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Graph Minor

Jianbing Niu

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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ABSTRACT

Graph Minor

Jianbing Niu

In this paper, we present three results: (1) Let G be a $(k+2)$ -connected *non*-($k-3$)-*apex* graph where $k \geq 2$. If G contains three k -cliques, say L_1, L_2, L_3 , such that $|L_i \cap L_j| \leq k-2$ ($1 \leq i < j \leq 3$), then G contains a K_{k+2} as a minor. (2) Let G be a 6-connected claw-free graph. If $\delta(G) \geq 7$ and G contains three disjoint 5-cliques, say L_1, L_2, L_3 , then G contains a K_7 as a minor. (3) There is a function $h : N \longrightarrow N$, such that, for every 4-connected graph G with minimum degree at least five embedded in a surface with Euler genus g and face-width at least $h(g)$, every longest circuit of the graph G has a chord.

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DEDICATION

To

my father, my mother

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Chapter 1

Introduction

In this dissertation, we study problems related to Hadwiger's Conjecture and Thomassen's Conjecture.

In 1943, Hadwiger made the following conjecture, which is a generalization of the Four Color Problem.

Conjecture 1.0.1 (Hadwiger [10]) *For all $k \geq 1$, every k -chromatic graph has the complete graph K_k on k vertices as a minor.*

A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and vertices and contracting edges.

In Chapter 2, we give the following result.

Theorem 2.1.12 *Let G be a $(k + 2)$ -connected, non- $(k - 3)$ -apex graph where $k \geq 2$. If G contains three k -cliques, say L_1, L_2, L_3 , such that $|L_i \cap L_j| \leq k - 2$ ($1 \leq i < j \leq 3$), then G contains a K_{k+2} as a minor.*

In Chapter 3, we give the following result.

Theorem 3.1.6 *Let G be a 6-connected claw-free graph. If $\delta(G) \geq 7$ and G contains three disjoint 5-cliques, say L_1, L_2, L_3 , then G contains a K_7 as a minor.*

In 1985, Thomassen made the following conjecture.

Conjecture 1.0.2 (Thomassen [1], p. 466) *Every longest circuit of a 3-connected graph must have a chord.*

An edge e is called a *chord* of a circuit if e is not an edge of the circuit and both end vertices of e are in the circuit.

Thomassen's conjecture has been verified for many families of graphs. In Chapter 4, we verify the conjecture for *locally planar graphs*, and we prove the following result.

Theorem 4.1.2 *There is a function $h : N \rightarrow N$, such that, for every 4-connected graph G with minimum degree at least five embedded in a surface with Euler genus g and face-width at least $h(g)$, every longest circuit of the graph G has a chord.*

Chapter 2

Cliques, Minors and Apex Graphs

2.1 Introduction

Hadwiger's Conjecture from 1943 suggests a far reaching generalization of the Four Color Problem, and it is one of the most famous problems in the theory of graph minors. Hadwiger's Conjecture states the following.

Conjecture 2.1.1 (Hadwiger [10]) *For all $k \geq 1$, every k -chromatic graph has the complete graph K_k on k vertices as a minor.*

For $k = 1, 2, 3$, this conjecture is easy to prove, and for $k = 4$, Hadwiger [10] and Dirac [9] proved it. For $k = 5$, however, it seems extremely difficult. In 1937, Wagner [35] proved that the case $k = 5$ is equivalent to the Four Color Theorem [2, 3, 24]. In 1993, Robertson, Seymour and Thomas [27] proved that a minimal counterexample to the case $k = 6$ is a graph G which has a vertex v such that $G - v$ is planar. Hence, assuming the Four Color Theorem, the case $k = 6$ of Hadwiger's Conjecture holds. This result is the deepest in this research area. So far, the cases $k \geq 7$ are open.

Motivated by Hadwiger's Conjecture, the following question has drawn the attention of many researchers.

Question 2.1.2 *What do K_k -minor-free graphs look like?*

One approach to this question is to consider the maximal size of graphs not having K_k as a minor. Wagner [36] showed that a sufficiently large chromatic number (which depends only on k) guarantees a K_k as a minor, and Mader [20] showed that a sufficiently large average degree will do. Kostochka [17] and Thomason [30] independently proved that $k\sqrt{\log k}$ is the correct order for the average degree because random graphs having no K_k -minor may have average degree of order $k\sqrt{\log k}$. (Recently, Thomason [31] gave a more exact “extremal” function.)

Another approach is due to Robertson and Seymour [25]. They considered how to construct graphs with no K_k -minor. If G contains a set X with at most $k - 5$ vertices such that $G - X$ is planar, G does not contain K_k as a minor since planar graphs cannot contain K_5 as a minor. Similarly, if G contains a set X with at most $k - 7$ vertices such that $G - X$ can be drawn in the projective plane, then clearly G does not contain K_k as a minor. (Since graphs drawn on the projective plane cannot contain K_7 as a minor.) Or if G contains a set X with at most $k - 8$ vertices such that $G - X$ can be drawn in the torus, then clearly G does not contain K_k as a minor. (Again, graphs drawn on the torus cannot contain K_8 as a minor.) These observations together with the concept “clique-sum” led Robertson and Seymour to one of their celebrated results of excluding complete graph minors, and this is the most important step in their proof of “Wagner’s Conjecture” [26].

Our motivation is the following question.

Question 2.1.3 *What do K_k -minor-free k -connected graphs look like?*

It does not seem that random graphs give an answer to this question because, as Thomason [31] pointed out, extremal graphs are more or less exactly vertex disjoint unions of suitable dense random graphs. It does not seem that Robertson and Seymour’s excluded minor theorem gives an answer either, because their characterization does not seem to guarantee high connectivity. In view of these observations, we still do not know what K_k -minor-free k -connected graphs look like.

The following question is motivated by Hadwiger’s Conjecture.

Question 2.1.4 *Is it true that a minimal counterexample to Hadwiger’s Conjecture for $k \geq 6$ has a set X of $k - 5$ vertices such that $G - X$ is planar?*

This is true for $k = 6$ as Robertson, Seymour and Thomas [27] showed. To consider a minimal counterexample to Hadwiger’s Conjecture, one can prove the following conjecture.

Conjecture 2.1.5 *A minimal counterexample to Hadwiger's Conjecture is k -connected.*

This is true for $k \leq 7$ as Mader proved in [21]. Note that Toft [34] proved that a minimal counterexample to Hadwiger's Conjecture is k -edge-connected. This is strong evidence for Conjecture 2.1.5.

Question 2.1.4 and Conjecture 2.1.5 lead us to the following question.

Question 2.1.6 *Is it true that a K_k -minor-free k -connected graph for $k \geq 6$ has a set X of $k - 5$ vertices such that $G - X$ is planar?*

The case $k = 6$ is a well-known conjecture due to Jorgensen [13], and is still open. If true, this would imply Hadwiger's Conjecture for the $k = 6$ case by Mader's result [20]. The case $k = 7$ was conjectured in [14] as well.

Even though the case $k = 6$ of the Question 2.1.6 is still open, Robertson, Seymour and Thomas [27] gave a result for searching for a K_6 -minor.

Theorem 2.1.7 (Robertson, Seymour and Thomas [27]) *Let G be a simple 6-connected non-apex graph. If G contains three 4-cliques, say, L_1, L_2, L_3 , such that $|L_i \cap L_j| \leq 2$ ($1 \leq i < j \leq 3$), then G contains a K_6 as a minor.*

Recently, Kawarabayashi and Toft [14] proved the following theorem.

Theorem 2.1.8 (Kawarabayashi and Toft [14]) *Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor.*

This settles the case $(6, 1)$ of the following conjecture known as the $(k - 1, 1)$ -Minor Conjecture, which is a relaxed version of Hadwiger's Conjecture.

Conjecture 2.1.9 (Chartrand, Geller, Hedetniemi [7]; Woodall [37]) *For all $k \geq 1$, every k -chromatic graph has either a K_k -minor or a $K_{\lfloor \frac{k+1}{2} \rfloor, \lceil \frac{k+1}{2} \rceil}$ -minor.*

In [14], the following result is the key lemma, which gives a result for searching for a K_7 -minor.

Theorem 2.1.10 (Kawarabayashi and Toft [14]) *Let G be a 7-connected graph. Suppose G contains three 5-cliques, say, L_1, L_2, L_3 , such that $|L_1 \cup L_2 \cup L_3| \geq 12$, then G contains a K_7 -minor.*

Recently, Kawarabayashi, R. Luo, J. Niu and C-Q Zhang [15] proved the following theorem.

Theorem 2.1.11 (Kawarabayashi, R. Luo, J. Niu and C-Q Zhang [15]) *Let G be a $(k+2)$ -connected graph where $k \geq 5$. If G contains three k -cliques, say L_1, L_2, L_3 , such that $|L_1 \cup L_2 \cup L_3| \geq 3k - 3$, then G contains a K_{k+2} as a minor.*

Our work is motivated by Theorem 2.1.7, and the main result of this Chapter is the following theorem which generalizes Theorem 2.1.7, Theorem 2.1.10 and Theorem 2.1.11.

Theorem 2.1.12 *Let G be a $(k+2)$ -connected, non- $(k-3)$ -apex graph where $k \geq 2$. If G contains three k -cliques, say L_1, L_2, L_3 , such that $|L_i \cap L_j| \leq k - 2$ ($1 \leq i < j \leq 3$), then G contains a K_{k+2} as a minor.*

We hoped our result could be used to prove some results on 7- and 8-chromatic graphs. In fact, in [16], Kawarabayashi proved that any 7-chromatic graph has K_7 or $K_{3,5}$ as a minor using Theorem 2.1.11. Maybe one can use this result to prove 8-chromatic case of Conjecture 2.1.9.

A k -connected graph may contain many $(k-2)$ -cliques, but not necessarily a K_k -minor. For example, the graph $K_{k-5} + G_1$, where G_1 is a 5-connected planar graph, is K_k -minor-free and contains many copies of $(k-2)$ -cliques. Theorem 2.1.12, implies that a k -connected K_k -minor-free graph can't contain three “nearly” disjoint $(k-2)$ -cliques.

The following was conjectured by Seymour and Thomas (private communication with R. Thomas.)

Conjecture 2.1.13 *For every $p \geq 1$, there exists a constant $N = N(p)$ such that every $(p-2)$ -connected graph on $n \geq N$ vertices and at least $(p-2)n - \frac{(p-1)(p-2)}{2} + 1$ edges has a K_p -minor.*

Note that the connectivity condition and the condition of the order of graphs are necessary because random graphs having no K_k -minor may have average degree $k\sqrt{\log k}$, but all these graphs are small. So if a graph is large enough and highly connected, we do not know any infinite family of counterexamples. This conjecture is true for $p \leq 9$. For $p \leq 7$, these conjecture were proved by Mader [21]. Jorgensen [13] proved the case of $p = 8$. Very recently, Song and Thomas [29] proved the case $p = 9$. Note that all of these results do not require the connectivity condition in this conjecture.

We hope that our result could give a weaker result since, as far as we know, the only known extremal graphs are $K_{k-5} + G_1$, where G_1 is a 5-connected planar graph. So this graph could contain $(k-2)$ -clique. On the other hand, our result implies that it cannot contain three nearly "disjoint" K_{k-2} . Hence one can prove a weaker bound on the number of edges.

2.2 Terminology and notations

All graphs considered in this paper are finite, undirected, and without loops or multiple edges. The complete graph (or, clique, as a subgraph) on n vertices is denoted by K_n and the complete bipartite graph such that one partite set has n vertices and the other partite set has m vertices is denoted by $K_{n,m}$.

A graph H is a *minor* of a graph G if H can be obtained from G by deleting edges and vertices and contracting edges.

For a vertex x of a subgraph H_1 of G , the neighborhood of x in H_1 is denoted by $N_{H_1}(x)$. For a vertex $v \in V(G)$ and a vertex subset (or a subgraph) Y of G , $d_Y(x) = |\{v \in Y : xv \in E(G)\}|$. A graph G is *k-chromatic* if G is vertex- k -colorable, but not vertex- $(k-1)$ -colorable. Let V_1 and V_2 be subsets of $V(G)$. The *symmetric difference* of V_1 and V_2 , denoted by $V_1 \Delta V_2$, is the set $(V_1 \cup V_2) - (V_1 \cap V_2)$.

We say a graph G is *k-apex* if $G - X$ is planar for some subset $X \subseteq V(G)$ with $|X| = k$. By the definition, a 0-apex graph is planar. A 1-apex graph is often referred as an apex graph if there is no confusion.

A subset $X \subseteq V(G)$ is a *fragment* of G if $X \neq \emptyset$ and X induces a connected subgraph of G . Subsets $X, Y \subseteq V(G)$ are *adjacent* in G if $X \cap Y = \emptyset$ and some $x \in X$ is adjacent in G to some $y \in Y$.

A *cluster* in G is a set of mutually adjacent fragments G , and it is a *p-cluster* if it has cardinality p . Thus, G has a K_p -minor if and only if it has a p -cluster. Given a subset $Y \subseteq V(G)$, a *p-cluster* $\wp$ is said to *traverse* Y if $\wp = \{X_1, X_2, \dots, X_p\}$ and $X_i \cap Y \neq \emptyset$ ($1 \leq i \leq p$).

Let v_1, v_2, v_3 be mutually adjacent vertices of a graph G . We say G is *triangular* with respect to v_1, v_2, v_3 if G is simple and either

(i) for some $i(1 \leq i \leq 3)$, $G - v_i$ has maximum degree at most 2, and either $G - v_i$ is a cycle or it has no cycle, or

(ii) all vertices of G have degree at most 3, there is at most one vertex v of degree 3 with $v \neq v_1, v_2, v_3$, and $G - v_1 - v_2 - v_3$ has no cycle, or

(iii) all vertices of G have degree at most 3, there is a triangle C in $G - v_1 - v_2 - v_3$, every vertex of degree 3 is in $\{v_1, v_2, v_3\} \cup V(C)$, and every cycle except for the two triangles $\{v_1, v_2, v_3\}$ and C contains both a vertex in $\{v_1, v_2, v_3\}$ and $V(C)$.

2.3 Lemmas

2.3.1 Good paths

One of the key lemmas in our proof is Mader's "*H*-Wege" Theorem, which was proved in [22].

