(g_{h_i}^{-1}(0))$ (where g_{h_i} is defined as in Lemma 2.29). For each $i \in \mathbb{N}$, let $Y_i = K(g_{h_i}^{-1}(0))$. Define $X = \varprojlim \{X_i, f_i\}_{i=1}^\infty$ and $p = (p_i)_{i=1}^\infty$. By Lemma 2.25, we have that X is arc-like. We claim that $C_p(X)$ is homeomorphic to $K(Z)$. The proof of this is as follows:

By Lemma 2.10, $C_p(X)$ is homeomorphic to $C_p^\infty(X)$. For each $i \in \mathbb{N}$, let $\zeta_i: C_{p_i}(X_i) \longrightarrow Y_i$ be a homeomorphism. By Lemma 2.11, $C_p^\infty(X)$ is homeomorphic to $Y = \varprojlim \{Y_i, \psi_i\}_{i=1}^\infty$ where $\psi_i = \zeta_i \circ f_i \circ \zeta_{i+1}^{-1}$. Let ϕ be a homeomorphism from $C_p^\infty(X)$ onto Y . Hence, it is enough to show that $K(Z)$ is homeomorphic to Y .

First, observe that if $s_n \in D(Z)$, then the subset C_{s_n} of $K(Z)$ defined by $C_{s_n} = \{(x, y) \in K(Z): x \in \overline{s_n}\}$ is a 2-cell if $s_n \in D_0(Z)$ and is an arc if $s_n \in D_1(Z)$.

FIGURE 2

FIGURE 3

Take a point s_n in $D(Z)$. Define $M_n = \{(x_i)_{s_i \in S_n} : x_{s_n} = 0\}$ and $N_n = X_n$. For each $1 \leq i < n$, define recursively $M_i = f_i(M_{i+1})$ and $N_i = f_i(N_{i+1})$. Also, for each $i > n$, define recursively as follows:

$$M_i = f_i^{-1}(M_{i-1}) \text{ and } N_i = (f_i^{-1}(N_{i-1})) \text{ if } s_i < s_{i-1}$$

and

$$M_i = \lambda_i(f_i^{-1}(M_{i-1})) \text{ and } N_i = \lambda_i(f_i^{-1}(N_{i-1})) \text{ if } s_{i-1} < s_i.$$

Let

$$M(n) = \varprojlim \{M_i, f_i \mid M_{i+1}\}_{i=1}^{\infty}$$

and let

$$N(n) = \varprojlim \{N_i, f_i \mid N_{i+1}\}_{i=1}^{\infty}.$$

If $h(s_n) = 0$, then N_n was constructed by applying Lemma 2.27 to M_n ; hence, $C(M_n, N_n)$ is a 2-cell. Therefore, for each $i > n$, $C(M_i, N_i)$ is a 2-cell. Note that, for each $i > n$, the boundary of $C(M_i, N_i)$ is map under ψ_{i-1} onto the boundary of $\zeta_{i-1}(C(M_{i-1}, N_{i-1}))$ (see Figure 4 on the next page). Hence, by Lemma 2.13, $C^\infty(M(n), N(n)) = \varprojlim \{C(M_i, N_i), \psi_i \mid C(M_{i+1}, N_{i+1})\}_{i=1}^{\infty}$ is a 2-cell.

If $h(s_n) = 1$, then N_n was constructed by applying Lemma 2.28 to M_n ; hence, $C(M_n, N_n)$ is an arc. Therefore, for each $i > n$, $C(M_i, N_i)$ is an arc. Note that, for each $i > n$, $\psi_{i-1} \mid C(M_i, N_i)$ is a monotone map. Hence, by Lemma 2.12, $C^\infty(M(n), N(n)) = \varprojlim \{C(M_i, N_i), \psi_i \mid C(M_{i+1}, N_{i+1})\}_{i=1}^{\infty}$ is an arc.

Therefore, we can define a homeomorphism ϕ_n from C_{s_n} onto $\phi(C^\infty(M(n), N(n)))$.

Now, we can define a homeomorphism Φ between $K(Z)$ and Y as follows:

If $x \in C_{s_i}$ for some $i \in \mathbb{N}$, then define $\Phi(x) = \phi_i(x)$.

The points in $K(Z)$ that are not in one of the C_{s_n} 's are limits of points in the boundary of the C_{s_n} 's; so, we define the homeomorphism Φ for those points to be the limit of the images of the points that converge to them.

Therefore, $K(Z)$ is homeomorphic to Y .

To complete the proof note that in either case X is arc-like; hence, by Theorem 12.4 of [18], X is atriodic. $\square$

Combining Theorem 2.4 with Theorem 2.18, we have the following characterization of $C_p(X)$ when X is an atriodic continuum.

FIGURE 4

COROLLARY 2.31. *If X is an atriodic and $p \in X$, then there exists a map $g_p: I \longrightarrow I$ such that $g_p(0) = 0 = g_p(1)$ and $C_p(X)$ is homeomorphic to $\{(t, r) \in I \times I: 0 \leq r \leq g_p(t)\}$ and, conversely, if $g: I \longrightarrow I$ is a map such that $g(0) = 0 = g(1)$, then there exists an arc-like continuum (hence, atriodic) X and $p \in X$ such that $C_p(X)$ is homeomorphic to $\{(t, r) \in I \times I: 0 \leq r \leq g(t)\}$.*

The following proposition will help us to reformulate Corollary 2.31 in a convenient manner.

PROPOSITION 2.32. *Assume Z is a nondegenerate continuum and consider the following three conditions:*

- (1) *Z is a planar AR.*
- (2) *Cut points of Z have component number two.*
- (3) *Any true cyclic element of Z contains at most two cut points of Z .*

Then there is a map $g: I \longrightarrow I$ such that $g(0) = 0 = g(1)$ and Z is homeomorphic to $K(g)$.

PROOF. First, note that, by Theorem 11 of [10] (p. 534), the true cyclic elements of Z are 2-cells.

In [20] (p. 247), Sam B. Nadler, Jr. and Thelma West proved the following.

- (*) Assume that Z is a nondegenerate continuum satisfying (1)-(3). Then there exist distinct points r and s of Z such that every cut point of Z separates r and s .

Take r and s as in (*). Let A be the set of cut points of Z . Define an order on $A \cup \{r, s\}$ as follows: $x < y$ if and only if $\{x\}$ separates $\{r\}$ and $\{y\}$ in Z , $r < x$ for each $x \in A \cup \{s\}$ and $x < s$ for each $x \in A \cup \{r\}$. It is not difficult to see that $A \cup \{r, s\}$ is a linearly ordered set with a maximal and minimal element. Hence, $A \cup \{r, s\}$ can be embedded in I such that r is sent to 0 and s is sent to 1. Let B denote the homeomorphic image of $A \cup \{r, s\}$ in I . By Proposition 2.21 it is enough to prove that $K(B)$ is homeomorphic to Z . But the points that are in $K(B) - B$ are points that are in 2-cells minus two boundary points which correspond to the points of $Z - (A \cup \{r, s\})$ that are in a true cyclic element minus two points. So, we can define a homeomorphism between $K(B)$ and Z . $\square$

We can now rephrase Corollary 2.31 to show all the possible models for $C_p(X)$ and all the size levels for an arc (as in [20]) are the same.

THEOREM 2.33. *Assume Z is a continuum and consider the following three conditions:*

(1) Z is a planar AR.

(2) Cut points of Z have component number two.

(3) Any true cyclic element of Z contains at most two cut points of Z .

If X is an atriodic continuum and $p \in X$, then $C_p(X)$ satisfies (1)-(3) and, conversely, if Z satisfies (1)-(3), then there exist an arc-like continuum (hence, atriodic) X and a point $p \in X$ such that $C_p(X)$ is homeomorphic to Z .

Arcwise connected continua

In this section we give some results concerning when the planarity of $C_p(X)$ implies the planarity of X .

DEFINITION 2.34. A *half-ray triod* is a continuum which is the union of a half-ray H and an arc A such that $H \cap A = \emptyset$ and $\overline{H} - H$ is a subarc or a point of A which contains neither non-cut point of A .

Note that in particular a half-ray triod is a triod.

Sam B. Nadler Jr. and J. Quinn proved in [19] (p. 224) the following theorem.

THEOREM 2.35. *A continuum X is arcwise connected and contains no half-ray triod if and only if X is either an arc or an arcwise connected circle-like continuum.*

Also, Sam B. Nadler Jr. proved in [16] (p. 233) the following theorem.

THEOREM 2.36. *If X is an arcwise connected circle-like continuum, then X is embeddable in the plane.*

THEOREM 2.37. *Suppose X is an arcwise connected continuum such that $C_p(X)$ is embeddable in the plane for each $p \in X$. Then X is embeddable in the plane.*

PROOF. Since $C(p, X)$ is embeddable in the plane for each $p \in X$, by Lemma 2.6, we have that X is atriodic. Hence; by Theorem 2.35, X is an arc or X is an arcwise connected circle like continuum.

Obviously an arc is embeddable in the plane. If X is an arcwise connected circle like continuum, then X is embeddable in the plane by Theorem 2.36. $\square$

The following example shows that the arcwise connectivity in Theorem 2.37 is essential.

FIGURE 5

EXAMPLE 2.38. *There is a continuum X such that $C_p(X)$ is embeddable in the plane for each $p \in X$, but X is not embeddable in the plane.*

PROOF. Let A be the convex arc in $\mathbb{R}^3$ from $(0, 0, 0)$ to $(0, 1, 0)$ and let B the convex arc in $\mathbb{R}^3$ from $(0, 2, 0)$ to $(0, 1, 0)$. Let $S = \{(x, y, 0): x^2 + y^2 = 1\}$. Define

$$C = \left\{ \frac{t}{t+1}(\cos t, \sin t, 0): t \in [0, \infty) \right\} \text{ and } D = \left\{ \frac{t+2}{t+1}(\sin t, \cos t, 0): t \in [0, \infty) \right\}.$$

Let $X = S \cup A \cup B \cup C \cup D$ (see Figure 5 above).

If $p \in S$, then there are two types of subcontinua that contain p : the subcontinua that are contained in S and the subcontinua that contain S . The subcontinua of X containing p that are contained in S are arcs; so, they are completely determined by their end points; hence, they form a 2-cell in $C_p(X)$. Next, the subcontinua of X containing p that contain S are composed of S and a part of the ray $X - S$; so, they are completely determined by their two end points in the ray; hence, they form a 2-cell in $C_p(X)$. Therefore, $C_p(X)$ is the join of two disjoint 2-cells at a boundary point of each.

If $p \in X - S$, then there are two types of subcontinua of X that contain p : the subcontinua that are contained in the ray $X - S$ and X . The subcontinua of X containing p that are

contained in the ray $X - S$ are arcs; so, they are completely determined by their end points; hence, $C_p(X)$ is a 2-cell. Therefore, $C_p(X)$ is embeddable in the plane for each $p \in X$.

If X were embeddable in the plane, then C and D would be in different components of $\mathbb{R}^2 - S$; hence, by the Jordan Curve Theorem, $(A \cup B) \cap S \neq \emptyset$, which is a contradiction. Therefore, X is not embeddable in the plane. $\square$

CHAPTER 3

Mutual aposyndesis of symmetric products

DEFINITION 3.1. Let X be a continuum. We say that X is:

- (1) *aposyndetic* provided that for any two different points $x, y \in X$ there is a subcontinuum K of X such that $x \in \text{int}K$ and $y \notin K$.
- (2) *mutually aposyndetic* provided that for any two different points $x, y \in X$ there exist disjoint subcontinua K and L of X such that $x \in \text{int}K$, $y \in \text{int}L$.

