2008

Cycles in graph theory and matroids

Ju Zhou
West Virginia University

Cycles in Graph Theory and Matroids

Ju Zhou

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

Hong-Jian Lai, Ph.D., Chair
John Goldwasser, Ph.D.
K. Subramani, Ph.D.
Jerzy Wojciechowski, Ph.D.
Cun-Quan Zhang, Ph.D.

Department of Mathematics

Morgantown, West Virginia
2008

Keywords: Hamiltonian, Hamiltonian-Connected, s-Hamiltonian, s-Hamiltonian Connected, Eulerian, Supereulerian, Line Graph, Claw-Free Graph

ABSTRACT

A circuit is a connected 2-regular graph. A cycle is a graph such that the degree of each vertex is even. A graph G is Hamiltonian if it has a spanning circuit, and Hamiltonian-connected if for every pair of distinct vertices $u, v \in V(G)$, G has a spanning (u, v)-path. A graph G is s-Hamiltonian if for any $S \subseteq V(G)$ of order at most s, $G - S$ has a Hamiltonian-circuit, and s-Hamiltonian connected if for any $S \subseteq V(G)$ of order at most s, $G - S$ is Hamiltonian-connected. In this dissertation, we investigated sufficient conditions for Hamiltonian and Hamiltonian related properties in a graph or in a line graph. In particular, we obtained sufficient conditions in terms of connectivity only for a line graph to be Hamiltonian, and sufficient conditions in terms of degree for a graph to be s-Hamiltonian and s-Hamiltonian connected.

A cycle C of G is a spanning eulerian subgraph of G if C is connected and spanning. A graph G is supereulerian if G contains a spanning eulerian subgraph. If G has vertices $v_1, v_2, \cdots, v_n$, the sequence $(d(v_1), d(v_2), \cdots, d(v_n))$ is called a degree sequence of G. A sequence $d = (d_1, d_2, \cdots, d_n)$ is graphic if there is a simple graph G with degree sequence d. Furthermore, G is called a realization of d. A sequence $d \in \mathcal{G}$ is line-hamiltonian if d has a realization G such that $L(G)$ is hamiltonian. In this dissertation, we obtained sufficient conditions for a graphic degree sequence to have a supereulerian realization or to be line hamiltonian.

In 1960, Erdös and Pósa characterized the graphs G which do not have two edge-disjoint circuits. In this dissertation, we successfully extended the results to regular matroids and characterized the regular matroids which do not have two disjoint circuits.

Cycles in Graph Theory and Matroids

Ju Zhou

Acknowledgements

First, I'd like to express my sincere thanks to my supervisor, Dr. Hong-Jian Lai, for his guidance, encouragement, help and support during all these years. It has been a great pleasure to work under his supervision and I am lucky enough to have had a chance to learn from him. What I learned from him will benefit me all my life.

Second, I'd also like to thank my committee members: Dr. John Goldwasser, Dr. Hong-Jian Lai, Dr. K. Subramani, Dr. Jerzy Wojciechowski, and Dr. Cun-Quan Zhang, for their help and support during my studies.

Third, I want to take this chance to thank my previous supervisor Dr. Jinjiang Yuan in China for his guidance during his supervision and his continuous care and encouragement about my progress after that.

Fourth, I'd also like to thank my parents for their life-long moral support, their expectations and encouragement and my husband for his care and love. It is the company of their love which makes me feel not lonely and full of energy on my way to pursuing knowledge.

And finally, I would like to thank the Department of Mathematics and Eberly College of Arts and Sciences at West Virginia University for providing me with an excellent study environment and continual support during my yeas as a graduate student.

Contents

DEDICATION

To

my parents Guoqiang Zhou and Huage Jia , my siblings Peng Zhou and Pan Zhou

and

my husband Taoye Zhang

Chapter 1

Introduction

1.1 Notation and Terminology

We use [2] for notations and terminology in graph theory not defined here, and consider finite loopless connected graphs. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$ and $\alpha(G)$ to denote its vertex set, edge set, minimal degree and independence number, respectively. In particular, $\kappa(G)$ and $\kappa'(G)$ represent the *connectivity* and *edge-connectivity* of a graph G.

A graph is *trivial* if it contains no edges. A vertex cut X of G is essential if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-connected* if G does not have an essential cut X with $|X| < k$. An edge cut Y of G is essential if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G.

A *circuit* is a connected 2-regular graph. A *cycle* is a graph such that the degree of each vertex is even. A graph G is *Hamiltonian* if it has a spanning circuit, and *Hamiltonian-connected* if for every pair of distinct vertices $u, v \in V(G)$, G has a spanning (u, v)-path. A graph G is *s-Hamiltonian* if for any $S \subseteq V(G)$ of order at most s, $G - S$ has a Hamiltonian-circuit, and *s-Hamiltonian connected* if for any $S \subseteq V(G)$ of order at most s, $G - S$ is Hamiltonian-connected. A cycle C of G is a *spanning eulerian subgraph* of G if C is connected and spanning. A graph G is *supereulerian* if G contains a spanning eulerian subgraph.

The *contraction* G/X is the graph obtained from G be identifying the two ends of each edge in X and then deleting the resulting loops. When $X = \{e\}$, we also use G/e for $G/\{e\}$. For an integer $i > 0$, define

$$D_i(G) = \{v \in V(G) : deg_G(v) = i\}.$$

For any $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Catlin in [3] introduced collapsible graphs. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. Note that when $R = \emptyset$, a spanning connected subgraph H with $O(H) = \emptyset$ is a spanning Eulerian subgraph of G. Thus every collapsible graph is supereulerian. Catlin ([3]) showed that any graph G has a unique subgraph H such that every component of H is a maximally collapsible subgraph of G and every nontrivial collapsible subgraph of G is contained in a component of H. The contraction G/H is called the *reduction* of G. A graph G is *reduced* if it is the reduction of itself.

If G has vertices $v_1, v_2, \cdots, v_n$, the sequence $(d(v_1), d(v_2), \cdots, d(v_n))$ is called a *degree sequence* of G. A sequence $d = (d_1, d_2, \cdots, d_n)$ is *nonincreasing* if $d_1 \geq d_2 \geq \cdots d_n$. A sequence $d = (d_1, d_2, \cdots, d_n)$ is *graphic* if there is a simple graph G with degree sequence d. Furthermore, G is called a *realization* of d. Let $\mathcal{G}$ be the set of all graphic degree sequences. A sequence $d \in \mathcal{G}$ is *line-hamiltonian* if d has a realization G such that $L(G)$ is hamiltonian. A sequence $d = (d_1, d_2, \cdots, d_n)$ is *collapsible* if d has a simple collapsible realization.

Let H_1, H_2 be subgraphs of a graph G. Then $H_1 \cup H_2$ is a subgraph of G with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$; and $H_1 \cap H_2$ is a subgraph of G with vertex set $V(H_1) \cap V(H_2)$ and edge set $E(H_1) \cap E(H_2)$. If V_1, V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1, H_2 are two vertex disjoint subgraphs of G, then we also write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$.

We use Oxley [18] or Welsh [24] for notations and terminology of matroids not defined here. In combinatorics, a matroid is a structure that captures the essence of a notion of independence that generalizes linear independence in vector spaces. One of the most valuable definitions is that in terms of independence. In this definition, a finite matroid M is a pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a collection of subsets of E (called the independent sets) with the following properties:

(1) The empty set is independent.

(2) Every subset of an independent set is independent.

(3) If A and B are two independent sets and A has more elements than B, then there exists an element in A which is not in B and when added to B still gives an independent set.

Besides the vector spaces of linear algebra, a second original source for the theory of matroids is graph theory. Every finite graph (or multigraph) G gives rise to a matroid as follows: take as E the set of all edges in G and consider a set of edges independent if and only if it does not contain a simple circuit. Such an edge set is called a forest in graph theory. This is called the *cycle matroid* or *graphic matroid* of G ; it is usually written $M(G)$. Any matroid that is equivalent to the cycle matroid of a (multi)graph, even if it is not presented in terms of graphs, is called a *graphic matroid.* The matroids that are graphic have been characterized by Tutte.

A subset of E that is not independent is called *dependent.* A *circuit* in a matroid M is a minimal dependent subset of E. A *cycle* in a matroid M is disjoint union of circuits M. A matroid is *regular* if it can be represented by a totally unimodular matrix (a matrix whose square submatrices all have determinants equal to 0, 1, or -1). It's not hard to verify that every graphic matroid is regular matroid.

1.2 Road Map

This dissertation consists of 5 chapters. Starting from chapter 2, each chapter will be a study on a specific topic. In Chapter 2, we will investigate the sufficient condition in terms of connectivity for a line graph to be Hamiltonian. In Chapter 3, we will investigate the sufficient conditions in terms of degree for a graph to be s-Hamiltonian or s-Hamiltonian connected. In Chapter 4, we investigate sufficient conditions for a graphic degree sequence to have a supereulerian realization or to be line hamiltonian. In Chapter 5, we extend the characterization of graphs without two edge-disjoint circuits to the characterization of matroids without two disjoint circuits.

1.3 Main Results

There are extensive researches about circuits in graph theory. One part of my work is related to problems in graph theory involving Hamiltonian, Hamiltonian-connected, s-Hamiltonian, S-Hamiltonian connected and supereulerian.

Bill Tutte once said: if a theorem about graphs can be expressed in terms of edges and circuits alone, it probably exemplifies a more general theorem about matroids. The other part of my work is to generalize some known results in graph theory to matroids.

1.3.1 Partial Results Towards Thomassen Conjecture

In 1986, Thomassen proposed the following conjecture.

Conjecture 1.3.1 *(Thomassen [23]) Every 4-connected line graph is hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a *claw-free* graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture.

Conjecture 1.3.2 *(Matthews and Sumner [16]) Every 4-connected claw-free graph is hamiltonian.*

The best result towards these conjectures so far were obtained by Zhan and Ryjáček.

Theorem 1.3.3 *(Zhan [25]) Every 7-connected line graph is hamiltonian connected.*

Theorem 1.3.4 *(Ryjáček [19])*
(i) Conjecture 1.1 and Conjecture 1.2 are equivalent.
(ii) Every 7-connected claw-free graph is hamiltonian.

It is well known that the line graph of the graph obtained by subdividing each edge of the Petersen graph exactly once is a 3-connected claw-free graph without a hamiltonian circuit. We consider the following problem: For 3-connected claw-free graphs, can high essential connectivity guarantee the existence of a hamiltonian circuit? This leads us to prove the following Theorem 1.3.5.

Theorem 1.3.5 *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Ryjáček [19] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian, where $cl(G)$ is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity of a graph, the following corollary is obtained.

Corollary 1.3.6 *Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.*

However, what is the smallest positive integer k such that every 3-connected, essentially k-connected claw-free graph is hamiltonian? This question remains to be answered. Corollary 1.3.6 suggests that $4 \leq k \leq 11$. We fail to construct examples to show that there exists a 3-connected essentially 4-connected non-hamiltonian claw-free graph, and we conjecture that $k = 4$.

1.3.2 s-Hamiltonian and s-Hamiltonian Connected

The following sufficient conditions to ensure the existence of a Hamiltonian circuit in a simple graph G of order $n \geq 3$ are well known.

Theorem 1.3.7 *(Dirac [6]) If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 1.3.8 *(Ore [17]) If $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is Hamiltonian.*

Theorem 1.3.9 *(Fan [9]) If G is a 2-connected graph and if $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices $u, v \in V(G)$ with $d(u, v) = 2$, then G is Hamiltonian.*

Theorem 1.3.10 *(Chen [4]) If G is a 2-connected graph and if $\max\{d(u), d(v)\} \geq n/2$ for each pair of vertices $u, v \in V(G)$ with $1 \leq |N(u) \cap N(v)| \leq \alpha(G) - 1$, then G is Hamiltonian.*

Theorem 1.3.11 *(Chen et al [5]) If G is a k-connected $(k \geq 2)$ graph and if $\max\{d(v) : v \in I\} \geq n/2$ for every independent set I of order k such that I has two distinct vertices x, y with $d(x, y) = 2$, then G is Hamiltonian.*

Zhao et al recently proved Theorem 1.3.12 below, which unified and extended the above theorems.