Lemma 2.3.1 (Mader [22]) *Let G be a graph, let $S \subseteq V(G)$ be an independent set, and $k \geq 0$ be an integer. Then exactly one of the following two statements holds.*

(1) *There are k paths of G , each with two distinct ends both in S , such that each $v \in V(G) - S$ is in at most one of the paths.*

(2) *There exists a vertex set $W \subseteq V(G) - S$ and a partition $Y_1, \dots, Y_n$ of $V(G) - (S \cup W)$, and a subset $X_i \subseteq Y_i$, $1 \leq i \leq n$, such that*

$$(a) |W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor < k,$$

(b) *no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and,*

(c) *every path of $G - W$ with distinct ends both in S has an edge with both ends in Y_i for some i .*

Let $Z_1, Z_2, \dots, Z_h$ be subsets of $V(G)$. A path P of G with ends u, v is said to be *good* if there exist distinct i, j with $1 \leq i, j \leq h$ such that $u \in Z_i$ and $v \in Z_j$.

As Robertson, Seymour and Thomas pointed out in [27], we can deduce the following lemma from Lemma 2.3.1.

Lemma 2.3.2 (Robertson, Seymour and Thomas [27]) *Let G be a graph, let $Z_1, Z_2, \dots, Z_h$ be subsets of $V(G)$, and let $k \geq 1$ be an integer. Then exactly one of the following two statements holds.*

- (1) *There are k mutually disjoint good paths of G .*
- (2) *There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, \dots, Y_n$ of $V(G) - W$, and a subset $X_i \subseteq Y_i$, for $1 \leq i \leq n$ such that*

$$(a) |W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor < k,$$

(b) *for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^h Z_j) \subseteq X_i$, and*

(c) *every good path P in $G - W$ has an edge with both ends in Y_i for some i .*

2.3.2 Cluster

Lemma 2.3.3 (Robertson, Seymour and Thomas [27], page 291) *Let v_1, v_2, v_3 be mutually adjacent vertices of a 4-connected simple non-planar graph G . Let $\mathfrak{S} \subseteq V(G)$ with $v_1, v_2, v_3 \in \mathfrak{S}$ such that $\mathfrak{S}$ is not triangular. Then there is a 5-cluster $\wp = \{\{v_1\}, \{v_2\}, \{v_3\}, X_1, X_2\}$ in G such that $\wp$ traverses $\mathfrak{S}$.*

The following lemma is an immediate corollary of a result by Robertson, Seymour and Thomas in [27] (page 288).

Lemma 2.3.4 *Let G be a 4-connected graph and $\mathfrak{S} \subseteq V(G)$ with $|\mathfrak{S}| = 4$. Then either*

- (i) there is a 4-cluster in G traversing $\mathfrak{S}$, or
- (ii) G can be drawn in a plane so that every vertex in $\mathfrak{S}$ is incident with the infinite region.

2.3.3 The 6-cluster lemma

The following lemma deals with an extreme case of our main theorem (Theorem 2.1.12). Since the proof of the lemma is relatively long and complicated, we present it here as an independent lemma and its proof in Section 2.5. Readers may postpone the reading of the proof of Lemma 2.3.5 until after the proof of the main theorem.

Lemma 2.3.5 *Let x_i, y_i, z_i ($1 \leq i \leq 3$) be distinct vertices of a 6-connected simple graph G , such that $\{x_1, y_1, z_2, z_3\}$, $\{x_2, y_2, z_3, z_1\}$, $\{x_3, y_3, z_1, z_2\}$ are 4-cliques. Suppose, that there is a partition Y_1, Y_2 of $V(G) - \{z_1, z_2, z_3\}$ with $x_1, x_2, x_3 \in Y_1$, and $y_1, y_2, y_3 \in Y_2$, such that $x_i y_i$ ($1 \leq i \leq 3$) are the only edges of G with one end in Y_1 and the other in Y_2 . Then G has a 6-cluster traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$.*

Note that, Robertson, Seymour and Thomas gave a result in page 293 of [27] similar to Lemma 2.3.5. However, in order to obtain a sharper and more general result in our main theorems (Theorem 2.1.12 and Theorem 2.4.1), we need a stronger result in Lemma 2.3.5 (for 6-cluster instead of 6-minor), which is approached differently from that in [27].

2.4 Proof of the main theorem

The main theorem (Theorem 2.1.12) is to be proved in this section. Here we prove a theorem that is slightly stronger than the main theorem (Theorem 2.1.12).

Theorem 2.4.1 *Let G be a $(k+2)$ -connected, non- $(k-3)$ -apex graph where $k \geq 2$. If G contains three k -cliques, say L_1, L_2, L_3 , such that $|L_i \cap L_j| \leq k-2$ ($1 \leq i < j \leq 3$), then one of the following holds,*

(1) G contains a $(k+2)$ -cluster traversing $L_1 \cup L_2 \cup L_3$, or

(2) (an exceptional case) $|T| = k - 2$ where $T = L_1 \cap L_2 \cap L_3$, and $G - T$ is a planar graph with all edges $L_i - T$ ($i = 1, 2, 3$) around the exterior face. In this case, G contains a $(k+2)$ -cluster $\{\{v_1\}, \dots, \{v_k\}, B, I\}$ where $L_1 = \{v_1, \dots, v_k\}$, B is the set of all vertices of $G - T$ around the exterior face except for those in L_1 , and I is the set of all interior vertices of $G - T$.

Note: readers might be confused by a non- $(k-3)$ -apex graph if $k = 2$. Recall that a graph H is a non- t -apex if $G - R$ is planar for some vertex subset R , then R must be of order at least $t + 1$. Hence, a non- $(k-3)$ -apex graph for $k = 2$ can be any graph, planar or non-planar.

Proof: Let G be a counterexample to the theorem with k as small as possible.

2.4.1 We claim that $k \geq 3$.

For otherwise, we may assume $k = 2$, G is 4-connected graph, and G contains three disjoint 2-cliques, say L_1, L_2, L_3 .

Since L_1 and L_2 are disjoint 2-cliques, $|L_1 \cup L_2| = 4$. Note that G is 4-connected, by Lemma 2.3.4. There are two cases:

(1) There is a 4-cluster in G traversing $L_1 \cup L_2$. In this case, by the definition of cluster, this 4-cluster in G also traverses $L_1 \cup L_2 \cup L_3$, hence we are done.

(2) G can be drawn in a plane so that every vertex in $L_1 \cup L_2$ is incident with the infinite region. In this case, since G is 4-connected, the edges of L_1 and L_2 must be around the exterior face. If one vertex v_1 of L_3 is not incident with the infinite region, then there are four internal vertex-disjoint paths from v_1 to $L_1 \cup L_2$, hence we get a 4-cluster traversing $L_1 \cup L_2 \cup L_3$. Therefore two vertices of L_3 must be incident with the infinite region. Note that G is 4-connected, G has a 4-cluster $\{\{v_1\}, \{v_2\}, B, I\}$ where $L_1 = \{v_1, v_2\}$, B is the set of all vertices of G around the exterior face except for those in L_1 , and I is the set of all interior vertices of G (it is easy to see that I and B both are connected). ■

2.4.2 We claim that $|L_1 \cap L_2 \cap L_3| = 0$.

For otherwise, we assume $|L_1 \cap L_2 \cap L_3| \neq 0$. Let $x \in L_1 \cap L_2 \cap L_3$ and $G' = G - \{x\}$, then G' is a $(k+1)$ -connected non $(k-4)$ -apex graph. By minimality of k , there are two cases:

Case (1): There is a $(k+1)$ -cluster $\wp_1$ of $G - \{x\}$ traversing $L_1 \cup L_2 \cup L_3 - \{x\}$. Let $\wp = \wp_1 \cup \{\{x\}\}$, then $\wp$ is a $(k+2)$ -cluster traversing $L_1 \cup L_2 \cup L_3$, a contradiction.

Case (2): $|T'| = k - 3$ where $T' = (L_1 \cap L_2 \cap L_3) - \{x\}$, and $G' - T'$ is a planar graph with all edges of $(L_i - \{x\}) - T'$ around the exterior face. G' contains a $(k+1)$ -cluster $\wp' = \{\{v_1\}, \dots, \{v_{k-1}\}, B, I\}$ where $\{x, v_1, \dots, v_{k-1}\} = L_1$ and B is the set of all vertices of $G' - T'$ around the exterior face except for those in L_1 , and I is the set of all interior vertices of $G' - T'$.

In order to show that $\wp = \{\{x\}, \{v_1\}, \dots, \{v_{k-1}\}, B, I\}$ is a $(k+2)$ -cluster of G , it is sufficient to prove that x is adjacent to every other fragment. (i) it is obvious that $v_i \in N(x)$ since $x \in L_1$; (ii) it is similar that $N(x) \cap B \neq \emptyset$ since $x \in L_2$ and $L_2 \cap B \neq \emptyset$; (iii) $N(x) \cap I \neq \emptyset$ for otherwise, the vertex x can be embedded into the exterior face of $G' - T'$ and therefore, $G - T'$ is planar. This contradicts that G is non- $(k-3)$ -apex.

2.4.3 We claim that $k \geq 4$.

For otherwise, by (2.4.1), we may assume $k = 3$. That is, G is a 5-connected non-planar graph, and G contains three 3-cliques, say L_1, L_2, L_3 , such that $|L_i \cap L_j| \leq 1$ ($1 \leq i < j \leq 3$). By (2.4.2), we have $|L_1 \cap L_2 \cap L_3| = 0$.

Let $Z = L_1 \cup L_2 \cup L_3$ and $v_1, v_2, v_3 \in L_1$. Then Z is not triangular with respect to v_1, v_2, v_3 . By Lemma 2.3.3, there is a 5-cluster $\wp$ in G such that $\wp$ traverses Z . ■

2.4.4 We claim that G does not contain a $(k+2)$ -cluster traversing $L_1 \cup L_2 \cup L_3$.

By (2.4.2), we have $|L_1 \cap L_2 \cap L_3| = 0$. And by (2.4.3), $k \geq 4$. Hence G is at least 6-connected, $|E(G)| \geq 3|V(G)|$. Therefore G is not a planar graph, that is, G is not the exceptional case. Hence the result follows.

2.4.5 We claim that $|L_i \cap L_j| \leq 1$ for $1 \leq i < j \leq 3$.

For otherwise, we may assume $|L_1 \cap L_2| \geq 2$. Let $B \subseteq L_1 \cap L_2$ with $|B| = 2$. By (2.4.2), $B \cap L_3 = \emptyset$ since $L_1 \cap L_2 \cap L_3 = \emptyset$. Since $G - B$ is k -connected, there exist k disjoint paths

from L_3 to $L_1 \cup L_2 - B$. Let $x, y \in B$ and $P_1, P_2, \dots, P_k$ be a set of disjoint paths from L_3 to $L_1 \cup L_2 - B$. Then $\wp = \{P_1, P_2, \dots, P_k, x, y\}$ is a $(k+2)$ -cluster that traverses $L_1 \cup L_2 \cup L_3$, a contradiction. ■

2.4.6 A path P of G with ends u, v is said to be good if there exist distinct i, j with $1 \leq i, j \leq 3$ such that $u \in L_i$ and $v \in L_j$.

2.4.7 We claim that there don't exist $(k+2)$ mutually disjoint good paths in G .

Let $P_1, P_2, \dots, P_{k+2}$ be a set of disjoint good paths of G . Then $\wp = \{P_1, P_2, \dots, P_{k+2}\}$ is a $(k+2)$ -cluster that traverse $L_1 \cup L_2 \cup L_3$. ■

By Lemma 2.3.2 and (2.4.7), we have the following structure of G :

2.4.8 There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, \dots, Y_n$ of $V(G) - W$, and a subset $X_i \subseteq Y_i$, for $1 \leq i \leq n$ such that

(a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor \leq k+1,$

(b) for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^3 L_j) \subseteq X_i$, and

(c) every good path P in $G - W$ has an edge with both ends in Y_i for some i .

Let $M = (L_1 \cap L_2) \cup (L_2 \cap L_3) \cup (L_3 \cap L_1)$, and choose W and $Y_1, X_1, \dots, Y_n, X_n$ such that $|W|$ is as large as possible. Without loss of generality, we can assume that $Y_i \neq \emptyset$ for any $i \in \{1, 2, \dots, n\}$. By the definition of W , M and (2.4.8)(c), we have the following immediate observations:

2.4.9

(a) $M \subseteq W$
by (2.4.8)(c).

(b) $|L_1 \cup L_2 \cup L_3| = |L_1| + |L_2| + |L_3| - |M|$
by definition of M and (2.4.2).

(c) $|M| \leq 3$
by (2.4.2) and (2.4.5).

(d) $|L_i \cup L_j| > k + 2$ for $1 \leq i < j \leq 3$.

(2.4.9)(d) is proved as follows: by (2.4.5) and $k \geq 4$ (by (2.4.3))

$$|L_i \cup L_j| = |L_i| + |L_j| - |L_i \cap L_j| = 2k - 1 > k + 2.$$

The following claim (e) follows from the assumption (2.4.8)(b).

(e) $W \cup X_1 \cup \dots \cup X_n \supseteq L_1 \cup L_2 \cup L_3$, and $|W| + \sum_{i=1}^n |X_i| \geq |L_1 \cup L_2 \cup L_3|$.

2.4.10 *We claim that $X_i \neq \emptyset$ for all i .*

Suppose that $X_i = \emptyset$ for some i . Since $|W| \leq k + 1$ (by (2.4.8)(a)) and $|L_1 \cup L_2 \cup L_3| \geq |L_1 \cup L_2| \geq k + 2$ (by (2.4.9)(d)), there is an integer j ($j \neq i$) such that $X_j \neq \emptyset$ (by (2.4.9)(e)). Hence $n \geq 2$. Since Y_i is not empty, W is a cutset that separates Y_i and non-empty X_j and is of cardinality at most $k + 1$. This contradicts that G is $(k + 2)$ -connected.

2.4.11 *We claim that $|X_i|$ is odd for all i .*

Suppose that $|X_1|$ is even, then by (2.4.10), $|X_1| \geq 2$. Let $v \in X_1$, $W^* = W \cup \{v\}$, $Y_1^* = Y_1 - v$, $X_1^* = X_1 - v$ and $X_i^* = X_i$, $Y_i^* = Y_i$ for $2 \leq i \leq n$. The partition $\{W^*, X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_n^*\}$ of $V(G)$ satisfies (2.4.8)(a)-(c), contradicting the choice that $|W|$ is as large as possible.