Aposyndesis was first studied in connection with hyperspaces by Jack T. Goodykoontz, Jr., who proved that 2^X and $C(X)$ are aposyndetic ([5], Theorem 1). Also, Alejandro Illanes has some results about aposyndesis in hyperspaces ([7]). Sergio Macías has recently proved that if X is a chainable continuum such that its second symmetric product is mutually aposyndetic, then X is homeomorphic to I ([11], Theorem 15); therefore, the second symmetric product of a continuum is not always mutually aposyndetic.

Illanes asked the author if $F_n(X)$ is mutually aposyndetic when $n \geq 3$. Our purpose in this chapter is to answer Illanes' question affirmatively.

We note that our result is the analogue for symmetric products of the following fact about Cartesian products: The Cartesian product of three nondegenerate continua is mutually aposyndetic ([6], Theorem 2).

Our proof involves a number of technical details. After the proof, we comment about another, natural approach which, unfortunately, does not work.

DEFINITION 3.2. Let X be a continuum and let $A_1, \dots, A_m$ be subsets of X . Let

$$\langle A_1, \dots, A_m \rangle = \{K \in F_n(X) : K \subset \bigcup_{i=1}^m A_i \text{ and } K \cap A_i \neq \emptyset \text{ for each } i \in \{1, \dots, m\}\}.$$

It is not difficult to see that the following lemma is true.

LEMMA 3.3. *Let X be a continuum. If $A_1, \dots, A_m$ are closed (open) subsets of X then $\langle A_1, \dots, A_m \rangle$ is closed (open) in $F_n(X)$.*

LEMMA 3.4. *Let X be a continuum and let $C_1, \dots, C_n$ be connected subsets of X . If $m \geq n$, then the set $\langle C_1, \dots, C_n \rangle$ is a connected subset of $F_m(X)$.*

PROOF. Take two different points $\{x_1, \dots, x_r\}, \{y_1, \dots, y_s\} \in \langle C_1, \dots, C_n \rangle$. For each $i \in \{1, \dots, n\}$, there is a point $x_{j_i} \in C_i$ for some $j_i \in \{1, \dots, r\}$ and there is a point $y_{k_i} \in C_i$ for some $k_i \in \{1, \dots, s\}$. We are going to prove the following fact:

(*) There is a connected subset of $\langle C_1, \dots, C_n \rangle$ which contains both $\{x_1, \dots, x_r\}$ and $\{x_{j_1}, \dots, x_{j_n}\}$.

To prove (*), assume $\{x_1, \dots, x_r\} \neq \{x_{j_1}, \dots, x_{j_n}\}$. Then there is a point $x_t \in \{x_1, \dots, x_r\} - \{x_{j_1}, \dots, x_{j_n}\}$. We know that $x_t \in C_u$ for some $u \in \{1, \dots, n\}$. Define $f: \{x_{j_1}\} \times \dots \times \{x_{j_n}\} \times C_u \longrightarrow \langle C_1, \dots, C_n \rangle$ by $f(x_{j_1}, \dots, x_{j_n}, c) = \{x_{j_1}, \dots, x_{j_n}, c\}$. It is easy to see that f is continuous. Consider $\mathcal{A} = f(\{x_{j_1}\} \times \dots \times \{x_{j_n}\} \times C_u)$. We have that $\mathcal{A}$ is a connected subset of $\langle C_1, \dots, C_n \rangle$ and that $\{x_{j_1}, \dots, x_{j_n}\}$ and $\{x_{j_1}, \dots, x_{j_n}, x_t\}$ are points of $\mathcal{A}$.

If $\{x_1, \dots, x_r\} = \{x_{j_1}, \dots, x_{j_n}, x_t\}$, then (*) is proved. So, assume that $\{x_1, \dots, x_r\} \neq \{x_{j_1}, \dots, x_{j_n}, x_t\}$. Then there is a point $x_v \in \{x_1, \dots, x_r\} - \{x_{j_1}, \dots, x_{j_n}, x_t\}$. Applying the previous argument to x_v , we can find a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{x_{j_1}, \dots, x_{j_n}, x_t\}$ and $\{x_{j_1}, \dots, x_{j_n}, x_t, x_v\}$.

In this fashion we can construct a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{x_{j_1}, \dots, x_{j_n}\}$ and $\{x_1, \dots, x_r\}$. Therefore (*) is proved.

Now, define $g_1: C_1 \times \{x_{j_2}\} \times \dots \times \{x_{j_n}\} \longrightarrow \langle C_1, \dots, C_n \rangle$ by $g_1(c, x_{j_2}, \dots, x_{j_n}) = \{c, x_{j_2}, \dots, x_{j_n}\}$. We know that g_1 is continuous. Consider $\mathcal{B}_1 = g_1(C_1 \times \{x_{j_2}\} \times \dots \times \{x_{j_n}\})$. We have that $\mathcal{B}_1$ is a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{x_{j_1}, \dots, x_{j_n}\}$ and $\{y_{k_1}, x_{j_2}, \dots, x_{j_n}\}$.

Define $g_2: \{y_{k_1}\} \times C_2 \times \{x_{j_3}\} \times \dots \times \{x_{j_n}\} \longrightarrow \langle C_1, \dots, C_n \rangle$ by $g_2(y_{k_1}, c, x_{j_3}, \dots, x_{j_n}) = \{y_{k_1}, c, x_{j_3}, \dots, x_{j_n}\}$. We know that g_2 is continuous. Consider $\mathcal{B}_2 = g_2(\{y_{k_1}\} \times C_2 \times \{x_{j_3}\} \times \dots \times \{x_{j_n}\})$. We have that $\mathcal{B}_2$ is a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{y_{k_1}, x_{j_2}, \dots, x_{j_n}\}$ and $\{y_{k_1}, y_{k_2}, x_{j_3}, \dots, x_{j_n}\}$. Similarly, we define $\mathcal{B}_3, \dots, \mathcal{B}_n$.

Using $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$, we can construct a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{x_{j_i}, \dots, x_{j_n}\}$ and $\{y_{k_1}, \dots, y_{k_n}\}$.

Similar to the proof of (*), we can construct a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{y_{k_1}, \dots, y_{k_n}\}$ and $\{y_1, \dots, y_s\}$.

Thus, we have constructed a connected subset of $\langle C_1, \dots, C_n \rangle$ that contains both $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_s\}$.

Therefore $\langle C_1, \dots, C_n \rangle$ is connected. □

LEMMA 3.5. *Let $m \geq 3$. Let X be a continuum, let U, W be nonempty proper open subsets of X and let x, y be two different points in X . Then the following set is a connected subset of $F_m(X)$:*

$$\langle \overline{U}, \overline{W}, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle.$$

PROOF. Let $\mathcal{A} = \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$. We will show, first, that $\mathcal{A}$ is connected. Given $q \in BdU$. Then $\{q, x, y\} \in \langle \{q\}, \{x\}, X \rangle \cap \langle \{x\}, \{y\}, X \rangle$ and, hence, $\langle \{q\}, \{x\}, X \rangle \cap \langle \{x\}, \{y\}, X \rangle \neq \emptyset$. Thus, since $\langle \{q\}, \{x\}, X \rangle$ and $\langle \{x\}, \{y\}, X \rangle$ are connected by Lemma 3.4, $\langle \{q\}, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$ is connected. Therefore, since

$$\mathcal{A} = \left(\bigcup \{ \langle \{q\}, \{x\}, X \rangle : q \in BdU \} \right) \cup \langle \{x\}, \{y\}, X \rangle,$$

we have that $\mathcal{A}$ is connected.

Now, let $\mathcal{B} = \langle BdU, BdW, X \rangle \cup \mathcal{A}$. We will show that $\mathcal{B}$ is connected. Given $q \in BdU$ and $r \in BdW$. Then, noticing that $\{q, r, x\} \in \langle BdU, \{x\}, X \rangle$, we see that $\langle \{q\}, \{r\}, X \rangle \cap \mathcal{A} \neq \emptyset$. Thus, since $\langle \{q\}, \{r\}, X \rangle$ is connected by Lemma 3.4 and $\mathcal{A}$ is connected (as already proved), $\langle \{q\}, \{r\}, X \rangle \cup \mathcal{A}$ is connected. Therefore, since

$$\mathcal{B} = \left(\bigcup \{ \langle \{q\}, \{r\}, X \rangle : q \in BdU, r \in BdW \} \right) \cup \mathcal{A},$$

we have that $\mathcal{B}$ is connected.

Finally, let $\mathcal{C} = \langle \overline{U}, \overline{W}, X \rangle \cup \mathcal{B}$. We will show that $\mathcal{C}$ is connected. Let C_1 be a component of $\overline{U}$ and let C_2 be a component of $\overline{W}$. Then, by the Boundary Bumping Theorem ([18], p. 73), we know that there exist $c_1 \in C_1 \cap BdU$ and $c_2 \in C_2 \cap BdW$; hence, $\{c_1, c_2\} \in \langle C_1, C_2, X \rangle \cap \mathcal{B}$. Thus, since $\langle C_1, C_2, X \rangle$ is connected by Lemma 3.4 and $\mathcal{B}$ is connected (as already proved), $\langle C_1, C_2, X \rangle \cup \mathcal{B}$ is connected. Therefore, since

$$\mathcal{C} = \left(\bigcup \{ \langle C_1, C_2, X \rangle : C_1 \text{ is a component of } \overline{U} \text{ and } C_2 \text{ is a component of } \overline{W} \} \right) \cup \mathcal{B},$$

we have that $\mathcal{C}$ is connected.

Therefore, $\langle \overline{U}, \overline{W}, X \rangle \cup \langle BdU, BdW, X \rangle \cup \langle BdU, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$ is connected. $\square$

LEMMA 3.6. *Let X be a continuum and let $U_1 \subset U_2 \subset \dots \subset U_n$ be nonempty proper open subsets of X . Then the following set is a connected subset of $F_m(X)$ for each $m \geq n+1$: $\mathcal{D} = \langle \overline{U_1} \rangle \cup \langle BdU_1, \overline{U_2} \rangle \cup \langle BdU_1, BdU_2, \overline{U_3} \rangle \cup \dots \cup \langle BdU_1, \dots, BdU_{n-1}, \overline{U_n} \rangle \cup \langle BdU_1, \dots, BdU_n, X \rangle$.*

PROOF. Take two elements $A, B \in \mathcal{D}$. We will construct a connected subset of $\mathcal{D}$ which contains A and B . We only consider the case when $A, B \in \langle \overline{U_1} \rangle$; the other cases can be reduced to this case by easy arguments.

