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Durability prediction of structural composites through a continuum damage mechanics approach

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Durability Prediction of Structural Composites through a Continuum Damage Mechanics Approach

Hermann E. Alcazar

Dissertation submitted to the
College of Engineering and Mineral Resources
at West Virginia University
in partial fulfillment of the requirements
for the degree of

**Doctor of Philosophy
in
Mechanical Engineering**

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ABSTRACT

Durability Prediction of Structural Composites through a Continuum Damage Mechanics Approach

Hermann E. Alcazar

The main objective of this research is to develop and implement a new analytical model that relies on a continuum-damage-mechanics approach to predict the evolution of plastic strain and mechanical damage until failure in a unidirectional fiber reinforced composite. The term “damage” is used only in the context of failure mechanisms associated with fracture, which are commonly associated with degradation in stiffness. Plastic and damage evolution are related to typical failure mechanisms in composite materials such as fiber, inter-fiber, and intra-fiber fracture.

The plastic strain surface is defined based on the Tsai-Wu failure criterion, while the stiffness degradation damage surface is defined based on the energy-release crack growth. The coefficients that characterize the damage and the plastic surfaces are obtained from known material properties. Data obtained from inter-fiber shear load/unload experimental results are used to define the plastic and damage anisotropic associative evolution. The plastic and damage thresholds are obtained by using nonlinear extrapolation. The mathematical equations and physical principles underlying this model are formulated in the tensorial three-dimensional space and tailored to the primary objective of modeling damage evolution.

This model is implemented as a new, user defined material in the commercial finite element analysis software ANSYS. The finite element results are validated by comparisons with published experimental data from shear load/unload in-plane tests, as well as with published experimental data from load/unload tension tests of a $[\pm 45^\circ]_{2S}$ composite laminate. The comparison shows a good correlation between the model predictions and the experimental data. Finally, the new material model is implemented in ANSYS to predict the durability of a composite beam subjected to four-point bending, where the evolution of fiber, inter-laminar, and intra-laminar types of damage is quantified.

DEDICATION

This dissertation is dedicated to my wife Patricia and daughter Andrea, for their sacrifice and unconditional support.

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LIST OF SYMBOLS, ABBREVIATIONS, AND NOMENCLATURE

Abbreviations

CDM	Continuum Damage Mechanics
NA	Neutral Axis
RVE	Representative Volume Element
w.r.t.	with respect to

List of symbols

$A^{(k)}$	k -th generalized thermodynamic force
$\mathbf{C}, C_{ijkl}$	4 th order stiffness tensor
D	Damage parameter
$\mathbf{D}, D_{ij}$	2 nd order anisotropic damage tensor
E	Stiffness, material property
F	Tsai-Wu parameters
G	Shear stiffness, material property
H	Damage surface parameters
I_{ij}	2 nd order identity symmetric tensor
$\mathbf{I}, I_{ijkl}$	4 th order identity symmetric tensor
$\mathbf{M}, M_{ijkl}$	4 th order effective damage tensor
R	Thermodynamic isotropic hardening plasticity force
$R, [R]$	Reuter 6x6 matrix
S_{ij}	Strength, material parameter
$\mathbf{Y}, Y_{ij}$	2 nd order thermodynamic damage force tensor
$a^{(k)}$	k -th generalized thermodynamic internal state variable
$\mathbf{d}, d_i$	Anisotropic damage referred to its principal directions, also damage eigenvalues
f	Yield thermodynamic function
f_{ij}, f_{ijkl}	Tensorial Tsai-Wu index failure polynomial coefficients
g	Yield surface function
h_{ijkl}	Tensorial damage surface coefficients
h	Constraint thermodynamic function (Chapter 3)
p	Isotropic hardening plastic evolution

q_i	Heat flux per unit area and unit time conducted into the body
s	Specific entropy
α, β	Material parameters used to define plastic and damage potentials as n -terms Prony or exponential series respectively
γ	Dissipative function (Chapter 3)
γ	Thermodynamic isotropic hardening damage force
δ	Isotropic hardening damage evolution
δ_{ij}	Kronecker delta
ϵ_α	Engineering strain
ϵ_{ij}	Eulerian strain, 2 nd order tensorial representation
θ	Temperature
λ	Lagrange multiplier
ν	Poisson ratio, material property
ρ	Specific density
σ_{ij}	Stress, 2 nd order tensorial representation
ψ	Specific Helmholtz free energy density potential
Ω_{ij}	2 nd order effective damage tensor

Nomenclature

$(\dot{})$	Referred to rate of change over time
$(\bar{})$	Referred to undamaged properties or configuration
$(\sim)$	Referred to ultimate or failure properties
$()_i$	1 st order tensor representation ($i = 1, 2, 3$)
$()_{ij}$	2 nd order tensor representation ($i, j = 1, 2, 3$)
$()_{ijkl}$	4 th order tensor representation ($i, j, k, l = 1, 2, 3$)
$()_\alpha$	Vector representation of 2 nd order tensor ($\alpha = 1, 2, \dots, 6$)
$()_{\alpha\beta}$	Matrix representation of 4 th order tensor ($\alpha, \beta = 1, 2, \dots, 6$)
$()^N$	Normal variable
$()^S$	Shear variable
$()_d$	Damage function or potential
$()_p$	Plastic function or potential
$()^d$	Damage variable
$()^e$	Elastic variable

- $()^i$ Inelastic variable
- $()^p$ Plastic variable
- $()_{Th-d}$ Damage threshold value
- $()_{Th-p}$ Plastic threshold value

- $()^{(local)}$ Local phenomena
- $()^{(heat)}$ Heat conduction phenomena

1. INTRODUCTION

1.1 Objective / Motivation

The use of composite materials is increasing rapidly. Traditional materials such as steel or aluminum are being supplanted by reinforced laminates called composites. Polymer matrix reinforced with carbon or glass fibers has excellent strength/weight ratios. A high strength to weight ratio is essential in aerospace applications. Figure 1-1 shows the extensive use of carbon fiber and carbon epoxy composites in the structure of the fighter planes F/A- 18C/D and F/A- E/F. Also, Boeing 777 is manufactured using a higher percentage of composite materials than their previous models.

Figure 1-1 Use of composites in aeronautical industry

Not only in aerospace industry is the use of composites increasing, but in other branches of transportation also; the evidence of this is clearly demonstrated in Figure 1-2 in the form of examples such as wheel hubs, chassis, pressure vessels, and other parts of the body. Other industries also forecast increased use.

The expected increase in the use of carbon fiber in various industries is depicted in Figure 1-3; the ones with the most potential are automobile and wind-energy industries.

Unlike conventional materials, composites offer an endless array of design variations; at the same time this flexibility is accompanied by complexities in modelling and analysis. The process of building a knowledge base pertaining to the behavior of composite materials is an ongoing endeavor. The present research attempts to add another piece to this giant puzzle. At the outset, a main objective and five associated objectives are presented.