2.4.12 *We claim that*

(a) $n \geq k - 2$.

(b) if $n = k - 2$ then

$$|W| = |M| \text{ and } W \cup X_1 \cup X_2 \cup \dots \cup X_n = L_1 \cup L_2 \cup L_3.$$

Since $|L_1 \cup L_2 \cup L_3| \geq |L_1 \cup L_2| \geq k + 2$ (by (2.4.9)(d)) and $|W| \leq k + 1$ (by (2.4.8)(a)), we have $n \geq 1$. By (2.4.8)(a), (2.4.9)(a) and (2.4.9)(b), we have

$$\begin{aligned} 2(k + 1) &\geq 2(|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor) = 2|W| + \sum_{1 \leq i \leq n} |X_i| - n \\ &\geq |W| + |L_1 \cup L_2 \cup L_3| - n \geq |M| + |L_1 \cup L_2 \cup L_3| - n \\ &= |L_1| + |L_2| + |L_3| - n = 3k - n. \end{aligned}$$

Thus,

$$n \geq k - 2$$

and if $n = k - 2$, then all equalities hold and therefore,

$$|W| = |M| \text{ and } |W| + \sum_{1 \leq i \leq n} |X_i| = |L_1 \cup L_2 \cup L_3|.$$

2.4.13 Definition of A_i (for $i = 1, 2, 3$)

Let G'' be the subgraph obtained from $G - W$ by deleting all edges contained in any Y_j . Let A_i be the union of the vertex subsets of all components of G'' containing some vertex of L_i for each $i \in \{1, 2, 3\}$.

2.4.14 Properties of $\{A_1, A_2, A_3\}$

Properties of $\{A_1, A_2, A_3\}$ are to be studied in this subsection. The first property is immediate by (2.4.8) and the definition of A_i .

(a) $L_i - W \subseteq A_i \subseteq V(G) - W$ for $i = 1, 2, 3$.

Note that each $Y_j - X_j$ is an independent set of G'' , and by (2.4.8)(b), we have the following properties.

(b) $A_i \subseteq X_1 \cup \cdots \cup X_n$ for $i = 1, 2, 3$.

(c) A_1, A_2, A_3 are disjoint by the definition of A_i and (2.4.8)(c).

(d) Every path of $G - W$ from A_i to A_{i^*} (for $1 \leq i < i^* \leq 3$) has at least two vertices in X_j for some j .

Proof of (d). Suppose there exists a path P from $v \in A_1$ to $u \in A_2$ in $G - W$. By the definition of A_1, A_2 , we can take two disjoint paths Q and R such that Q is a path from some vertex $x \in L_1$ to v in $G[A_1]$ and R is a path from some vertex $y \in L_2$ to u in $G[A_2]$. Both Q and R have no edges with both ends in Y_j for any j by definition of A_i . Then we have a path S from x to y by using Q, P, R . Since S is a good path by (2.4.8)(c), S has an edge $e = x_1 y_1 \in Y_j$ for some j . Note that $e \notin E(Q)$ and $e \notin E(R)$. This implies $e \in E(P)$ and $x_1, y_1 \in V(P)$. Note that, by (2.4.14)(b), both v and u belong to $X_1 \cup \dots \cup X_n$. By (2.4.8)(b), the part of P from v to x_1 must contain a vertex from X_j , and likewise the part of P from y_1 to u . ■

(e) $|A_i| \leq k + 1 - |W|$ for $1 \leq i \leq 3$.

Proof of (e). Suppose $|A_1| \geq k + 2 - |W|$. It is obvious that $|W| \leq k + 1$ (by (2.4.8)(a)). Hence, $A_1 \neq \emptyset$. We also have that $L_2 \cup L_3 - W \neq \emptyset$ since $|L_2 \cup L_3| \geq k + 2$ (by (2.4.9)(d)) and $|W| \leq k + 1$ (by (2.4.8)(a)).

Since $|L_2 \cup L_3| \geq k + 2$ (by (2.4.9)(d)), we have that $|L_2 \cup L_3 - W| \geq k + 2 - |W|$. Note that $G - W$ is $(k + 2 - |W|)$ -connected, there are $(k + 2 - |W|)$ disjoint paths from A_1 to $L_2 \cup L_3 - W$ each of which is of order at least $k + 2 - |W|$. By (2.4.14)(d), every path P_j contains at least two vertices of X_i for some i . Hence, $\sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor \geq k + 2 - |W|$. This is a contradiction to (2.4.8)(a). The other cases follow by similar argument. ■

2.4.15 We claim that $|W| \leq 3$. If equality holds then $M = W$ and $|A_i| = k + 1 - |W|$.

This claim is to be proved in two steps in this subsection. First we show that

$$(a) \quad \sum_{i=1}^3 |L_i \cap W| \leq |W| + 3.$$

Note that $\sum_{i=1}^3 |L_i \cap W| \leq |W| + |M|$. Hence, $\sum_{i=1}^3 |L_i \cap W| \leq |W| + 3$ since $|M| \leq 3$ by (2.4.9)(c). ■

(b) By (2.4.14)(a), (2.4.14)(e) and (2.4.15)(a), we have the following inequality:

$$3k = \sum_{i=1}^3 |L_i| \leq \sum_{i=1}^3 (|A_i| + |L_i \cap W|) \leq 3(k + 1 - |W|) + |W| + 3 = 3k + 6 - 2|W|.$$

Hence, $|W| \leq 3$. And if $|W| = 3$ then $M = W$ and $|A_i| = k + 1 - |W|$. ■

2.4.16 *We claim that, for $1 \leq j \leq n$, if $|W \cup X_j| < (k + 2)$ then $X_j = Y_j$.*

Suppose that $X_j \neq Y_j$. First we claim that $V(G) - Y_j - W$ is not empty. Since $|W \cup X_j| \leq k + 2$ and $|L_1 \cup L_2 \cup L_3| = 3k - |M| \geq 3k - 3 \geq k + 3$ (by (2.4.9)(b) and (2.4.9)(c)), $(L_1 \cup L_2 \cup L_3) - (W \cup X_j) \neq \emptyset$. Hence $V(G) - Y_j - W$ which contains $(L_1 \cup L_2 \cup L_3) - (W \cup X_j)$ is not empty.

Note that G is $(k + 2)$ -connected and by (2.4.8)(b), $W \cup X_j$ is a vertex-cut separating $Y_j - X_j$ and $V(G) - Y_j - W$ neither of which is empty. It follows that $|W \cup X_j| \geq (k + 2)$, as required. ■

2.4.17 *We claim that, for $1 \leq j \leq n$, if $|X_j| < 3$ then $X_j = Y_j$.*

By (2.4.16), it is obvious that $X_j = Y_j$ if $|X_j| < 3$ since $k \geq 4$ (by (2.4.3)) and $|W| \leq 3$ (by (2.4.15)).

2.4.18 *Let $Z = (X_1 \cup \dots \cup X_n) - (L_1 \cup L_2 \cup L_3)$*

2.4.19 *Some vertex-cuts of G*

Suppose that $X_i \cap L_j \neq \emptyset$ for some $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, 3\}$. By (2.4.8)(c), (2.4.14)(a) and (2.4.14)(d), any path joining $X_i \cap L_j$ and $L_1 \cup L_2 \cup L_3 - W - L_j$ must use a vertex of W or Z or $X_i \Delta L_j$. Therefore, $(X_i \Delta L_j) \cup W \cup Z$ is a cutset of G separating $X_i \cap L_j$ from $L_1 \cup L_2 \cup L_3 - W - L_j$.

2.4.20 *We claim that $|X_i| \geq 3$ for $1 \leq i \leq n$.*

This claim is to be proved in several steps in this subsection.

(a) First we show that, **for $1 \leq i \leq 3$, $1 \leq j \leq n$, if $|X_j| = 1$, then $A_i \cap X_j = \emptyset$.**

Suppose $A_1 \cap X_j \neq \emptyset$. Let $X_j = \{v\}$ and $N = N_G(v)$. Since G is $(k+2)$ -connected, $|N| \geq k+2$. Hence $|N - W| \geq k+2 - |W|$. Note that $|A_1| \leq k+1 - |W|$ by (2.4.14)(e), this implies $N - A_1 - W \neq \emptyset$. Take a vertex $x \in N - A_1 - W$. Since $|X_j| = 1$, we have $X_j = Y_j = \{v\}$ by (2.4.17). Note that $xv \in E(G)$, x is in A_1 by the definition of A_1 , a contradiction. Hence $A_1 \cap X_j = \emptyset$. ■

(b) Second we show that, **for** $1 \leq i \leq 3$, $1 \leq j \leq n$, **if** $|X_j| = 1$, **then** $A_i \cap N_G(X_j) = \emptyset$.

Suppose that $|X_1| = 1$ and $x \in A_1 \cap N_G(X_1)$. Hence, by (2.4.14)(b), $x \in X_i$ for some $i \neq 1$. Since $|X_1| = 1$, by the definition of A_1 (defined in (2.4.13)), $X_1 \subseteq A_1$. This contradicts (2.4.20)(a) since $|X_1| = 1$. ■

(c) Since $|X_i|$ is odd for each i (by (2.4.11)), let m be an integer such that $m \leq n$ with $|X_i| = 1$ for $1 \leq i \leq m \leq n$ and $|X_j| \geq 3$ for $m < j \leq n$.

By the definition of A_i and (2.4.9), we have

$$\sum_{i=1}^3 |A_i| \geq |L_1 \cup L_2 \cup L_3| - |W| = 3k - |M| - |W| \quad (\text{I})$$

Also, by (2.4.8)(a),

$$\sum_{m < j \leq n} |X_j| \leq 3 \sum_{m < j \leq n} \lfloor \frac{1}{2} |X_j| \rfloor \leq 3 \sum_{1 \leq j \leq n} \lfloor \frac{1}{2} |X_j| \rfloor \leq 3(k+1 - |W|) \quad (\text{II})$$

Assume $X = X_1 \cup X_2 \cup \dots \cup X_m$ and $N = N_G(X) - X$. Then we can get the followings.

(i) $N \subseteq W \cup X_{m+1} \cup \dots \cup X_n$ by (2.4.8)(b) and (2.4.17).

(ii) $N \cap A_1 = N \cap A_2 = N \cap A_3 = \emptyset$ by (2.4.20)(b).

(iii) $|N| \geq k+2$ since N separates X from $A_1 \cup A_2 \cup A_3$ (by (2.4.20)(a) and (2.4.20)(b)) and G is $(k+2)$ -connected.

Hence, we have

$$|N| + |A_1| + |A_2| + |A_3| \leq |W| + \sum_{i=m+1}^n |X_i| \quad (\text{III})$$

By (iii), (I), (II), and (III), we have

$$\begin{aligned} (k+2) + (3k - |M| - |W|) &\leq |W| + 3(k+1 - |W|) \\ &= 3k + 3 - 2|W|. \end{aligned}$$

Hence,

$$|W| \leq 1 + |M| - k.$$

By (2.4.9)(a),

$$|W| \leq 1 + |W| - k.$$

That is,

$$k \leq 1.$$

This contradicts $k \geq 4$ (2.4.3) and completes the proof of (2.4.20). ■

2.4.21 *We prove some inequalities for $|Z|$.*

(i)

$$|Z| \leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|,$$

and the equality holds if and only if $|X_j| = 3$ for every $j \in \{1, 2, \dots, n\}$.

(ii)

$$|Z| \leq 3 + |M| - 2|W|,$$

and the equality holds if and only if $|X_j| = 3$ for every $j \in \{1, 2, \dots, n\}$ and $W \subseteq L_1 \cup L_2 \cup L_3$.

Let $s = |Z|$. Then, by (2.4.18),

$$|X_1 \cup \dots \cup X_n| = s + |L_1 \cup L_2 \cup L_3 - W|.$$

But, by (2.4.20), $|X_j| \leq 3 \lfloor \frac{1}{2}|X_j| \rfloor$ for $1 \leq j \leq n$, and therefore

$$3 \sum_{1 \leq j \leq n} \lfloor \frac{1}{2}|X_j| \rfloor \geq \sum_{1 \leq j \leq n} |X_j| = s + |L_1 \cup L_2 \cup L_3 - W|,$$

with equality if and only if $|X_j| = 3$ for any $j \in \{1, 2, \dots, n\}$. By (2.4.8)(a), we have

$$3(k+1 - |W|) \geq s + |L_1 \cup L_2 \cup L_3 - W|.$$

That is,

$$s \leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|,$$

and the equality holds if and only if $|X_j| = 3$ for any $j \in \{1, 2, \dots, n\}$. That completes the proof of (2.4.21)(i).

Note that, by (2.4.9)(b), we have

$$|L_1 \cup L_2 \cup L_3 - W| \geq |L_1 \cup L_2 \cup L_3| - |W| = 3k - |M| - |W|,$$

and the equality holds if and only if $W \subseteq L_1 \cup L_2 \cup L_3$. Hence, by (2.4.21)(i),

$$\begin{aligned} s &\leq 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W| \leq 3k + 3 - 3|W| - (3k - |M| - |W|) \\ &= 3 + |M| - 2|W|, \end{aligned}$$

and the equality holds if and only if $W \subseteq L_1 \cup L_2 \cup L_3$ and $|X_j| = 3$ for every $j \in \{1, 2, \dots, n\}$. This completes the proof of (2.4.21)(ii). ■

2.4.22 (i) $|A_i \cap X_j| < \frac{1}{2}|X_j|$ for $1 \leq j \leq n$ and $1 \leq i \leq 3$.