Take $A, B \in \langle \overline{U_1} \rangle$, $A = \{a_{01}, \dots, a_{0n}\}$, $B = \{b_{01}, \dots, b_{0n}\}$; take $C_{11}, \dots, C_{1n}$ and $K_{11}, \dots, K_{1n}$ to be the components of $\overline{U_1}$ such that $a_{0i} \in C_{1i}$ and $b_{0i} \in K_{1i}$ for each $i \in \{1, \dots, n\}$. Consider

$$\langle C_{11}, \dots, C_{1n} \rangle \subset \langle \overline{U_1} \rangle \text{ and } \langle K_{11}, \dots, K_{1n} \rangle \subset \langle \overline{U_1} \rangle.$$

By Lemma 3.4, $\langle C_{11}, \dots, C_{1n} \rangle$ and $\langle K_{11}, \dots, K_{1n} \rangle$ are connected. By the Boundary Bumping Theorem ([18], p. 73), we can take $\{a_{11}, \dots, a_{1n}\}$ and $\{b_{11}, \dots, b_{1n}\}$ such that $a_{1i} \in C_{1i} \cap BdU_1$ and $b_{1i} \in K_{1i} \cap BdU_1$ for each $i \in \{1, \dots, n\}$; take $C_{22}, \dots, C_{2n}$ and $K_{22}, \dots, K_{2n}$ to be the components of $\overline{U_2}$ such that $a_{1i} \in C_{2i}$ and $b_{1i} \in K_{2i}$ for each $i \in \{2, \dots, n\}$. Consider

$$\langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle \subset \langle BdU_1, \overline{U_2} \rangle \text{ and } \langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle \subset \langle BdU_1, \overline{U_2} \rangle.$$

By Lemma 3.4, $\langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle$ and $\langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle$ are connected; thus, since

$$\{a_{11}, \dots, a_{1n}\} \in \langle C_{11}, \dots, C_{1n} \rangle \cap \langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle$$

and

$$\{b_{11}, \dots, b_{1n}\} \in \langle K_{11}, \dots, K_{1n} \rangle \cap \langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle,$$

we have that the following two sets are connected:

$$\langle C_{11}, \dots, C_{1n} \rangle \cup \langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle,$$

$$\langle K_{11}, \dots, K_{1n} \rangle \cup \langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle.$$

Now, by the Boundary Bumping Theorem ([18], p. 73), we can take $\{a_{22}, \dots, a_{2n}\}$ and $\{b_{22}, \dots, b_{2n}\}$ such that $a_{2i} \in C_{2i} \cap BdU_2$ and $b_{2i} \in K_{2i} \cap BdU_2$ for each $i \in \{2, \dots, n\}$; take $C_{33}, \dots, C_{3n}$ and $K_{33}, \dots, K_{3n}$ to be the components of $\overline{U_3}$ such that $a_{2i} \in C_{3i}$ and $b_{2i} \in K_{3i}$ for each $i \in \{3, \dots, n\}$. Consider

$$\langle \{a_{11}\}, \{a_{22}\}, C_{33}, \dots, C_{3n} \rangle \subset \langle BdU_1, BdU_2, \overline{U_3} \rangle$$

and

$$\langle \{b_{11}\}, \{b_{22}\}, K_{33}, \dots, K_{3n} \rangle \subset \langle BdU_1, BdU_2, \overline{U_3} \rangle.$$

By Lemma 3.4, $\langle \{a_{11}\}, \{a_{22}\}, C_{33}, \dots, C_{3n} \rangle$ and $\langle \{b_{11}\}, \{b_{22}\}, K_{33}, \dots, K_{3n} \rangle$ are connected; thus, since

$$\langle a_{11}, a_{22}, a_{23}, \dots, a_{2n} \rangle \in \langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle \cap \langle \{a_{11}\}, \{a_{22}\}, C_{33}, \dots, C_{3n} \rangle$$

and

$$\{b_{11}, b_{22}, b_{23}, \dots, b_{2n}\} \in \langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle \cap \langle \{b_{11}\}, \{b_{22}\}, K_{33}, \dots, K_{3n} \rangle$$

we have that the following two sets are connected:

$$\langle C_{11}, \dots, C_{1n} \rangle \cup \langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle \cup \langle \{a_{11}\}, \{a_{22}\}, C_{33}, \dots, C_{3n} \rangle,$$

$$\langle K_{11}, \dots, K_{1n} \rangle \cup \langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle \cup \langle \{b_{11}\}, \{b_{22}\}, K_{33}, \dots, K_{3n} \rangle.$$

Repeating the procedure indicated above, we can find connected sets $\mathcal{C}_1$ and $\mathcal{K}_1$ which contain A and B , respectively, of the form

$$\mathcal{C}_1 = \langle C_{11}, \dots, C_{1n} \rangle \cup \langle \{a_{11}\}, C_{22}, \dots, C_{2n} \rangle \cup \dots \cup \langle \{a_{11}\}, \dots, \{a_{n-1n-1}\}, C_{nn} \rangle$$

and

$$\mathcal{K}_1 = \langle K_{11}, \dots, K_{1n} \rangle \cup \langle \{b_{11}\}, K_{22}, \dots, K_{2n} \rangle \cup \dots \cup \langle \{b_{11}\}, \dots, \{b_{n-1n-1}\}, K_{nn} \rangle$$

where $a_{ii}, b_{ii} \in BdU_i$ for each $i \in \{1, \dots, n-1\}$, $C_{11}, \dots, C_{1n}, K_{11}, \dots, K_{1n}$ are components of $\overline{U_1}$, $C_{22}, \dots, C_{2n}, K_{22}, \dots, K_{2n}$ are components of $\overline{U_2}$, ..., and C_{nn}, K_{nn} are components of $\overline{U_n}$. Now, by the Boundary Bumping Theorem ([18], p. 73), we can take points a_{nn} and b_{nn} such that $a_{nn} \in C_{nn} \cap BdU_n$ and $b_{nn} \in K_{nn} \cap BdU_n$. Consider

$$\mathcal{C}_2 = \langle \{a_{11}\}, \dots, \{a_{nn}\}, X \rangle \subset \langle BdU_1, \dots, BdU_n, X \rangle$$

and

$$\mathcal{K}_2 = \langle \{b_{11}\}, \dots, \{b_{nn}\}, X \rangle \subset \langle BdU_1, \dots, BdU_n, X \rangle.$$

By Lemma 3.4, $\langle \{a_{11}\}, \dots, \{a_{nn}\}, X \rangle$ and $\langle \{b_{11}\}, \dots, \{b_{nn}\}, X \rangle$ are connected; thus, since

$$\{a_{11}, \dots, a_{nn}\} \in \langle \{a_{11}\}, \dots, \{a_{nn}\}, X \rangle \cap \langle \{a_{11}\}, \dots, \{a_{n-1n-1}\}, C_{nn} \rangle$$

and

$$\{b_{11}, \dots, b_{nn}\} \in \langle \{b_{11}\}, \dots, \{b_{nn}\}, X \rangle \cap \langle \{b_{11}\}, \dots, \{b_{n-1n-1}\}, C_{nn} \rangle$$

we have that $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{K}_1 \cup \mathcal{K}_2$ are connected. Consider

$$\mathcal{J}_1 = \langle \{b_{11}\}, \{a_{22}\}, \dots, \{a_{nn}\}, X \rangle \subset \langle BdU_1, \dots, BdU_n, X \rangle,$$

$$\mathcal{J}_2 = \langle \{b_{11}\}, \{b_{22}\}, \{a_{33}\}, \dots, \{a_{nn}\}, X \rangle \subset \langle BdU_1, \dots, BdU_n, X \rangle,$$

$\vdots$

$$\mathcal{J}_{n-1} = \langle \{b_{11}\}, \dots, \{b_{n-1n-1}\}, \{a_{nn}\}, X \rangle \subset \langle BdU_1, \dots, BdU_n, X \rangle;$$

by Lemma 3.4, $\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_{n-1}$ are connected; also, since

$$\begin{aligned} \{a_{11}, \dots, a_{nn}, b_{11}\} &\in \mathcal{C}_2 \cap \mathcal{J}_1 \\ \{a_{22}, \dots, a_{nn}, b_{11}, b_{22}\} &\in \mathcal{J}_1 \cap \mathcal{J}_2 \\ &\vdots \\ \{a_{nn}, b_{11}, \dots, b_{nn}\} &\in \mathcal{J}_{n-1} \cap \mathcal{K}_2 \end{aligned}$$

we have that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{J}_1 \cup \dots \cup \mathcal{J}_{n-1} \cup \mathcal{K}_2 \cup \mathcal{K}_1$ is a connected set. Hence, we have found a connected set that contains both A and B and is contained in $\mathcal{D}$.

Therefore $\mathcal{D}$ is connected. $\square$

THEOREM 3.7. *Let X be a continuum. Then $F_n(X)$ is mutually aposyndetic for each $n \geq 3$.*

PROOF. Take two different elements $A, B \in F_n(X)$. We need to construct two disjoint subcontinua $\mathcal{A}$ and $\mathcal{B}$ of $F_n(X)$ such that $A \in \text{int}\mathcal{A}$ and $B \in \text{int}\mathcal{B}$. We will consider two cases.

Case 1. Suppose that $A \cap B = \emptyset$. Then we can find $U_1, U_2, \dots, U_{n-1}$ and $V_1, V_2, \dots, V_{n-1}$ open subsets of X such that $A \subset U_1$, $B \subset V_1$, $\overline{U_{n-1}} \cap \overline{V_{n-1}} = \emptyset$ and for each $i \in \{1, \dots, n-2\}$, $\overline{U_i} \subset U_{i+1}$ and $\overline{V_i} \subset V_{i+1}$. Consider the following two sets: $\mathcal{A} = \langle \overline{U_1} \rangle \cup \langle BdU_1, \overline{U_2} \rangle \cup \langle BdU_1, BdU_2, \overline{U_3} \rangle \cup \dots \cup \langle BdU_1, \dots, BdU_{n-2}, \overline{U_{n-1}} \rangle \cup \langle BdU_1, \dots, BdU_{n-1}, X \rangle$ and $\mathcal{B} = \langle \overline{V_1} \rangle \cup \langle BdV_1, \overline{V_2} \rangle \cup \langle BdV_1, BdV_2, \overline{V_3} \rangle \cup \dots \cup \langle BdV_1, \dots, BdV_{n-2}, \overline{V_{n-1}} \rangle \cup \langle BdV_1, \dots, BdV_{n-1}, X \rangle$. By Lemma 3.3, we have that $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $F_n(X)$; hence, by Lemma 3.6, $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $F_n(X)$. We will show that $\mathcal{A}$ and $\mathcal{B}$ are disjoint as follows:

First, since $\overline{U_i} \cap \overline{V_j} = \emptyset$ for each $i, j \in \{1, \dots, n-1\}$, it follows that for each $k, l \in \{1, \dots, n-1\}$,

$$\langle BdU_1, \dots, BdU_k, \overline{U_{k+1}} \rangle \cap \langle BdV_1, \dots, BdV_l, \overline{V_{l+1}} \rangle = \emptyset.$$

Second, for each $k \in \{1, \dots, n-2\}$, since $\overline{U_{k+1}} \cap \overline{V_j} = \emptyset$ for each $j \in \{1, \dots, n-1\}$, we have that

$$\langle BdU_1, \dots, BdU_k, \overline{U_{k+1}} \rangle \cap \langle BdV_1, \dots, BdV_{n-1}, X \rangle = \emptyset.$$

And third, since $\overline{U_{n-1}} \cap \overline{V_{n-1}} = \emptyset$ and for each $i \in \{1, \dots, n-2\}$, $\overline{U_i} \cap BdU_{i+1} = \emptyset$ and $\overline{V_i} \cap BdV_{i+1} = \emptyset$, it follows that

$$\langle BdU_1, \dots, BdU_{n-1}, X \rangle \cap \langle BdV_1, \dots, BdV_{n-1}, X \rangle = \emptyset.$$

Therefore $\mathcal{A} \cap \mathcal{B} = \emptyset$. Also, since $A \subset U_1$, $B \subset V_1$ and U_1, V_1 are open subsets of X , we have that $A \in \langle U_1 \rangle \subset \mathcal{A}$ and $B \in \langle V_1 \rangle \subset \mathcal{B}$. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are disjoint subcontinua of $F_n(X)$ such that $A \in \text{int}\mathcal{A}$ and $B \in \text{int}\mathcal{B}$.