Theorem 1.3.12 *(Zhao et al [14]) If G is a k-connected $(k \geq 2)$ graph of order n and if $\max\{d(v) : v \in I\} \geq n/2$ for every independent set I of order k such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G) - 1$, then G is Hamiltonian.*

We shall obtain sufficient conditions for s-Hamiltonian graphs and s-Hamiltonian connected graphs, respectively, as shown below.

Theorem 1.3.13 *Let k, s be two integers with $k \geq s+2$ and $0 \leq s \leq n-3$. If G is a k-connected graph of order n and if max$\{d(v) : v \in I\} \geq (n+s)/2$ for every independent set I of order $k-s$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)+s-1$, then G is s-Hamiltonian.*

Theorem 1.3.14 *Let k, s be two integers with $k \geq s+3$ and $0 \leq s \leq n-2$. If G is a k-connected graph of order n and if max$\{d(v) : v \in I\} \geq (n+s+1)/2$ for every independent set I of order $k-s-1$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)+s$, then G is s-Hamiltonian connected.*

Note that Theorem 1.3.12 is a special case of Theorem 1.3.13 when $s = 0$. Applying Theorem 1.3.14 to the case when $s = 0$, we get the following corollary.

Corollary 1.3.15 *If G is a k-connected ($k \geq 3$) graph of order n and if max$\{d(v) : v \in I\} \geq (n+1)/2$ for every independent set I of order $k-1$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)$, then G is Hamiltonian-connected.*

1.3.3 Degree Sequence and Supereulerian Graphs

In [26], Zhang et al. proved the following theorem.

Theorem 1.3.16 *[26] Every bipartite graphic sequence with the minimum degree $\delta \geq 2$ has a realization that admits a nowhere-zero 4-flow.*

We first get the following result.

Theorem 1.3.17 *If $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ is a nonincreasing sequences with $d_n \geq 2$, then d has a supereulerian realization.*

In [12], Jaeger proved the following result.

Theorem 1.3.18 *[12] Every supereulerian graph admits a nowhere-zero 4-flow.*

Combining Theorem 4.1.3, we get a result analogous to Theorem 1.3.16.

Theorem 1.3.19 *If $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ is a nonincreasing sequences with $d_n \geq 2$, then d has a realization that admits a nowhere-zero 4-flow.*

Furthermore, we get a result about line-hamiltonian sequence as follows.

Theorem 1.3.20 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing graphic sequence with $n \geq 3$, then the following are equivalent.*

(i) d is line-hamiltonian.

(ii) $d \in \mathcal{G}$ and either $d_1 = n - 1$, or

$$\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2). \tag{1.1}$$

(iii) d has a realization G such that $G - D_1(G)$ is supereulerian.

1.3.4 Regular Matroids without Disjoint Circuits

In 1960, Erdös and Pósa consider the problem of determining all connected graphs that do not have edge-disjoint circuits. We view the complete graph K_3 as a plane graph and let K_3^* denote the geometric dual of the plane graph K_3.

Theorem 1.3.21 *(Erdös and Pósa [8], also see Theorem 3.1, Theorem 3.2 of Bollobás [1]) Let G be a graph with $\delta(G) \geq 3$. The following are equivalent.*
(i) G does not have edge-disjoint circuits.
(ii) $G \in \{K_{3,3}, K_3^*, K_4\}$.

Since a graph G does not have disjoint circuits if and only if any subdivision of G does not have disjoint circuits, the following corollary follows immediately.

Corollary 1.3.22 *(Erdös and Pósa [8], also see Corollary 3.3 of Bollobás [1]) Let G be a simple graph of order $n \geq 3$.*
(i) If $|E(G)| \geq n + 4$, then G has 2 edge-disjoint circuits.
(ii) The graph G with $|E(G)| = n + 3$ does not have edge-disjoint circuits if and only if G can be obtained from a subdivision G_0 of $K_{3,3}$ by adding a forest and exactly one edge, joining each tree of the forest to G_0.

Theorem 1.3.21 can be viewed as a result on cosimple graphic matroids. Thus we consider generalizing Theorem 1.3.21 to matroids. Our main results of this note are the following.

Theorem 1.3.23 *Let M be a connected cosimple regular matroid. The following are equivalent.*
(i) M does not have disjoint circuits.
(ii) $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

Corollary 1.3.24 *Let M be a regular matroid. Then M has no disjoint circuits if and only if one of the following holds:*
(i) $M = U_{m,m}$, for some integer $m > 0$, or
(ii) M is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$, or
(iii) $M = M_1 \bigoplus M_2$ is the direct sum of two matroids M_1 and M_2, where M_1 is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$ and where $M_2 \cong U_{m,m}$, for some $m = |E(M)| - |E(M_1)| \geq 1$.

Chapter 2

Partial Result towards Thomassen Conjecture

2.1 The Problem and the Main Results

In 1986, Thomassen proposed the following conjecture.

Conjecture 2.1.1 *(Thomassen [23]) Every 4-connected line graph is hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a *claw-free* graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture.

Conjecture 2.1.2 *(Matthews and Sumner [16]) Every 4-connected claw-free graph is hamiltonian.*

The best result towards these conjectures so far were obtained by Zhan and Ryjáček.

Theorem 2.1.3 *(Zhan [25]) Every 7-connected line graph is hamiltonian connected.*

Theorem 2.1.4 *(Ryjáček [19])*

(i) Conjecture 1.1 and Conjecture 1.2 are equivalent.

(ii) Every 7-connected claw-free graph is hamiltonian.

It is well known that the line graph of the graph obtained by subdividing each edge of the Petersen graph exactly once is a 3-connected claw-free graph without a hamiltonian circuit. In this chapter, we consider the following problem: For 3-connected claw-free graphs, can high essential connectivity guarantee the existence of a hamiltonian circuit? This leads us to prove the following Theorem 2.1.5.

Theorem 2.1.5 *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Ryjáček [19] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian, where $cl(G)$ is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity of a graph, The following corollary is obtained.

Corollary 2.1.6 *Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.*

However, what is the smallest positive integer k such that every 3-connected, essentially k-connected claw-free graph is hamiltonian? This question remains to be answered. Thus Corollary 2.1.6 below suggests that $4 \leq k \leq 11$. We fail to construct examples to show that there exists a 3-connected essentially 4-connected non-hamiltonian claw-free graph, and we conjecture that $k = 4$.

2.2 Reductions

We shall introduce some of the reduction techniques to be used in the proof.

Theorem 2.2.1 *Let G be a connected graph and let G' denote its reduction. Let $F(G)$ denote the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Each of the following holds.*
(i) (Catlin [3]) If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.
(ii) (Catlin, Theorem 8 of [3]) If G is reduced and if $|E(G)| \geq 3$, then $\delta(G) \leq 3$, and $2|V(G)| - |E(G)| \geq 4$.
(iii) (Catlin, Theorem 5 of [3]) A graph G is reduced if and only if G contains no nontrivial collapsible subgraphs. As circuits of length less than 4 are collapsible, a reduced graph does not have a circuit of length less than 4.

Let G be a connected, essentially 3-edge-connected graph such $L(G)$ is not a complete graph. The *core* of this graph G, denoted by G_0, is obtained by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 2.2.2 *(Shao [22]) Let G be a connected, essentially 3-edge-connected graph G.*
(i) G_0 is uniquely defined, and $\kappa'(G_0) \geq 3$.
(ii) If G_0 is supereulerian, then $L(G)$ is hamiltonian.

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-circuit is called a *2-path* or a P_2 *subgraph* of G. An edge cut X of G is a P_2*-edge-cut* of G if at least two components of $G - X$ contain 2-paths. By the definition of a line graph, for a graph G, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2 edge cut with size less than k. Since the core G_0 is obtained from G by contractions (deleting a pendant edge is equivalent to contracting the same edge), every P_2-edge-cut of G_0 is also a P_2-edge-cut of G. Hence we have the following.

Lemma 2.2.3 *Let $k > 2$ be an integer, and let G be a connected, essentially 3-edge-connected graph. If $L(G)$ is essentially k-connected, then every P_2-edge-cut of G_0 has size at least k.*

2.3 Proof of Theorem 2.1.5

Throughout this section, we assume that G is a graph such that $L(G)$ is 3-connected, essentially 11-connected, and that $L(G)$ is not a complete graph. Let G_0 denote the core of G and G_0' denote the reduction of G_0. We shall show that $G_0' = K_1$, and so G_0 is collapsible, which implies that G_0 is supereulerian. Hence by Lemma 2.2.2, $L(G)$ is hamiltonian.

By contradiction, we assume that G_0' is a nontrivial graph. By Theorem 2.2.1(iii),

$$G_0' \text{ does not have a circuit of length less than 4.} \tag{2.1}$$

Since $L(G)$ is 3-connected, G is essentially 3-edge-connected. By Lemma 2.2.2, G_0' is 3-edge-connected. By Theorem 2.2.1(ii), $D_3(G_0') \neq \emptyset$.

Lemma 2.3.1 *For each $u, v, w \in V(G_0')$ such that $P = uvw$ is 2-path in $V(G_0')$, the edge cut $X = [\{u, v, w\}, V(G_0') - \{u, v, w\}]_{G_0'}$ is a P_2-edge-cut of G_0' and $|X| \geq 11$.*

Proof: Suppose that $G_0' - X$ has components $H_1, H_2, \cdots, H_c$ with $c \geq 2$ and with $H_1 = G_0'[\{u, v, w\}]$ denoting a 2-path of G_0'. To show that X is a P_2-edge-cut of G_0', it suffices to show that for some $i \geq 2$,

$$|V(H_i)| \geq 2 \text{ and } |E(H_i)| \geq 2. \tag{2.2}$$

Suppose first that for some $i \geq 2$, $|E(H_i)| = 1$. Since H_i is a component of $G_0' - X$, $|[\{u, v, w\}, V(H_i)]_{G_0'}| \geq \kappa'(G_0') \geq 3$, and so G_0' would have a circuit of length at most 3, contrary to (2.1). Similarly, suppose that for some $i \geq 2$, we have $E(H_i) = \{xy\}$. Then by $\kappa'(G_0') \geq 3$, each of x and y has degree at least 3 in G_0' and so $|[\{u, v, w\}, V(H_i)]_{G_0'}| \geq 4$. It follows again that G_0' would have a circuit of length at most 3, contrary to (2.1). This proves (2.2).

Thus X is a P_2-edge-cut of G_0'. Since $L(G)$ is essentially 11-connected, $|X| \geq 11$. □

Lemma 2.3.2 *Every component of $G'_0[D_3(G'_0)]$ contains at most 2 vertices.*

Proof: By contradiction, we assume that one component of $G'_0[D_3(G'_0)]$ contains at least 3 vertices, and so this component has three vertices u, v, w such that $G'_0[\{u, v, w\}]$ is connected. Thus $X = [\{u, v, w\}, V - \{u, v, w\}]$ is a P_2-edge-cut of G'_0. Since $u, v, w \in D_3(G'_0)$, $|X| \leq 5$, contrary to Lemma 2.3.1. □

Define a real valued function

$$f(x) = \frac{x-4}{x}, \text{ over the interval } [3, \infty).$$

For each $v \in G'_0$, define $l(v) = f(deg_{G'_0}(v))$. Note that (i) of Lemma 2.3.3 below is a fact from Calculus and (ii) of Lemma 2.3.3 follows from (i) of Lemma 2.3.3.

Lemma 2.3.3 *Each of the following holds.*
(i) $f(x)$ is an increasing function.
(ii) If $deg_{G'_0}(v) \geq k$, then $l(v) \geq f(k)$.

Lemma 2.3.4 *Suppose that $v \in D_3(G'_0)$ is an isolated vertex of $G'_0[D_3(G'_0)]$ such that v_1, v_2, v_3 are the vertices adjacent to v in G'_0. Then $l(v_1) + l(v_2) + l(v_3) \geq 1$.*

Proof: Since v is an isolated vertex in $D_3(G'_0)$, $v_i \notin D_3(G'_0)$. Relabelling the vertices if needed, we may assume that

$$4 \leq deg_{G'_0}(v_1) \leq deg_{G'_0}(v_2) \leq deg_{G'_0}(v_3). \tag{2.3}$$

For $i, j \in \{1, 2, 3\}$, by Lemma 2.3.1, $deg_{G'_0}(v_i) + deg_{G'_0}(v_j) - 2 + 1 = |[\{v, v_i, v_j\}, V(G'_0) - \{v, v_i, v_j\}]| \geq 11$, and so

$$deg_{G'_0}(v_i) + deg_{G'_0}(v_j) \geq 12. \tag{2.4}$$

If $deg_{G'_0}(v_1) \geq 6$, then by (2.3) and by Lemma 2.3.3(ii), $l(v_1) + l(v_2) + l(v_3) \geq 3f(6) = 1$. Suppose then that $deg_{G'_0}(v_1) = 5$. Then by (2.4), both $deg_{G'_0}(v_2) \geq 7$ and $deg_{G'_0}(v_3) \geq 7$.