Suppose that $|A_1 \cap X_1| \geq \frac{1}{2}|X_1|$. Since $|X_1| \geq 3$ by (2.4.20), there exists a vertex $v \in A_1 \cap X_1$. Since $|L_2 \cup L_3 - W| \geq |L_2 \cup L_3| - |W| \geq k + 2 - |W|$ by (2.4.9)(d), and $G - W$ is $(k + 2 - |W|)$ -connected, there are $(k + 2 - |W|)$ paths of $G - W$ between A_1 and $L_2 \cup L_3 - W$, disjoint except possibly for v . Choose them with no internal vertex in A_1 . By (2.4.14)(d), each has at least two vertices in X_j for some j , but at most $\lfloor \frac{1}{2}|X_j| \rfloor$ of them have two vertices in X_j for each $j \neq 1$. Note that by (2.4.8)(a), we have

$$\sum_{2 \leq j \leq n} \lfloor \frac{1}{2}|X_j| \rfloor \leq k + 1 - |W| - \lfloor \frac{1}{2}|X_1| \rfloor.$$

Thus, at least $1 + \lfloor \frac{1}{2}|X_1| \rfloor$ of them have two vertices in X_1 . But each has only one vertex in A_1 , and so has a vertex in X_1 which does not belong to A_1 , and all these vertices in $X_1 - A_1$ are different. Hence $|X_1 - A_1| \geq 1 + \lfloor \frac{1}{2}|X_1| \rfloor$, a contradiction. ■

(ii) $|L_i \cap X_j| < \frac{1}{2}|X_j|$ for $1 \leq j \leq n$ and $1 \leq i \leq 3$ by (2.4.14)(a) and (2.4.22)(i).

2.4.23 We eliminate the case of $k = 4$ and $|W| = 3$.

In this case, by (2.4.12)(a), $n \geq k - 2 = 2$.

Moreover, by (2.4.8)(a), we have

$$|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor \leq k + 1 = 5.$$

That is,

$$\sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor \leq 2.$$

We have $n = 2$ and $|X_1| = |X_2| = 3$ (by (2.4.20)). Hence, $Z = \emptyset$ by (2.4.12)(b).

Next we claim that $|L_i \cap X_j| \neq 0$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. For otherwise, we may assume that $L_1 \cap X_1 = \emptyset$. Then by (2.4.22)(ii), $|L_2 \cap X_1| \leq 1$ and $|L_3 \cap X_1| \leq 1$. This implies that $Z \supseteq X_1 - (L_1 \cup L_2 \cup L_3) = X_1 - L_2 \cup L_3 \neq \emptyset$. This contradicts that $Z = \emptyset$.

Therefore, by (2.4.22)(ii), we have $|L_i \cap X_j| = 1$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$. And by (2.4.12) (b), we have $W = M$ and $W \cup X_1 \cup X_2 = L_1 \cup L_2 \cup L_3$. Hence, $|W \cap L_i| = 2$ for $i \in \{1, 2, 3\}$.

Next, we will apply Lemma (2.3.5) to find a 6-cluster in G traversing $L_1 \cup L_2 \cup L_3$.

Let $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{y_1, y_2, y_3\}$, $W = \{z_1, z_2, z_3\}$, let $\{x_i, y_i\} \subseteq L_i$ for $i \in \{1, 2, 3\}$, and $z_1 \in L_2 \cap L_3$, $z_2 \in L_1 \cap L_3$, $z_3 \in L_1 \cap L_2$ (by (2.4.2)).

Hence, we get the description of graph G as in Lemma (2.3.5): $x_i, y_i, z_i (1 \leq i \leq 3)$ are distinct vertices of 6-connected graph G , and $L_1 = \{x_1, y_1, z_2, z_3\}$, $L_2 = \{x_2, y_2, z_3, z_1\}$, $L_3 = \{x_3, y_3, z_1, z_2\}$ are 4-cliques, and there is a partition Y_1, Y_2 of $V(G) - \{z_1, z_2, z_3\}$ with $X_1 \subseteq Y_1$, $X_2 \subseteq Y_2$, and $x_i y_i (1 \leq i \leq 3)$ are the only edges of G with one end in Y_1 and the other in Y_2 .

Therefore by Lemma (2.3.5), G has a 6-cluster traversing $L_1 \cup L_2 \cup L_3$. This is a contradiction.

2.4.24 We claim that, for $1 \leq j \leq n$, if $|X_j| = 3$ then $X_j = Y_j$.

By (2.4.15), $|W| \leq 3$. By (2.4.16), it is obvious that $X_j = Y_j$ if $k \geq 5$ or $|W| < 3$. Hence, the only remaining case is $k = 4$ and $|W| = 3$, which was eliminated in (2.4.23).

2.4.25 (i) We claim that if $v \in A_i \cap X_j$ for some $i \in \{1, 2, 3\}$ and some $j \in \{1, 2, \dots, n\}$, then $d_{Y_j - A_i}(v) \geq 2$, and the equality holds if and only if $d_G(v) = k + 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

By the definition of A_i (2.4.13), we have

$$N_G(v) - (Y_j - A_i) \subseteq A_i \cup W - \{v\}.$$

Since G is $(k + 2)$ -connected and $|A_i| \leq k + 1 - |W|$ (by (2.4.14)(e)), we have:

$$|N_G(v) \cap (Y_j - A_i)| \geq (k + 2) - |A_i \cup W - \{v\}| \geq (k + 2) - (k + 1 - |W| + |W| - 1) = 2$$

and the equality holds if and only if $d(v) = k + 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

(ii) We claim that if $v \in A_i \cap X_j$ and $|X_j| = 3$ for some $i \in \{1, 2, 3\}$ and some $j \in \{1, 2, \dots, n\}$, then $d_{X_j}(v) = 2$, $W \cup A_i \subseteq N_G(v) \cup \{v\}$ and $|A_i| = k + 1 - |W|$.

Note that $|X_j| = 3$. By (2.4.24), we have $Y_j = X_j$, and therefore,

$$d_{Y_j - A_i}(v) = d_{X_j - A_i}(v) \leq 2.$$

On the other hand, by (2.4.25)(i), we have $d_{Y_j-A_i}(v) \geq 2$. Hence, $d_{Y_j-A_i}(v) = 2$. By (2.4.25)(i) again, we are done.

2.4.26 *We claim that if $|X_j| = 3$ for some j , then*

- (i) $|X_j \cap A_i| = 1$ for each $i \in \{1, 2, 3\}$,
- (ii) X_j induces a clique of G , and
- (iii) $W \subseteq N_G(v)$ for any $v \in X_j$.

Proof of (i): For otherwise, we may assume that $|X_j \cap A_1| \neq 1$. By (2.4.22)(i), $|X_j \cap A_1| = 0$, $|A_2 \cap X_j| \leq 1$ and $|A_3 \cap X_j| \leq 1$. This implies that there exists $x \in X_j$, and $x \notin A_1 \cup A_2 \cup A_3$. Since $|X_j| = 3$, we have $X_j = Y_j$ by (2.4.24), and by the definition of A_i (2.4.13), we have $N_G(x) \subseteq W \cup (Z - \{x\}) \cup (X_j - \{x\})$. Note that, by (2.4.21)(ii), (2.4.15), we have

$$|W| + |Z - \{x\}| + |X_j - \{x\}| \leq |W| + (3 + |M| - 2|W| - 1) + 2 = 4 + |M| - |W|.$$

Note that $|M| \leq |W|$ by (2.4.9)(a). Hence, we have $|N_G(x)| \leq 4$. This contradicts that G is $(k+2)$ -connected where $k \geq 4$ (by (2.4.3)).

Proof of (ii) and (iii). (ii) and (iii) are immediate corollaries of (2.4.26)(i) and (2.4.25)(ii).

2.4.27 *We claim that there exists some $j \in \{1, 2, \dots, n\}$ such that $|X_j| \geq 5$.*

By (2.4.20), we may assume $|X_j| = 3$ for all $j \in \{1, 2, \dots, n\}$.

By (2.4.26)(i)-(ii), every vertex of $(L_1 - W) \cap X_j$ (for all $j \in \{1, 2, \dots, n\}$) is adjacent to a vertex of $A_2 \cap X_j$ and a vertex of $A_3 \cap X_j$. By (2.4.26)(i) and (2.4.26)(iii), every vertex of $L_1 \cap W$ is adjacent to some vertex of A_2 and some vertex of A_3 , hence, $\wp = \{u_1, u_2, \dots, u_k, A_2, A_3\}$ is a $(k+2)$ -cluster that traverses $L_1 \cup L_2 \cup L_3$ where $\{u_1, u_2, \dots, u_k\} \in L_1$. This is a contradiction.

2.4.28 *We claim that $|X_j| \geq 5$ for every $j \in \{1, 2, \dots, n\}$.*

For otherwise, by (2.4.20), we may assume $|X_1| = 3$. By (2.4.26)(i), $|A_i \cap X_1| = 1$ for each $i \in \{1, 2, 3\}$. Hence, by (2.4.25)(ii), $|A_i| = k + 1 - |W|$ for each $i \in \{1, 2, 3\}$.

Furthermore, by (2.4.14)(b), (2.4.14)(c), we have

$$|Z| \geq |A_1| + |A_2| + |A_3| - |L_1 \cup L_2 \cup L_3 - W| = (3k + 3 - 3|W|) - |L_1 \cup L_2 \cup L_3 - W|.$$

However, by (2.4.21)(i), we have

$$|Z| = 3k + 3 - 3|W| - |L_1 \cup L_2 \cup L_3 - W|.$$

The equality of (2.4.21)(i) implies that $|X_i| = 3$ for all $i \in \{1, 2, \dots, n\}$. This contradicts (2.4.27). ■

2.4.29 *We show some inequalities for n .*

By (2.4.28) and (2.4.8)(a),

$$5n \leq \sum_{1 \leq j \leq n} |X_j| \leq 2 * (k + 1 - |W|) + n = 2k + 2 + n - 2|W|. \quad (\text{IV})$$

The inequality (IV) can be simplified as

$$2n \leq k + 1 - |W|. \quad (\text{V})$$

Note that the equality of (IV) (and (V), as well) holds if and only if $|X_i| = 5$ for every i .

2.4.30 *We claim that $n = k - 2$.*

For otherwise, since $n \geq k - 2$ by (2.4.12)(a), we may assume that $n \geq k - 1$.

By (V), we have

$$2k - 2 \leq 2n \leq k + 1 - |W|. \quad (\text{VI})$$

That is,

$$k \leq 3 - |W|.$$

Note that $k \geq 4$ by (2.4.3). This is a contradiction.

2.4.31 *The final step of the proof.*

By (2.4.30), $n = k - 2$. By (2.4.12)(b), we have

$$W = M \quad \text{and} \quad L_1 \cup L_2 \cup L_3 = W \cup X_1 \cup \cdots \cup X_n.$$

Hence,

$$Z = \emptyset. \tag{VII}$$

By (V) of (2.4.29), we have

$$2k - 4 = 2n \leq k + 1 - |W|.$$

That is,

$$k \leq 5 - |W| \tag{VIII}$$

Note that $k \geq 4$ by (2.4.3). Therefore, there are only two cases: $k = 5$ and $k = 4$ (by (VII) and (VIII)).

Case 1: $k = 5$. In this case, $|W| = 0$. By (2.4.30), $n = k - 2 = 3$. The equality of (V) of (2.4.29) implies that

$$|X_1| = |X_2| = |X_3| = 5.$$

By (2.4.22)(ii), without loss of generality, we assume $|L_1 \cap X_1| = 2$. By (2.4.19), $(X_1 \Delta L_1)$ is a vertex-cut of order at most 6 since $Z = \emptyset$ and $W = \emptyset$. This contradicts that G is $(k + 2)$ -connected where $k = 5$.

Case 2: $k = 4$. In this case, $n = k - 2 = 2$ (by (2.4.30)). There are two subcases: $|W| = 1$ and $|W| = 0$ (by VIII).

Subcase 1: $|W| = 1$. The equality of (V) of (2.4.29) implies that

$$|X_1| = |X_2| = 5.$$

Without loss of generality, we assume $W \subseteq L_1$ and $|L_1 \cap X_1| = 2$. By (2.4.19), $(X_1 \Delta L_1)$ is a vertex-cut of order at most 5 since $Z = \emptyset$ and $W \subseteq L_1$. This contradicts that G is $(k + 2)$ -connected where $k = 4$.

Subcase 2: $|W| = 0$.

Since $Z = \emptyset$ by (VII), we have

$$\sum_{j=1}^n |X_j| = |L_1 \cup L_2 \cup L_3| = 3k = 12. \tag{IX}$$

Therefore, the only possibility in this subcase is that $|X_1| = 5$ and $|X_2| = 7$ (by (2.4.11) and (2.4.28)).

Without loss of generality, we assume $|L_1 \cap X_1| = 2$. By (2.4.19), $(X_1 \Delta L_1)$ is a vertex-cut of order at most 5 since $Z = \emptyset$ and $W = \emptyset$. This contradicts that G is $(k+2)$ -connected where $k = 4$.

This completes the proof of main theorem (Theorem 2.1.12).

2.5 Proof of Lemma 2.3.5

The following two lemmas will be useful in the proof of Lemma 2.3.5.

Lemma 2.5.1 (Whitney [8]) *Any two planar embeddings of a 3-connected graph are equivalent.*

Let $G = (V, E)$ is a graph and $A \subseteq V(G)$, we denote the set of vertices on $V(G) - A$ which are adjacent to some vertex in A by $\Upsilon(A)$. That is $\Upsilon(A) = N(A) - A$.

Lemma 2.5.2 (Seymour [28], Thomassen [32]) *Let s_1, t_1, s_2, t_2 be distinct vertices of a graph $G = (V, E)$. Then just one of the following is true:*

- (i) *there are paths joining s_1 to t_1 and s_2 to t_2 respectively, vertex-disjoint.*
- (ii) *for some $k \geq 0$ there are pairwise disjoint sets $A_1, \dots, A_k \subseteq V(G) - \{s_1, s_2, t_1, t_2\}$ such that*
 - (a) *for $i \neq j$, $\Upsilon(A_i) \cap A_j = \emptyset$,*
 - (b) *for $1 \leq i \leq k$, $|\Upsilon(A_i)| \leq 3$,*
 - (c) *if $\tilde{G}$ is the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $\Upsilon(A_i)$, and also for $j = 1, 2$ adding an edge e_j joining*

s_j to t_j , then $\tilde{G}$ may be drawn in the plane with no pairs of edges crossing except e_1, e_2 which cross once.

Proof of Lemma 2.3.5: Let G be a counterexample to the lemma with least number of vertices. Let $W = \{z_1, z_2, z_3\}$, $X_1 = \{x_1, x_2, x_3\}$, $X_2 = \{y_1, y_2, y_3\}$. Let $\mathfrak{S}_{ij} = \{z_i\} \cup X_j$ and G_{ij} be the subgraph induced by $\{z_i\} \cup Y_j$ where $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Let f_1, f_2, f_3 be the edges with ends z_2z_3, z_3z_1 , and z_1z_2 respectively.