Case 2. Suppose that $A \cap B \neq \emptyset$. Since $A \neq B$, we can suppose, without loss of generality, that there exists a point $p \in A - B$. Clearly, there exists a point $q \in A$ different from p . Since X is metric, we can find open subsets U, W and $V_1, V_2, \dots, V_{n-1}$ of X and two different points $x, y \in X$ with the following properties: $p \in U$; $q \in W$; $B \subset V_1$; $\overline{U} \cap \overline{V_{n-1}} = \emptyset$; $x, y \notin (\overline{U} \cup \overline{W} \cup \overline{V_{n-1}})$; for each $i \in \{1, \dots, n-1\}$, $\overline{W} \cap \text{Bd}V_i = \emptyset$; and for each $i \in \{1, \dots, n-2\}$, $\overline{V_i} \subset V_{i+1}$. Consider $\mathcal{A} = \langle \overline{U}, \overline{W}, X \rangle \cup \langle \text{Bd}U, \text{Bd}W, X \rangle \cup \langle \text{Bd}U, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle$ and $\mathcal{B} = \langle \overline{V_1} \rangle \cup \langle \text{Bd}V_1, \overline{V_2} \rangle \cup \langle \text{Bd}V_1, \text{Bd}V_2, \overline{V_3} \rangle \cup \dots \cup \langle \text{Bd}V_1, \dots, \text{Bd}V_{n-2}, \overline{V_{n-1}} \rangle \cup \langle \text{Bd}V_1, \dots, \text{Bd}V_{n-1}, X \rangle$. By Lemma 3.3, we have that $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $F_n(X)$; hence, by Lemma 3.5 and Lemma 3.6 (correspondingly), $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $F_n(X)$. We will show that $\mathcal{A}$ and $\mathcal{B}$ are disjoint as follows:

First, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$, it follows that for each $k \in \{1, \dots, n-2\}$,

$$(\langle \overline{U}, \overline{W}, X \rangle \cup \langle \text{Bd}U, \text{Bd}W, X \rangle \cup \langle \text{Bd}U, \{x\}, X \rangle) \cap \langle \text{Bd}V_1, \dots, \text{Bd}V_k, \overline{V_{k+1}} \rangle = \emptyset.$$

Second, since $x, y \notin \overline{V_{n-1}}$, we have that

$$\langle \{x\}, \{y\}, X \rangle \cap \langle \text{Bd}V_1, \dots, \text{Bd}V_k, \overline{V_{k+1}} \rangle = \emptyset.$$

Third, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$, for each $i \in \{1, \dots, n-1\}$, $\overline{W} \cap \text{Bd}V_i = \emptyset$ and, for each $i \in \{1, \dots, n-2\}$, $\overline{V_i} \cap \text{Bd}V_{i+1} = \emptyset$, it follows that

$$\langle \overline{U}, \overline{W}, X \rangle \cap \langle \text{Bd}V_1, \dots, \text{Bd}V_{n-1}, X \rangle = \emptyset$$

and

$$\langle \text{Bd}U, \text{Bd}W, X \rangle \cap \langle \text{Bd}V_1, \dots, \text{Bd}V_{n-1}, X \rangle = \emptyset.$$

And fourth, since $\overline{U} \cap \overline{V_{n-1}} = \emptyset$ and $x, y \notin \overline{V_{n-1}}$, we have that

$$(\langle \text{Bd}U, \{x\}, X \rangle \cup \langle \{x\}, \{y\}, X \rangle) \cap \langle \text{Bd}V_1, \dots, \text{Bd}V_{n-1}, X \rangle = \emptyset.$$

Therefore $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Next, note that $p \in U$, $q \in W$, $B \subset V_1$ and U, W, V_1 are open subsets of X ; hence, we have that $A \in \langle U, W, X \rangle \subset \mathcal{A}$ and $B \in \langle V_1 \rangle \subset \mathcal{B}$. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are disjoint subcontinua of $F_n(X)$ such that $A \in \text{int}\mathcal{A}$ and $B \in \text{int}\mathcal{B}$.

Therefore $F_n(X)$ is mutually aposyndetic. $\square$

Symmetric products are related to Cartesian products; in particular, there is a natural map $\pi_n: X^n \rightarrow F_n(X)$ given by $\pi_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$. If π_n is open for $n \geq 3$, a simple proof of our theorem might be possible using Theorem 2 of [6]. It is known that π_2 is open ([11], Lemma 9). However, we now show, π_n never is open for $n \geq 3$:

PROPOSITION 3.8. *Let X be a continuum. Then π_n is not open for $n \geq 3$.*

PROOF. Let p, q be two different points of X and take sequences $\{p_k\}_{k \in \mathbb{N}}$ and $\{q_k\}_{k \in \mathbb{N}}$ such that $p_k \rightarrow q$, $q_k \rightarrow q$, and $p_k \neq q_l$ for every $k, l \in \mathbb{N}$. Let $\varepsilon > 0$ such that $B_\varepsilon(p) \cap B_\varepsilon(q) = \emptyset$. Let $U = B_\varepsilon(p) \times B_\varepsilon(p) \times \cdots \times B_\varepsilon(p) \times B_\varepsilon(q) \subset X^n$. Then U is an open subset of X^n that contains $(p, p, \dots, p, q)$.

Suppose that π_n is open. Then $\pi_n(U)$ is an open subset of $F_n(X)$ which contains $\{p, q\}$; since $p_n \rightarrow q$ and $q_n \rightarrow q$, there exists $N \in \mathbb{N}$ such that $\{p, p_N, q_N\} \in \pi_3(U)$ and $p_N, q_N \in B_\varepsilon(q)$. Note that there are $n!$ points in $\pi_n^{-1}(\{p, p_N, q_N\})$; each of those points has two coordinates in $B_\varepsilon(q)$. Hence, $\{p, p_N, q_N\} \notin \pi_n(U)$ which is a contradiction.

Therefore π_n is not open. □

CHAPTER 4

Non-confluence of the natural map of products onto symmetric products

DEFINITION 4.1. If f is a map from a continuum X onto Y and k is a positive integer, then f is k -confluent if each continuum M in Y is the union of the images of k or fewer components of $f^{-1}(M)$.

Note that 1-confluent maps are also called weakly confluent maps.

As we mention in Chapter 2, symmetric products are related to Cartesian products; in particular, there is a natural map $\pi_n: X^n \rightarrow F_n(X)$ given by $\pi_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$.

We know that for $n \geq 3$, π_n is not open (Proposition 3.8). In a conversation with Professor Illanes, he suggested the problem of whether or not this function was k -confluent for any positive integer k . In this chapter we answer this question negatively. We also show that there is a non- k -confluent map of some continuum onto $F_2(X)$ for any positive integer k (even though $\pi_2: X^2 \rightarrow F_2(X)$ is itself an open map ([11], Lemma 9) and, thus, is confluent).

Non- k -confluence of π_n when $n \geq 3$

This section is devoted to proving the following theorem. We ask the reader to recall Definition 3.2, Lemma 3.3 and Lemma 3.4, which we use in the proof.

THEOREM 4.2. *Let X be a continuum. If $n \geq 3$, then π_n is not k -confluent for all $k \in \mathbb{N}$.*

PROOF. Let $k \in \mathbb{N}$. Take $(n-2) + k$ different points, $p_1, \dots, p_{n-2}, q_1, \dots, q_k$, of X . Since X is metric, we can find open subsets $U, W_1, \dots, W_k$ of X with the following properties: $p_1 \in U$, $q_i \in W_i$ for each $i \in \{1, \dots, k\}$, $\overline{U} \cap \{p_2, \dots, p_{n-1}\} = \emptyset$, $\overline{U} \cap \overline{W_i} = \emptyset$ for each $i \in \{1, \dots, k\}$ and, for each $j \in \{1, \dots, k\}$, $\overline{W_j} \cap (\overline{W_i} \cup \{p_2, \dots, p_{n-1}\}) = \emptyset$ for each $i \in \{1, \dots, j-1, j+1, \dots, k\}$. Let R be the component of $\overline{U}$ such that $p_1 \in R$ and, for each $i \in \{1, \dots, k\}$, let M_i be the component of $\overline{W_i}$ such that $q_i \in M_i$. By the Boundary Bumping Theorem ([18], p. 73), there are points,

$r, m_1, \dots, m_k$ such that $r \in R \cap BdU$ and, for each $i \in \{1, \dots, k\}$, $m_i \in M_i \cap BdW_i$. Since X is a metric space and $p_1 \notin BdU$, there is an open subset W of X such that $p_1 \in W \subset U$ and $\overline{W} \cap BdU = \emptyset$. Let N be the component of $\overline{W}$ such that $p_1 \in N$. By the Boundary Bumping Theorem ([18], p. 73), there is a point $n \in N \cap BdW$. Consider the following $k+1$ sets:

$$\begin{aligned} \mathcal{C}_1 &= \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_1\}, M_1 \rangle, \\ \mathcal{C}_2 &= \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_2\}, M_2 \rangle \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_2\}, N \rangle, \\ \mathcal{C}_3 &= \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_3\}, M_3 \rangle \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_3\}, N \rangle, \\ &\vdots \\ \mathcal{C}_k &= \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_k\}, M_k \rangle \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_k\}, N \rangle, \\ \mathcal{C}_{k+1} &= \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_1\}, R \rangle \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{r\}, X \rangle \cup \\ &\quad \langle \{p_1\}, \dots, \{p_{n-2}\}, \{m_2\}, X \rangle \cup \dots \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{m_k\}, X \rangle. \end{aligned}$$

We show that $\mathcal{C}_1, \dots, \mathcal{C}_k$ are connected. By Lemma 3.4, $\mathcal{C}_1$ is connected. Let $i \in \{2, \dots, k\}$. Then, by Lemma 3.4, the following sets are connected:

$$\begin{aligned} &\langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_i\}, M_i \rangle, \\ &\langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_i\}, N \rangle; \end{aligned}$$

thus, since

$$\{p_1, \dots, p_{n-2}, q_i\} \in \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_i\}, M_i \rangle \cap \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_i\}, N \rangle,$$

$\mathcal{C}_i$ is connected. Therefore, we have proved that $\mathcal{C}_1, \dots, \mathcal{C}_k$ are connected.

Now, we show that $\mathcal{C}_{k+1}$ is connected. By Lemma 3.4,

$$\langle \{p_1\}, \dots, \{p_{n-1}\}, \{q_1\}, R \rangle \text{ and } \langle \{p_1\}, \dots, \{p_{n-1}\}, \{r\}, X \rangle \text{ are connected;}$$

thus, since

$$\{p_1, \dots, p_{n-2}, q_1, r\} \in \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_1\}, R \rangle \cap \langle \{p_1\}, \dots, \{p_{n-2}\}, \{r\}, X \rangle,$$

we have that $\langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_1\}, R \rangle \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{r\}, X \rangle$ is connected. Also, for each $i \in \{2, \dots, k\}$, by Lemma 3.4,

$$\langle \{p_1\}, \dots, \{p_{n-2}\}, \{r\}, X \rangle \text{ and } \langle \{p_1\}, \dots, \{p_{n-2}\}, \{m_i\}, X \rangle \text{ are connected;}$$

thus, since

$$\{p_1, \dots, p_{n-2}, r, m_i\} \in \langle \{p_1\}, \dots, \{p_{n-2}\}, \{r\}, X \rangle \cap \langle \{p_1\}, \dots, \{p_{n-2}\}, \{m_i\}, X \rangle,$$

we have that $\langle \{p_1\}, \dots, \{p_{n-2}\}, \{r\}, X \rangle \cup \langle \{p_1\}, \dots, \{p_{n-2}\}, \{m_i\}, X \rangle$ is connected. Therefore, we have proved that $\mathcal{C}_{k+1}$ is connected.