- (a) Wheel hubs (<http://evilism.files.wordpress.com/2009/06/weds-sport-carbon-fiber-wheel.jpg>)
- (b) Chassis (<http://www.carbonfiberglass.com/wp-content/uploads/2009/03/3-aston-martin-one-77-carbon-fiber-frame.jpg>)
- (c) Pressure vessel (<http://www.myh2hummer.com/stills/hummerh2h-pics2.jpg>)
- (d) Drive shaft (http://www.cratemotordrivetrain.com/user/carbon_45.jpg)

Figure 1-2 Use of composites in automotive industry

The main objective is

- To develop a model that predicts the plastic and damage evolution until failure of a unidirectional fiber reinforced composite using a continuum damage mechanics approach, considering a 3D state of stresses and non-associated model.

The associated objectives are:

- To define a proper set of state variables, a specific Helmholtz free-energy potential, an associated plastic surface, and an associated damage surface.
- To define and evaluate the proper plastic and damage surface coefficients.
- To express explicitly all the variables that are used.
- To implement the model in Matlab and compare it with data obtained in the literature.
- To implement the model in ANSYS using a user-defined subroutine, and compare the results with existing data in the literature for several cases.

(<http://www.zoltek.com/carbonfiber/future.php>)

Figure 1-3 Projected carbon fiber production by Zoltek

1.2 Dissertation Outline

Chapter 1 presents the introduction and objectives. Chapter 2 deals with the literature survey summarizing research done in areas of failure of isotropic and composite materials and finally that of damage mechanics.

Chapter 3 explains in detail the thermomechanics formulation for small deformations.

Chapter 4 defines plasticity, damage, and finally a plasticity/damage failure mechanism. A step by step damage-plasticity definition of the Jacobian, discretization, and material parameter evaluation is made. The plastic and damage surface parameters are defined.

The model setup is discussed in detail in Chapter 5. The isotropic hardening evolution parameters for damage and plastic behavior are obtained from load/unload stress-strain inplane shear experiment data. The model is implemented in Matlab and preliminary results are obtained. A correction for the normal hardening damage parameters is analyzed, for load/unload stress-strain inplane shear experiment data.

Chapter 6 contains the results when the model is implemented inside ANSYS based on a userdefined Fortran subroutine. First an inplane shear analysis is made, then tension in the fiber, and tension in the transverse directions are considered. In all these three analysis critical values are obtained which are very similar to those used as input values. Then a tensile load in the x -direction of a $[\pm 45]_{2s}$ laminate made of carbon fiber composite material T300/914 is modeled and compared with experimental results from Ladavez and Dantec (1992). A four-point bending analysis of a $[0/90]_s$ laminate is also done, where quantitative damage values are obtained.

Finally the conclusions, contributions to the state to the art and recommendations are stated.

In the Appendix A a complete discussion of the compressed (or matrix) representation of second and fourth-order tensors is done. An explicit definition of all variables of the problem is given in Appendix B, while Appendix C shows a flow chart of the damage-plasticity model.

2. LITERATURE REVIEW – ASSESSMENT OF CURRENT KNOWLEDGE AND TOOLS

“Changes in the real world are always irreversible processes because of friction, which results in the production of entropy and thus a permanent change in the universe”- DeHoff (1993)

Materials in general usually fail under progressive deterioration, as a result of nucleation, growth, and coalescence of microdamages. Experimental observations have shown that microdamages have a tendency to form macroscopically localized damage, which is a precursor to failure. This progressive physical process of degradation in the mechanical properties up to complete failure is commonly referred to as damage.

Metallographic studies, Chaboche (2003), show that damage is basically characterized by three mechanisms of microdamage growth: (i) nucleation of microscopic voids that initiate at inclusions and second-phase particles and failure of particles or microcracking of the matrix surrounding the inclusion, (ii) growth of the microvoids by means of plastic strain and hydrostatic stress, and (iii) coalescence or microcracks linking the growing microvoids with adjacent ones, thus leading to complete failure of the material, as the damage density approaches unity (see Figure 2-1).

A damage parameter D is introduced, which represents the damage evolution. When $D = 0$ no damage is observed, then the damaged area and effective undamaged area (effective area without damage, defined extensively in 2.1) are the same. On the other hand, when $D = 1$ the effective area is zero, fracture of the representative volume element (RVE) occurs, Figure 2-1. It should be noted that the damage parameter goes from 0 (virgin material) to 1 (failure), $0 \leq D \leq 1$.

This evolution of damage is well known when compared to fracture mechanics and fatigue approaches, as seen in Figure 2-1. Some examples of damage in materials, as stated by Kachanov (1986), Lemaitre *et al.* (1990), and others in the literature, with some early damage morphologies are:

- *Creep damage*: At high temperatures and under the action of stress an accumulation and growth of microvoids in metal grains take place (ductile transgranular creep failure). At the same time there occurs an accumulation and growth of microcracks on intergranular boundaries (brittle intergranular creep failure).

- *Ductile Plastic Damage*: Nucleation and growth of microvoids and microcracks as the result of large plastic strains, with the process leading to plastic fracture.
- *Fatigue Damage*: Accumulation and growth of micro and macro cracks taking place under the action of cyclic loading (low or high cycle fatigue).
- *Others*, such as brittle damage, embrittlement of steels, chemical damage, environmental degradation, damage of concrete.

Figure 2-1 Damage in terms of crack evolution

Two types of constitutive relations are found in the literature to model damage, **microscopic** (void growth and coalescence) and **macroscopic** (phenomenological) approaches.

The former approach requires microscopic experimental studies to define the model's parameters and a transition from non-homogeneous microscopic material to macroscopic material, which is not always obvious. Here the void geometry is analyzed. Lee *et al.* (1997) considered elliptical microcracks, while Fan *et al.* (2004) considered circular voids. The distribution of the voids and cracks is also taken into account, Araki and Sato (2002), to relate damage with the void-crack nucleation and coalescence, Horstemeyer and Gokhale (1999). The statistical location of the voids and cracks is also considered.

Desrumaux *et al.* (2001) develop a model to predict the overall behavior of a random discontinuous fiber composite.

The second approach is based on phenomenological observations and thermodynamic considerations. It is motivated by microscopic considerations but not deduced from them; it only requires simple macroscopic experiments.

2.1 Various failure mechanisms – Effects leading to failure

A number of failure mechanisms have been studied and stated in the literature for lamina failure. In principle, these failure mechanisms can be divided in two areas, the one that takes into account failure modes and those that do not.

The most known failure criteria not associated with failure modes is proposed by Tsai and Wu (1971), which is an extension of the von Mises distortion energy theory for anisotropic materials with different compression and tension strengths.

The composite failure modes are generally identified as fiber tension breakage, fiber compression (kinking), tensile matrix cracking (intrafiber tensile), compressive matrix (intrafiber compression), and delamination (interlamina).

The maximum stress criterion and the maximum strain criterion are the simplest failure associated criteria, where the maximum strain or strength is compared with the actual strain or stress. Among several other failure theories have been defined considering the interaction among different stresses that act over the lamina.