It follows by Lemma 2.3.3(ii) that $l(v_1) + l(v_2) + l(v_3) \geq f(5) + 2f(7) \geq 1$. Finally, we assume that $deg_{G'_0}(v_1) = 4$. Then by (4), both $deg_{G'_0}(v_2) \geq 8$ and $deg_{G'_0}(v_3) \geq 8$. It follows by Lemma 2.3.3(ii) that $l(v_1) + l(v_2) + l(v_3) \geq f(4) + 2f(8) = 1$. □

Lemma 2.3.5 *Suppose that $v, w \in D_3(G'_0)$ and $vw \in E(G'_0)$. If v_1, v_2, w are the vertices adjacent to v in G'_0 and if v_3, v_4, v are the vertices adjacent to w in G'_0, then*
(i) v_1, v_2, v_3, v_4 are mutually distinct vertices, and
(ii) both $l(v_1) + l(v_2) \geq 1$ and $l(v_3) + l(v_4) \geq 1$.

Proof: If $|\{v_1, v_2, v_3, v_4\}| \leq 4$, then G'_0 could contain a circuit of length at most 3, contrary to Theorem 2.2.1(iii). Thus Lemma 2.3.5(i) follows.

For $i \in \{1, 2, 3, 4\}$, by Lemma 2.3.1, $deg_{G'_0}(v_i) - 1 + 3 = |[\{v, w, v_i\}, V - \{v, w, v_i\}]| \geq 11$, and so

$$deg_{G'_0}(v_i) \geq 9. \tag{2.5}$$

It follows by (2.5) and Lemma 2.3.3(ii) that both $l(v_1) + l(v_2) \geq 2f(9) \geq 1$ and $l(v_3) + l(v_4) \geq 2f(9) \geq 1$. □

Let $d_i = |D_i(G'_0)|$, for each $i \geq 3$. By Lemmas 2.3.2 Lemma 2.3.4 and Lemma 2.3.5, we have

$$\begin{aligned} d_3 &= \sum_{v\in D_3} 1 \leq \sum_{v\in D_3} \sum_{uv\in E, u\notin D_3} l(u) = \sum_{u\notin D_3} \sum_{uv\in E, v\in D_3} l(u) \\ &= \sum_{i\geq 4} \sum_{u\in D_i} \sum_{uv\in E, v\in D_3} l(u) \leq \sum_{i\geq 4} \sum_{u\in D_i} i \cdot f(i) \\ &= \sum_{i\geq 4} \sum_{u\in D_i} (i-4) = \sum_{i\geq 4} (i-4) \cdot d_i. \end{aligned} \tag{2.6}$$

It follows by (2.6) that

$$2(2|V(G)| - |E(G)|) = 4|V(G)| - 2|E(G)| = \sum_{i\geq 3} (4-i) \cdot d_i = d_3 - \sum_{i\geq 4} (i-4) \cdot d_i \leq 0,$$

contrary to Theorem 2.2.1. Thus $G'_0 = K_1$ and G_0 is supereulerian. By Lemma 2.2.2, $L(G)$ is hamiltonian. This completes the proof of Theorem 2.1.5.

Chapter 3

s-Hamiltonian and s-Hamiltonian Connected

3.1 The Problem and the Main Results

Let G be a graph. If $v \in V(G)$ and H is a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v)$ for $d_G(v)$ and $N(v)$ for $N_G(v)$. If C and H are subgraphs of G, then $N_C(H) = \cup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of G induced by $V(G) - V(C)$. Let $P = x_1x_2 \cdots x_m$ denote a path of order m. To emphasize the end vertices of the path P, we also say that P is an (x_1, x_m)-path. Define $N_P^+(u) = \{x_{i+1} \in V(P) : x_i \in N_P(u)\}$. So if $x_m \in N_P(u)$, then $|N_P^+(u)| = |N_P(u)| - 1$. Two vertices are consecutive in P if they are the ends of an edge in $E(P)$. Thus, each pair of vertices x_i, x_{i+1} are consecutive in P for any $i \in \{1, \cdots, m-1\}$. When $1 \leq i < j \leq m$, we use $[x_i, x_j]$ to denote the section $x_ix_{i+1} \cdots x_j$ of P and $[x_j, x_i]$ to denote the section $x_jx_{j-1} \cdots x_i$ of P. If there is an (x_1, x_m)-path P^* in G such that $V(P) \subset V(P^*)$ and $|V(P^*)| > |V(P)|$, then we say that P^* extends P. Let $C = x_1 \cdots x_mx_1$ be a circuit. Define $N_C^+(H) = \{x_{i+1} \in V(C) : x_i \in N_C(u)\}$, where the subscriptions are taken by modulo m. Two vertices are consecutive in C if they are the ends of an edge in $E(C)$. If there is a circuit C^* in G such that $V(C) \subset V(C^*)$ and $|V(C^*)| > |V(C)|$, then we say that C^* extends C.

The following sufficient conditions to ensure the existence of a Hamiltonian circuit in a simple graph G of order $n \geq 3$ are well known.

Theorem 3.1.1 *(Dirac [6]) If* $\delta(G) \geq n/2$, *then* G *is Hamiltonian.*

Theorem 3.1.2 *(Ore [17]) If* $d(u) + d(v) \geq n$ *for each pair of nonadjacent vertices* $u, v \in V(G)$, *then* G *is Hamiltonian.*

Theorem 3.1.3 *(Fan [9]) If* G *is a 2-connected graph and if* $\max\{d(u), d(v)\} \geq n/2$ *for each pair of vertices* $u, v \in V(G)$ *with* $d(u, v) = 2$, *then* G *is Hamiltonian.*

Theorem 3.1.4 *(Chen [4]) If* G *is a 2-connected graph and if* $\max\{d(u), d(v)\} \geq n/2$ *for each pair of vertices* $u, v \in V(G)$ *with* $1 \leq |N(u) \cap N(v)| \leq \alpha(G) - 1$, *then* G *is Hamiltonian.*

Theorem 3.1.5 *(Chen et al [5]) If* G *is a* k*-connected* $(k \geq 2)$ *graph and if* $\max\{d(v) : v \in I\} \geq n/2$ *for every independent set* I *of order* k *such that* I *has two distinct vertices* x, y *with* $d(x, y) = 2$, *then* G *is Hamiltonian.*

Zhao et al recently proved Theorem 3.1.6 below, which unified and extended the above theorems.

Theorem 3.1.6 *(Zhao et al [14]) If* G *is a* k*-connected* $(k \geq 2)$ *graph of order* n *and if* $\max\{d(v) : v \in I\} \geq n/2$ *for every independent set* I *of order* k *such that* I *has two distinct vertices* x, y *with* $1 \leq |N(x) \cap N(y)| \leq \alpha(G) - 1$, *then* G *is Hamiltonian.*

In this chapter, we shall obtain sufficient conditions for s-Hamiltonian graphs and s-Hamiltonian connected graphs, respectively, as shown below.

Theorem 3.1.7 *Let k, s be two integers with $k \geq s+2$ and $0 \leq s \leq n-3$. If G is a k-connected graph of order n and if $max\{d(v) : v \in I\} \geq (n+s)/2$ for every independent set I of order $k-s$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)+s-1$, then G is s-Hamiltonian.*

Theorem 3.1.8 *Let k, s be two integers with $k \geq s+3$ and $0 \leq s \leq n-2$. If G is a k-connected graph of order n and if $max\{d(v) : v \in I\} \geq (n+s+1)/2$ for every independent set I of order $k-s-1$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)+s$, then G is s-Hamiltonian connected.*

Note that Theorem 3.1.6 is a special case of Theorem 3.1.7 when $s = 0$. Applying Theorem 3.1.8 to the case when $s = 0$, we get the following corollary.

Corollary 3.1.9 *If G is a k-connected $(k \geq 3)$ graph of order n and if $max\{d(v) : v \in I\} \geq (n+1)/2$ for every independent set I of order $k-1$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)$, then G is Hamiltonian-connected.*

The following Lemma 3.1.10 is very important for the proof of the main theorems. A proof can also be found in [15].

Lemma 3.1.10 *Let G be a connected graph, $F = x_1 \cdots x_m(x_1)$ be a longest path (or circuit) in G and H be a component of $G - V(F)$. If $x_i, x_j \in N_F(H)$ with $1 \leq i < j < m$, then*
(i) $x_{i+1}x_{j+1} \notin E(G)$;
(ii) $N(x_{i+1}) \cap V(H) = \emptyset$;
(iii) $N_F^+(H) \cup \{x\}$ is an independent set of G, where $x \in V(H)$.

Theorem 3.1.7 and Theorem 3.1.8 will be proved in the following two sections, respectively.

3.2 Proof of Theorem 3.1.7

Throughout this section, let k, s denote two integers with $k \geq s+2$ and $0 \leq s \leq n-3$.

Lemma 3.2.1 *[7] Let G be a graph and $P = x_1 \cdots x_n$ be a Hamiltonian-path of G. If $d(x_1) + d(x_n) \geq n$, then G contains a Hamiltonian-circuit.*

Lemma 3.2.2 *Let G be a k-connected graph of order n, $S \subseteq V(G)$ be a vertex set of order s, $C = x_1 \cdots x_m x_1$ be a circuit of $G - S$ with $|V(C)| < n - s$ and H be a component of $G - S - V(C)$. Then $G - S$ contains a circuit C^* extending C, if one of the following holds:*
(i) there exist two distinct vertices $x_i, x_j \in V(C)$ with $x_{i+1}, x_{j+1} \in N_C^+(H)$ such that $d(x_{i+1}) \geq (n+s)/2$ and $d(x_{j+1}) \geq (n+s)/2$, or
(ii) there exists a vertex $x_{i+1} \in N_C^+(H)$ and a vertex $y \in V(H)$ such that $d(x_{i+1}) \geq (n+s)/2$ and $d(y) \geq (n+s)/2$.

Proof Since the proof when (ii) holds is similar to the proof when (i) holds, we only present the proof of the lemma assuming (i) holds. Let $x'_i, x'_j \in V(H)$ (possibly $x'_i = x'_j$) be such that $x'_i x_i, x'_j x_j \in E(G)$ and let P be an (x'_j, x'_i)-path in H. Then $G[V(C \cup P)]$ has a Hamiltonian-path $P^* = [x_{i+1}, x_j]P\ [x_i, x_1][x_m, x_{j+1}]$. Let $H' = G - V(S \cup C \cup H)$. If $N_{H'}(x_{i+1}) \cap N_{H'}(x_{j+1}) \neq \emptyset$, let $z \in N_{H'}(x_{i+1}) \cap N_{H'}(x_{j+1})$ and then $G - S$ has a circuit $C^* = z[x_{i+1}, x_j]P\ [x_i, x_1][x_m, x_{j+1}]z$ extending C. Now suppose that $N_{H'}(x_{i+1}) \cap N_{H'}(x_{j+1}) = \emptyset$ and so $d_{H'}(x_{i+1}) + d_{H'}(x_{j+1}) \leq |V(H')|$. If $N_{H-P}(x_{i+1}) \cup N_{H-P}(x_{j+1}) \neq \emptyset$, without loss of generality, let $y \in N_{H-P}(x_{i+1}) \cup N_{H-P}(x_{j+1})$ and $yx_{i+1} \in E(G)$ and let P'' be an (x'_i, y)-path in H. So $G - S$ has a circuit $C^* = x_i P''[x_{i+1}, x_m][x_1, x_i]$ extending C. Now we can suppose that $N_{H-P}(x_{i+1}) \cup N_{H-P}(x_{j+1}) = \emptyset$ and so $d_{H-P}(x_{i+1}) + d_{H-P}(x_{j+1}) = 0$. By (i) of Lemma 3.2.2, both $d(x_{i+1}) \geq (n+s)/2$ and $d(x_{j+1}) \geq (n+s)/2$. Thus,

$$\begin{aligned} d_{P^*}(x_{i+1}) + d_{P^*}(x_{j+1}) &= d(x_{i+1}) + d(x_{j+1}) \\ &\quad -(d_{S \cup H' \cup (H-P)}(x_{i+1}) + d_{S \cup H' \cup (H-P)}(x_{j+1})) \\ &\geq n + s - 2s - |V(H')| \geq |V(P^*)|. \end{aligned}$$

By Lemma 3.2.1, $G[V(C \cup P)]$ contains a Hamiltonian-circuit C^* extending C. □

Lemma 3.2.3 *Suppose that G satisfies the hypothesis of Theorem 3.1.7. Let $S \subseteq V(G)$ be a vertex set with $|S| = s' \leq s$, $C = x_1 \cdots x_m x_1$ be a longest circuit of $G - S$ with $|V(C)| < n - s'$ and H be a component of $G - S - V(C)$. Then*
(i) $|N_C(H)| \geq k - s$;
(ii) if $x \in V(H), x_i \in V(C)$ are such that $xx_i \in E(G)$, then $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) + s - 1$;
(iii) $d(x) \geq (n + s)/2$ for each $x \in V(H)$ with $|N_C(x)| \geq 1$.