2.5.1 *We claim that $G - \{f_1, f_2, f_3\}$ is not planar.*

Since G is 6-connected, $|E(G)| \geq 3|V(G)|$. Therefore $|E(G - \{f_1, f_2, f_3\})| \geq 3|V(G)| - 3$. Note that a planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges. Hence $G - \{f_1, f_2, f_3\}$ is not planar.

2.5.2

The strategy of the proof is to prove that $G' = G - \{f_1, f_2, f_3\}$ is planar (hence contradicts 2.5.1). The planarity of G' is yielded by showing that each $G_i = G'[W \cup Y_i]$ ($1 \leq i \leq 2$) has a planar embedding with vertices $z_1, x_2, z_3, x_1, z_2, x_3$ for $i = 1$ (or $z_1, y_2, z_3, y_1, z_2, y_3$ for $i = 2$, respectively) around its exterior face (Lemma 2.5.2 is applied here). And the planar embedding of G_i is constructed from the unique embedding of the 3-connected graph $H_i = Y_i \cup E_i$ ($1 \leq i \leq 2$) where $E_1 = \{x_1x_2, x_1x_3, x_2x_3\}$, and $E_2 = \{y_1y_2, y_1y_3, y_2y_3\}$, and planar embeddings of other subgraphs of $G_i \cup E_i$.

2.5.3 *We claim that $|Y_1|, |Y_2| \geq 4$.*

For otherwise, we may assume $|Y_1| = 3$. That is, $Y_1 = X_1$. Since G is 6-connected, $d_G(x_1) \geq 6$. Hence $W \cup \{y_1, x_2, x_3\} \subseteq N_G(x_1)$. With the similar argument, we have $W \cup \{y_3, x_1, x_2\} \subseteq N_G(x_3)$ and $W \cup \{y_2, x_1, x_3\} \subseteq N_G(x_2)$. Therefore, $\wp = \{\{x_1\}, \{x_2\}, \{x_3\}, \{z_1\}, \{z_2\}, \{z_3\}\}$ is a 6-cluster traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$, a contradiction.

2.5.4 *We claim that the subgraph induced by Y_i is connected for $i = 1, 2$.*

For otherwise, we assume that the subgraph induced by Y_1 is not connected. Let D be a component not containing x_1 , then $\{x_2, x_3, z_1, z_2, z_3\}$ is a 5-cutset separating D and Y_2 , it contradicts that G is 6-connected.

2.5.5 *We claim that there is no 4-cluster in G_{ij} traversing $\mathfrak{S}_{ij}$ where $1 \leq i \leq 3, 1 \leq j \leq 2$.*

For otherwise, we may assume that there is a 4-cluster $\wp_1$ in G_{11} traversing $\mathfrak{S}_{11}$. Let $\wp = \wp_1 \cup \{\{z_2, z_3\}, \{Y_2\}\}$, then $\wp$ is a 6-cluster in G traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$, a contradiction.

2.5.6 *We claim that both X_1 and X_2 are independent sets of G .*

For otherwise, without loss of generality, we assume that $x_1x_2 \in E(G)$. There is a vertex $x \in Y_1 - X_1$ by (2.5.3) and there are 4 internally disjoint paths $P_{x_1}, P_{x_2}, P_{x_3}, P_{z_3}$ of $G - \{z_1, z_2\}$ since $G - \{z_1, z_2\}$ is 4-connected where P_u is the path joining x and $u \in \{x_1, x_2, x_3, z_3\}$.

Now since x_iy_i ($1 \leq i \leq 3$) are the only edges of G with one end in Y_1 and the other in Y_2 , we have $P_u \cap Y_2 = \emptyset$ where $u \in \{x_1, x_2, x_3, z_3\}$. Hence $\wp = \{P_{x_1} - x, P_{x_2} - x, P_{x_3} - x, P_{z_3} - x\}$ is a 4-cluster in G_{31} traversing $\mathfrak{S}_{31}$. This contradicts (2.5.5).

2.5.7 *We claim that there are no two vertex-disjoint paths in G_{11} joining z_1 to x_1 and x_2 to x_3 respectively.*

For otherwise, let P_1 and P_2 be the paths joining z_1 to x_1 and x_2 to x_3 respectively. Let A_1 and A_2 be disjoint fragments of the subgraph induced by Y_1 with $P_1 - \{z_1\} \subseteq A_1$, $P_2 \subseteq A_2$, and $A_1 \cup A_2$ maximal. Since the subgraph induced by Y_1 is connected by (2.5.4), A_1 and A_2 are adjacent. Therefore $\wp = \{\{z_1\}, \{z_2\}, \{z_3\}, A_1, A_2, Y_2\}$ is a 6-cluster traversing $\{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}$, a contradiction.

2.5.8 (i) *We claim that G_{11} can be drawn in a plane so that every vertex in $\mathfrak{S}_{11}$ is incident with the infinite region and z_1, x_2, x_1, x_3 are around its exterior face in this order, and this embedding is denoted by π_{11} .*

By Lemma 2.5.2 and (2.5.7), there exists some $k \geq 0$ and there are pairwise disjoint sets $A_1, \dots, A_k \subseteq V(G) - \{z_1, x_2, x_1, x_3\}$ such that

(a) for $i \neq j$, $\Upsilon(A_i) \cap A_j = \emptyset$,

(b) for $1 \leq i \leq k$, $|\Upsilon(A_i)| \leq 3$,

(c) if $\widetilde{G_{11}}$ is the graph obtained from G_{11} by deleting A_i and adding new edges joining every pair of distinct vertices in $\Upsilon(A_i)$, and adding an edge e_1 joining z_1 to x_1 and an edge e_2 joining x_2 to x_3 , then $\widetilde{G_{11}}$ may be drawn in the plane with no pairs of edges crossing except e_1, e_2 which cross once.

We claim that $k = 0$. For otherwise, $\Upsilon(A_k) \cup \{z_2, z_3\}$ is a cut set of order at most 5 separating A_k and Y_2 , it contradicts that G is 6-connected.

Hence we have $G_{11} = \widetilde{G_{11}} - \{e_1, e_2\}$, and result follows.

With the similar argument, we have

(ii) G_{21} can be drawn in a plane so that z_2, x_1, x_2, x_3 are around its exterior face in this order, and this embedding is denoted by π_{21} .

(iii) G_{31} can be drawn in a plane so that z_3, x_1, x_3, x_2 are around its exterior face in this order, and this embedding is denoted by π_{31} .

2.5.9 Let G_i be the graph obtained from the subgraph induced by $W \cup Y_i$ and deleting edges f_1, f_2, f_3 for $1 \leq i \leq 2$, and let G_1^+ be the graph obtained from G_1 by adding edges x_1x_2, x_1x_3, x_2x_3 , let G_2^+ be the graph obtained from G_2 by adding edges y_1y_2, y_1y_3, y_2y_3 .

In next a few subsections, we are to show that G_1^+ has a planar embedding such that the triangle $x_1x_2x_3$ is a facial circuit.

2.5.10

(i) Let H_{h1} be the graph obtained from G_1^+ by deleting z_i and z_j where $\{h, i, j\} = \{1, 2, 3\}$. Obviously H_{h1} is also the graph obtained from G_{h1} by adding edges x_1x_2, x_1x_3, x_2x_3 . By (2.5.8), G_{h1} has an embedding π_{h1} with $\{x_i, x_h, x_j, z_h\}$ in its exterior face. Hence π_{h1} can also

be considered as an embedding of H_{h1} with the triangle $x_1x_2x_3$ as the exterior face. We denote this embedded graph by $\pi_{h1}(H_{h1})$.

(ii) Let H_1 be the graphs obtained from the subgraph induced by Y_1 by adding edges x_1x_2 , x_1x_3 , x_2x_3 . Obviously, H_1 can also be obtained from H_{h1} by deleting z_h for $h \in \{1, 2, 3\}$. Hence π_{h1} can also be considered as an embedding of H_1 , and we denote this embedded graph by $\pi_{h1}(H_1)$.

2.5.11 *We claim that H_1 is a 3-connected planar graph.*

Planarity is an immediate conclusion from (2.5.10) (ii). Next we will show that H_1 is 3-connected.

For otherwise, we assume that H_1 is not 3-connected. Let A be a cut set of order less than 3. Since $\{x_1, x_2, x_3\}$ induces a clique in the graph H_1 , let D be a component of $H_1 - A$ with $x_i \notin D$ for every $i \in \{1, 2, 3\}$. Then $A \cup \{z_1, z_2, z_3\}$ is a cutset of G of order at most 5 separating D and Y_2 , it is a contradiction.

2.5.12 *By lemma (2.5.1), H_1 has only one embedding π_0 . That is, $\pi_0(H_1) = \pi_{h1}(H_1)$ for each $h \in \{1, 2, 3\}$.*

2.5.13 *(i) We claim that G_1^+ has a planar embedding such that the triangle $x_1x_2x_3x_1$ is a facial circuit.*

Let C_{ij} be the facial circuit of the embedded graph $\pi_0(H_1)$ containing the edge x_ix_j other than the triangle $x_1x_2x_3x_1$. Let F_{ij} be the face of $\pi_0(H_1)$ bounded by C_{ij} where $1 \leq i < j \leq 3$.

By (2.5.12), $\pi_0(H_1) = \pi_{11}(H_1)$, and by (2.5.10) (ii), $\pi_{11}(H_1)$ is obtained from $\pi_{11}(H_{11})$ by deleting the vertex z_1 , hence z_1 must be inside the face F_{23} of $\pi_0(H_1)$, and all neighborhoods of z_1 must be in the facial circuit C_{23} . Similarly, for each $\{h, i, j\} = \{1, 2, 3\}$, all neighborhoods of z_h must be in the facial circuit C_{ij} .

Note that F_{ij} 's are distinct for $1 \leq i < j \leq 3$ since H_1 is 3-connected by (2.5.11). Now we can get a planar embedding of G_1^+ by adding the vertex z_h and edges z_hu into the face F_{ij} where $u \in N_G(z_h) \cap Y_1$.

With the similar argument, we have

(ii) G_2^+ has a planar embedding such that the triangle $y_1y_2y_3y_1$ is a facial circuit.

2.5.14 *The final step of the proof.*

By (2.5.13) and (2.5.9), G_i is planar with triangle $z_1z_2z_3$ as the exterior face.

Now we identify z_i of G_1 to z_i of G_2 where $1 \leq i \leq 3$, and join x_i of G_1 to y_i of G_2 , we get planar graph $G - \{f_1, f_2, f_3\}$, it contradicts (2.5.1).

This completes the proof of this lemma. ■

Chapter 3

Hadwiger's Conjecture for Claw-free Graphs

3.1 Introduction

Claw-free graphs have been a subject of interest to many researchers in recent years. In this chapter, we give some results on claw-free graphs.

If H is a graph, then we say that G is H -free if G does not contain a copy of H as an induced subgraph. By the *claw*, we mean the complete bipartite graph $K_{1,3}$. Thus G is said to be *claw-free* if it does not contain an induced subgraph that is isomorphic to claw.

If G is a graph, then the *line graph* of G , usually denoted by $L(G)$, is obtained by associating one vertex to each edge of G , and two vertices of $L(G)$ being joined by an edge if and only if the corresponding edges in G are adjacent. Beineke [4, 5] and Robertson [23] (see also page 74 of [12]) independently gave a characterization of line graphs in terms of forbidden induced subgraphs and the claw is one (among nine) of those subgraphs. Thus every line graph is claw-free.

As stated in Chapter 2, Hadwiger's Conjecture is as follows:

Conjecture 3.1.1 (Hadwiger [10]) *For all $k \geq 1$, every k -chromatic graph has the complete*

graph K_k as a minor.

Our motivation is the following proposition.

Proposition 3.1.2 *Hadwiger's Conjecture is true for line graphs of simple graphs.*

Note that every line graph is claw-free, we may ask the following question.

Question 3.1.3 *Is Hadwiger's Conjecture true for claw-free graphs?*

Hadwiger's Conjecture is open for $k \geq 7$ as stated in Chapter 2, we may focus on the special case $k = 7$ of Hadwiger's Conjecture, and we may ask the following question.

Question 3.1.4 *Let G be a claw-free 7-chromatic graph, does G contain the complete graph K_7 as a minor?*

Question 3.1.5 *Let G be a claw-free 6-connected 7-chromatic graph, does G contain the complete graph K_7 as a minor?*

In this chapter, we prove the following result,

Theorem 3.1.6 *Let G be a 6-connected claw-free graph. If $\delta(G) \geq 7$ and G contains three disjoint 5-cliques, say L_1, L_2, L_3 , then G contains a K_7 as a minor.*

We hope our result would be used to prove the Question 3.1.5.

3.2 Proof of the main theorem

Prove by way of contradiction. Assume G doesn't contain a K_7 as a minor, and the following assertion is obvious by Menger's Theorem.

3.2.1 *The graph G contains no clique of order 6.*

3.2.2 *A path P of G with ends u, v is said to be good if there exist distinct i, j with $1 \leq i, j \leq 3$ such that $u \in L_i$ and $v \in L_j$.*

3.2.3 *We claim that there don't exist 7 mutually disjoint good paths in G .*

Let $P_1, P_2, \dots, P_7$ be a set of disjoint good paths of G . Let G' be the graph obtained by contracting P_i to a new vertex v_i for all $i \in \{1, 2, \dots, 7\}$. The subgraph Q of G' induced by v_i ($1 \leq i \leq 7$) is a K_7 -clique and corresponds to a K_7 -minor in G . ■

By Lemma 2.3.2 and (3.2.3), we have the following structure of G :

3.2.4 *There exists a vertex set $W \subseteq V(G)$ and a partition $Y_1, \dots, Y_n$ of $V(G) - W$, and a subset $X_i \subseteq Y_i$, for $1 \leq i \leq n$ such that*

(a) $|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor \leq 6,$

(b) **for any i with $1 \leq i \leq n$, no vertex in $Y_i - X_i$ has a neighbor in $V(G) - (W \cup Y_i)$ and $Y_i \cap (\cup_{j=1}^3 L_j) \subseteq X_i$, and**

(c) **every good path P in $G - W$ has an edge with both ends in Y_i for some i .**

We choose $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n$ such that

(i) $|W|$ is as large as possible.