Hashin (1973) and then Hashin (1980) associate the matrix failure with the transverse and inplane shear stresses. Puck (1998) and Puck (2002) establish fracture modes A, B, and C interfiber matrix failures. Mode A is related to the tension matrix failure and Modes B and C are related to a compression matrix failure. When the tension transverse is weighted with respect to the inplane shear, then Mode B would appear, otherwise Mode C will happen. The orientation of the fracture plane for Mode B is between 45° and 55° ($53^\circ \pm 2^\circ$, Davila *et al.* (2005)), while in Mode C it is 0° .

After the World Wide Failure Exercise, Davila *et al.* (2005) proposed six phenomenological failure criteria for fiber unidirectional composites denoted LaRC03. Continuing with this work, Pinho *et al.* (2005) proposed the LaRC04, that extends to 3-D stress states including shear non-linearity.

Extensive research has been performed in the analysis of interlamina damage or delamination. Camanho *et al.* (2001), Lonetti *et al.* (2003), Turon *et al.* (2007), Davila *et al.* (2008), and Bruno *et al.* (2008) are several approaches which study delamination.

2.2 Different approaches

Several continuum damage mechanics approaches have been developed, since the early work of Kachanov (1958) who define the first damage concepts. Initially these works referred to isotropic materials: Fung (1994), Krajcinovic (1996), Luccioni and Oller (2003), Lemaitre and Demorat (2005), and Voyiadjis and Kattan (2005). When anisotropic materials are considered, the same basic concept of progressive damage modeling is applied.

When the material is a fiber unidirectional reinforced composite, different approaches are considered. In essence, the concepts of thermomechanics or continuum mechanics are used with those related to anisotropic or orthotropic materials. The main goal is the basic definition of the damage algorithm and the treatment of the stress fields.

A simple plane-stress model has been proposed in the beginning by Ladeveze and Dantec (1992). This model considers plasticity and damage evolution as failure mechanisms.

Williams *et al.* (2003) predict damage growth based on the sub laminate response, taking into account of ply interactions through the stacking sequence. A plane-stress behavior is considered, while the potential damage function is assumed to be an equivalent strain function of local strain components.

Liu and Zheng (2008) model a progressive failure analysis in cylindrical coordinates, considering a damage surface limited by the quadratic Tsai-Wu failure criterion. Normal, transverse, and shear failure modes are defined here. The Helmholtz free energy is defined for elastic and damage hardening.

The LaC04 failure criterion is used as failure limits by Maimi *et al.* (2007). Damage evolutions in the longitudinal, transverse, and shear are considered. Also four failures modes (fiber tension/compression and matrix tension/compression) are used as damage activation functions. The implementation of this model is stated in the second part of this paper in Maimi *et al.* (2007).

Damage evolution in fiber reinforced laminates, also are studied by Schuecker and Pettermann (2006). They present an intralaminar progressive damage modeling without considering plastic evolution. The main difference in this approach is the use of the Puck failure criterion, being able to divide several failure characteristics in unidirectional composites. Schuecker and Pettermann (2008), extend their previous work to take into account inplane shear plastic evolution.

Inplane plastic and damage evolution laws are studied by Edlung and Volgers (2004). They consider a plane-stress analysis, with damage evolution in the longitudinal, transverse, and shear directions. Also a plane stress model is identified by Lapczyk and Hurtado (2007), where four failure modes (fiber tension/compression and matrix tension/compression) are modeled separately using the Hahn criterion as surface damage

limits. No plasticity is considered and the Gibbs free energy is used as the thermodynamic potential.

Barbero and Lonetti (2001) used the Tsai-Wu failure criteria as limiting damage case for only damage evolution in three orthotropic directions for a general 3D case. Barbero and Lonetti (2002) extended this case for plastic evolution. This model is implanted into a 3D finite element model based on the first-order shear-deformation theory by Robbins, *et al.* (2005). A more detailed formulation is made in Barbero (2008) for an associated plastic and damage evolution for a complete 3D case.

3. THERMOMECHANICS FORMULATION

“Thermodynamics is the study of the macroscopic consequences of myriads of atomic coordinates, which by virtue of the statistical averaging, do not appear explicitly in a macroscopic description of a system”- Callen (1960)

3.1 Classical Thermodynamics

Thermodynamics deals with systems which are at rest from a phenomenological or macroscopic point of view. Let the state of such a system be described by a set of mutually independent *kinematic variables, coordinates, or parameters* $a^{(k)}$ ($k = 0, 1, \dots, n$) [Coleman and Gurtin (1967), Germain *et al* (1983), and Ziegler (1983)]. The parameter $a^{(k)}$ could be a scalar, vector, tensor, or function as defined by the problem to be analyzed.

The *first fundamental law of thermodynamics*, often called the law of conservation of energy, states that there is a state function called internal energy, defined in terms of the kinematic variables $U(a^{(0)}, a^{(1)}, \dots, a^{(n)})$, for a closed system

$$dU = \delta Q + \delta W \quad (3.1)$$

where dU is total differential increase of the internal energy of the system, δQ is an infinitesimal amount of heat supplied to the system, and δW is the infinitesimal elementary work done on the system. The use of δ denotes an inexact differential, i.e., there is no function Q or W that can be differentiated to yield δQ or δW .

The term state function is commonly defined as a property of the system that depends on the current state, that is dependent of the mutually independent kinematic variables $\{a^{(0)}, a^{(1)}, \dots, a^{(n)}\}$. This state function is path-independent or not dependent on the way the system has reached the current state and describes quantitatively the equilibrium state of thermodynamic systems. Also it should be pointed out that state functions are extensive quantities (i.e., a physical quantity whose values are proportional to the size of the system it describes).

Rudolf Clausius formulated in 1865 the *second fundamental law of thermodynamics* and coined the term entropy (from the Greek *entropē*, change). He stated that entropy, a measure of the amount of energy which is unavailable to do work, of any isolated thermodynamic system tends to increase over time, approaching a maximum value. Entropy is an extensive state function, whose change between two equilibrium states dS could be determined quantitatively by the transferred reversible heat δQ and the absolute temperature θ , as

$$dS = \frac{\delta Q}{\theta} \quad (3.2)$$

When a mechanical system is considered, a small increment in energy or mechanical work is the product of a force times a small displacement, when both are measured in the same direction. An extension of this concept is applied to thermodynamics, so an increment in the total change of energy or work can be expressed as the sum of the products of certain *generalized forces* $A^{(k)}$, and their corresponding small *generalized displacements* $a^{(k)}$. Each pair of generalized force and associated displacement is called *conjugate variables*, where intensive and extensive quantities are considered for generalized forces and displacements correspondingly. Then the infinitesimal elementary work done on the system is of the form

$$\delta W = A^{(k)} da^{(k)} \quad (3.3)$$

wherein the summation convention is implied. All work functions are path-dependent, so from the thermodynamics point of view they are not state functions. It is a matter of experience when various work effects should be taken into account. In some cases their contribution may be so small as to be neglected. Table 3-1 shows various work interactions in terms of the intensive and extensive factors, as well as the overall differential expressions.