Proof (i) Since $C = x_1 \cdots x_m x_1$ is a longest circuit of $G - S$ with $|V(C)| < n - s'$, it follows that $H \neq \emptyset$ and $V(C) - N_C(H) \neq \emptyset$. By the facts that $N_C(H) \cup S$ separates H and $G - H - (S \cup N_C(H))$ and that G is k-connected, we have $|N_C(H)| + |S| \geq k$ and so $|N_C(H)| \geq k - s' \geq k - s$.

(ii) By Lemma 3.1.10 (iii), $N_C^+(H) \cup \{x\}$ is an independent set and so $|N_C(H)| = |N_C^+(H)| \leq \alpha(G) - 1$. It follows that $1 \leq |N(x) \cap N(x_{i+1})| \leq |N_C(H) \cup S| \leq \alpha(G) + s' - 1 \leq \alpha(G) + s - 1$.

(iii) Suppose, to the contrary, that there exists an $x \in V(H)$ with $|N_C(x)| \geq 1$ and with $d(x) < (n + s)/2$. Let $x_i \in N_C(x)$. By Lemma 3.1.10 (iii) and by the fact that $|N_C^+(H)| = |N_C(H)| \geq k - s$, G has an independent set $J = J' \cup \{x\}$ of order $k - s$ with $x_{i+1} \in J' \subseteq N_C^+(H)$. By (ii), $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) + s - 1$. Hence by the hypothesis of Theorem 3.1.7 and by the fact that $d(x) < (n + s)/2$, there must exist an $x_{l+1} \in J'$ satisfying $d(x_{l+1}) \geq (n + s)/2$. By (i), $|N_C^+(H)| = |N_C(H)| \geq k - s \geq 2$, and so there exists an $x_{j+1} \in N_C^+(H) - \{x_{l+1}\}$. Since $x_{j+1} \in N_C^+(H)$, $x_j \in N_C(H)$ and we may assume $y \in V(H)$ with $yx_j \in E(G)$ (possible $y = x$). By (ii), we have $1 \leq |N(y) \cap N(x_{j+1})| \leq \alpha(G) + s - 1$. Similarly, G has an independent set $J_1 = J_1' \cup \{y\}$ of order $k - s$, where $x_{j+1} \in J_1' \subseteq N_C^+(H) - \{x_{l+1}\}$. By the hypothesis of Theorem 3.1.7, there exists a $z \in J_1$ such that $d(z) \geq (n+s)/2$. Consequently, either $z \in N_C^+(H)$, whence by Lemma 3.2.2 (i), $G - S$ has a circuit C^* extending C; or $z = y$, whence by Lemma 3.2.2 (ii), $G - S$ has a circuit C^* extending C. In either case, a contradiction to the assumption that C is a longest circuit of $G - S$ is obtained. □

Proof of Theorem 3.1.7 Let G be a graph satisfying the hypothesis of Theorem 3.1.7. Suppose, to the contrary, that G is not s-Hamiltonian. Then there exists a vertex set

$S \subseteq V(G)$ with $|S| = s' \leq s$ such that $G - S$ does not have a Hamiltonian-circuit. By the fact that $k - s' \geq k - s \geq 2$, $G - S$ is 2-connected. We may assume that

$$C = x_1 \cdots x_m x_1 \text{ is a longest circuit in } G - S. \tag{3.1}$$

Then $|V(C)| < n - s'$. Let H be a component of $G - S - V(C)$. By Lemma 3.2.3 (i), we have $|N_C(H)| \geq k - s \geq 2$. Choose $x_i, x_j \in N_C(H)$ to be such that

$$X \cap N_C(H) = \emptyset, \text{ and } |X| \text{ is minimum}, \tag{3.2}$$

where $X = \{x_{i+1}, \cdots, x_{j-1}\}$. Then $|X| > 0$. Otherwise, there exist $y_i, y_{i+1} \in V(H)$ such that $x_i y_i \in E(G), x_{i+1}y_{i+1} \in E(G)$ (y_i and y_{i+1} might be the same vertex). Let $P_H[y_i, y_{i+1}]$ be a (y_i, y_{i+1})-path in H. Then $C^* = [x_1, x_i]P_H[y_i, y_i + 1][x_{i+1}, x_m]x_1$ is a circuit extending C, contrary to (3.1). By Lemma 3.2.3 (iii), for each vertex $x \in V(H)$ with $|N_C(x)| \geq 1$, $d(x) \geq (n + s)/2$. Since $N(x) \cup \{x\} \subseteq V(H) \cup N_C(H) \cup S$ for each $x \in V(H)$, $|V(H)| + |N_C(H)| + |S| \geq (n + s)/2 + 1$, and then

$$|V(H)| + |N_C(H)| \geq \frac{n - s'}{2} + 1. \tag{3.3}$$

Claim 1. $G - S - V(C)$ has only one component $H = G - S - V(C)$ and $|X| < |V(H)|$.

Proof. Suppose, to the contrary, that $G - S - V(C)$ has at least two components. Assume that H is the component with the smallest order and let $H^* = G - S - V(C \cup H)$. Since $|V(H)|$ is minimized, $|V(H)| \leq |V(H^*)|$. It follows by (3.3) and $|N_C(H)| \geq 2$ that

$$\begin{aligned} |X| &\leq \frac{|V(C)| - |N_C(H)|}{|N_C(H)|} = \frac{n - |V(H^*)| - s' - (|V(H)| + |N_C(H)|)}{|N_C(H)|} \\ &\leq \frac{(n - s')/2 - 1 - |V(H^*)|}{|N_C(H)|} \leq \frac{|V(H)| + |N_C(H)| - 2 - |V(H^*)|}{|N_C(H)|} \\ &= \frac{|V(H)| - |V(H^*)|}{|N_C(H)|} + \frac{|N_C(H)| - 2}{|N_C(H)|}. \end{aligned} \tag{3.4}$$

Then as $|V(H)| \leq |V(H^*)|$, (4) implies $|X| < 1$, contrary to the fact that $|X| > 0$. Hence, H is the only component of $G - S - V(C)$. Since $|N_C(H)| \geq 2$, we have that $|X| < |V(H)|$.
□

Choose $x'_i, x'_j \in V(H)$ with $x_i x'_i \in E(G), x_j x'_j \in E(G)$ to be such that $|V(P')|$ is as large as possible, where P' is an (x'_i, x'_j)-path in H. Then $C' = [x_1, x_i]P'[x_j, x_m]x_1$ is a circuit such that

$$V(C) \setminus X \subseteq V(C') \text{ and } |V(C')| \text{ is maximized.} \tag{3.5}$$

By (3.5), C' is a longest path containing $V(C) \setminus X$ and so by applying Lemma 3.2.3 and the argument on C to C', it follows that $G - S - V(C')$ has only one component H' and that $H' = G[X \cup V(H - P')]$. By (3.2) and the fact that $|X| > 0$, $H - P' = \emptyset$. Otherwise, H' is connected while $G[X \cup (H - P')]$ is disconnected, a contradiction. Therefore P' is a path of order $|V(H)|$. By the fact that $|X| < |V(H)|$, we have $|V(C')| = |V(C)| - |X| + |V(H)| > |V(C)|$, contrary to (3.1). This completes the proof of Theorem 3.1.7. □

3.3 Proof of Theorem 3.1.8

Lemma 3.3.1 *Let G be a graph and $P = x_1 \cdots x_n$ be a Hamiltonian-path of G. If $d(x_1) + d(x_n) \geq n + 1$, then for any edge $e = x_i x_{i+1} \in E(P)$, G has a Hamiltonian-circuit C such that $e \in (C)$.*

Proof Let $T = \{x_j |\ x_1 x_{j+1} \in E, x_{j+1} \in V(P)\}$. Then

$$|T \cap N(x_n)| = |T| + |N(x_n)| - |T \cup N(x_n)| \geq n + 1 - (n - 1) = 2.$$

That means there exists $x_j \in T \cap N(x_n) - \{x_i\}$, and so G has a Hamiltonian-circuit $C = [x_1, x_j][x_n, x_{j+1}]x_1$. Clearly, $E(P) - \{x_j x_{j+1}\} \subseteq E(C)$, and so $e = x_i x_{i+1} \in E(C)$. Thus the lemma holds. □

Lemma 3.3.2 *Let G be a k-connected graph of order n, $S \subseteq V(G)$ be a vertex set with $|S| = s' \leq s$, $P = x_1 \cdots x_m$ be a path of $G - S$ with $|V(P)| < n - s$ and H be a component of $G - S - V(P)$. Then $G - S$ contains a path P^* extending P, if one of the following holds:*

(i) there exist two distinct vertices $x_i, x_j \in V(P)$ with x_{i+1}, x_{j+1} in $N_P^+(H)$ such that $d(x_{i+1}) \geq (n + s + 1)/2$ and $d(x_{j+1}) \geq (n + s + 1)/2$, or

(ii) there exists a vertex $x_{i+1} \in N_P^+(H)$ *and a vertex* $y \in V(H)$ *such that* $d(x_{i+1}) \geq (n+s+1)/2$ *and* $d(y) \geq (n+s+1)/2$.

Proof Since the proof when (ii) holds is similar to the proof when (i) holds, we shall only present the proof of the Lemma 3.3.2 assuming (i) holds. Let $x_i', x_j' \in V(H)$ with $x_i'x_i, x_j'x_j \in E(G)$ and let P' be an (x_j', x_i')-path in H. Define G_1 to be the graph obtained from G by adding a new edge x_1x_m if $x_1x_m \notin E(G)$ and to be G if $x_1x_m \in E(G)$. Then we have an (x_{i+1}, x_{j+1})-path $P_1 = [x_{i+1}, x_j]P'[x_i, x_1][x_m, x_{j+1}]$ with $V(P_1) = V(P) \cup V(P')$ in G_1. Moreover, x_1x_m is an edge of P_1. Let $H^* = G - V(S \cup P \cup H)$. If $N_{H^*}(x_{i+1}) \cap N_{H^*}(x_{j+1}) \neq \emptyset$, let $z \in N_{H^*}(x_{i+1}) \cap N_{H^*}(x_{j+1})$ and then $G[V(P_1) \cup \{z\}]$ has a Hamiltonian-circuit C such that $x_1x_m \in E(C)$. Therefore, $C - \{x_1x_m\}$ is an (x_1, x_m)-path in $G - S$ which extends P. Now suppose that $N_{H^*}(x_{i+1}) \cap N_{H^*}(x_{j+1}) = \emptyset$ and so we have $d_{H^*}(x_{i+1}) + d_{H^*}(x_{j+1}) \leq |V(H^*)|$. If $N_{H-P'}(x_{i+1}) \cup N_{H-P'}(x_{j+1}) \neq \emptyset$, without loss of generality, let $y \in N_{H-P'}(x_{i+1}) \cup N_{H-P'}(x_{j+1})$ and $yx_{i+1} \in E(G)$ and let P'' be an (x_i', y)-path in H. So $G - S$ has a path $P^* = [x_1, x_i]P''[x_{i+1}, x_m]$ extending P. Now we can suppose that $N_{H-P'}(x_{i+1}) \cup N_{H-P'}(x_{j+1}) = \emptyset$ and so $d_{H-P'}(x_{i+1}) + d_{H-P'}(x_{j+1}) = 0$. Since $d(x_{i+1}) \geq (n+s+1)/2$ and $d(x_{j+1}) \geq (n+s+1)/2$, we have

$$\begin{aligned} d_{P_1}(x_{i+1}) + d_{P_1}(x_{j+1}) &= d(x_{i+1}) + d(x_{j+1}) \\ &\quad -(d_{S\cup H^*\cup(H-P')}(x_{i+1}) + d_{S\cup H^*\cup(H-P')}(x_{j+1})) \\ &\geq n + s + 1 - 2s - |V(H^*)| \geq |V(P_1)| + 1. \end{aligned}$$

By Lemma 3.3.1, $G_1[V(P_1)]$ contains a Hamiltonian-circuit C such that $x_1x_m \in E(C)$, and then $C - \{x_1x_m\}$ is an (x_1, x_m)-path P^* in $G - S$ extending P. □

By a proof similar to that for Lemma 3.2.3, we obtain the following lemma.