(ii) Subject to (i), n is the largest number such that $Y_i \neq \emptyset$ for any $i \in \{1, 2, \dots, n\}$.

By the definition of W , (3.2.4)(b) and (3.2.4)(c), we have the following immediate observations:

3.2.5 $W \cup X_1 \cup \dots \cup X_n \supseteq L_1 \cup L_2 \cup L_3$, and $|W| + \sum_{i=1}^n |X_i| \geq |L_1 \cup L_2 \cup L_3|$.

3.2.6 We claim that $n \geq 3 + |W|$.

Since $|L_1 \cup L_2 \cup L_3| = 15$ and $|W| \leq 6$ (by (3.2.4)(a)), by (3.2.5), we have $n \geq 1$. By (3.2.4)(a), and (3.2.5), we have

$$12 \geq 2(|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2} |X_i| \rfloor) \geq 2|W| + \sum_{1 \leq i \leq n} |X_i| - n \geq |W| + |L_1 \cup L_2 \cup L_3| - n \geq |W| + 15 - n.$$

Thus,

$$n \geq 3 + |W|.$$

3.2.7 Definition of A_i (for $i = 1, 2, 3$)

Let G'' be the subgraph obtained from $G - W$ by deleting all edges contained in any Y_j . Let A_i be the union of the vertex subsets of all components of G'' containing some vertex of L_i for each $i \in \{1, 2, 3\}$.

3.2.8 Properties of $\{A_1, A_2, A_3\}$

Properties of $\{A_1, A_2, A_3\}$ are to be studied in this subsection. The first property is immediate by (3.2.4) and the definition of A_i .

(a) $L_i - W \subseteq A_i \subseteq V(G) - W$ for $i = 1, 2, 3$.

Note that each $Y_j - X_j$ is an independent set of G'' , and by (3.2.4)(b), we have the following properties.

(b) $A_i \subseteq X_1 \cup \dots \cup X_n$ for $i = 1, 2, 3$.

(c) A_1, A_2, A_3 are disjoint by the definition of A_i and (3.2.4)(c).

3.2.9 We claim that $|W| \leq 5$.

For otherwise, by (3.2.4)(a), we may assume that $|W| = 6$.

By (3.2.4)(a), $|X_i| \leq 1$ for $1 \leq i \leq n$. By the choice of $|W|$ and n , Y_i is a connected subgraph.

For $1 \leq j \leq 3$, let $\overline{A_j}$ be the union of all Y_i s if $Y_i \cap A_j \neq \emptyset$ for $1 \leq i \leq n$.

By (3.2.4)(b), (3.2.4)(c) and the fact that $|X_i| \leq 1$ for $1 \leq i \leq n$, $\overline{A_j}$ and $\overline{A_k}$ are disconnected in $G - W$ for $1 \leq j < k \leq 3$. Hence, W is a vertex cut of G separating $\overline{A_1}$, $\overline{A_2}$, and $\overline{A_3}$.

Note that G is 6-connected and $|W| = 6$, W is a minimal vertex cut of G .

If $\overline{A_j} \neq \emptyset$ for each $j \in \{1, 2, 3\}$, then every vertex w of the minimal cut W must be adjacent to at least one vertex of each $\overline{A_j}$. This contradicts that G is claw-free.

Hence, without loss of generality, we may assume that $\overline{A_1} = \emptyset$.

(i) We claim that $\overline{A_j} \neq \emptyset$ for $j \in \{2, 3\}$. For otherwise, we may assume that $\overline{A_2} = \emptyset$, then both L_1 and L_j are contained in W . this contradicts that L_1 and L_j are disjoint and $|W| = 6$.

(ii) We claim that $W \subseteq N_G(\overline{A_j})$ for $j \in \{2, 3\}$. For otherwise, $|N_G(\overline{A_j}) \cap W| < |W| = 6$. By (3.2.4)(b) and (3.2.4)(c), $N_G(\overline{A_j}) \cap W$ is a vertex cut separating $\overline{A_j}$ from $V(G) - \overline{A_j} - W$. It contradicts that G is 6-connected.

(iii) Since $\overline{A_1} = \emptyset$, $L_1 \subseteq W$. By (3.2.9)(i) and (ii), we have $W \subseteq N_G(\overline{A_j})$ for $j \in \{2, 3\}$. Hence, we contract $\overline{A_2} \cup (W - L_1)$ to new vertex u and contract $\overline{A_3}$ to new vertex v , $L_1 \cup \{u, v\}$ is a 7-clique. It is a contradiction.

3.2.10 *We claim that $W \cup X_i$ is a vertex cut of G if $Y_i - X_i \neq \emptyset$.*

Since $n \geq 3$ by (3.2.6), $V(G) - W - Y_i \neq \emptyset$. Hence, by (3.2.4)(b), $W \cup X_i$ separates $Y_i - X_i$ from $V(G) - W - Y_i$.

3.2.11 *We claim that $X_i \neq \emptyset$ for all i .*

Without loss of generality, we assume that $X_1 = \emptyset$. Note that $Y_1 \neq \emptyset$. Hence, by (3.2.10), W is a vertex cut of G , but $|W| \leq 5$ by (3.2.9), it contradicts that G is 6-connected.

3.2.12 *We claim that $|X_i|$ is odd for all i .*

Suppose that $|X_1|$ is even, then by (3.2.11), $|X_1| \geq 2$. Assume $v \in X_1$, let $W^* = W \cup \{v\}$, $Y_1^* = Y_1 - v$, $X_1^* = X_1 - v$ and $X_i^* = X_i$, $Y_i^* = Y_i$ for $2 \leq i \leq n$. Hence, the partition $\{W^*, X_1^*, \dots, X_n^*, Y_1^*, \dots, Y_n^*\}$ of $V(G)$ satisfies (3.2.4)(a)- (c), contradicting the choice that $|W|$ is as large as possible.

3.2.13 *We claim that, for $1 \leq j \leq n$, if $|W \cup X_j| < 6$ then $X_j = Y_j$.*

This is an immediate corollary of (3.2.10).

3.2.14 *Some definitions.*

(1) **Without loss of generality, we assume that $|X_1| \geq |X_2| \geq \dots \geq |X_n|$, and let m be the greatest integer such that $|X_m| \geq 3$.**

(2) **$B_i = A_i \cap (X_1 \cup X_2 \cup \dots \cup X_m)$ for $1 \leq i \leq 3$.**

(3) **Without loss of generality, we assume that $|B_1| \leq |B_2| \leq |B_3|$.**

3.2.15 *Properties of $\{B_1, B_2, B_3\}$.*

(a) **$B_i \subseteq A_i \subseteq X_1 \cup \dots \cup X_n$ for $i = 1, 2, 3$ by (3.2.8)(b).**

(b) **B_1, B_2, B_3 are disjoint by the definition of B_i and (3.2.8)(c).**

3.2.16 *We claim that, if $A_i - B_i \neq \emptyset$, then $W \cup B_i$ is a vertex cut of G separating $A_i - B_i$ from $V(G) - A_i - W$.*

Without loss of generality, we assume that $A_1 - B_1 \neq \emptyset$, and without loss of generality, it is sufficient to show that $W \cup B_1$ is a vertex cut of G separating $A_1 - B_1$ from $V(G) - A_1 - W$.

By the definition of A_i (3.2.7), we have $|V(G) - A_1| \geq |A_2 \cup A_3 \cup W| \geq |L_2 \cup L_3| = 10$. By (3.2.9), we have $|W| \leq 5$. Therefore $|V(G) - A_1 - W| = |V(G) - A_1| - |W| \geq 5$. That is, $V(G) - A_1 - W \neq \emptyset$.

By (3.2.4)(b) and (3.2.4)(c), $W \cup B_1$ is a vertex cut separating $A_1 - B_1$ from $V(G) - A_1 - W$.

3.2.17 *We claim that, if $A_i - B_i \neq \emptyset$ and $|W \cup B_i| = 6$, then $W \subseteq N_G(A_i - B_i)$ and $W \subseteq N_G(V(G) - A_i - W)$.*

By (3.2.16), $W \cup B_i$ is a vertex cut of G separating $A_i - B_i$ from $V(G) - A_i - W$.

Note that G is 6-connected and $|W \cup B_i| = 6$, $W \cup B_i$ is a minimal cut of G . Hence, every vertex of $W \cup B_i$ is adjacent to some vertex of $A_i - B_i$, that is, $W \subseteq N_G(A_i - B_i)$. With the similar argument, we have $W \subseteq N_G(V(G) - A_i - W)$.

3.2.18 *We claim that $|B_1 \cup B_2 \cup B_3| \leq 18 - 3|W|$, if equality holds then $|X_1| = |X_2| = \dots = |X_m| = 3$ and $m = 6 - |W|$.*

By (3.2.4)(a), (3.2.15)(a), (3.2.15)(b) and (3.2.14)(1), we have

$$|B_1 \cup B_2 \cup B_3| \leq \sum_{1 \leq j \leq m} |X_j| \leq 3 \sum_{1 \leq j \leq m} \lfloor \frac{1}{2} |X_j| \rfloor = 3 \sum_{1 \leq j \leq n} \lfloor \frac{1}{2} |X_j| \rfloor \leq 3(6 - |W|).$$

And if equality holds then $|X_1| = |X_2| = \dots = |X_m| = 3$ and $m = 6 - |W|$.

3.2.19 *We claim that $|B_i| \geq 5 - |W|$ for $1 \leq i \leq 3$.*

By (3.2.9), we have $|W| \leq 5$. It is obvious if $|W| = 5$. Therefore, we may assume $|W| \leq 4$, and it is sufficient to show $|B_1| \geq 5 - |W|$ by (3.2.14)(3).

For otherwise, if $|B_1| \leq 4 - |W|$, then $|B_1 \cup W| \leq 4$. By (3.2.8)(a), we have $|A_1| \geq |L_1 - W| \geq 5 - |W|$, therefore $|A_1 - B_1| \geq 1$.

Therefore, by (3.2.16), $W \cup B_1$ is a vertex-cut. Note that $|B_1 \cup W| \leq 4$. It contradicts the fact that G is 6-connected.

3.2.20 *We claim that, if $|B_1| = 6 - |W|$ then $|W| \geq 2$.*

By (3.2.19), (3.2.14)(3) and (3.2.18), we have $|B_1| = |B_2| = |B_3| = 6 - |W|$, $m = 6 - |W|$ and $|X_1| = \dots = |X_m| = 3$.

For otherwise, we assume that $|W| \leq 1$. Then, by (3.2.13), $Y_i = X_i$ for $1 \leq i \leq n$.

(i) We claim that $|B_i \cap X_j| \leq 1$ for $1 \leq i \leq 3$, $1 \leq j \leq m$. For otherwise, we may assume that $|B_1 \cap X_1| \geq 2$. Note that $(B_1 \Delta X_1) \cup W$ separates $(B_1 \cap X_1) \cup (A_1 - B_1)$ from $V(G) - A_1 - X_1 - W$ by (3.2.4)(b) and (3.2.4)(c), and both $(B_1 \cap X_1) \cup (A_1 - B_1)$ and $V(G) - B_1 - X_1$ are non-empty. Since G is 6-connected, $|(B_1 \Delta X_1) \cup W| \geq 6$, but $|(B_1 \Delta X_1) \cup W| = |B_1 \cup W| + |X_1| - 2|B_1 \cap X_1| \leq 6 + 3 - 4 = 5$. It is a contradiction.

(ii) Note that $m = 6 - |W|$, $|B_i| = 6 - |W|$ and $|B_i \cap X_j| \leq 1$ for $1 \leq i \leq 3$, $1 \leq j \leq m$ by (3.2.20)(i), we have that $|B_i \cap X_j| = 1$ for $1 \leq i \leq 3$, $1 \leq j \leq m$.

(iii) By (3.2.20)(ii), let $\{x_{ij}\} = B_i \cap X_j$ for $1 \leq i \leq 3$, $1 \leq j \leq m$. We claim that $N_G(x_{ij}) \cap X_j \neq \emptyset$. For otherwise, we assume that $N_G(x_{11}) \cap X_1 = \emptyset$, then let $W^* = W$, $Y_1^* = Y_1$, $X_1^* = X_1 - \{x_{11}\}$, $X_i^* = X_i$, $Y_i^* = Y_i$ for $2 \leq i \leq n$ and $Y_{n+1}^* = X_{n+1}^* = \{x_{11}\}$. Hence, the partition $\{W^*, X_1^*, \dots, X_n^*, X_{n+1}^*, Y_1^*, \dots, Y_n^*, Y_{n+1}^*\}$ of $V(G)$ satisfies (3.2.4)(a)-(c), contradicting the choices of $|W|$ and n .

(iv) We claim that $|N_G(x_{ij}) \cap A_i| \leq 4$. For otherwise, we assume that $|N_G(x_{11}) \cap A_1| \geq 5$. Note that $N_G(x_{11}) \cap X_1$ is nonempty by (3.2.20)(iii), and there are no edges between $N_G(x_{11}) \cap A_1$ and $N_G(x_{11}) \cap X_1$. Since G is claw-free, $G - W$ is claw-free. Hence, $N_G(x_{11}) \cap A_1$ is a clique. That is, $\{x_{11}\} \cup (N_G(x_{11}) \cap A_1)$ is a clique of order at least 6. It contradicts (3.2.1).

(v) We claim that $|W| \neq 0$ and $W \subseteq N_G(x_{ij})$. It is obvious since $N_G(x_{ij}) \subseteq W \cup A_i \cup (X_j - \{x_{ij}\})$, $|N_G(x_{ij})| \geq 7$ and $|(N_G(x_{ij}) \cap A_i) \cup (X_j - \{x_{ij}\})| \leq 6$ by (3.2.20)(iv).

(vi) Let $\{x\} = W$, then, by (3.2.20)(v), we have $xx_{11}, xx_{22}, xx_{33} \in E(G)$ and there are no edges between x_{11}, x_{22}, x_{33} by (3.2.4)(b). Hence, we get a claw centered at x . It is a contradiction.