Table 3-1 Generalized work interactions

System	Generalized force $A^{(k)}$	Generalized displacement $a^{(k)}$	Work [W]
Mechanical	Force, F [N]	Displacement, s [m]	$F ds$
Fluid mechanics	Pressure, p [Pa]	Volume, V [m ³]	$-p dV$
Elasticity	Stress, σ_{ij} [Pa]	Strain, ε_{ij}	$V_0 \sigma_{ij} d\varepsilon_{ij}$
Capacitor	Electric field, E_i	Electric dipole moment, P_i	$VE_i dP_i$
Magnetic	Magnetic field strength, H_i	Magnetization, M_i	$\mu_0 H_i dM_i$

Maugin (1999) expresses the Corollary of Carnot's theorem where a thermodynamic system can always be described by the state variables $\{a^{(0)} = S, a^{(1)}, \dots, a^{(n)}\}$ in such a

way that Eqs. (1.2) and (1.3) are considered. Then the internal energy is a function of the kinematic variables $U(S, a^{(1)}, \dots, a^{(n)})$, so when the differential is performed the following relation is obtained

$$dU = \frac{\partial U}{\partial S} dS + \sum_{k=1}^n \frac{\partial U}{\partial a^{(k)}} da^{(k)} \quad (3.4)$$

When Eqs. (3.2) and (3.3) are substituted into Eq. (3.1), we get

$$dU = \theta dS + \sum_{k=1}^n A^{(k)} da^{(k)} \quad (3.5)$$

The comparison of Eqs. (3.4) and (3.5) gives

$$\theta = \frac{\partial U}{\partial S}, \quad A^{(k)} = \frac{\partial U}{\partial a^{(k)}}, \quad k = 1, 2, \dots, n \quad (3.6)$$

in such a manner that θ and $A^{(k)}$ are positively homogeneous functions of degree zero, with both as intensive variables of state.

3.2 Internal State Variables

3.2.1 Classical thermodynamics

Classical thermodynamics makes use of four basic quantities, two intensive variables (pressure and temperature) and two extensive variables (volume and entropy). Temperature and entropy are considered conjugate variables as defined earlier. In classical thermodynamics, pressure and volume are also conjugate variables, as seen in Table 3-2.

Table 3-2 Conjugate thermodynamic variables

Generalized force (<i>intensive</i> variable)	Generalized displacement (<i>extensive</i> variable)
Pressure, p	Volume [*] , V
Temperature, θ	Entropy [*] , S

Four quantities called thermodynamic potentials are well defined state functions and used in the classical thermodynamics to characterize various macroscopic systems. They are

the internal energy u , Helmholtz free energy ψ , enthalpy h , and Gibbs free energy g , with all four having units of energy. All four are related through the Legendre transformation, which, for a general case, describes a convex function using a different set of variables, namely the gradients of the original function. Several authors such as Housby and Puzrin (2000), Gelfand and Fomin (1963), and Maugin and Morro (1989) define this transformation. So, given the function $A(x^{(1)}, \dots, x^{(m)}; \alpha^{(1)}, \dots, \alpha^{(n)})$ it can be transformed to its dual function $B(x^{(1)}, \dots, x^{(m)}; \beta^{(1)}, \dots, \beta^{(n)})$ using

$$B(x^{(1)}, \dots, x^{(m)}; \beta^{(1)}, \dots, \beta^{(n)}) = \min \left(\pm A(x^{(1)}, \dots, x^{(m)}; \alpha^{(1)}, \dots, \alpha^{(n)}) \mp \sum_{k=1}^n \alpha^{(k)} \beta^{(k)} \right) \quad (3.7)$$

where,

$$\beta^{(k)} = \frac{\partial A}{\partial \alpha^{(k)}}, \quad \alpha^{(k)} = \mp \frac{\partial B}{\partial \beta^{(k)}}, \quad k = 1, 2, \dots, n \quad (3.8)$$

and,

$$X^{(i)} = \frac{\partial A}{\partial x^{(i)}} = \pm \frac{\partial B}{\partial x^{(i)}} \quad i = 1, 2, \dots, m \quad (3.9)$$

The variables $x^{(i)}$ play a passive role in this transformation and are treated as constant parameters. In the case of thermodynamics the variables $\alpha^{(k)}$ and $\beta^{(k)}$ are conjugate variables, as $x^{(i)}$ and $X^{(i)}$. The use of this dual function provides the most convenient description of the system in study, i.e., the one that best determines the stability of a system and how the system evolves towards equilibrium.

Table 3-3 Thermodynamic potentials in classical thermodynamics

	Internal energy	Helmholtz free energy	Enthalpy	Gibbs free energy
<i>Natural variables</i>	$U = U(V, S)$	$\Psi = \Psi(V, \theta)$ $\Psi = U - S\theta$	$H = H(p, S)$ $H = U + pV$	$Z = Z(p, \theta)$ $Z = H - S\theta = \Psi + pV$
	$p = -\frac{\partial U}{\partial V}$	$p = -\frac{\partial \Psi}{\partial V}$	$V = \frac{\partial H}{\partial p}$	$V = \frac{\partial Z}{\partial p}$
	$\theta = \frac{\partial U}{\partial S}$	$S = -\frac{\partial \Psi}{\partial \theta}$	$\theta = \frac{\partial H}{\partial S}$	$S = -\frac{\partial Z}{\partial \theta}$

The thermodynamic potentials in classical thermodynamics are expressed as functions of so-called natural variables, as expressed in Table 3-3. When the work differential equation for pressure and volume is taken as $-p dV$, as expressed in Table 3-1, Maxwell's relations arise from further partial differentiation of expressions in the last row of Table 3-3; they can also be defined using a Legendre theorem as expressed above.

For an ideal gas, considering a homogeneous, quasi-static, and reversible closed system process, the elementary work done when a pressure produces a change in volume (neglecting all other electrical, chemical, etc. processes) is defined as (see Figure 3-1)

$$\delta W = -p dV \quad (3.10)$$

Figure 3-1 Ideal gas inside a cylinder

When Eqs. (3.2) and (3.10) are substituted into Eq. (3.1), the differential internal energy is then defined as

$$dU = -p dV + \theta dS \quad (3.11)$$

and when divided by its mass, the specific internal energy is derived as

$$\begin{aligned} \frac{dU}{m} &= -\frac{p dV}{m} + \frac{\theta dS}{m} \\ du &= -p dv + \theta ds \end{aligned} \quad (3.12)$$

The specific internal energy defined as a function of the specific volume and the specific entropy, $u(v, s)$, its derivative could be found using the chain rule:

$$du(v, s) = \frac{\partial u}{\partial v} dv + \frac{\partial u}{\partial s} ds \quad (3.13)$$

3.2.2 Uniaxial strains

It has been seen previously that the increase in the internal energy is related to the pressure and change in volume. In the case of solids, this relation is between stress and the change in strain and known as *strain energy*.