Define $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_{k+1}$. Since

$$\{p_1, \dots, p_{n-2}, q_1\} \in \mathcal{C}_1 \cap \langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_1\}, R \rangle \subset \mathcal{C}_1 \cap \mathcal{C}_{k+1}$$

and, for each $i \in \{2, \dots, k\}$,

$$\{p_1, \dots, p_{n-2}, m_i, q_i\} \in \mathcal{C}_i \cap \langle \{p_1\}, \dots, \{p_{n-2}\}, \{m_i\}, X \rangle \subset \mathcal{C}_i \cap \mathcal{C}_{k+1},$$

we have that $\mathcal{C}$ is connected. By Lemma 3.3, we have that $\mathcal{C}$ is a closed subset of $F_n(X)$; therefore, $\mathcal{C}$ is a subcontinuum of $F_n(X)$.

Let

$$\begin{aligned} A_1 &= \{p_1\}, \dots, A_{n-2} = \{p_{n-2}\}, A_{n-1} = \{q_1\}, A_n = R, \\ B_1 &= \{p_1\}, \dots, B_{n-2} = \{p_{n-2}\}, B_{n-1} = \{r\}, B_n = X; \end{aligned}$$

also, for each $i \in \{1, \dots, k\}$, let

$$\begin{aligned} A_1^i &= \{p_1\}, \dots, A_{n-2}^i = \{p_{n-2}\}, A_{n-1}^i = \{q_i\}, A_n^i = M_i, \\ B_1^i &= \{p_1\}, \dots, B_{n-2}^i = \{p_{n-2}\}, B_{n-1}^i = \{q_i\}, B_n^i = N, \end{aligned}$$

and, for each $i \in \{2, \dots, k\}$,

$$C_1^i = \{p_1\}, \dots, C_{n-2}^i = \{p_{n-2}\}, C_{n-1}^i = \{m_i\}, C_n^i = X.$$

In what follows, Σ is the set of all permutations of $\{1, \dots, n\}$.

Let

$$\begin{aligned} D_2 &= \pi_n^{-1}(\langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_2\}, N \rangle) = \bigcup \left\{ B_{\sigma(1)}^2 \times \dots \times B_{\sigma(n)}^2 : \sigma \in \Sigma \right\}, \\ &\quad \vdots \\ D_k &= \pi_n^{-1}(\langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_k\}, N \rangle) = \bigcup \left\{ B_{\sigma(1)}^k \times \dots \times B_{\sigma(n)}^k : \sigma \in \Sigma \right\}, \\ D_{k+1} &= \pi_n^{-1}(\mathcal{C}_{k+1}) \cup \left(\bigcup_{s=2}^k (\pi_n^{-1}(\langle \{p_1\}, \dots, \{p_{n-2}\}, \{q_s\}, M_s \rangle)) \right) = \\ &\quad \pi_n^{-1}(\mathcal{C}_{k+1}) \cup \left(\bigcup_{s=2}^k \left(\bigcup \left\{ A_{\sigma(1)}^s \times \dots \times A_{\sigma(n)}^s : \sigma \in \Sigma \right\} \right) \right). \end{aligned}$$

Note that

$$\begin{aligned}
\pi_n^{-1}(\mathcal{C}_1) &= \bigcup \left\{ A_{\sigma(1)}^1 \times \cdots \times A_{\sigma(n)}^1 : \sigma \in \Sigma \right\}, \\
\pi_n^{-1}(\mathcal{C}_2) &= \left(\bigcup \left\{ A_{\sigma(1)}^2 \times \cdots \times A_{\sigma(n)}^2 : \sigma \in \Sigma \right\} \right) \cup D_2, \\
&\vdots \\
\pi_n^{-1}(\mathcal{C}_k) &= \left(\bigcup \left\{ A_{\sigma(1)}^k \times \cdots \times A_{\sigma(n)}^k : \sigma \in \Sigma \right\} \right) \cup D_k, \\
\pi_n^{-1}(\mathcal{C}_{k+1}) &= \bigcup \left(\left\{ A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} : \sigma \in \Sigma \right\} \cup \right. \\
&\quad \left. \left\{ B_{\sigma(1)} \times \cdots \times B_{\sigma(n)} : \sigma \in \Sigma \right\} \cup \right. \\
&\quad \left. \left\{ C_{\sigma(1)}^1 \times \cdots \times C_{\sigma(n)}^1 : \sigma \in \Sigma \right\} \cup \cdots \cup \right. \\
&\quad \left. \left\{ C_{\sigma(1)}^k \times \cdots \times C_{\sigma(n)}^k : \sigma \in \Sigma \right\} \right).
\end{aligned}$$

We have that

$$\{p_1, \dots, p_{n-2}, q_1, m_1\} \in \mathcal{C}_1 - \left(\bigcup_{i=2}^{k+1} \pi_n(D_i) \right)$$

and, for each $j \in \{2, \dots, k\}$,

$$\{p_1, \dots, p_{n-2}, q_j, n\} \in D_j - \left(\mathcal{C}_1 \cup \left(\bigcup_{i \neq j} \pi_n(D_i) \right) \right) \subset \mathcal{C}_j - \left(\mathcal{C}_1 \cup \left(\bigcup_{i \neq j} \pi_n(D_i) \right) \right).$$

Since $\overline{U} \cap \{p_2, \dots, p_{n-1}\} = \emptyset$, $\overline{U} \cap \overline{W_i} = \emptyset$ for each $i \in \{1, \dots, k\}$ and, for each $j \in \{1, \dots, k\}$, $\overline{W_j} \cap (\overline{W_i} \cup \{p_2, \dots, p_{n-1}\}) = \emptyset$ for each $i \in \{1, \dots, j-1, j+1, \dots, k\}$, it follows that

$$\pi_n^{-1}(\mathcal{C}_1) \cap D_i = \emptyset \text{ for each } i \in \{2, \dots, k+1\}$$

and, for each $j \in \{2, \dots, k\}$,

$$D_j \cap D_i = \emptyset \text{ for each } i \in \{2, \dots, j-1, j+1, \dots, k+1\}.$$

Thus, we have that $\{\pi_n^{-1}(\mathcal{C}_1), D_2, \dots, D_k, D_{k+1}\}$ is a collection of $k+1$ mutually disjoint subsets of $\pi_n^{-1}(\mathcal{C})$ such that the image under π_n of the union of any proper subcollection is contained properly in $\mathcal{C}$. Therefore, π_n is not k -confluent. $\square$

Symmetric products and $\text{Class}(\mathbf{W}_k)$

DEFINITION 4.3. If X is a continuum and k is a positive integer, then X is said to be in $\text{Class}(W_k)$ provided that any continuous function from any continuum onto X is k -confluent.

Note that $\text{Class}(\mathbf{W}_1)$ is also called $\text{Class}(\mathbf{W})$.

In this section we prove some general properties of the continua that belong to $\text{Class}(\mathbf{W}_k)$. Then we show that symmetric products do not belong to $\text{Class}(\mathbf{W}_k)$.

LEMMA 4.4. *Let X be a continuum. If X is in $\text{Class}(W_k)$, then X is not a $(k+2)$ -od.*

PROOF. Assume X is a $(k+2)$ -od. Then there is a subcontinuum Z such that $X - Z = \bigcup_{i=1}^n U_i$, $U_i \neq \emptyset$ for each $i \in \{1, \dots, k+2\}$, and $\overline{U_i} \cap U_j = \emptyset$ whenever $i \neq j$. For each $i \in \{2, \dots, k+2\}$, define $C_i = Z \cup U_1 \cup U_i$. Also, for each $i \in \{2, \dots, n\}$, let $h_i: C_i \rightarrow I^{\mathbb{N}}$ be an embedding such that $\bigcap_{i=2}^{k+2} h_i(C_i) = \{p\}$ and $h_2(q) = \dots = h_{k+2}(q) = p$ where $q \in U_1$. Define $C = \bigcup_{i=2}^{k+2} h_i(C_i)$; clearly C is a continuum. Consider $f: C \rightarrow X$ defined by letting $f(x) = h_i^{-1}(x)$ if $x \in h_i(C_i)$. Note that for each $i \in \{2, \dots, k+2\}$, f is continuous on $h_i(C_i)$ and $h_i^{-1}(p) = q$. Hence, f is a map.

Now, consider $M = Z \cup \left(\bigcup_{i=2}^{k+2} U_i \right)$. By the Boundary Bumping Theorem ([18], p. 73), M is a subcontinuum of X . Observe that

$$f^{-1}(M) = \left(\bigcup_{i=2}^{k+2} h_i(Z) \right) \cup \left(\bigcup_{i=2}^{k+2} h_i(U_i) \right) = \bigcup_{i=2}^{k+2} (h_i(Z) \cup h_i(U_i)).$$

For each $i \in \{2, \dots, k+2\}$, take $x_i \in U_i$. We have that, for each $j \in \{2, \dots, k+2\}$,

$$x_j \in f(h_j(U_j)) - \left(\bigcup_{i \neq j} f(h_i(Z) \cup h_i(U_i)) \right).$$

Also, if $i \neq j$, then

$$\left(\overline{h_i(Z) \cup h_i(U_i)} \right) \cap (h_j(Z) \cup h_j(U_j)) = \emptyset.$$

Thus, $\{h_2(Z) \cup h_2(U_2), \dots, h_{k+2}(Z) \cup h_{k+2}(U_{k+2})\}$ is a collection of $k+1$ mutually disjoint subsets of $f^{-1}(M)$ such that the image under f of the union of any proper subcollection is contained properly in M . Therefore, f is not k -confluent. $\square$

DEFINITION 4.5. Let X be a continuum. Let $\mathcal{C} = \{(A, B): A \text{ and } B \text{ are subcontinua of } X \text{ and } A \cup B = X\}$. For each $(A, B) \in \mathcal{C}$, let $r(A, B)$ denote one less than the number of components of $A \cap B$ (if $A \cap B$ has infinitely many components, let $r(A, B) = \infty$). Then, the multicoherence degree of X , denote by $r(X)$, is defined by letting $r(X) = \sup\{r(A, B): (A, B) \in \mathcal{C}\}$ or $r(X) = \infty$ depending on whether $\{r(A, B): (A, B) \in \mathcal{C}\}$ is bounded above or not.