Lemma 3.3.3 *Suppose that G satisfies the hypothesis of Theorem 3.1.8. Let* $S \subseteq V(G)$ *be a vertex set with* $|S| = s' \leq s$, $P = x_1 \cdots x_m$ *be a longest path of* $G - S$ *with* $|V(P)| < n - s'$ *and* H *be a component of* $G - S - V(P)$. *Then*

(i) $|N_P(H)| \geq k - s$;

(ii) if $x \in V(H), x_i \in V(P)$ *with* $xx_i \in E$, *then* $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) + s$;

(iii) $d(x) \geq (n+s+1)/2$ *for each* $x \in V(H)$ *with* $|N_P(x)| \geq 1$.

Proof of Theorem 3.1.8 Let G be a graph satisfying the hypothesis of Theorem 3.1.8. Suppose, to the contrary, that $G-S$ is not Hamiltonian-connected for some vertex set $S \subseteq V(G)$ with $|S| = s' \leq s$. Then there exists a pair of vertices, say x and y, such that $G-S$ does not have a Hamiltonian (x,y)-path. Since $k-s' \geq k-s \geq 3$, $G-S$ is 3-connected and we can choose

$$P = x_1x_2\cdots x_m \text{ to be a longest } (x,y)\text{-path in } G-S, \tag{3.6}$$

where $x = x_1, y = x_m$. Then $|V(P)| < n-s'$. Let H be a component of $G-S-V(P)$. By Lemma 3.3.3 (i), we have $|N_P(H)| \geq k-s \geq 3$. Choose $x_i, x_j \in N_P(H)$ to be such that

$$X \cap N_P(H) = \emptyset \text{ and } |X| \text{ is minimum,} \tag{3.7}$$

where $X = \{x_{i+1}, \cdots, x_{j-1}\}$. Then $|X| > 0$. Otherwise, there exist $y_i, y_{i+1} \in V(H)$ such that $x_iy_i \in E(G), x_{i+1}y_{i+1} \in E(G)$ (y_i and y_{i+1} might be the same vertex). Let $P_H[y_i, y_i+1]$ be a (y_i, y_{i+1})-path in H. Then $P^* = [x_1, x_i]P_H[y_i, y_{i+1}][x_{i+1}, x_m]$ is an (x_1, x_m)-path extending P, contrary to (3.6). By Lemma 3.3.3 (iii), for each vertex $x \in V(H)$ with $|N_C(x)| \geq 1$, $d(x) \geq (n+s+1)/2$. Since for each $x \in V(H)$, $N(x) \cup \{x\} \subseteq V(H) \cup N_P(H) \cup S$,

$$|V(H)| + |N_P(H)| \geq (n-s')/2 + 3/2. \tag{3.8}$$

By a proof similar to that for the Claim 1 in the proof of Theorem 1.7, we get the following.

Claim 2. $G-S-V(P)$ has only one component $H = G-S-V(P)$ and $|X| < |V(H)|$.

Choose $x'_i, x'_j \in V(H)$ with $x'_i, x'_j \in V(H)$ to be such that $|V(P')|$ is as large as possible, where P' is an (x'_i, x'_j)-path in H. Then $P^* = [x_1, x_i]P'[x_j, x_m]$ is a path such that

$$V(P) \setminus X \subseteq V(P^*) \text{ and } |V(P^*)| \text{ is maximized.} \tag{3.9}$$

By (3.9), P^* is a longest path containing $V(P)\backslash X$ and so by applying Lemma 3.3.3 and the argument on P to P^*, it follows that $G-S-V(P^*)$ has only one component H' and that $H' = G[X \cup V(H-P')]$. By (3.7) and the fact that $|X| > 0$, $H - P' = \emptyset$. Otherwise, H' is connected while $X \cup (H-P')$ is disconnected, a contradiction. Therefore, P' is a path of order $|V(H)|$. By the fact that $|X| < |V(H)|$, we have $|V(P^*)| = |V(P)| - |X| + |V(H)| > |V(P)|$, contrary to (3.6). This completes the proof of Theorem 3.1.8. $\square$

Chapter 4

Degree Sequence and Supereulerian Graphs

4.1 The Problem and the Main Results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex $v \in V(G)$ is called a *pendent vertex* if $d(v) = 1$ and denote the set of all pendent vertices of G by $D_1(G)$. An edge $e \in E(G)$ is called a *pendent edge* if one of its endpoints is a pendent vertex. If $v \in V(G)$, then $N(v) = \{u : uv \in E(G)\}$. If $T \subseteq V(G)$, then $N(T) = \{u \in V(G) \setminus T : uv \in E(G) \text{ and } v \in T\}$.

In [26], Zhang et al. proved the following theorem.

Theorem 4.1.1 *[26] Every bipartite graphic sequence with the minimum degree $\delta \geq 2$ has a realization that admits a nowhere-zero 4-flow.*

In this paper, we first get the following result.

Theorem 4.1.2 *If $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ is a nonincreasing sequences with $d_n \geq 2$, then d has a supereulerian realization.*

In [12], Jaeger proved the following result.

Theorem 4.1.3 *[12] Every supereulerian graph admits a nowhere-zero 4-flow.*

Combining Theorem 4.1.3, we get a result analogous to Theorem 4.1.1.

Theorem 4.1.4 *If* $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ *is a nonincreasing sequences with* $d_n \geq 2$*, then* d *has a realization that admits a nowhere-zero 4-flow.*

Furthermore, we get a result about line-hamiltonian sequence as follows.

Theorem 4.1.5 *If* $d = (d_1, d_2, \cdots, d_n)$ *is a nonincreasing graphic sequence with* $n \geq 3$*, then the following are equivalent.*

(i) d *is line-hamiltonian.*

(ii) $d \in \mathcal{G}$ *and either* $d_1 = n - 1$*, or*

$$\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2} (d_j - 2). \tag{4.1}$$

(iii) d *has a realization* G *such that* $G - D_1(G)$ *is supereulerian.*

4.2 Collapsible Sequences

Theorem 4.2.1 *Let* G *be a connected graph. Each of the following holds.*
(i) (Catlin, Corollary of Lemma 3, [3]) If H *is a collapsible subgraph of* G*, then* G *is collapsible if and only if* G/H *is collapsible.*
(ii) (Catlin, Corollary 1, [3]) If G *contains a spanning tree* T *such that each edge of* T *is contained in a collapsible subgraph of* G*, then* G *is collapsible.*
(iii) (Caltin, Theorem 7, [3]) C_2, K_3 *are collapsible.*
(iv) (Caltin, Theorem 2, [3]) If G *is collapsible, then* G *is supereulerian.*

Theorem 4.2.1(ii) and (iii) imply Corollary 4.2.2 (i); Theorem 4.2.1(i) and (iii) imply Corollary 4.2.2(ii).

Corollary 4.2.2 *(i) If every edge of a spanning tree of G lies in a K_3, then G is collapsible.*

(ii) If $G - v$ is collapsible and if v has degree at least 2 in G, then G is collapsible.

Corollary 4.2.3 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing graphic sequence with $d_1 = n - 1$ and $d_n \geq 2$, then every realization of d is collapsible.*

Proof. Let G be a realization of d with $N(v_1) = \{v_2, \cdots, v_n\}$ and let T be the spanning tree with $E(T) = \{v_1v_i : 2 \leq i \leq n\}$. Since $d_n \geq 2$ and $N(v_1) = \{v_2, \cdots, v_n\}$, for any $v_i \in \{v_1v_i : 2 \leq i \leq n\}$, there is $v_j \in \{v_1v_i : 2 \leq i \leq n\} \setminus \{v_i\}$ such that $v_iv_j \in E(G)$. So every edge of T lies in a K_3, and by Theorem 4.2.1(ii), G is collapsible.□

Lemma 4.2.4 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing graphic sequence with $d_3 = \cdots = d_n = 3$, then d is collapsible.*

Proof. Let v_1, v_2 be two vertices and let

$$S = \begin{cases} \{s_1, s_2, \cdots, s_{d_2}\} & : \text{ if } d_2 \text{ is even} \\ \{s_1, s_2, \cdots, s_{d_2-1}\} & : \text{ if } d_2 \text{ is odd} \end{cases}$$

be a set of vertices other than $\{v_1, v_2\}$ and let $T = \{t_1, t_2, \cdots, t_{d_1-d_2}\}$ be a set of $d_1 - d_2$ vertices other than $S \cup \{v_1, v_2\}$. Let H denote the graph obtained from $\{v_1, v_2\} \cup S \cup T$ by joining v_2 to each vertex of S and joining v_1 to each vertex of $S \cup T$ (if d_2 is odd, then we also join v_1 and v_2). Note that $d_H(v_1) = d_2 + d_1 - d_2 = d_1, d_H(v_2) = d_2$, $d_H(s) = 2$ for $s \in S$ and $d_H(t) = 1$ for $t \in T$.

Let $C = t_1t_2 \cdots t_{d_2-d_1}t_1$ be a cycle passing all vertices of T and let $H' = H \cup E(C)$. As $|S|$ is even, we join all vertices of S in pairs (i.e., $s_1s_2, s_3s_4, \cdots$) in H' and denote the resulting graph by H''. Note that $d_{H''}(v_1) = d_1, d_{H''}(v_2) = d_2$ and $d_{H''}(v) = 3$ for $v \in S \cup T$.

Also note that

$$|V(H'')| = \begin{cases} 2 + d_1 & : \text{ if } d_2 \text{ is even} \\ 1 + d_1 & : \text{ if } d_2 \text{ is odd.} \end{cases}$$

Let $m = n - |V(H'')|$ and so

$$m = \begin{cases} n - (2 + d_1) & : \text{ if } d_2 \text{ is even} \\ n - (1 + d_1) & : \text{ if } d_2 \text{ is odd} \end{cases}$$

is even. By the construction of H'', H'' contains a triangle $v_1 s_1 s_2$. We subdivide $v_1 s_1$ and $v_1 s_2$ $\frac{m}{2}$ times, respectively, and let $x_1, x_2, \cdots, x_{\frac{m}{2}}$ and $y_1, y_2, \cdots, y_{\frac{m}{2}}$ be the subdivision vertices of $v_1 s_1$ and $v_1 s_2$, respectively. Then for $1 \le j \le \frac{m}{2}$, we join $x_j y_j$ and denote the resulting graph by G (see Figure 1). Hence, by the construction of G, G is a realization of d.