An elemental definition of the strain energy is that it is the increase in energy associated with the deformation of a bar subjected to a tensile force, Beer *et al.* (2006). For the case of a bar under tension, Figure 3-2, the elementary work is equal to the load F times a small (differential) change in the elongation $d(L\varepsilon)$, or

$$\delta W = -F d(L\varepsilon) \quad (3.14)$$

Figure 3-2 Bar under tension

Following the same procedure as in the case of ideal gases, the specific internal energy is then found to be

$$\begin{aligned} du &= \frac{F d(L\varepsilon)}{m} + \theta ds \\ &= \frac{F d(L\varepsilon)}{\rho V} + \theta ds \\ &= \frac{1}{\rho} \frac{F}{A} \frac{d(L\varepsilon)}{L} + \theta ds \\ du &= \frac{1}{\rho} \sigma d\varepsilon + \theta ds \end{aligned} \quad (3.15)$$

or, when the specific internal energy is defined as a function of the specific variables volume and entropy, $u(\varepsilon, s)$, it follows that its Legendre relations are

$$\begin{cases} \sigma = \rho \frac{\partial u(\varepsilon, s)}{\partial \varepsilon} \\ \theta = \frac{\partial u(\varepsilon, s)}{\partial s} \end{cases} \quad (3.16)$$

3.2.3 Small deformations in continuum mechanics

A more general case must consider the balance of some physical quantities. These balance laws are related to the conservation of mass, linear momentum and angular

momentum, balance of energy, and entropy generation. The balances are defined initially in the context of global relationships and develop into field equations [Chadwick (1976), Spencer (1983), Marsden and Hughes (1983), Fung (1994), Mase and Mase (1999), and Reddy (2002)].

In order to obtain these field equations from its balance law, certain considerations must be taken. Only small deformations are to be considered from here on, so current, spatial, or Eulerian configuration is to be used. That is, the Lagrange finite strain tensor and the Eulerian strain tensor are assumed the same and equal to a single infinitesimal strain tensor ε_{ij} defined as,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3.17)$$

The physical quantities under consideration are those that act over the continuum, some over the surface dS (defined by its normal n_i) and others through the volume dV , as seen in Figure 3-3. The mechanical effects are the internal forces per unit mass b_i and the surface tractions $t_i^{(\hat{n})}$, that act over the volume and surface respectively. The heat supply, per unit mass r is produced by internal sources which act through the volume; while the heat flux per unit area and per unit time q_i conducted into the body act over the surface when thermal effects are considered. The density of the continuum ρ is also considered and defined through the volume and per unit volume. It will be considered that all quantities are function of time and position (t, x_i) and constant over the differential element.

Figure 3-3 Mechanical and thermal effects over a continuum

Conservation of mass: The mass of the body remains constant in time, invariant under motion, regardless of the used configuration, so

$$\begin{aligned}\dot{m} &= \frac{d}{dt} \int_V \rho dV = 0 \\ \Rightarrow \quad \dot{\rho} + \rho v_{i,i} &= 0 \quad (\text{continuity equation})\end{aligned} \quad (3.18)$$

Balance of linear momentum: The rate of change of the linear momentum equals the net force (surface and body forces) exerted over the body, so

$$\begin{aligned}\frac{d}{dt} \int_V \rho v_i dV &= \int_V \rho b_i dV + \oint_S \rho t_i^{(\hat{n})} dS \\ \Rightarrow \quad \sigma_{ji,j} + \rho b_i &= \rho \dot{v}_i \quad (\text{Equation of motion})\end{aligned} \quad (3.19)$$

Balance of moment of momentum: The rate of change of the moment of momentum with respect to any given point equals the moment exerted by the surface and body forces with respect the same point, then

$$\begin{aligned}\frac{d}{dt} \int_V \varepsilon_{ijk} x_j \rho v_k dV &= \int_V \varepsilon_{ijk} x_j \rho b_k dV + \oint_S \varepsilon_{ijk} x_j \rho t_k^{(\hat{n})} dS \\ \Rightarrow \quad \sigma_{ij} &= \sigma_{ji}\end{aligned} \quad (3.20)$$

First law of thermodynamics: For this effect, a balance of energy is performed for only mechanical and thermal considerations, not including electrical, chemical, magnetic, or other energies. So, the change of total energy of a material region or the sum of rates of the specific internal energy and specific kinetic energy is equal to the sum of rates of power of work of the mechanical power and thermal energy.

$$\begin{aligned}\frac{d}{dt} \int_V \rho \left(u + \frac{1}{2} v_i v_i \right) dV &= \int_V \rho (b_i v_i + r) dV + \oint_S \rho (t_i^{(\hat{n})} v_i - q_i n_i) dS \\ \Rightarrow \quad \rho \dot{u} - \sigma_{ij} \dot{\varepsilon}_{ij} - \rho r + q_{i,i} &= 0 \quad (\text{Energy equation})\end{aligned} \quad (3.21)$$

The rate of work is due to the internal forces per unit mass b_i and the surface tractions $t_i^{(\hat{n})}$, right side of Eq. (3.21). The heat supply per unit mass r produced by internal sources and the heat flux per unit area and per unit time q_i conducted into the body are acting over the volume and surface respectively, defining the rate of thermal energy. These effects of mechanical and thermal sources can be observed in Figure 3-3.

Second law of thermodynamics: The rate of change of entropy or entropy production is equal or bigger than the change in entropy due to the heat produced by internal sources and flux entering the continuum over its absolute temperature [see Eq. (3.2)], then

$$\frac{d}{dt} \int_V \rho s dV \geq \int_V \frac{\rho r}{\theta} dV - \oint_S \rho \frac{q_i n_i}{\theta} dS$$

$$\Rightarrow \quad \rho \theta \dot{s} - \rho r + q_{i,i} - q_i \frac{\theta_{,i}}{\theta} \geq 0 \quad (\text{Clausius-Duhem equation}) \quad (3.22)$$

Collins and Houlsby (1997), Housby and Puzrin (2000), and Basaran and Nie (2004) observe a dissipation process from the Clausius-Duhem equation, Eq. (3.22), whenever an irreversible process is taken into account. Additionally, since each dissipation must be greater than zero, then it is widely accepted that each dissipation component must be also non-negative, or

$$\begin{cases} \rho \theta \dot{s} - \rho r + q_{i,i} \geq 0 \\ -q_i \frac{\theta_{,i}}{\theta} \geq 0 \end{cases} \quad (3.23)$$

When a reversible process is taken into account, no dissipation is expected. Then, the dissipation observed in Eq. (3.23a) should be zero, and employing the definition of the rate of internal energy, Eq. (3.21), then

$$\begin{aligned} \rho \theta \dot{s} - \rho r + q_{i,i} &= 0 \\ \rho \theta \dot{s} - \rho \dot{u} + \sigma_{ij} \dot{\varepsilon}_{ij} &= 0 \\ \dot{u} &= \frac{\sigma_{ij}}{\rho} \dot{\varepsilon}_{ij} + \theta \dot{s} \end{aligned} \quad (3.24)$$