PROPOSITION 4.6. *Let X be a continuum. If X is in $\text{Class}(W_k)$, then $r(X) \leq k-1$.*

PROOF. Assume $r(X) \geq k$. Then there are subcontinua A and B of X such that $A \cup B = X$ and the number of components of $A \cap B$ is greater than or equal to $k+1$. Let $C_1, \dots, C_{k+1}$

be $k + 1$ different components of $A \cap B$. By the Boundary Bumping Theorem ([18], p. 73), there are subcontinua $D_1, \dots, D_{k+1}$ of B such that, for each $i \in \{1, \dots, k + 1\}$, $C_i \subsetneq D_i$ and $D_i \cap D_j = \emptyset$ if $i \neq j$. Let $h_1: A \longrightarrow I^{\mathbb{N}}$ and $h_2: B \longrightarrow I^{\mathbb{N}}$ be embeddings of A and B respectively such that $h_1(A) \cap h_2(B) = \{p\}$ and $h_1(q) = h_2(q) = p$ where $q \in C_1$. Define $C = h_1(A) \cup h_2(B)$; clearly, C is a continuum. Consider $f: C \longrightarrow X$ defined by letting $f(x) = h_1^{-1}(x)$ if $x \in h_1(A)$ or $f(x) = h_2^{-1}(x)$ if $x \in h_2(B)$. Note that f is continuous on $h_1(A)$ and on $h_2(B)$ and $h_1^{-1}(p) = h_2^{-1}(p) = q$. Hence, f is a map.

Now, consider $M = A \cup D_1 \cup \dots \cup D_{k+1}$; clearly, M is a subcontinuum of X . Observe that

$$f^{-1}(M) = h_1(A) \cup h_2(D_1) \cup h_2(D_2) \cup \dots \cup h_2(D_{k+1}).$$

For each $i \in \{1, \dots, k + 1\}$, take $x_i \in D_i - C_i$. We have that, for each $j \in \{1, \dots, k + 1\}$,

$$x_j \in h_2(D_j) - \left(h_1(A) \cup \left(\bigcup_{i \neq j} h_2(D_i) \right) \right).$$

Also, if $i \neq j$, then $h_2(D_i) \cap h_2(D_j) = \emptyset$ and, for each $i \in \{2, \dots, k + 1\}$, $h_1(A) \cap h_2(D_i) = \emptyset$. Thus, $\{h_1(A) \cup h_2(D_1), h_2(D_2), \dots, h_2(D_{k+1})\}$ is a collection of $k + 1$ mutually disjoint subsets of $f^{-1}(M)$ such that the image under f of the union of any proper subcollection is contained properly in M . Therefore, f is not k -confluent. $\square$

LEMMA 4.7. *Let X be a continuum. Then $F_2(X)$ is an n -od for all $n \geq 3$.*

PROOF. Take $n \geq 3$. Since X is a continuum, there are $n - 2$ nonempty open subsets $U_1, \dots, U_{n-2}$ of X such that $\overline{\bigcup_{i=1}^{n-2} U_i} \neq X$ and $\overline{U_i} \cap U_j = \emptyset$ if $i \neq j$. Define

$$\mathcal{Z} = \left\langle Bd \left(\bigcup_{i=1}^{n-2} U_i \right), X \right\rangle$$

$$\begin{aligned}
 \mathcal{U}_1 &= \left\langle \bigcup_{i=1}^{n-2} U_i \right\rangle, \\
 \mathcal{U}_2 &= \left\langle X - \overline{\bigcup_{i=1}^{n-2} U_i} \right\rangle, \\
 \mathcal{U}_3 &= \left\langle U_1, X - \overline{\bigcup_{i=1}^{n-2} U_i} \right\rangle, \\
 &\vdots \\
 \mathcal{U}_n &= \left\langle U_{n-2}, X - \overline{\bigcup_{i=1}^{n-2} U_i} \right\rangle.
 \end{aligned}$$

First, we are going to prove that $\mathcal{Z}$ is connected. Take $A, B \in \mathcal{Z}$, then there is $a \in A \cap Bd\left(\bigcup_{i=1}^{n-2} U_i\right)$ and there is $b \in B \cap Bd\left(\bigcup_{i=1}^{n-2} U_i\right)$. Hence, $\langle\{a\}, X\rangle \cup \langle\{b\}, X\rangle \subset \mathcal{Z}$. By Lemma 3.4, $\langle\{a\}, X\rangle$ and $\langle\{b\}, X\rangle$ are connected subsets of $F_2(X)$. Since $\{a, b\} \in \langle\{a\}, X\rangle \cap \langle\{b\}, X\rangle$, we have that $\langle\{a\}, X\rangle \cup \langle\{b\}, X\rangle$ is a connected set that contains A and B and is contained in $\mathcal{Z}$. Thus, $\mathcal{Z}$ is connected. Therefore, $\mathcal{Z}$ is a subcontinuum of $F_2(X)$.

Clearly $F_2(X) - \mathcal{Z} = \bigcup_{j=1}^n \mathcal{U}_j$. Since $U_1, \dots, U_{n-2}$ are nonempty and $\overline{\bigcup_{i=1}^{n-2} U_i} \neq X$, $\mathcal{U}_j \neq \emptyset$ for each $j \in \{1, \dots, n\}$. Also, we have that $\{\mathcal{U}_1, \dots, \mathcal{U}_n\}$ is a collection of pairwise disjoint nonempty open subsets of $F_2(X)$. Hence, $\overline{\mathcal{U}_i} \cap \mathcal{U}_j = \emptyset$ whenever $i \neq j$. Therefore $F_2(X)$ is an n -od. $\square$

THEOREM 4.8. *Let X be a continuum. If $n \geq 2$, then $F_n(X)$ is not in $\text{Class}(W_k)$ for all $k \in \mathbb{N}$.*

PROOF. By Lemma 4.7, $F_2(X)$ is a $(k+2)$ -od for any $k \in \mathbb{N}$. Therefore, by Lemma 4.4, $F_2(X)$ is not in $\text{Class}(W_k)$ for all $k \in \mathbb{N}$. Now, if $n \geq 3$, by Theorem 4.2, π_n is not k -confluent for any $k \in \mathbb{N}$. Therefore, if $n \geq 3$, $F_n(X)$ is not in $\text{Class}(W_k)$ for all $k \in \mathbb{N}$. $\square$

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CHAPTER 5

$C(X)$ -coselection spaces and Z -sets in hyperspaces

DEFINITION 5.1. If (Y, d) is a metric space, a closed subset A of Y is a *Z -set in Y* if and only if for every $\varepsilon > 0$, there exist a continuous function $f_\varepsilon: Y \longrightarrow Y - A$ such that $d(y, f_\varepsilon(y)) < \varepsilon$.

DEFINITION 5.2. Let X be a continuum. A map $f: X \longrightarrow C(X)$ is called a *$C(X)$ -coselection for X* provided that $x \in f(x)$ for each $x \in X$. If f is a $C(X)$ -coselection for X where $f(x) \in C(X) - (F_1(X) \cup \{X\})$ for all $x \in X$, then f is called a *non-trivial $C(X)$ -coselection for X* . If f is a $C(X)$ -coselection for X , then the *mesh of f* , denoted $\text{mesh}(f)$, is defined by $\text{mesh}(f) = \sup \{\text{diam}(f(x)): x \in X\}$. If X is a continuum such that given any $\varepsilon > 0$ there exist a non-trivial $C(X)$ -coselection for X of mesh less than ε , then X is called a *$C(X)$ -coselection space*.

In [12] (p. 240) Sergio Macías and Sam B. Nadler, Jr., asked several questions, three of which are as follows:

QUESTION 5.3. *Let X be a continuum. If $C(X)$ is contractible, then is $F_1(X)$ a Z -set in $C(X)$? In particular, if X is contractible, then is $F_1(X)$ a Z -set in $C(X)$?*

QUESTION 5.4. *If X is a continuum such that $F_1(X)$ is a Z -set in 2^X , then is $F_1(X)$ a Z -set in $C(X)$?*

QUESTION 5.5. *Let X be a continuum. If $\{x\}$ is a Z -set in $C(X)$ for all $x \in X$, then is $F_1(X)$ a Z -set in $C(X)$?*

In this chapter, we answer the first question affirmatively (Corollary 5.15) and we give examples that answer the second and third questions negatively (Example 5.17 and Example 5.18). In answering the first question we obtain a partial answer to a question of Nadler about coselections ([17], p. 266).

$C(X)$ -coselection spaces

In [17] (p. 266), Professor Nadler asked the following question:

QUESTION 5.6. *When do $C(X)$ -coselections exist?*

We give a sufficient condition in Theorem 5.9 and a necessary condition in terms of Z -sets in Theorem 5.13. Then we answer Question 5.3.

The following two lemmas are from [8] (p. 135 and p. 406, respectively).

LEMMA 5.7. *Let X be a continuum and let $\mu: 2^X \longrightarrow I$ be a Whitney map. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $A, B \in 2^X$, $A \subset B$, and $\mu(B) - \mu(A) < \delta$, then $H(A, B) < \varepsilon$.*

LEMMA 5.8. *Let X be a continuum. The hyperspace $C(X)$ is contractible if and only if for each Whitney map μ for $C(X)$ with $\mu(X) = 1$, there exist a map $G: C(X) \times I \longrightarrow C(X)$ such that:*

- (a) $G(A, 0) = A$ and $G(A, 1) = X$ for each $A \in C(X)$,
- (b) $G(A, s) \subset G(A, t)$ if $A \in C(X)$ and $0 \leq s \leq t \leq 1$, and
- (c) for each $t \in I$, $G(\mu^{-1}([0, t]) \times \{t\}) = \mu^{-1}(t)$ and for each $A \in \mu^{-1}([t, 1])$, $G(A, t) = A$.

THEOREM 5.9. *If $C(X)$ is contractible, then X is a $C(X)$ -coselection space.*

PROOF. Let $\varepsilon > 0$ and let $\mu: C(X) \longrightarrow I$ be a Whitney map with $\mu(X) = 1$. Then, by Lemma 5.7, there exists $\delta > 0$ such that if $A, B \in 2^X$, $A \subset B$, and $\mu(B) - \mu(A) < \delta$, then $H(A, B) < \frac{\varepsilon}{2}$. Since $C(X)$ is contractible, let $G: C(X) \times I \longrightarrow C(X)$ be a map satisfying (a), (b) and (c) of Lemma 5.8.

Define $f_\varepsilon: X \longrightarrow C(X)$ by letting $f_\varepsilon(x) = G(\{x\}, \frac{\delta}{2})$. Clearly, f_ε is continuous. Also, f_ε maps X to $C(X) - (F_1(X) \cup \{X\})$ since $\mu(f_\varepsilon(x)) = \mu(G(\{x\}, \frac{\delta}{2})) = \frac{\delta}{2}$. Now, we prove that $x \in f_\varepsilon(x)$ and $\text{diam}(f_\varepsilon(x)) < \varepsilon$ for each $x \in X$ as follows.

Take $x \in X$. Then, $\{x\} = G(\{x\}, 0) \subset G(\{x\}, \frac{\delta}{2}) = f_\varepsilon(x)$ and $\mu(f_\varepsilon(x)) - \mu(\{x\}) = \mu(G(\{x\}, \frac{\delta}{2})) - \mu(\{x\}) = \frac{\delta}{2} < \delta$; hence, $x \in f_\varepsilon(x)$ and $H(\{x\}, f_\varepsilon(x)) < \frac{\varepsilon}{2}$ by Lemma 5.7; which implies that $\text{diam}(f_\varepsilon(x)) < \varepsilon$.

Therefore, X is a $C(X)$ -coselection space. □

DEFINITION 5.10. Let X be a continuum. A nonempty proper closed subset A of X is said to be an R^3 -set of X provided that there exist an open subset U of X and a sequence of components $\{C_n\}_{n=1}^\infty$ of U such that $A \subset U$ and $A = \liminf C_n$.