3.2.21 We claim that $|B_1| = 5 - |W|$.

For otherwise, by (3.2.19), (3.2.14)(3) and (3.2.18), we have $|B_1| = 6 - |W|$. Therefore, by (3.2.14)(3), and (3.2.18), $|B_1| = |B_2| = |B_3| = 6 - |W|$, $m = 6 - |W|$ and $|X_1| = \dots = |X_m| = 3$.

(i) We claim that at most one of $A_1 - B_1$, $A_2 - B_2$, $A_3 - B_3$ is non-empty. For otherwise, we assume that $A_2 - B_2 \neq \emptyset$ and $A_3 - B_3 \neq \emptyset$, let $x_2 \in A_2 - B_2$, $x_3 \in A_3 - B_3$, if $A_1 - B_1 \neq \emptyset$, then by (3.2.17) and (3.2.20), we have a claw centered in W . Hence, $A_1 = B_1$, that is $L_1 \subseteq B_1 \cup W$. Note that $|W| \geq 2$ by (3.2.20), therefore, we can assume that $x \in L_1 \cap W$, and $x_1 \in L_1 \cap B_1$. Hence, we get a claw centered at x , a contradiction.

(ii) Without loss of generality, we assume that $A_1 - B_1 = \emptyset$ and $A_2 - B_2 = \emptyset$. Therefore $(L_1 \cup L_2) \subseteq W \cup B_1 \cup B_2$. That is, $10 \leq 12 - |W|$. Hence, $|W| \leq 2$. By (3.2.20), we have $|W| = 2$. Therefore $|B_1| = |B_2| = 6 - |W| = 4$. Therefore $|W \cap L_1| = |W \cap L_2| = 1$. With the similar argument as above, we have $A_3 - B_3 \neq \emptyset$. By (3.2.13), $Y_i = X_i$ for $1 \leq i \leq m$. Note that $\delta(G) \geq 7$ and $|N_G(x)| \leq |W| + |A_1 - x| + 2 = 7$ for $x \in B_1$, hence, $W \subseteq N_G(x)$ for $x \in B_1$. Let $v \in W \cap L_2$, $x_1 \in B_1$. Hence, $vx_1 \in E(G)$. Without loss of generality, we may assume that $x_1 \in B_1 \cap X_1$. Since $|B_2| = 4$ and $|X_1| = 3$, we may select $x_2 \in (B_2 - X_1)$. Let $x_3 \in A_3 - B_3$, we have $vx_3 \in E(G)$ by (3.2.17). Hence, we have a claw centered at v . It is a contradiction.

3.2.22 We claim that, if $|B_i| = 5 - |W|$ for some $1 \leq i \leq 3$, then $A_i = B_i$ and $L_i = W \cup B_i$.

First we show that $A_i = B_i$. For otherwise, we assume that $A_i - B_i \neq \emptyset$. By (3.2.16), $W \cup B_i$ is a vertex cut of G . Note that $|W \cup B_i| = 5$, it contradicts the fact that G is 6-connected.

Since $L_i \subseteq W \cup A_i = W \cup B_i$, and $|W \cup B_i| = 5 = |L_i|$, we have $L_i = W \cup B_i$.

3.2.23 We claim that, if $|W| \neq 0$ then $|B_2| = 6 - |W|$.

By (3.2.21), we have $|B_1| = 5 - |W|$. By (3.2.18), (3.2.19), and (3.2.21), $|B_2|$ is either $6 - |W|$ or $5 - |W|$.

If $|B_2| = 5 - |W|$. By (3.2.22), we have $L_2 = W \cup B_2$. Note that $|B_1| = 5 - |W|$ by (3.2.21), we have $L_1 = W \cup B_1$ by (3.2.22). Since L_1 and L_2 are disjoint, $|W| = 0$, it is a contradiction.

3.2.24 We claim that $|W| \leq 1$.

For otherwise, we may assume that $|W| \geq 2$. By (3.2.21), we have $|B_1| = 5 - |W|$. By (3.2.22), we have $A_1 = B_1$ and $L_1 = W \cup B_1$. By (3.2.23), we have $|B_2| = 6 - |W|$. By (3.2.18), (3.2.21), and (3.2.23), we have $6 - |W| \leq |B_3| \leq 7 - |W|$.

(i) We claim that, for $i \in \{2, 3\}$, if $|B_i| = 6 - |W|$, then $A_i - B_i \neq \emptyset$ and $W \subseteq N_G(A_i - B_i)$. Since $|W| \geq 2$, we have $|B_i| = 6 - |W| \leq 4$, note that $|L_i| = 5$, $L_i - B_i \neq \emptyset$. Therefore $A_i - B_i \neq \emptyset$. Note that $|W \cup B_i| = 6$ and $A_i - B_i \neq \emptyset$, by (3.2.17), we have $W \subseteq N_G(A_i - B_i)$.

(ii) We claim that $|W| \leq 4$. For otherwise, by (3.2.9), we may assume that $|W| = 5$. Note that $L_1 = W \cup B_1$ and $|B_1| = 5 - 5 = 0$, therefore $W = L_1$. Since $|B_2| = 6 - |W|$, $A_2 - B_2 \neq \emptyset$ and $W \subseteq N_G(A_2 - B_2)$ by (3.2.24)(i). By (3.2.17), we have $W \subseteq N_G(V(G) - W - A_2)$. Hence,

we contract $(A_2 - B_2)$, $B_2 \cup (V(G) - W - A_2)$ into new vertices u, v respectively, $L_1 \cup \{u, v\}$ is a K_7 clique. It is a contradiction.

(iii) We claim that $|B_3| = 7 - |W|$. For otherwise, we assume that $|B_3| = 6 - |W|$. By (3.2.24)(i), for $i = 2, 3$, we have $A_i - B_i \neq \emptyset$ and $W \subseteq N_G(A_i - B_i)$. Since $|L_1| = 5$ and $|W| \leq 4$ (by (3.2.24)(ii)), we have $L_1 - W \neq \emptyset$. Let $x \in W$, $x_1 \in (L_1 - W)$, and $x_2 \in (A_2 - B_2) \cap N_G(x)$, $x_3 \in (A_3 - B_3) \cap N_G(x)$. By the definition of A_i and B_i , there is no edge among x_1, x_2, x_3 , we get a claw centered at x , it is a contradiction.

(iv) We claim that, if $|B_3| \leq 4$, then $A_3 - B_3 \neq \emptyset$ and $N_G(A_3 - B_3) \cap W \neq \emptyset$. Since $|A_3| \geq |L_3| = 5$ and $|B_3| \leq 4$, $A_3 - B_3 \neq \emptyset$. By (3.2.16), $W \cup B_3$ is a vertex cut of G separating $A_3 - B_3$ from $V(G) - A_3 - W$. If $N_G(A_3 - B_3) \cap W = \emptyset$, then B_3 is a vertex cut of G . Note that $|B_3| \leq 4$, it contradicts the fact that G is 6-connected.

(v) We claim that $|W| = 2$. For otherwise, $|B_3| = 7 - |W| \leq 4$, then, by (3.2.24)(iv), $A_3 - B_3 \neq \emptyset$, $N_G(A_3 - B_3) \cap W \neq \emptyset$. Let $x \in N_G(A_3 - B_3) \cap W$, and $x_1 \in L_1 - W$, $x_2 \in (A_2 - B_2) \cap N_G(x)$, $x_3 \in (A_3 - B_3) \cap N_G(x)$, there is no edge among x_1, x_2, x_3 by (3.2.4)(b). Hence, we get a claw, it is a contradiction.

(vi) By (3.2.18), (3.2.24)(iii) and (3.2.24)(v), we have $m = 6 - |W| = 4$ and $|X_1| = \dots = |X_m| = 3$. Hence, by (3.2.13), we have $Y_i = X_i$ for $1 \leq i \leq n$. Let $x \in L_1 - W$, we may assume that $x \in X_1$, therefore $N_G(x) \subseteq (A_1 - \{x\}) \cup W \cup (X_1 - L_1) = (B_1 - \{x\}) \cup W \cup (X_1 - L_1)$, that is $|N_G(x)| \leq (5 - |W| - 1) + |W| + 2 = 4 + 2 = 6$. It contradicts that $\delta(G) \geq 7$.

3.2.25 *We claim that, if $|X_i| \leq 3$ for some $1 \leq i \leq n$, then $X_i = Y_i$.*

By (3.2.13) and (3.2.24), it is obvious.

3.2.26 *We claim that, if $|B_i| = 5 - |W|$ for some $1 \leq i \leq 3$, then $|B_i \cap X_j| = 0$ for each X_j with $|X_j| = 3$.*

For otherwise, let $x \in B_i \cap X_j$. Since $|X_j| = 3$, by (3.2.25), we have $X_j = Y_j$. Note that $|B_i| = 5 - |W|$, by (3.2.22), we have $A_i = B_i$. Hence, $N_G(x) \subseteq (X_j - \{x\}) \cup W \cup (B_i - \{x\})$ by (3.2.4)(b) and (3.2.4)(c). That is, $|N_G(x)| \leq |(X_j - \{x\}) \cup W \cup (B_i - \{x\})| \leq 6$. It contradicts that $\delta(G) \geq 7$.

3.2.27 *We claim that*

- (i) $m \geq 2|W| + |X_1 \cup X_2 \cup \dots \cup X_m| - 12 \geq 2|W| + |B_1 \cup B_2 \cup B_3| - 12;$
- (ii) There exists X_i with $|X_i| \geq 5$ for $1 \leq i \leq m;$
- (iii) $m \leq 5 - |W|.$

Proof of (i): By (3.2.4)(a), and (3.2.12), we have

$$12 \geq 2(|W| + \sum_{1 \leq i \leq n} \lfloor \frac{1}{2}|X_i| \rfloor) = 2(|W| + \sum_{1 \leq i \leq m} \lfloor \frac{1}{2}|X_i| \rfloor) = 2|W| + \sum_{1 \leq i \leq m} |X_i| - m.$$

Thus,

$$m \geq 2|W| + |X_1 \cup X_2 \cup \dots \cup X_m| - 12.$$

The second part of the inequality follows by (3.2.14)(1)-(3).

Proof of (ii): For otherwise, we assume that $|X_i| = 3$ for $1 \leq i \leq m$. Note that $|B_1| = 5 - |W|$ by (3.2.21). By (3.2.26), $B_1 \cap X_i = \emptyset$ for $1 \leq i \leq m$. Hence, $B_1 = \emptyset$. It contradicts that $|B_1| = 5 - |W| \geq 4$ (by (3.2.21) and (3.2.24)).

Proof of (iii): By (3.2.27)(i) and (3.2.27)(ii), we have,

$$m \geq 2|W| + |X_1 \cup X_2 \cup \dots \cup X_m| - 12 \geq 2|W| + (3m + 2) - 12 = 2|W| + 3m - 10.$$

Thus, $m \leq 5 - |W|.$

3.2.28 *We claim that $|W| = 0$.*

For otherwise, we may assume that $|W| = 1$ by (3.2.24). By (3.2.21), we have $|B_1| = 5 - |W| = 4$. By (3.2.23), we have $|B_2| = 6 - |W| = 5$. Hence, $5 \leq |B_3| \leq 6$ by (3.2.14)(3) and (3.2.18). Therefore $|B_1 \cup B_2 \cup B_3|$ is either 14 or 15.

By (3.2.27)(iii), we have $m \leq 5 - |W| = 4$.

By (3.2.27)(i), we have

$$m \geq 2|W| + |X_1 \cup X_2 \cup \dots \cup X_m| - 12 \geq 2|W| + |B_1 \cup B_2 \cup B_3| - 12 \geq 2 + 14 - 12 = 4.$$

Hence, $m = 4$. Furthermore, the fact that all equalities hold implies that $|X_1 \cup X_2 \cup \dots \cup X_m| = |B_1 \cup B_2 \cup B_3| = 14$. Therefore, the only possibility is $\{|X_1|, |X_2|, \dots, |X_4|\} = \{5, 3, 3, 3\}$. In this case, by (3.2.26) and $|B_1| = 5 - |W|$, we have $B_1 \subseteq X_1$. Hence, we get a cut-set $(X_1 - B_1) \cup W$ separating $B_1 \cup (Y_1 - X_1)$ from $V(G) - W - Y_1$. Note that $|(X_1 - B_1) \cup W| \leq 2$, it is a contradiction.

3.2.29 *We claim that*

- (i) $|B_2| = 5$;
- (ii) $5 \leq |B_3| \leq 6$.

Proof of (i): By (3.2.21), (3.2.14)(3) and (3.2.18) and (3.2.28), we have, $|B_1| = 5$, $5 \leq |B_2| \leq 6$, and $|B_1| + |B_2| + |B_3| \leq 18$.

Assume that $|B_2| = 6$. Thus, we have $6 \leq |B_3|$ (by (3.2.14)(3)). Therefore, $|B_1 \cup B_2 \cup B_3| \geq 17$.

By (3.2.27)(iii), and (3.2.28), we have $m \leq 5$.

By (3.2.27)(i), we have $m \geq 2|W| + |X_1 \cup X_2 \cup \dots \cup X_m| - 12 \geq 2|W| + |B_1 \cup B_2 \cup B_3| - 12 \geq 17 - 12 = 5$.

Hence, $m = 5$, and the fact that all equalities hold implies that $|X_1 \cup X_2 \cup \dots \cup X_m| = |B_1 \cup B_2 \cup B_3| = 17$ and $|B_3| = 6$. Therefore, the only possibility is $\{|X_1|, |X_2|, \dots, |X_5|\} = \{5, 3, 3, 3, 3\}$. By (3.2.26), we have $B_1 \subseteq X_1$. By the definition of A_i, B_i (see (3.2.7), (3.2.14)(ii)), the graph $A_1 \cup X_1$ is not connected to other part of G , a contradiction.

(ii) can be proved by the similar argument as (3.2.29)(i).

3.2.30 *We claim that $|X_i| \geq 5$ for $1 \leq i \leq m$.*

For otherwise, we may assume that $|X_m| = 3$. By (3.2.21), (3.2.29), (3.2.28) and (3.2.26), we have $B_1 \cap X_m = B_2 \cap X_m = \emptyset$. There are four cases:

Case 1: $X_m \subseteq B_3$. In this case, $B_3 - X_m$ is a vertex-cut separating X_m from $V(G) - B_3$ by (3.2.4)(b), note that $|B_3 - X_m| \leq 3$, it is a contradiction that G is 6-connected.