Table 3-4 Specific thermodynamic potentials for small deformations

Internal energy	Helmholtz free energy	Enthalpy	Gibbs free energy
$u = u(\varepsilon_{ij}, s)$	$\psi = \psi(\varepsilon_{ij}, \theta)$ $\psi = u - s\theta$	$h = h(\sigma_{ij}, s)$ $h = u - \sigma_{ij}\varepsilon_{ij}$	$\zeta = \zeta(\sigma_{ij}, \theta)$ $g = h - s\theta$ $= \psi - \sigma_{ij}\varepsilon_{ij}$
$\sigma_{ij} = \rho \frac{\partial u}{\partial \varepsilon_{ij}}$	$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}}$	$\varepsilon_{ij} = \rho \frac{\partial h}{\partial \sigma_{ij}}$	$\varepsilon_{ij} = \rho \frac{\partial \zeta}{\partial \sigma_{ij}}$
$\theta = \frac{\partial u}{\partial s}$	$s = -\frac{\partial \psi}{\partial \theta}$	$\theta = \frac{\partial h}{\partial s}$	$s = -\frac{\partial \zeta}{\partial \theta}$

So, when the internal energy is defined with respect to the internal state variables, strain, and entropy, $u(\varepsilon, s)$, then its derivative is as follows

$$du(\varepsilon_{ij}, s) = \frac{\partial u}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial u}{\partial s} ds \quad (3.25)$$

of the previous concepts are employed in order to include some generalized state variables. First, the additive decomposition of strain is considered, this is done in order to define the internal energy as a function of the total strain. So from Eq. (3.26) the elastic strain is a function of the total and plastic strain, then the internal energy would be a function of both, as $u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})$, and its equations of state are then reformulated considering,

$$\begin{aligned} \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial \varepsilon_{ij}} &= \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial (\varepsilon_{ij} - \varepsilon_{ij}^p)} : \frac{\partial (\varepsilon_{ij} - \varepsilon_{ij}^p)}{\partial \varepsilon_{ij}} = \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial (\varepsilon_{ij} - \varepsilon_{ij}^p)} : \mathbf{I} \\ &= \frac{\partial u(\varepsilon_{ij}^e, s, a^{(k)})}{\partial \varepsilon_{ij}^e} = \frac{\sigma_{ij}}{\rho} \end{aligned} \quad (3.27a)$$

and

$$\begin{aligned} \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial \varepsilon_{ij}^p} &= \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial (\varepsilon_{ij} - \varepsilon_{ij}^p)} : \frac{\partial (\varepsilon_{ij} - \varepsilon_{ij}^p)}{\partial \varepsilon_{ij}^p} = \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial (\varepsilon_{ij} - \varepsilon_{ij}^p)} : (-\mathbf{I}) \\ &= -\frac{\partial u(\varepsilon_{ij}^e, s, a^{(k)})}{\partial \varepsilon_{ij}^e} = -\frac{\sigma_{ij}}{\rho} \end{aligned} \quad (3.27b)$$

so, when its derivative is obtained as

$$\begin{aligned} du(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)}) &= \frac{\partial u}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial u}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial u}{\partial s} ds + \sum_{k=1}^n \frac{\partial u}{\partial a^{(k)}} da^{(k)} \\ &= \frac{\partial u}{\partial \varepsilon_{ij}^e} d\varepsilon_{ij} - \frac{\partial u}{\partial \varepsilon_{ij}^e} d\varepsilon_{ij}^p + \frac{\partial u}{\partial s} ds + \sum_{k=1}^n \frac{\partial u}{\partial a^{(k)}} da^{(k)} \\ &= \frac{\partial u}{\partial \varepsilon_{ij}^e} d(\varepsilon_{ij} - \varepsilon_{ij}^p) + \frac{\partial u}{\partial s} ds + \sum_{k=1}^n \frac{\partial u}{\partial a^{(k)}} da^{(k)} \\ &= \frac{\partial u}{\partial \varepsilon_{ij}^e} d\varepsilon_{ij}^e + \frac{\partial u}{\partial s} ds + \sum_{k=1}^n \frac{\partial u}{\partial a^{(k)}} da^{(k)} \end{aligned} \quad (3.28)$$

then it can be seen that the derivative of the internal energy is function of the elastic strain:

$$\left\{ \begin{array}{l} \sigma_{ij} = \rho \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial \varepsilon_{ij}^e} \\ \theta = \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial s} \\ A^{(k)} = -\rho \frac{\partial u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})}{\partial a^{(k)}} \end{array} \right. \quad (3.29)$$

The thermodynamic forces $A^{(k)}$ are taken as the negative of the density, times the partial derivative of its internal variable $a^{(k)}$ by extension from the previous exposed concepts. The nature of this equation of state is explained in detail when dissipation phenomena is analyzed. Based on the equations of state defined in Eq. (3.29), Table 3-5 is completed without considering the enthalpy potential as it does not have physical meaning when continuum mechanics is considered.

Table 3-5 Specific thermodynamic potentials for small deformations in continuum mechanics

Internal energy	Helmholtz free energy	Gibbs free energy
$u = u(\varepsilon_{ij}, \varepsilon_{ij}^p, s, a^{(k)})$	$\psi = \psi(\varepsilon_{ij}, \varepsilon_{ij}^p, \theta, a^{(k)})$ $= u - s\theta$	$\zeta = \zeta(\sigma_{ij}, \theta, a^{(k)})$ $= \psi - \sigma_{ij}\varepsilon_{ij}^e$
$\sigma_{ij} = \rho \frac{\partial u}{\partial \varepsilon_{ij}^e}$	$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}^e}$	$\varepsilon_{ij}^e = -\rho \frac{\partial \zeta}{\partial \sigma_{ij}}$
$\theta = \frac{\partial u}{\partial s}$	$s = -\frac{\partial \psi}{\partial \theta}$	$s = -\frac{\partial \zeta}{\partial \theta}$
$A^{(k)} = -\rho \frac{\partial u}{\partial a^{(k)}}$	$A^{(k)} = -\rho \frac{\partial \psi}{\partial a^{(k)}}$	$A^{(k)} = -\rho \frac{\partial \zeta}{\partial a^{(k)}}$