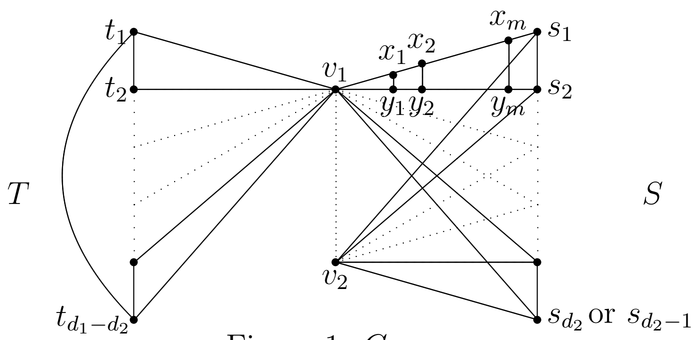

Figure 1: G

By Theorem 4.2.1(iii), K_3 is collapsible. If we contract $v_1 x_1 y_1$, then we get a triangle $v_1 x_2 y_2$ and if we contract $v_1 x_2 y_2$, then we get a triangle $v_1 x_3 y_3$ and so on until we get $v_1 s_1 s_2$. After contracting $v_1 t_1 t_2$ we get a graph in which each edge lies in a triangle. By Theorem 4.2.2(i), G is collapsible. □

Theorem 4.2.5 *(Ex. 1.5.7(a) on page 11, [2]) Let* $d = (d_1, d_2, \cdots, d_n)$ *be a nonincreasing sequence. Then* d *is graphic if and only if* $d' = (d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n)$ *is graphic.*

Lemma 4.2.6 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing sequence with $n \geq 4$ and $d_n = 3$, then d is graphic if and only if $d' = (d_1 - 1, d_2 - 1, d_3 - 1, d_4, \cdots, d_{n-1})$ is graphic.*

Proof. Let G be a realization of d with $d(v_i) = d_i$ for $1 \leq i \leq n$. If $N(v_n) = \{v_1, v_2, v_3\}$, then $G - v_n$ is a realization of d'. So it is suffices to prove the following claim.

Claim 1 *There is a realization G with $d(v_i) = d_i$ for $1 \leq i \leq n$ and $N(v_n) = \{v_1, v_2, v_3\}$.*

Proof. Choose G to be a realization of d such that $|N(v_n) \cap \{v_1, v_2, v_3\}|$ is as large as possible. If $|N(v_n) \cap \{v_1, v_2, v_3\}| = 3$, then we are done. Suppose that $|N(v_n) \cap \{v_1, v_2, v_3\}| < 3$. Then $v_n v_i \notin E(G)$ for some $i \in \{1, 2, 3\}$. As $d(v_n) = 3$, there exists $x \in N(v_n)$ such that $x \notin \{v_1, v_2, v_3\}$. Then there must exist $v_i' \in N(v_i)$ such that $v_i'x \notin E(G)$, otherwise $|N(x)| \geq |N(v_i) \cup \{v_n\}| = d_i + 1$, contrary to the fact that $d(x) \leq d_3 \leq d_i$. Let $G' = G - \{v_i v_i', v_n x\} + \{v_i v_n, v_i' x\}$. Then $|N_{G'}(v_n) \cap \{v_1, v_2, v_3\}| > |N_G(v_n) \cap \{v_1, v_2, v_3\}|$, contradicting the choice of G. □

Conversely, if G' is a realization of d', then can get a realization G of d by adding a new vertex u to G' and joining u to the vertices of degree $d_1 - 1, d_2 - 1, d_3 - 1$ in G', respectively. □

Theorem 4.2.7 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing graphic sequence with $n \geq 4$ and $d_n \geq 3$, then d has a collapsible realization.*

Proof. We argue by induction on n. First we assume that $n = 4$. Then the realization of d must be a K_4. By Theorem 4.2.1, K_4 is collapsible.

Next we assume that $n \geq 5$. If $d_n \geq 4$, then $d_2 - 1 \geq d_3 - 1 \geq \cdots \geq d_{d_1+1} - 1 \geq 3$ and $d_{d_1+2} \geq \cdots \geq d_n \geq 3$. By Theorem 4.2.5 and the induction hypothesis, $(d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1, d_{d_1+2}, \cdots, d_n)$ has a collapsible realization H and assume that $v_2, v_3, \cdots, v_{d_1+1}$ have degree $d_2 - 1, d_3 - 1, \cdots, d_{d_1+1} - 1$, respectively. Then we can get a realization H' of d from H by adding a new vertex v_1 and joining v_1 to $v_2, v_3, \cdots, v_{d_1+1}$, respectively. By Corollary 4.2.2(ii) H' is collapsible. Now we may assume that $d_n = 3$.

Case 1. If $d_3 = 3$, then by Lemma 4.2.4, $(d_1, d_2, 3, \cdots, 3)$ is collapsible.

Case 2. If $d_3 \geq 4$, then $d_1 - 1 \geq d_2 - 1 \geq d_3 - 1 \geq 3$ and $d_4 \geq \cdots \geq d_n = 3$. By Lemma 4.2.6, $(d_1 - 1, d_2 - 1, d_3 - 1, d_4, \cdots, d_{n-1})$ is graphic. By the induction hypothesis, $(d_1 - 1, d_2 - 1, d_3 - 1, d_4, \cdots, d_{n-1})$ has a collapsible realization K and assume that u_1, u_2, u_3 has degree $d_1 - 1, d_2 - 1, d_3 - 1$, respectively. Then we can get a realization K' of d from K by adding a new vertex u and joining u to u_1, u_2, u_3, respectively. By Corollary 4.2.2(ii) K' is collapsible. □

4.3 Supereulerian Sequence and Hamiltonian Line Graph

Lemma 4.3.1 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing graphic sequence with $d_n \geq 2$, then there exists a 2-edge-connected realization of d.*

Proof. Choose G to be a realization of d such that G has as few components as possible. Therefore,the following claim holds.

Claim 2 *G is connected.*

Proof. Suppose, to the contrary, that G has more than one components. Let G_1, G_2 be two components of G and $e_1 = u_1v_1 \in E(G_1), e_2 = u_2v_2 \in E(G_2)$. Then $G - \{e_1, e_2\} + \{u_1u_2, v_1v_2\}$ is a realization of d with fewer components than G, contradicting the choice of G. □

If $d_1 = d_2 = \cdots = d_n = 2$, then C_n is a 2-edge-connected realization of d. Now suppose that $d_1 > 2$. Then the following claim holds.

Claim 3 *There is a a 2-edge-connected realization of d.*

Choose G to be a realization of d with $\kappa'(G)$ as large as possible. By Claim 2, $\kappa'(G) \geq 1$. Suppose, to the contrary, that $\kappa'(G) = 1$ and furthermore, we can choose

G to be a realization of d with as few cut edges as possible. Let $e = uv$ be a cut edge such that one of the component G_1 of $G - e$ is 2-edge-connected. Assume $u \in V(G_1)$. Then $d(u) \geq 3$. Suppose that $uv \cdots w$ is a path of G such that the internal vertices on this path are of degree 2 and so $d(w) \geq 3$ (it is possible that $w = v$). Then there are $uu_1, uu_2 \in E(G_1)$ and $ww_1, ww_2 \in E(G_2)$. Now we can get G' from G by deleting uu_1, uu_2, ww_1, ww_2 and adding u_1w_1, u_2w_2, and get G'' from G' by first dividing u_1w_1 into u_1u' and $u'w_1$, dividing u_2w_2 into u_1w' and $w'w_2$ and then identifying u and u', w and w'. Then G'' is a realization of d with fewer cut edges than G, contradicting the choice of G. □

Lemma 4.3.2 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing sequence with $d_1 = n-1, d_n = 2$, then $d = (d_1, d_2, \cdots, d_n)$ is graphic if and only if (i) $d' = (d_1 - 1, d_2 - 1, \cdots, d_{n-1})$ is graphic when $d_2 \geq n-2$ and (ii) $d'' = (d_1 - 2, d_2, \cdots, d_{n-2})$ is graphic or $d''' = (d_1 - 1, d_2, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_{n-1})$ for some $d_i \geq 3$ is graphic when $d_2 \leq n-3$.*

Proof. Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing sequence with $d_1 = n-1, d_n = 2$. Suppose that $d = (d_1, d_2, \cdots, d_n)$ is graphic. We consider the following three cases.

Case 1. $d_2 = n - 1$.

Let G be a realization d with $d(v_i) = d_i$ for $1 \leq i \leq n$. Since $d_1 = d_2 = n-1$, $v_1v_n \in E(G)$ and $v_2v_n \in E(G)$ and so $G - v_n$ is a realization of $d' = (d_1 - 1, d_2 - 1, \cdots, d_{n-1})$.

Case 2. $d_2 = n - 2$.

In this case, the following claim holds.

Claim 4 *d has a realization G such that $N(v_n) = \{v_1, v_2\}$.*

Proof Let G be a realization of d. If $N(v_n) = \{v_1, v_2\}$, we are done. Otherwise, since $d(v_1) = n-1$, $N(v_n) = \{v_1, v_i\}$. Then $d(v_i) \leq n-2$ and there exists v_j such that $v_iv_j \notin E(G)$. Since $d(v_2) = n-2$ and $v_n \notin N(v_2)$, $v_j \neq v_2$ and $v_j \in N(v_2)$. So $G - \{v_2v_j, v_iv_n\} + \{v_2v_n, v_iv_j\}$ is a realization of d with $N(v_n) = \{v_1, v_2\}$. □

Let G be a realization of d with $d(v_i) = d_i$ for $1 \leq i \leq n$ and $N(v_n) = \{v_1, v_2\}$. Then $G - v_n$ is a realization of $d' = (d_1 - 1, d_2 - 1, \cdots, d_{n-1})$.

Case 3. $d_2 \leq n - 3$.

In this case, there exists a realization G of d with $N(v_n) = \{v_{n-1}, v_1\}, N(v_{n-1}) = \{v_1, v_n\}$ or $N(v_n) = \{v_1, v_i\}$ and $d(v_i) \geq 3$. In the former case, $G - \{v_n, v_{n-1}\}$ is a realization of $d'' = (d_1 - 2, d_2, \cdots, d_{n-2})$. In the latter case, $G - v_n$ is a realization of $d''' = (d_1 - 1, d_2, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_{n-1})$.

Conversely, if $d' = (d_1 - 1, d_2 - 1, \cdots, d_{n-1})$ is graphic, then there is a realization G' of d' and so $G' + \{v_1v_n, v_2v_n\}$ is a realization of d; if $d'' = (d_1 - 2, d_2, \cdots, d_{n-2})$ is graphic, then there is a realization G'' of d'' and so $G'' + \{v_1v_n, v_nv_{n-1}, v_{n-1}v_1\}$ is a a realization of d; if $d''' = (d_1 - 1, d_2, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_{n-1})$ is graphic, then there is a realization G''' of d''' and so $G''' + \{v_1v_n, v_iv_n\}$ is a realization of d. □

Lemma 4.3.3 *If $d = (d_1, d_2, \cdots, d_n)$ is a nonincreasing sequence with $d_1 \leq n - 2$ and $d_n = 2$, then $d = (d_1, d_2, \cdots, d_n)$ is graphic if and only if (i) $d' = (d_1, d_2, \cdots, d_{n-1})$ is graphic or $d'' = (d_1, d_2, \cdots, d_i - 1, \cdots, d_j - 1, \cdots, d_{n-1})$ for some $d_i \geq 3$ and $d_j \geq 3$.*

Proof. Let $d = (d_1, d_2, \cdots, d_n)$ be a nonincreasing sequence with $d_1 \leq n - 2$ and $d_n = 2$. Suppose that $d = (d_1, d_2, \cdots, d_n)$ is graphic. Then there exists a 2-edge-connected realization G of d with $d(v_i) = d_i$ for $1 \leq i \leq n$. Suppose that $N(v_n) = \{v_i, v_j\}$. If $v_iv_j \notin E(G)$, then $G - v_n + \{v_iv_j\}$ is a realization of $(d_1, d_2, \cdots, d_{n-1})$. Now suppose that $v_iv_j \in E(G)$ and we distinguish the following two cases:

Case 1: $\{v_n, v_i, v_j\} \cup N(v_i) \cup N(v_j) \neq V(G)$.