Case 2: $|X_m \cap B_3| = 2$. By (3.2.29)(ii), $5 \leq |B_3| \leq 6$. Note that $|B_3| \neq 5$ by (3.2.26) and (3.2.28). Thus $|B_3| = 6$. By (3.2.27)(i), (3.2.14)(1)-(3) and (3.2.28), note that $B_1 \cap X_m =$

$B_2 \cap X_m = \emptyset$, we have $m \geq 2|W| + |X_1 \cup X_2 \cup \cdots \cup X_m| - 12 \geq |B_1 \cup B_2 \cup B_3| + |X_m - B_3| = 16 + 1 - 12 = 5$. Note that $m \leq 5$ by (3.2.27)(iii), hence, $m = 5$. Therefore, all possibilities are $\{|X_1|, |X_2|, \dots, |X_5|\} = \{3, 3, 3, 3, 3\}$ and $\{|X_1|, |X_2|, \dots, |X_5|\} = \{5, 3, 3, 3, 3\}$. $\{|X_1|, |X_2|, \dots, |X_5|\} = \{3, 3, 3, 3, 3\}$ is impossible by (3.2.26) and $|B_1| = 5 - |W| > 0$. Next we focus on $\{|X_1|, |X_2|, \dots, |X_5|\} = \{5, 3, 3, 3, 3\}$, in this subcase, by (3.2.26) and $|B_1| = |B_2| = 5$, we have $B_1 \subseteq X_1$, and $B_2 \subseteq X_1$. Note that B_1 and B_2 are disjoint, hence, $B_1 \cup B_2 \subseteq X_1$ and $|B_1 \cup B_2| = 10$, it is a contradiction.

Case 3: $|X_m \cap B_3| = 1$. By (3.2.29)(ii), $5 \leq |B_3| \leq 6$. Note that $|B_3| \neq 5$ by (3.2.26) and (3.2.28). Thus $|B_3| = 6$. By (3.2.27)(i), (3.2.14)(1)-(3) and (3.2.28), note that $B_1 \cap X_m = B_2 \cap X_m = \emptyset$, we have $m \geq 2|W| + |X_1 \cup X_2 \cup \cdots \cup X_m| - 12 \geq |B_1 \cup B_2 \cup B_3| + |X_m - B_3| = 16 + 2 - 12 = 6$. Note that $m \leq 5$ by (3.2.27)(iii), it is a contradiction.

Case 4: $X_m \cap B_3 = \emptyset$. By (3.2.27)(i), (3.2.14)(1)-(3) and (3.2.28), note that $B_1 \cap X_m = B_2 \cap X_m = \emptyset$, we have $m \geq 2|W| + |X_1 \cup X_2 \cup \cdots \cup X_m| - 12 \geq |B_1 \cup B_2 \cup B_3| + |X_m| \geq 15 + 3 - 12 = 6$. Note that $m \leq 5$ by (3.2.27)(iii), it is a contradiction.

3.2.31 We claim that $m = 3$, $|X_1 \cup X_2 \cup \cdots \cup X_m| = 15$, $|B_3| = 5$.

By (3.2.30), (3.2.27)(i), and (3.2.28), we have

$$m \geq 2|W| + |X_1 \cup X_2 \cup \cdots \cup X_m| - 12 \geq 5m - 12.$$

Thus $m \leq 3$.

By (3.2.21), and (3.2.14)(3), and (3.2.28), we have $|B_1 \cup B_2 \cup B_3| \geq 15 - 3|W| = 15$. Therefore by (3.2.27)(i) and (3.2.28), we have $m \geq 2|W| + |X_1 \cup X_2 \cup \cdots \cup X_m| - 12 \geq 2|W| + |B_1 \cup B_2 \cup B_3| - 12 \geq 15 - 12 = 3$.

Hence, we have $m = 3$, and the fact that all equalities hold implies that $|X_1 \cup X_2 \cup \cdots \cup X_m| = |B_1 \cup B_2 \cup B_3| = 15$. and $|B_3| = 5$.

3.2.32 The final step.

By (3.2.31), we have $m = 3$. By (3.2.30) and (3.2.31), the only possibility is $\{|X_1|, |X_2|, |X_3|\} = \{5, 5, 5\}$.

By (3.2.21), (3.2.29)(i) and (3.2.31), we have $|B_i| = 5$ for $1 \leq i \leq 3$. By (3.2.28), $|W| = 0$. Hence, by (3.2.22), $A_i = B_i = L_i$. By (3.2.13), $Y_i = X_i$ for $1 \leq i \leq 3$. Thus $V(G) = L_1 \cup L_2 \cup L_3 = X_1 \cup X_2 \cup X_3 = B_1 \cup B_2 \cup B_3$.

(i) We claim that $|B_i \cap X_j| \leq 2$ for $1 \leq i, j \leq 3$. For otherwise, we may assume that $|B_1 \cap X_1| \geq 3$. Let $x \in B_1 \cap X_1$, by (3.2.4)(b), $N_G(x) \subseteq (B_1 - \{x\}) \cup (X_1 - B_1)$. That is, $|N_G(x)| \leq |(B_1 - \{x\}) \cup (X_1 - B_1)| \leq 4 + 2 = 6$. It contradicts $\delta(G) \geq 7$.

(ii) Note that $|X_i| = 5$, by (3.2.32)(i), $|B_i \cap X_j| \geq 1$ and $|X_j - B_i| \geq 3$ for $1 \leq i, j \leq 3$.

(iii) We claim that, if $x \in B_i \cap X_j$ for $1 \leq i, j \leq 3$, then $|N_{(X_j - B_i)}(x)| \geq 3$. By (3.2.4)(b) and the definition of B_i ((3.2.14)(2)), we have $N_G(x) \subseteq (B_i - \{x\}) \cup N_{(X_j - B_i)}(x) \subseteq (B_i - \{x\}) \cup (X_j - B_i)$, that is, $|N_G(x)| \leq |B_i - \{x\}| + |N_{(X_j - B_i)}(x)| = 4 + |N_{(X_j - B_i)}(x)|$. Note that $\delta(G) \geq 7$, hence, we have $|N_{(X_j - B_i)}(x)| \geq 3$.

(iv) We claim that X_i induces a clique for $1 \leq i \leq 3$. If $x, y \in X_i$ but $xy \notin E(G)$. Without loss of generality, let $x \in B_1$, $y \in B_2$. Next, we show that $|X_i - B_1| \geq 4$, for otherwise, assume $|X_i - B_1| \leq 3$. Hence, by (3.2.32)(ii) and (3.2.32)(iii), we have $|X_i - B_1| = 3$ and $|N_{(X_i - B_1)}(x)| = 3$. Note that $y \in B_2 \cap X_i$, x is adjacent to y , it is a contradiction. Symmetrically, we have $|X_i - B_2| \geq 4$. Those together imply that $|B_3 \cap X_i| \geq 3$. This contradicts (3.2.32)(i).

Hence, we contract L_2, L_3 to two new vertices u, v respectively, then $L_1 \cup \{u, v\}$ is a 7-clique, a contradiction.

This completes the proof of this theorem.

Chapter 4

Chords of Longest Circuits in Locally Planar Graphs

4.1 Introduction

An edge e is called a *chord* of a circuit if e is not an edge of the circuit and both endvertices of e are in the circuit.

The following conjecture was proposed by Thomassen.

Conjecture 4.1.1 (Thomassen [1], p. 466) *Every longest circuit of a 3-connected graph must have a chord.*

Conjecture 4.1.1 has been verified for following families of graphs.

1. (C.-Q. Zhang [39]) *3-connected cubic planar graphs and 3-connected planar graphs with minimum degree at least four.*
2. (C. Thomassen [33]) *3-connected cubic graphs*

3. (X. Li and C.-Q. Zhang [18]) *3-connected projective graphs with minimum degree at least four*
4. (X. Li and C.-Q. Zhang [19]) *4-connected graphs embedded in the torus of the Klein bottle.*

etc.

In this chapter, we will verify the conjecture for *locally planar graphs*.

Theorem 4.1.2 *There is a function $h : N \longrightarrow N$, such that, for every 4-connected graph G with minimum degree at least five embedded in a surface with Euler genus g and face-width at least $h(g)$, every longest circuit of the graph G has a chord.*

4.2 Terminology and notations

All graphs considered in this paper are finite, undirected, and without loops or multiple edges. For a simple graph G , the vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively.

Let $u, v \in V(G)$. The vertex u is a *neighbor* of v if $uv \in E(G)$. The set of all neighbors of v is denoted by $N(v)$. The degree of a vertex v , denoted by $d(v)$, is the number of neighbors of v .

A surface S is a compact arc-wise connected Hausdorff topological space which is locally homeomorphic to a disc (which means that, for every point p on S , there is an open set in S containing p which is homeomorphic to an open disc in the plane). A surface can be obtained in the following way: Take a collection of pairwise disjoint convex polygons (and their interior) of side length 1 in the plane. Identify each side with precisely one other side (possibly in the same polygon). This results in a topological space S and a simple graph G (whose nodes are the corners of the polygons and whose edges are the sides).

If G is connected, then also S is arc-wise connected. If, in addition, S is locally homeomorphic to a disc at every node of G , then S is a surface. We say that G is a *2-cell embedding* of S .

If all the polygons are triangles (and no side of a triangle is identified with a side of the same triangle) then G is a *triangulation* of S and S is a *triangulated surface*. The simplest

triangulated surface is the (boundary of the) tetrahedron. A *face* of a surface with a 2-cell embedding is the interior of one of the polygons.

We shall now define the (triangulated) surfaces S_g and N_h .

S_0 is the sphere. If we cut out two disjoint discs in S_0 and identify their boundaries such that the clockwise orientations of these boundaries disagree, then we have added a *handle* to S_0 . If we add g handles we obtain S_g . If instead we cut out an open disc in S_0 and identify any two opposite points on the boundary, then we have added a *crosscap* to S_0 . If we add h crosscaps ($h \geq 1$), then we obtain N_h . S_1 , N_1 , N_2 are the *torus*, the *projective plane* and the *Klein bottle*, respectively. It is easy to see that all surfaces S_g and N_h are triangulated. (They can be obtained from the boundary of the tetrahedron by cutting out triangles instead of discs).

The surfaces S_g are called *orientable* while the surfaces N_h are called *nonorientable*.

All embedding of graphs in surfaces considered in this paper are 2-cell embedding. If G is embedded in a surface S with f faces, then the number $g = 2 - |V(G)| + |E(G)| - f$ is called the *Euler genus* of S . The *edge-width* $\mathbf{ew}(\mathbf{G})$ of a graph G embedded in a non-simply connected surface is defined as the length of a shortest non-contractible circuit in G . The *face-width* or *representativeness*, denoted by $\mathbf{fw}(\mathbf{G})$, is the minimum k such that every non-contractible simple closed curve on the surface intersects G in at least k points.

4.3 Lemmas

The following is the key lemma for the proof of the main theorem of this chapter.

Lemma 4.3.1 (T. Böhme, B. Mohar and C. Thomassen [6]) *There is a function $f : \mathbb{N} \times \mathbb{R}^+ \rightarrow \mathbb{N}$ such that for every $\epsilon > 0$ and every 4-connected graph G of order n embedded in a surface with Euler genus g and face-width at least $f(g, \epsilon)$, G contains a circuit C with $|V(C)| > (1 - \epsilon)n$.*

Lemma 4.3.2 *Let G be a connected graph with $|V(G)| \geq 3$ embedded in a surface S with Euler genus g , then $|E(G)| < 3|V(G)| + 3g$.*

Proof. Assume G is embedded in S with f faces. Since each face is bounded by at least 3 edges, $3f \leq 2|E(G)|$. By Euler formula,

$$2 - g = |V(G)| - |E(G)| + f \leq |V(G)| - |E(G)| + \frac{2|E(G)|}{3}.$$

That is,

$$2 - g \leq |V(G)| - \frac{|E(G)|}{3}.$$

Hence,

$$|E(G)| \leq 3|V(G)| - 6 + 3g < 3|V(G)| + 3g.$$

■

4.4 Proof of the main theorem

The main theorem (Theorem 4.1.2) is to be proved in this section.

Choose $\epsilon \leq \frac{1}{32}$. By Lemma 4.3.1, there is a function $f : N \times R^+ \rightarrow N$ such that, for every 4-connected graph G embedded in a surface with Euler genus g and face-width at least $f(g, \epsilon)$, every longest circuit of G is of length at least $(1 - \epsilon)|V(G)| \geq \frac{31}{32}|V(G)|$.

Let $h(g) = \max\{f(g, \epsilon), 4g\}$. And let G be a 4-connected graph with minimum degree at least five embedded in a surface with Euler genus g and face-width at least $h(g)$. We will verify the conjecture for the graph G by way of contradiction.

Let C be a longest circuit of G without chord and let $W = V(G) - V(C)$, and let $AE(C) = \{uv \in E(G) \mid uv \notin E(C) \text{ and } \{u, v\} \cap V(C) \neq \emptyset\}$. It is obvious that

$$|AE(C)| = \sum_{v \in V(C)} (d(v) - 2)$$

since C is chordless. Furthermore,

$$|AE(C)| \leq |E(G) - E(C)| \leq \sum_{v \in W} d(v).$$

That is,

$$\sum_{v \in V(C)} (d(v) - 2) \leq \sum_{v \in W} d(v).$$

Hence,

$$\sum_{v \in V(C)} (2d(v) - 2) \leq \sum_{v \in V(G)} d(v).$$

So,

$$\sum_{v \in V(C)} (d(v) - 1) \leq E(G).$$

Since the minimum degree G is at least 5, we have that

$$4|V(C)| \leq E(G).$$

Therefore, by Lemma 4.3.2,

$$4|V(C)| \leq E(G) < 3|V(G)| + 3g.$$

Since the face-width of G is at least $h(g) \geq 4g$, we have that $|V(G)| \geq 4g$. Therefore,

$$4|V(C)| < 3|V(G)| + \frac{3|V(G)|}{4} = \frac{15|V(G)|}{4}.$$

That is,

$$|V(C)| < \frac{15|V(G)|}{16}.$$

Note that C is a longest circuit of G . By Lemma 4.3.1, $|V(C)| > (1 - \epsilon)|V(G)| \geq \frac{31|V(G)|}{32}$. This is a contradiction.

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