3.3 Entropy Generation

A model is better defined using the specific Helmholtz free energy as the thermodynamic potential than the internal energy because it is difficult to use the entropy as a state variable; though using the temperature is more suitable. So, taking the time derivative of the Helmholtz free energy when it is defined as a function of the internal energy, or

$$\dot{\psi} = \dot{u} - s\dot{\theta} - \dot{s}\theta \quad (3.30)$$

and substituting Eq. (3.30) into the definition made previously for the first and second law of thermodynamics, Eqs. (3.21) and (3.22), the Clausius-Duhem inequality becomes,

$$\rho \theta \dot{s} - \rho r + q_{i,i} - q_i \frac{\theta_{,i}}{\theta} \geq 0$$

$$\rho \theta \dot{s} - \rho \dot{u} + \sigma_{ij} \dot{\varepsilon}_{ij} - q_i \frac{\theta_{,i}}{\theta} \geq 0$$

$$\begin{aligned}
\sigma_{ij} \dot{\varepsilon}_{ij} - \rho(\dot{u} - \theta \dot{s}) - q_i \frac{\theta_{,i}}{\theta} &\geq 0 \\
\sigma_{ij} \dot{\varepsilon}_{ij} - \rho(\dot{\psi} + s \dot{\theta}) - q_i \frac{\theta_{,i}}{\theta} &\geq 0
\end{aligned} \tag{3.31}$$

As considered initially, the total strain is the addition of the elastic and plastic strains. Thus, the total strain rate is also the addition of the time derivatives of the plastic and elastic strains, or

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \tag{3.32}$$

Since the Helmholtz free energy is defined as stated in Table 3-5, $\psi(\varepsilon_{ij}, \varepsilon_{ij}^p, \theta, a^{(k)})$ then its derivative is obtained, similarly as the internal energy using Eqs. (3.27a), (3.27b), and (3.28), as

$$d\psi(\varepsilon_{ij}, \varepsilon_{ij}^p, \theta, a^{(k)}) = \frac{\partial \psi}{\partial \varepsilon_{ij}^e} d\varepsilon_{ij}^e + \frac{\partial \psi}{\partial \theta} d\theta + \sum_{k=1}^n \frac{\partial \psi}{\partial a^{(k)}} da^{(k)} \tag{3.33}$$

and substituting Eqs. (3.32) and (3.33) into Eq. (3.31)

$$\begin{aligned}
\sigma_{ij} (\dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p) - \rho \left(\frac{\partial \psi}{\partial \varepsilon_{ij}^e} \dot{\varepsilon}_{ij}^e + \frac{\partial \psi}{\partial \theta} \dot{\theta} + \sum_{k=1}^n \frac{\partial \psi}{\partial a^{(k)}} \dot{a}^{(k)} + s \dot{\theta} \right) - q_i \frac{\theta_{,i}}{\theta} &\geq 0 \\
\left(\sigma_{ij} - \rho \frac{\partial \psi}{\partial \varepsilon_{ij}^e} \right) \dot{\varepsilon}_{ij}^e - \rho \left(s + \frac{\partial \psi}{\partial \theta} \right) \dot{\theta} + \sigma_{ij} \dot{\varepsilon}_{ij}^p - \sum_{k=1}^n \rho \frac{\partial \psi}{\partial a^{(k)}} \dot{a}^{(k)} - q_i \frac{\theta_{,i}}{\theta} &\geq 0
\end{aligned} \tag{3.34}$$

Each of these terms implies a dissipation process, which means that each of these terms must be bigger than zero in order to be thermodynamically admissible. This definition is consistent with the different approaches seen in the literature, since early works done by Coleman and Gurtin (1967), Germain *et al.* (1983), and Serrin (1996); and more recent works done by Houlsby and Puzrin (2000), Puzrin and Houlsby (2001), Chaboche (2003), and Santaoja (2004).

The authors agreed that the first two terms are related to reversible processes, so they cannot generate entropy or dissipation. This means that these two first terms must be zero. When it is done, the equations of state obtained previously are consistent with this appreciation. Also when the generalized thermodynamic forces $A^{(k)}$ are replaced by the negative of the partial derivative the following inequality is obtained:

$$\sigma_{ij} \dot{\varepsilon}_{ij}^p + \sum_{k=1}^n A^{(k)} \dot{a}^{(k)} - q_i \frac{\theta_{,i}}{\theta} \geq 0 \tag{3.35}$$

All the terms of Eq. (3.35) must be bigger than zero in order to maintain a thermodynamical admissible criterion. This is because the entropy generation in time is always increasing, then each of its components that produce entropy must be also nonnegative. So, always

$$\left\{ \begin{array}{l} \sigma_{ij} \dot{\epsilon}_{ij}^p \geq 0 \\ A^{(k)} \dot{a}^{(k)} \geq 0 \\ -q_i \frac{\theta_{,i}}{\theta} \geq 0 \end{array} \right. \quad (3.36)$$

must be satisfied. If the model is simple, an analytical proof of Eq. (3.36) could be done. If this proof is not trivial, a numerical validation of Eq. (3.36) must be done. In our case, the validation will be done numerically.

3.4 Normality Rule and Homogeneity

Once the entropy generation has been defined as the dissipative rate, several values of this dissipative rate are to be obtained according to the evolution of its internal variables. Since a physical parameter is varying in time, at every step it is moving to a state of minimum energy. Also, a damage surface would be a limit of this energy state. Both considerations must be satisfied at every step in time.

The dissipative rate given by Eq. (3.35) could be separated in two phenomena, one local and other due to heat conduction, as

$$\gamma = \gamma^{(local)} + \gamma^{(heat)} \geq 0 \quad (3.37)$$

where

$$\left\{ \begin{array}{l} \gamma^{(local)} = \sigma_{ij} \dot{\epsilon}_{ij}^p + \sum_{k=1}^n A^{(k)} \dot{a}^{(k)} \geq 0 \\ \gamma^{(heat)} = -q_i \frac{\theta_{,i}}{\theta} \geq 0 \end{array} \right. \quad (3.38)$$

In order to obtain the minimum energy value at every step in time, two concepts will be included in this development. One is the Lagrange multiplier and the other is the yield or failure surface or condition.

3.4.1 Lagrange multiplier

The Lagrange multiplier is a method for finding the local *extrema* of a function of several variables subject to one or more constraints. This method reduces a problem in n variables with m constraints to a solvable problem in $n+m$ variables without constraints.

For a simple case, observe Figure 3-5. Here both the dissipation function $\gamma^{(local)}(x, y)$ and a constraint function $h(x, y) = 0$ are defined with two variables x , and y . According to each function value, $0 < d_1 < d_2 < d_3$, several functions in x and y can be built, where the one with minimum energy satisfying its constraint will be the first function that touches the constraint, $\gamma^{(local)} = d_2$. The normal unit vector at this point is the same for both, the function and the constraint.

Figure 3-5 Lagrange multipliers with two variables

An extension of this concept is considered when the extremum of the local dissipative function $\gamma^{(local)}(a^{(k)})$ is constrained by a function $h(a^{(k)})=0$. For this case, the gradient of $\gamma^{(local)}$ must line up with the gradient of h , then

$$\nabla \gamma^{(local)} + \dot{\lambda} \nabla h = 0 \quad (3.39)$$

so,

$$\begin{cases} \frac{\partial \gamma^{(local)}}{\partial a^{(k)}} + \dot{\lambda} \frac{\partial h}{\partial a^{(k)}} = 0, & k = 1, 2, \dots, n \\ h = 0 \end{cases} \quad (3.40)$$

which leads to $n+1$ equations.