The following example shows that the converse of Theorem 5.9 is not true.

EXAMPLE 5.11. *There is a continuum X such that X is a $C(X)$ -coselection space but $C(X)$ is not contractible.*

PROOF. Let

$$\begin{aligned} Y_1 &= \{0\} \times [-3, 1], \\ Y_2 &= \left\{ \left(x, -2 + \sin\left(\frac{1}{x}\right) \right) : x \in [-1, 0) \right\}, \\ Y_3 &= \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x \in (0, 1] \right\}. \end{aligned}$$

Define $Y = Y_1 \cup Y_2 \cup Y_3$. Let $X = Y \times I$.

Clearly, $\{(0, -1)\} \times I$ is an R^3 -set of X . Hence, by Theorem 78.15 of [8], $C(X)$ is not contractible. We show that X is a $C(X)$ -coselection space.

Let $\varepsilon > 0$. Define $f_\varepsilon: X \longrightarrow C(X)$ by

$$f(x, t) = \begin{cases} \{x\} \times [0, \frac{\varepsilon}{2}] & \text{if } t \in [0, \frac{\varepsilon}{4}] \\ \{x\} \times [t - \frac{\varepsilon}{4}, t + \frac{\varepsilon}{4}] & \text{if } t \in [\frac{\varepsilon}{4}, 1 - \frac{\varepsilon}{4}] \\ \{x\} \times [1 - \frac{\varepsilon}{2}, 1] & \text{if } t \in [1 - \frac{\varepsilon}{4}, 1]. \end{cases}$$

It is easy to see that f_ε is a nontrivial $C(X)$ -coselection such that, for each $(x, t) \in X$, $\text{diam}(f_\varepsilon(x, t)) = \frac{\varepsilon}{2} < \varepsilon$. $\square$

The following lemma is from [12] (p. 228)

LEMMA 5.12. *Let X be a continuum. Let $\mathcal{A}$ be either $C(X)$ or 2^X . Then, $F_1(X)$ is a Z -set in $\mathcal{A}$ if and only if for each $\varepsilon > 0$, there exists a map $f_\varepsilon: F_1(X) \longrightarrow \mathcal{A} - F_1(X)$ such that $H(\{x\}, f_\varepsilon(\{x\})) < \varepsilon$ for each $x \in X$.*

THEOREM 5.13. *If X is a $C(X)$ -coselection space, then $F_1(X)$ is a Z -set in $C(X)$.*

PROOF. Let $\varepsilon > 0$. Since X is a $C(X)$ -coselection space, there exists a non-trivial $C(X)$ -coselection, g_ε , for X of mesh less than ε . Define $f_\varepsilon: F_1(X) \longrightarrow C(X)$ by $f_\varepsilon(\{x\}) = g_\varepsilon(x)$. Clearly f_ε is continuous. Also, since g_ε is a non-trivial $C(X)$ -coselection, f_ε maps $F_1(X)$ to $C(X) - F_1(X)$. Finally, since for each $x \in X$, $x \in g_\varepsilon(x)$ and $\text{diam}(g_\varepsilon(x)) < \varepsilon$, we have that $H(\{x\}, f_\varepsilon(\{x\})) < \varepsilon$ for each $x \in X$.

Therefore, $F_1(X)$ is a Z -set in $C(X)$ by Lemma 5.12. $\square$

We give an example to show that the converse of Theorem 5.13 does not hold.

EXAMPLE 5.14. *There is a continuum X such that $F_1(X)$ is a Z -set in $C(X)$ but X is not a $C(X)$ -coselection space.*

PROOF. Let A be the convex arc in $\mathbb{R}^2$ from $(-1, 0)$ to $(1, 0)$. For each positive integer n , let A_n be the convex arc in $\mathbb{R}^2$ from $(-1, 0)$ to $(0, \frac{1}{n})$, and let B_n be the convex arc in $\mathbb{R}^2$ from $(1, 0)$ to $(0, -\frac{1}{n})$. Let $X = A \cup \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \left(\bigcup_{n=1}^{\infty} B_n \right)$.

It is shown in [12] (Example 4.7) that for each positive integer m , $F_m(X)$ is a Z -set in 2^X and in $C_m(X)$. We show that X is not a $C(X)$ -coselection space.

Suppose, to the contrary, that X is a $C(X)$ -coselection space. Let $\varepsilon > 0$ be given such that $\varepsilon < \frac{1}{2}$. Then there exists a non-trivial $C(X)$ -coselection, g_ε , for X of mesh less than ε . For each positive integer n , let $x_n = (0, \frac{1}{n})$ and $y_n = (0, -\frac{1}{n})$. Since $\text{mesh}(g_\varepsilon) < \varepsilon$, we have that $g_\varepsilon(0, 0) \subset A$ and, for each positive integer n , $g_\varepsilon(x_n) \cap A = \emptyset$ and $g_\varepsilon(y_n) \cap A = \emptyset$. Observe that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge to $(0, 0)$. Also, note that for each positive integer n , $\sup(\pi_1(g_\varepsilon(x_n))) \leq 0$ and $\inf(\pi_1(g_\varepsilon(y_n))) \geq 0$ (where $\pi_1: X \rightarrow [-1, 1]$ is the natural projection). Hence, by continuity, on one hand $\sup(\pi_1(g_\varepsilon((0, 0)))) \leq 0$ and on the other hand, $\inf(\pi_1(g_\varepsilon((0, 0)))) \geq 0$. Thus, $g_\varepsilon((0, 0)) = \{(0, 0)\}$, which is a contradiction.

Therefore X is not a $C(X)$ -coselection space. $\square$

As a consequence of our results we can answer Question 5.3:

COROLLARY 5.15. *If $C(X)$ is contractible, then $F_1(X)$ is a Z -set in $C(X)$.*

PROOF. The corollary follows immediately from Theorem 5.9 and Theorem 5.13 $\square$

Examples

We first note that Lemma 5.12 can be modified as follows:

LEMMA 5.16. *Let X be a continuum. Let $\mathcal{A}$ be either $C(X)$ or 2^X and let $p \in X$. Then, $\{\{p\}\}$ is a Z -set in $\mathcal{A}$ if and only if for each $\varepsilon > 0$, there exists a map $f_\varepsilon: F_1(X) \rightarrow \mathcal{A} - \{\{p\}\}$ such that $H(\{x\}, f_\varepsilon(\{x\})) < \varepsilon$ for each $x \in X$.*

The following example gives us a negative answer for Question 5.4:

EXAMPLE 5.17. *There is continuum X such that $F_1(X)$ is a Z -set in 2^X but not in $C(X)$.*

PROOF. Let

$$\begin{aligned} Y_1 &= \{0\} \times [-3, 1], \\ Y_2 &= \left\{ \left(x, -2 + \sin\left(\frac{1}{x}\right) \right) : x \in [-1, 0) \right\}, \\ Y_3 &= \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : x \in (0, 1] \right\}. \end{aligned}$$

Define $Y = Y_1 \cup Y_2 \cup Y_3$ and let C be the Cantor middle-third subset of the interval I . Let $X = (Y \times C) \cup (\{0\} \times \{0\} \times I)$.

We show that $F_1(X)$ is not a Z -set in $C(X)$. Suppose, to the contrary, that $F_1(X)$ is a Z -set in $C(X)$. Let $\varepsilon > 0$ be given such that $\varepsilon < \frac{1}{2}$. Then, by Lemma 5.12, there is a map $g_\varepsilon: F_1(X) \rightarrow C(X) - F_1(X)$ such that $H(\{x\}, g_\varepsilon(\{x\})) < \varepsilon$. Since $Y_2 \times \{0\}$ is arcwise connected and $d((-1, -2 + \sin(-1), 0), p) > \varepsilon$ for each $p \in (Y_1 \cup Y_3) \times C$, we have that for each $p \in Y_2 \times \{0\}$, $g_\varepsilon(\{p\}) \subset Y_2 \times \{c_1\}$ for some $c_1 \in C$. Similarly, for each $p \in Y_3 \times \{0\}$, $g_\varepsilon(\{p\}) \subset Y_3 \times \{c_2\}$ for some $c_2 \in C$. By continuity, $c_1 = c_2$ and $g_\varepsilon(\{(0, -1, 0)\}) \subset \{0\} \times [-3, 1] \times \{c_1\}$. Now, observe that the sequences $\left\{ \left(\frac{2}{\pi(3+4m)}, -1, 0 \right) \right\}_{m \in \mathbb{N}}$ and $\left\{ \left(\frac{-2}{\pi(3+4m)}, -1, 0 \right) \right\}_{m \in \mathbb{N}}$ converge to $(0, -1, 0)$. Also, note that for each $m \in \mathbb{N}$,

$$\inf \left(\pi_2 \left(g_\varepsilon \left(\left\{ \left(\frac{2}{\pi(3+4m)}, -1, 0 \right) \right\} \right) \right) \right) \geq -1$$

and

$$\sup \left(\pi_2 \left(g_\varepsilon \left(\left\{ \left(\frac{-2}{\pi(3+4m)}, -1, 0 \right) \right\} \right) \right) \right) \leq -1$$

(where $\pi_2: Y \times \{c_1\} \rightarrow [-3, 1]$ is defined by $\pi_2(x, y, c_1) = y$). Hence, by continuity, on one hand $\inf(\pi_2(g_\varepsilon(\{(0, -1, 0)\}))) \geq -1$ and on the other hand, $\sup(\pi_2(g_\varepsilon(\{(0, -1, 0)\}))) \leq -1$. Thus, $g_\varepsilon(\{(0, -1, 0)\}) = \{(0, -1, c_1)\}$, which is a contradiction. Therefore, $F_1(X)$ is not a Z -set in $C(X)$.

Now, we show that $F_1(X)$ is a Z -set in 2^X . Let $\varepsilon > 0$. Since C is compact and totally disconnected, by Lemma 7.11 of [18], C can be written as the union of n nonempty, mutually disjoint, compact subsets $C_1, \dots, C_n$ with $\text{diam}(C_i) < \frac{\varepsilon}{2}$ for each $i \in \{1, \dots, n\}$. For each $i \in \{1, \dots, n\}$, let $D_i = \{x \in I : p \leq x \leq q \text{ for some } p, q \in C_i\}$. Note that $I - \bigcup_{i=1}^n D_i$ is the union of finitely many open intervals whose end-points are contained in C . For each $i \in \{1, \dots, n\}$, take two different points $c_{i1}, c_{i2} \in C_i$. Take an open interval $(a, b) \subset I - \bigcup_{i=1}^n D_i$. We have that $a \in D_r$ and $b \in D_s$ for some $r, s \in \{1, \dots, n\}$ with $r \neq s$. Let $0 < \eta < \frac{\varepsilon}{2}$ such that $\eta < \frac{|a-b|}{3}$.