Let $T = V(G) \setminus (\{v_n, v_i, v_j\} \cup N(v_i) \cup N(v_j))$. If there is $v_s \in T$ such that $N(v_s) \cap (N(v_i) \triangle N(v_j)) \neq \emptyset$, then we assume that $v_t \in N(v_s) \cap (N(v_i) \setminus N(v_j))$. Now we can get a realization G' of $d' = (d_1, d_2, \cdots, d_{n-1})$ from G by deleting v_n, splitting v_sv_t to $v_sv_{j'}, v_{j'}v_{i'}, v_{i'}v_t$ and then identifying v_i and $v_{i'}$, v_j and $v_{j'}$. Otherwise, for any vertex $v \in T$, $N(v) \cap (N(v_i) \triangle N(v_j)) = \emptyset$, which implies $N(T) \subseteq N(v_i) \cap N(v_j)$. Since G is 2-edge-connected, then there are $v_p, v_q \in N(v_i) \cap N(v_j)$ (it is possible that $v_p = v_q$) and

$v_s, v_t \in T$ (it is possible that $v_s = v_t$) such that $v_s v_p, v_t v_q \in E(G)$. Then we can get a realization G' of $(d_1, d_2, \cdots, d_{n-1})$ from G by deleting v_n, splitting $v_s v_p$ into $v_s v_{i'}, v_{i' v_p}$, splitting $v_s v_q$ into $v_s v_{j'}, v_{j' v_q}$ and then identifying v_i and $v_{i'}$, v_j and $v_{j'}$.

Case 2: $\{v_n, v_i, v_j\} \cup N(v_i) \cup N(v_j) = V(G)$.

In this case, $d_i \geq 3$ and $d_j \geq 3$. Otherwise, $\triangle(G) = n-1$, a contradiction. So $G - v_n$ is a realization of $d'' = (d_1, d_2, \cdots, d_i - 1, \cdots, d_j - 1, \cdots, d_{n-1})$.

Conversely, if $d' = (d_1, d_2, \cdots, d_{n-1})$ is graphic, then there is a realization G' of $d'$and so we can get a realization G of d by choosing an edge $e = v_i v_j \in E(G')$ and dividing it into $v_i v_n$ and $v_n v_j$. If $d'' = (d_1, d_2, \cdots, d_i - 1, \cdots, d_j - 1, \cdots, d_{n-1})$ is graphic, then there is a realization G'' of d'' and so we can get a realization G of d by adding vertex v_n and edges $v_i v_n, v_j v_n$ to G''. □

Theorem 4.3.4 *If $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ is a nonincreasing sequences with $d_n \geq 2$, then d has a supereulerian realization.*

Proof of Theorem 4.1.2. By induction on n.

If $n = 3$, then $(2, 2, 2) \in \mathcal{G}$, K_3 is supereulerian.

Suppose the theorem holds for all nonincreasing graphic degree sequences with fewer than n entries. Let $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ be a nonincreasing sequences with $d_n \geq 2$. If $d_n \geq 3$, then by Theorem 4.2.7, d has a collapsible realization G. By Corollary 4.2.2 (iii), G is supereulerian. If $d_1 = d_2 = \cdots = d_n = 2$, then C_n is a supereulerian realization of d.

In the following, we assume that $d_1 > d_n = 2$. We consider two cases.

Case 1: $d_1 \leq n - 2$.

By Lemma 4.3.3, $d' = (d_1, d_2, \cdots, d_{n-1})$ is graphic or $d'' = (d_1, d_2, \cdots, d_i - 1, \cdots, d_j - 1, \cdots, d_{n-1})$ with $d_i \geq 3$ and $d_j \geq 3$ is graphic. If d' is graphic, by the induction hypothesis, there is a supereulerian realization G' of d'. Let C' be a spanning eulerian subgraph of G' and $e = uv$ be an edge of C'. Then by splitting e of G' into $uv_n, v_n v$, we get a supereulerian realization of d. If d'' with $d_i \geq 3$ and $d_j \geq 3$ is graphic, then by the induction hypothesis, there is a supereulerian realization G'' of d''. Let C'' be a spanning eulerian subgraph of G''. If $v_iv_j \in E(G'')$, then let $C_1 = v_iv_jv_n$ and so $G = G'' + \{v_iv_n, v_jv_n\}$ is a supereulerian realization of d. If $v_iv_j \notin E(G'')$, then we can get a realization G of d from $G'' + \{v_iv_j\}$ by splitting an edge $e = uv$ of C' into uv_n and v_nv.

Case 2: $d_1 = n - 1$.

By Lemma 4.3.2, $d' = (d_1 - 1, d_2 - 1, \cdots, d_{n-1})$ or $d'' = (d_1 - 2, d_2, \cdots, d_{n-3})$ or $d''' = (d_1 - 1, \cdots, d_{i-1}, d_i - 1, d_{i+1}, \cdots, d_{n-1})$ is graphic. If d'(or d'' or d''') is graphic, then by the inductive hypothesis, there is a supereulerian realization G' of d' (or d'' or d'''). Let C' be a spanning eulerian subgraph of G'. Let $C_1 = v_nv_1v_2v_n$ (or $C_1 = v_1v_nv_{n-1}v_1$ or $C_1 = v_1v_nv_iv_1$). Since $v_1v_2 \in E(G')$ (or $v_{n-1}, v_n \notin V(G')$ or $v_1v_i \in E(G')$), $G' \triangle C_1$ is a supereulerian realization of d with a spanning eulerian subgraph $C' \triangle C_1$. □

Note that if G is supereulerian, then $\delta(G) \geq 2$ and so $d_n \geq 2$, we have the following corollary.

Corollary 4.3.5 *If $d = (d_1, d_2, \cdots, d_n) \in \mathcal{G}$ is a nonincreasing sequences, then d has a supereulerian realization if and only if $d_n \geq 2$.*

Theorem 4.3.6 *(Harry and Nash-Williams, [11]) Let $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Proof of Theorem 4.1.5. $(i) \Rightarrow (ii)$. Let G be a realization of d such that $L(G)$ is hamiltonian. By Theorem 4.3.6, G has a dominating eulerian subgraph H. If $d_1 = n-1$ and $\sum_{d_i=1} d_i > \sum_{d_j \geq 2}(d_j - 2)$, then $G = K_{1,n-1}$. Assume that G is not $K_{1,n-1}$. Then H is nontrivial. For any v_i with $d(v_i) = 1$, v_i must be adjacent to a vertex v_j in H and so $d_{G-E(H)}(v_j)$ is no less than the number of degree 1 vertices adjacent to v_j. Furthermore, since H is eulerian and nontrivial, $d_H(v_j) \geq 2$ and so $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$.

$(ii) \Rightarrow (iii)$ Suppose $d \in \mathcal{G}$ is a nonincreasing sequence such that $d_n \geq 1$ and $\sum_{d_i=1} d_i \leq \sum_{d_j \geq 2}(d_j - 2)$. If $d_n \geq 2$, then by Theorem 4.3.4, d has a supereulerian realization. So we assume that $d_n = 1$.

Claim 5 *Any realization of d contains a cycle.*

Suppose that there exists a realization G of d such that G is a tree. We may assume that $d_i \geq 2$ for $1 \leq i \leq k$ and $d_j = 1$ for $k+1 \leq j \leq n$. Then

$$\sum_{i=1}^{k} d_i + (n-k) = \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} d_i = \sum_{i=1}^{n} d_i = 2|E(G)| = 2(n-1),$$

and so

$$\sum_{i=1}^{k} (d_i - 2) + (n-k) = 2(n-1) - 2k.$$

Hence

$$\sum_{d_j \geq 2} (d_j - 2) = \sum_{i=1}^{k} (d_i - 2) = 2(n-1) - 2k - (n-k) = n-k-2 < n-k = \sum_{d_i = 1} d_i,$$

contrary to (4.1). This completes Claim 5 and we assume G is a realization of d containing a nontrivial cycle C.

Claim 6 *There is a realization G of d such that $\delta(G - D_1(G)) \geq 2$.*

As G contains a nontrivial cycle C, $G - D_1(G)$ is not empty. Let $S = N(D_1(G))$. It suffices to show that for each $s \in S$, $N_{G-D_1(G)}(s) \geq 2$. Suppose, to the contrary, that there is $s \in S$ such that $N_{G-D_1(G)}(s) = 2$. Choose G to be a graph such that the elements in $P(G) = \{s : s \in S \text{ with } d_G(s) = d_t \geq 2 \text{ such that } N_{G-D_1(G)}(s) = 1\}$ is as few as possible. Let $x \in P(G)$. Then $x \notin C$. Choose $e \in E(C)$ and we subdivide e and let v_e denote the subdivision vertex. And we delete $d_t - 1$ pendent edges of x, add $d_t - 2$ pendent edges to v_e and denote the resulting graph G_x (Note that if $d_t - 2 = 0$, then we subdivide e without adding any pendent edges). So $d_{G_x}(v_e) = 2 + d_t - 2 = d_t$ and $|D_1(G_x)| = |(D_1(G) - N_1(x)) \cup \{x\}| + d_t - 2 = |D_1(G)| - (d_t - 1) + 1 + d_t - 2 = |D_1(G)|$ but $|P(G_x)| < |P(G)|$, contradicting the choice of G.

(iii) $\Rightarrow$ *(i)* If G is a realization of d such that $\delta(G - D_1(G))$ is supererulerian, then by Theorem 4.3.6, $L(G)$ is hamiltonian. □

Chapter 5

Regular Matroids without Disjoint Circuits

5.1 The Problem and the Main Results

If G is a graph and if V_1, V_2 are two disjoint vertex subsets of G, then $[V_1, V_2]$ denote the set of edges in G with one end in V_1 and the other end in V_2. For a vertex $v \in V(G)$, let

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v\}.$$

Let M and N denote two matroids. If $\{e, f\}$ is a circuit of M^* and if $M/f = N$, then M is a *serial extension* of N. In this case, we say that f is serial to e. Note that being serial is an equivalence relation on $E(M)$ for a matroid M. The corresponding equivalence classes are the *serial classes* of M. Dually, two elements e, f are *parallel* in M if they are serial in M^*; being parallel is an equivalence relation on $E(M)$ and the equivalence classes are the *parallel classes* of M. An equivalence class is *nontrivial* if it has more than one elements.

In 1960, Erdös and Pósa consider the problem of determining all connected graphs that do not have edge-disjoint circuits. We view the complete graph K_3 as a plane graph and let K_3^* denote the geometric dual of the plane graph K_3.

Theorem 5.1.1 *(Erdös and Pósa [8], also see Theorem 3.1, Theorem 3.2 of Bollobás [1]) Let G be a graph with $\delta(G) \geq 3$. The following are equivalent.*
(i) G does not have edge-disjoint circuits.
(ii) $G \in \{K_{3,3}, K_3^*, K_4\}$.

Since a graph G does not have disjoint circuits if and only if any subdivision of G does not have disjoint circuits, the following corollary follows immediately.

Corollary 5.1.2 *(Erdös and Pósa [8], also see Corollary 3.3 of Bollobás [1]) Let G be a simple graph of order $n \geq 3$.*
(i) If $|E(G)| \geq n + 4$, then G has 2 edge-disjoint circuits.
(ii) The graph G with $|E(G)| = n + 3$ does not have edge-disjoint circuits if and only if G can be obtained from a subdivision G_0 of $K_{3,3}$ by adding a forest and exactly one edge, joining each tree of the forest to G_0.

Theorem 5.1.1 can be viewed as a result on cosimple graphic matroids. Thus we consider generalizing Theorem 5.1.1 to matroids. Our main results of this note are the following.

Theorem 5.1.3 *Let M be a connected cosimple regular matroid. The following are equivalent.*
(i) M does not have disjoint circuits.
(ii) $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

Corollary 5.1.4 *Let M be a regular matroid. Then M has no disjoint circuits if and only if one of the following holds:*
(i) $M = U_{m,m}$, for some integer $m > 0$, or
(ii) M is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$, or
(iii) $M = M_1 \bigoplus M_2$ is the direct sum of two matroids M_1 and M_2, where M_1 is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$ and where $M_2 \cong U_{m,m}$, for some $m = |E(M)| - |E(M_1)| \geq 1$.