FIGURE 1

We describe a map on X . First, we show how the map behaves on $D_r \cup D_s \cup (a, b)$ as follows (we illustrate what we say in Figure 1 above):

If $(x, y, z) \in Y \times D_r$, then send (x, y, z) to the doubleton of the corresponding points in the levels c_{r_1} and c_{r_2} , i.e., the map sends (x, y, z) to $\{(x, y, c_{r_1}), (x, y, c_{r_2})\}$. Similarly, if $(x, y, z) \in Y \times D_s$, then the map sends (x, y, z) to $\{(x, y, c_{s_1}), (x, y, c_{s_2})\}$. Thus, we have mapped $\{0\} \times \{0\} \times D_r$ to $\{(0, 0, c_{r_1}), (0, 0, c_{r_2})\}$ and $\{0\} \times \{0\} \times D_s$ to $\{(0, 0, c_{s_1}), (0, 0, c_{s_2})\}$. In particular the map sends $(0, 0, a)$ to $\{(0, 0, c_{r_1}), (0, 0, c_{r_2})\}$ and $(0, 0, b)$ to $\{(0, 0, c_{s_1}), (0, 0, c_{s_2})\}$. Now, we use the interval (a, b) to move $\{(0, 0, c_{r_1}), (0, 0, c_{r_2})\}$ to $\{(0, 0, c_{s_1}), (0, 0, c_{s_2})\}$ in such a way that a point and its image are close enough.

We do what was described above on all X ; here is a formula:

Define $h_{(a,b)}: (a, b) \longrightarrow F_2(\{0\} \times \{0\} \times I)$ by

$$h_{(a,b)}(x) = \begin{cases} \left\{ \left(0, 0, \frac{a+\eta-x}{\eta}c_{r_1} + \frac{x-a}{\eta}a \right), \left(0, 0, \frac{a+\eta-x}{\eta}c_{r_2} + \frac{x-a}{\eta}(a+\eta) \right) \right\}, \\ \text{if } x \in (a, a+\eta] \\ \{ (0, 0, x), (0, 0, x-\eta) \}, \\ \text{if } x \in [a+\eta, b-\eta] \\ \left\{ \left(0, 0, \frac{b-x}{\eta}(b-2\eta) + \frac{x-(b-\eta)}{\eta}c_{s_1} \right), \left(0, 0, \frac{b-x}{\eta}(b-\eta) + \frac{x-(b-\eta)}{\eta}c_{s_2} \right) \right\}, \\ \text{if } x \in [b-\eta, b). \end{cases}$$

Now, define $f_\varepsilon: F_1(X) \longrightarrow 2^X - F_1(X)$ as follows:

Take $\{(x, y, z)\} \in F_1$. If $z \in D_j$ for some $j \in \{1, \dots, n\}$, define

$$f_\varepsilon(\{(x, y, z)\}) = \{(x, y, c_{j_1}), (x, y, c_{j_2})\}.$$

Otherwise, $z \in (a, b) \subset I - \bigcup_{i=1}^n D_i$, so define $f_\varepsilon(\{(x, y, z)\}) = h_{(a,b)}(z)$.

It follows from the way f_ε is defined that it is continuous. Also, we have that, for each $A \in F_1(X)$, $f_\varepsilon(A)$ is made up of two points. Thus, for each $A \in F_1(X)$, $f_\varepsilon(A) \in 2^X - F_1(X)$. Finally, it is easy to see that, for each $A \in F_1(X)$, $H(A, f_\varepsilon(A)) \leq \eta + \frac{\varepsilon}{2} < \varepsilon$. Therefore, by Lemma 5.12, $F_1(X)$ is a Z -set in 2^X . $\square$

Finally, the previous example also give us a negative answer for Question 5.5:

EXAMPLE 5.18. *There is a continuum X such that $\{\{x\}\}$ is a Z -set in $C(X)$ for all $x \in X$, but $F_1(X)$ is not a Z -set in $C(X)$.*

PROOF. Let X be the continuum in Example 5.17. We have shown that $F_1(X)$ is not a Z -set in $C(X)$. Fix a point $(x, y, z) \in X$. We show that $\{(x, y, z)\}$ is a Z -set in $C(X)$. Let $\varepsilon > 0$. Since C is compact and totally disconnected, by Lemma 7.11 of [18], C can be written as the union of n nonempty, mutually disjoint, compact subsets $C_1, \dots, C_n$ with $\text{diam}(C_i) < \frac{\varepsilon}{2}$ for each $i \in \{1, \dots, n\}$. For each $i \in \{1, \dots, n\}$, let $D_i = \{x \in I: p \leq x \leq q \text{ for some } p, q \in C_i\}$. Note that $I - \bigcup_{i=1}^n D_i$ is the union of finitely many open intervals whose end-points are contained in C .

Suppose that $z \in D_j$ for some $j \in \{1, \dots, n\}$. Let $q = \min D_j$ and let $r = \max D_j$. Let (p, q) and (r, s) denote the open intervals of $I - \bigcup_{i=1}^n D_i$ whose end points are q and r respectively (if either of these two points is 0 or 1 then there is only one interval). Take a point $c_j \in C_j$ different from z . We can suppose without loss of generality that $z < c_j$. Let $0 < \eta < \frac{\varepsilon}{4}$ such that $\eta < \max \left\{ \frac{|p-q|}{4}, \frac{|r-s|}{4} \right\}$.

We describe a map on X . First, we show how the map behaves on $D_j \cup (p, q) \cup (r, s)$. Take $(t, u, v) \in Y \times D_j$. We consider three cases:

Case 1: $(x, y, z) \in X - \{0\} \times \{0\} \times I$. Then send (t, u, v) to the singleton of the corresponding point in the level c_j , i.e., the map sends (t, u, v) to $\{(t, u, c_j)\}$. Thus, we have mapped $\{0\} \times \{0\} \times D_j$ to $\{(0, 0, c_j)\}$. In particular the map sends both $(0, 0, q)$ and $(0, 0, r)$ to $\{(0, 0, c_j)\}$. Now, we use the interval (p, q) to move $\{(0, 0, c_j)\}$ to $\{(0, 0, p)\}$ and the interval (r, s) to move $\{(0, 0, c_j)\}$ to $\{(0, 0, s)\}$ in such a way that a point and its image are close enough. In the rest of X , the map is like the identity map.

We do what was described above as follows:

If $(x, y, z) \in X - \{0\} \times \{0\} \times I$, define $f_\varepsilon: F_1(X) \longrightarrow C(X) - \left\{ \{(x, y, z)\} \right\}$ by

$$f_\varepsilon(\{(t, u, v)\}) = \begin{cases} \{(t, u, v)\}, & \text{if } v \in I - (D_j \cup (q - \eta, q) \cup (r, r + \eta)) \\ \{(t, u, c_j)\}, & \text{if } v \in D_j \\ \left\{ \left(0, 0, \frac{q-v}{\eta}(q - \eta) + \frac{v-(q-\eta)}{\eta}c_j \right) \right\}, & \text{if } v \in (q - \eta, q) \\ \left\{ \left(0, 0, \frac{r+\eta-v}{\eta}c_j + \frac{v-r}{\eta}(r + \eta) \right) \right\}, & \text{if } v \in (r, r + \eta). \end{cases}$$

Case 2: $(x, y, z) \in \{0\} \times \{0\} \times I$. Then send (t, u, v) to the singleton of the corresponding point in the levels c_j , i.e., the map sends (t, u, v) to $\{(t, u, c_j)\}$. Thus, we have mapped $\{0\} \times \{0\} \times D_j$ to $\{(0, 0, c_j)\}$. In particular the map sends both $(0, 0, q)$ and $(0, 0, r)$ to $\{(0, 0, c_j)\}$. Now, we use the interval (r, s) to move $\{(0, 0, c_j)\}$ to $\{(0, 0, s)\}$ and the interval (p, q) to inflate $\{(0, 0, c_j)\}$ to an interval, move that interval towards $\{(0, 0, p)\}$ and contract it to $\{(0, 0, p)\}$ in such a way that a point and its image are close enough. In the rest of X , the map is like the identity map.

We do what was described above as follows:

Let $A = \{0\} \times \{0\}$. If $(x, y, z) \in \{0\} \times \{0\} \times I$, define $f_\varepsilon: F_1(X) \longrightarrow C(X) - \left\{ \{(x, y, z)\} \right\}$ by

$$f_\varepsilon(\{(t, u, v)\}) = \begin{cases} \{(t, u, v)\}, & \text{if } v \in I - (D_j \cup (q - 3\eta, q) \cup (r, r + \eta)) \\ \{(t, u, c_j)\}, & \text{if } v \in D_j \\ \left\{ \left(0, 0, \frac{r+\eta-v}{\eta}c_j + \frac{v-r}{\eta}(r + \eta) \right) \right\}, & \text{if } v \in (r, r + \eta) \\ A \times \left[c_j, \frac{q-v}{\eta}z + \frac{v-(q-\eta)}{\eta}c_j \right], & \text{if } v \in [q - \eta, q) \\ A \times \left[\frac{(q-\eta)-v}{\eta}(q - \eta) + \frac{v-(q-2\eta)}{\eta}z, \frac{(q-\eta)-v}{\eta}q + \frac{v-(q-2\eta)}{\eta}c_j \right], & \text{if } v \in [q - 2\eta, q - \eta) \\ A \times \left[\frac{(q-2\eta)-v}{\eta}(q - 3\eta) + \frac{v-(q-3\eta)}{\eta}(q - \eta), \frac{(q-2\eta)-v}{\eta}(q - 3\eta) + \frac{v-(q-2\eta)}{\eta}q \right], & \text{if } v \in (q - 3\eta, q - 2\eta] \end{cases}$$

Case 3: $z \notin D_i$ for all $i \in \{1, \dots, n\}$. Then there is an open interval $(a, b) \subset I - \bigcup_{i=1}^n D_i$ such that $z \in (a, b)$. Let $0 < \eta < \frac{\varepsilon}{4}$ such that $\eta < \max \left\{ \frac{|z-a|}{2}, \frac{|z-b|}{2} \right\}$.

We can describe the map that we want to define in (a, b) as follows:

We use the interval (a, b) to inflate $\{(0, 0, a)\}$ to an interval, move that interval towards $\{(0, 0, b)\}$ and contract it to $\{(0, 0, b)\}$ in such a way that a point and its image are close enough. In the rest of X , the map is like the identity map.

We do what was described above as follows:

If $z \notin D_i$ for all $i \in \{1, \dots, n\}$, then define $f_\varepsilon: F_1(X) \longrightarrow C(X) - \left\{ \{(x, y, z)\} \right\}$

$$\text{by } f_\varepsilon(\{(t, u, v)\}) = \begin{cases} \{(t, u, v)\}, & \text{if } v \in I - (a, b) \\ \{0\} \times \{0\} \times \left[\frac{b-v}{|z-b|}(z - \eta) + \frac{v-z}{|z-b|}b, \frac{b-v}{|z-b|}(z + \eta) + \frac{v-z}{|z-b|}b \right], & \text{if } v \in [z, b) \\ \{0\} \times \{0\} \times \left[\frac{v-a}{|z-a|}(z - \eta) + \frac{z-v}{|z-a|}a, \frac{v-a}{|z-a|}(z + \eta) + \frac{z-v}{|z-a|}a \right], & \text{if } v \in (a, z]. \end{cases}$$

In any of the cases above, it follows that f_ε is continuous. Also, we have that for each $A \in F_1(X)$, $f_\varepsilon(A)$ is a singleton different from $\{x, y, z\}$ or a nondegenerate interval. Thus, for each $A \in F_1(X)$, $f_\varepsilon(A) \in C(X) - \left\{ \{(x, y, z)\} \right\}$. Finally, it is easy to see that, for each

$A \in F_1(X)$, $H(A, f_\varepsilon(A)) \leq 3\eta + \frac{\varepsilon}{2} < \varepsilon$. Therefore, by Lemma 5.16, $\left\{\{(x, y, z)\}\right\}$ is a Z -set in $C(X)$. $\square$

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