5.2 Proof of the Main Results

We follow Seymour [21] to introduce the notion of binary matroid sums. Given two sets X and Y, the symmetric difference of X and Y, is

$$X \Delta Y = (X \cup Y) - (X \cap Y).$$

Let M_1 and M_2 be two binary matroids where $E(M_1)$ and $E(M_2)$ may intersect. Define $M_1 \Delta M_2$ to be the binary matroid on $E = E(M_1) \Delta E(M_2)$ whose cycles are all subsets of E of the form $C_1 \Delta C_2$, where C_1 is a cycle of M_1 and C_2 is a cycle of M_2. The binary matroid sums are defined as follows.
(i) If $E(M_1) \cap E(M_2) = \emptyset$, then $M_1 \Delta M_2$ is the *1-sum* of M_1 and M_2(also referred as a direct sum).
(ii) If $E(M_1) \cap E(M_2) = \{e_0\}$, such that, for each $i \in \{1, 2\}$, the element e_0 is neither a loop nor a coloop of M_i, then $M_1 \Delta M_2$ is the *2-sum* of M_1 and M_2.
(iii) If $E(M_1) \cap E(M_2) = C$, where C is a 3-circuit of both M_1 and M_2, such that C includes no cocircuit of either M_1 or M_2, and such that for $i \in \{1, 2\}$, $|E(M_i)| \geq 7$, then $M_1 \Delta M_2$ is the *3-sum* of M_1 and M_2.

For $k = 1, 2, 3$, we also use $M_1 \bigoplus_k M_2$ to denote the k-sum of two matroids M_1 and M_2. If each of M_1 and M_2 is isomorphic to a proper minor of $M_1 \bigoplus_k M_2$, then we say that M is a *proper k-sum* of M_1 and M_2. For the case k=1, we also use $M_1 \bigoplus M_2$ for $M_1 \bigoplus_1$ to denote the direct sum of M_1 and M_2.

Let A denote the matrix below

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and let R_{10} denote the binary matroid $M_2[A]$.

Seymour's regular matroid decomposition theorem can be applied to cosimple matroids in the following form.

Theorem 5.2.1 *(Seymour [20]) Let M be a cosimple connected regular matroid. Then one of the following holds.*
(i) M is cosimple and graphic.
(ii) M is cosimple and cographic.
(iii) M is isomorphic to R_{10}.
(iv) For $i \in \{2,3\}$, $M = M_1 \bigoplus_k M_2$ is the proper 2-sum or 3-sum of two cosimple regular matroids M_1 and M_2, where both M_1 and M_2 are isomorphic to proper minors of M.

The following lemma is straightforward.

Lemma 5.2.2 *Let G be a graph. If $M(G)$ is cosimple, then $\delta(G) \geq 3$.*

Proof: Note that any edge incident with a degree 1 vertex in G must be a loop of $M^*(G)$, and that the edges incident with a degree 2 vertex in G must be in a parallel class of $M^*(G)$. Since $M(G)$ is cosimple, $M^*(G)$ does not have loops or nontrivial parallel classes. Hence we must have $\delta(G) \geq 3$. □

Proof of Theorem 5.1.3 We first show that Theorem 5.1.3(i) implies Theorem 5.1.3(ii), and so we assume the M is a connected cosimple regular matroid with no disjoint circuits. By Theorem 5.2.1, one of the conclusions in Theorem 5.2.1 must hold.

If M is graphic, then we may assume that for some connected graph G, $M = M(G)$. By Lemma 5.2.2, $\delta(G) \geq 3$. Since G has no disjoint circuits, by Theorem 5.1.1, $G \in \{K_{3,3}, K_3^*, K_4\}$, and so Theorem 5.1.3(ii) holds.

If M is cographic, then we may assume that for some graph G, $M = M^*(G)$, where G is a connected graph with $n = r(M) + 1$ vertices. Since M is cosimple, G is a simple graph, and so G is a spanning subgraph of K_n, the complete graph on n vertices. Let $V(G) = \{v_1, v_2, \cdots v_n\}$. If $G \neq K_n$, then we may assume that $v_1v_2 \notin E(G)$. In this case, $E_G(v_1) \cap E_G(v_2) = \emptyset$, contrary to Theorem 5.1.3(i). Therefore, we must have $G = K_n$, and so $M \in \{M^*(K_n), n \geq 3\}$.

If M is isomorphic to R_{10}, then it is well known that R_{10} is a disjoint union of a 4-circuit and a 6-circuit, contrary to Theorem 5.1.3(i). Thus $M \cong R_{10}$ is impossible.

Now suppose that Theorem 5.2.1(iv) holds. We argue by induction on $|E(M)|$. Since any matroid with at most 3 elements must be graphic, we assume that $|E(M)| = n \geq 4$, and Theorem 5.2.1(ii) holds for any matroid M satisfying Theorem 5.1.3(i) with $|E(M)| < n$.

Since Theorem 5.2.1(iv) holds, for some $i \in \{2, 3\}$, $M = M_1 \bigoplus_i M_2$ is the proper i-sum of two cosimple regular matroids M_1 and M_2, where both M_1 and M_2 are proper minors of M.

If one of M_1 or M_2 has two disjoint circuits, then by the definition of binary matroid sums, M would also have disjoint circuits, contrary to Theorem 5.1.3(i). Therefore, for each i, M_i does not have disjoint circuits. Since M_i is a proper minor of M, by induction, $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

If $i = 2$, then we may assume that $e_0 \in E(M_1) \cap E(M_2)$. By the definition of 2-sum and by the fact that $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$, $\exists C_1 \in \mathcal{C}(M_1)$ and $C_2 \in \mathcal{C}(M_2)$ such that $e_0 \notin C_i$. It follows that $C_1 \cap C_2 = \emptyset$ and so Theorem 5.1.3(i) is violated. Thus this is impossible.

Now assume that $i = 3$, and $Z = E(M_1) \cap E(M_2)$ is a 3 element circuit of both M_1 and M_2. Recall that $M_1, M_2 \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$. By the definition of a 3-sum, for any $i \in \{1, 2\}$, $|E(M_i)| \geq 7$ and so $M_i \notin \{M^*(K_3), M^*(K_4)\}$. Since there is no 3-circuits in either $M(K_{3,3})$ or a $M^*(K_n)$ with $n > 4$, it is impossible that both $|Z| = 3$ and $Z \in \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$. This contradiction shows that this case is also impossible.

Thus if Theorem 5.1.3(i) holds, then we must have $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$.

Conversely, suppose $M \in \{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$. Since $K_{3,3}$ is a bipartite simple graph, any circuit of $K_{3,3}$ has length at least 4. Suppose that $K_{3,3}$ has two disjoint circuits C_1 and C_2, then since $K_{3,3}$ is 3-regular, we must have $V(C_1) \cap V(C_2) = \emptyset$, and so $6 = |V(K_{3,3})| \geq |V(C_1)| + |V(C_2)| \geq 8$, a contradiction. Hence $M(K_{3,3})$ cannot have disjoint circuits. Suppose that $M = M^*(K_n), n \geq 3$ and write $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Suppose that C_1 and C_2 are two circuits of $M^*(K_n)$. Then C_1 is an edge cut of K_n and so $C_1 = [V_1, V_2]$, for some proper vertex subset $V_1 \subseteq V(G)$ and $V_2 = V(G) - V_1$.

Similarly, $C_2 = [W_1, W_2]$, where $\emptyset \neq W_1 \subseteq V(G)$ and $W_2 = V(G) - W_1 \neq \emptyset$. We may assume that $v_1 \in V_1 \cap W_1$. If $V_2 \cap W_2 \neq \emptyset$, say $v_2 \in V_2 \cap W_2$, then $v_1v_2 \in C_1 \cap C_2$. If $V_2 \cap W_2 = \emptyset$, then we have $W_2 \subseteq V_1, V_2 \subseteq W_1$. Since $\emptyset \neq [V_2, W_2] \subseteq [V_2, V_1] = C_1$ and $\emptyset \neq [V_2, W_2] \subseteq [W_1, W_2] = C_2$, then $C_1 \cap C_2 \neq \emptyset$. This proves that $M^*(K_n)$ does not have disjoint circuits. □

Proof of Corollary 5.1.4 It suffices to show, by induction on $|E(M)|$, that if M has no disjoint circuits, then one of (i), (ii) and (iii) holds. Let M be a regular matroid that does not have disjoint circuits.

We first assume that M is connected. If M has a loop or a coloop, then since M is connected, we must have $M \in \{U_{0,1}, U_{1,1}\}$, and so Corollary 5.1.4 (i) or (ii) must hold. Thus we assume that M is loopless and coloopless.

If M is connected and cosimple, then by Theorem 5.1.3, M is a member of $\{M(K_{3,3})\} \cup \{M^*(K_n), n \geq 3\}$ and so Corollary 5.1.4(ii) holds. Otherwise, M has nontrivial serial classes. Let $\{e_1, e_2\}$ be a pair of serial elements in M. Since the intersection of any circuit and any cocircuit in a matroid M cannot have exactly one element, any circuit in M containing e_1 must also contain e_2. This implies that M has no disjoint circuits if and only if M/e_2 has no disjoint circuits. By induction, M/e_2 is a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$. Since M is a serial extension of M/e_2, M is also a serial extension of a member in $\{M(K_{3,3}), U_{0,1}\} \cup \{M^*(K_n), n \geq 3\}$.

Now suppose that M is not connected. Then $M = M_1 \bigoplus M_2 \bigoplus \cdots \bigoplus M_k$, where $M_1, M_2, \cdots, M_k$ are connected components of M. If $\forall i$, M_i contains no circuits, then Corollary 5.1.4(i) holds. Otherwise, since M has no disjoint circuits, exactly one connected component, say M_1, has at least one circuit. It follows that $M_2 \bigoplus \cdots \bigoplus M_k \cong U_{n,n}$ and so Corollary 5.1.4 (iii) must hold. □

Bibliography

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press Inc., London, (1978).

[2] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, Macmillan, London and Elsevier, New York, 1976.

[3] P. A. Catlin, *A reduction method to find spanning eulerian subgraphs*, J. Graph Theory 12 (1988) 29-45.

[4] G. Chen, *Hamiltonian graphs involving neighborhood intersections*, Disc. Math. 112 (1993), 253-258 .

[5] G. Chen, Y. Egawa, X. Liu, and Saito, *Essential independent set and Hamiltonian cycles*, J. Graph Theory 21 (1996), 243-250.

[6] G. A. Dirc, *Some theorems on abstract graphs*, Proc. London Math. Soc. 2 (1952), 69-81.

[7] M. H. ElZahar, *On circuits in gaphs*, Disc. Math. 50 (1984), 227-230.

[8] P. Erdös and L. Pósa, *On independent circuits contained in a graph*, Canad. J. Math. (1965), 347-352.

[9] G. Fan, *New sufficient conditions for cycles in graphs*, J. Combin. Theory Ser. B 37 (1984), 221-227.

[10] R. J. Gould, *Advances on the Hamiltonian problem- A survey*, Graphs and Combinatorics 19 (2003), 7-52.

[11] F. Harary and C. St. J. A. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull. 8 (1965), 701-709.

[12] F. Jaeger, *On interval hypergraphs and nowhere-zero flow in graphs*, PResearch Report of Mathematics Aplication and Information, Universite Scientifique et Medicale et Institut National Polytechnique de Grenoble, No. 126, Juillet (1978).

[13] F. Jaeger, *A note on subeulerian graphs*, J. Graph Theory 3 (1979) 91-93.

[14] K. Zhao, H.-J. Lai and Y. Shao, *New sufficient condition for hamiltonian graphs*, Applied Math. Letters, to appear.

[15] K. Zhao, H.-J. Lai and J. Zhou, *Hamiltonian-connected graphs with large neighborhoods and degrees*, in submission.

[16] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory, 8(1984) 139-146.

[17] O. Ore, *Note on Hamiltonian circuits*, Amer. Math. Monthly 67 (1960), 55.

[18] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[19] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory Ser. B, 70 (1997), 217-224.

[20] P. D. Seymour, *Decomposition of regular matroids* J. Combin. Theory Ser. B 28 (1980), 305-359.

[21] P. D. Seymour, *Matroids and multicommodity flows*, European J. Combin. Theory Ser. B. 2 (1981), 257-290.

[22] Y. Shao, *Claw-Free Graphs and Line Graphs*, Ph. D. Dissertation, West Virginia University, (2005).

[23] C. Thomassen, *Reflections on graph theory*, J. Graph Theory, 10 (1986), 309-324.

[24] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, (1976).

[25] S. M. Zhan, *On hamiltonian line graphs and connectivity*, Discrete Math. 89 (1991) 89-95.

[26] R. Luo, W. A. Zang and C-Q Zhang, *Nowhere-zero 4-flows, simultaneous edge-colorings, and critical partial latin squares*, Combinatorica 24 (4) (2004), 641-657.