Several constraint functions can be considered. For multiple constraints $h^{(1)}, \dots, h^{(m)}$ there would be as many Lagrange multipliers as constraint functions, or

$$\begin{cases} \frac{\partial \gamma^{(local)}}{\partial a^{(k)}} + \sum_{i=1}^m \dot{\lambda}^{(i)} \frac{\partial h^{(i)}}{\partial a^{(k)}} = 0 \\ h^{(i)} = 0 \end{cases}, \quad \text{for } \begin{cases} k = 1, 2, \dots, n \\ i = 1, 2, \dots, m \end{cases} \quad (3.41)$$

which leads to $n+m$ equations.

3.4.2 Yield function

The following analysis is done for local dissipative functions. The analysis for heat conduction is not included, but could be done in a similar way. From Eq. (3.35)

$$\dot{\gamma}^{(local)}(\dot{\varepsilon}_{ij}^p, \dot{a}^{(k)}; \varepsilon_{ij}^p, a^{(k)}) = \sigma_{ij} \dot{\varepsilon}_{ij}^p + \sum_{k=1}^n A^{(k)} \dot{a}^{(k)} \quad (3.42)$$

and performing the partial derivative with respect to its internal variables respectively, the following relations are obtained

$$\begin{cases} \frac{\partial \dot{\gamma}^{(local)}}{\partial \dot{\varepsilon}_{ij}^p} = \sigma_{ij} \\ \frac{\partial \dot{\gamma}^{(local)}}{\partial \dot{a}^{(k)}} = A^{(k)}, \quad k = 1, 2, \dots, n \end{cases} \quad (3.43)$$

A constraint function is defined with the same internal variables as the dissipative rate function, and thus

$$h = h(\dot{\varepsilon}_{ij}^p, \dot{a}^{(k)}; \varepsilon_{ij}^p, a^{(k)}) \quad (3.44)$$

and when the Lagrange multiplier method is used, using Eqs. (3.40) and (3.43), the following relations are obtained

$$\begin{cases} \frac{\partial \dot{\gamma}^{(local)}}{\partial \dot{\varepsilon}_{ij}^p} = -\dot{\lambda} \frac{\partial h}{\partial \dot{\varepsilon}_{ij}^p} = \sigma_{ij} \\ \frac{\partial \dot{\gamma}^{(local)}}{\partial \dot{a}^{(k)}} = -\dot{\lambda} \frac{\partial h}{\partial \dot{a}^{(k)}} = A^{(k)}, \quad k = 1, 2, \dots, n \\ h(\dot{\varepsilon}_{ij}^p, \dot{a}^{(k)}; \varepsilon_{ij}^p, a^{(k)}) = 0 \end{cases} \quad (3.45)$$

A dual constraint function f is now introduced. This dual function is obtained using the Legendre transformation as defined in Eq. (3.7), so

$$f\left(\hat{\sigma}_{ij}, \hat{A}^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) = g\left(\dot{\varepsilon}_{ij}^p, \dot{a}^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) - \left[\hat{\sigma}_{ij} \dot{\varepsilon}_{ij}^p + \sum_{k=1}^n \hat{A}^{(k)} \dot{a}^{(k)} \right] \quad (3.46)$$

where the following relations are obtained, using Eqs. (3.8) and (3.45a,b),

$$\begin{cases} \hat{\sigma}_{ij} = \frac{\partial h}{\partial \dot{\varepsilon}_{ij}^p} = -\frac{1}{\lambda} \sigma_{ij} \\ \hat{A}^{(k)} = \frac{\partial h}{\partial \dot{a}^{(k)}} = -\frac{1}{\lambda} A^{(k)}, \quad k = 1, 2, \dots, n \end{cases} \quad (3.47)$$

When Eqs. (3.45c) is introduced into (3.46), the dual function g is redefined as,

$$f\left(\tilde{\sigma}_{ij}, \tilde{A}^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) = -\tilde{\sigma}_{ij} \dot{\varepsilon}_{ij}^p - \sum_{k=1}^n \tilde{A}^{(k)} \dot{a}^{(k)} \quad (3.48)$$

or introducing relation (3.47),

$$\begin{aligned} f\left(-\frac{1}{\lambda} \sigma_{ij}, -\frac{1}{\lambda} A^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) &= -\left(-\frac{1}{\lambda} \sigma_{ij}\right) \dot{\varepsilon}_{ij}^p - \sum_{k=1}^n \left(-\frac{1}{\lambda} A^{(k)}\right) \dot{a}^{(k)} \\ &= \sigma_{ij} \left(\frac{1}{\lambda} \dot{\varepsilon}_{ij}^p\right) + \sum_{k=1}^n A^{(k)} \left(\frac{1}{\lambda} \dot{a}^{(k)}\right) \end{aligned} \quad (3.49)$$

and employing the definition in Eq. (3.48),

$$\begin{aligned} f\left(-\frac{1}{\lambda} \sigma_{ij}, -\frac{1}{\lambda} A^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) &= -\frac{1}{\lambda} \left[-\sigma_{ij} \dot{\varepsilon}_{ij}^p - \sum_{k=1}^n A^{(k)} \dot{a}^{(k)} \right] \\ &= -\frac{1}{\lambda} f\left(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) \end{aligned} \quad (3.50)$$

In mathematics, a homogeneous function is a function with multiplicative scaling behavior. If some of the internal variables of a function ϕ is multiplied by a factor, then the result is multiplied by some power of this factor, or

$$\phi\left(r x^{(1)}, \dots, r x^{(n)}; \alpha^{(1)}, \dots, \alpha^{(m)}\right) = r^w \phi\left(x^{(1)}, \dots, x^{(n)}; \alpha^{(1)}, \dots, \alpha^{(m)}\right) \quad (3.51)$$

where Euler's theorem on homogeneous function reads,

$$w \phi\left(r x^{(1)}, \dots, r x^{(n)}; \alpha^{(1)}, \dots, \alpha^{(m)}\right) = \sum_{k=1}^n x^{(k)} \frac{\partial \phi\left(x^{(1)}, \dots, x^{(n)}; \alpha^{(1)}, \dots, \alpha^{(m)}\right)}{\partial x^{(k)}} \quad (3.52)$$

By inspection it is seen that the definition of a homogeneous function, Eq. (3.50), and the definition of a dual function g in Eq. (3.50) are the same for $r = -1/\dot{\lambda}$ and $w = 1$. So Euler's theorem can be rewritten as

$$f\left(-\frac{1}{\dot{\lambda}}\sigma_{ij}, -\frac{1}{\dot{\lambda}}A^{(k)}; \varepsilon_{ij}^p, a^{(k)}\right) = \sigma_{ij} \frac{\partial f(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, \dots, a^{(k)})}{\partial \sigma_{ij}} + \sum_{k=1}^n A^{(k)} \frac{\partial f(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)})}{\partial A^{(k)}} \quad (3.53)$$

Finally, comparing Eqs. (3.53) and (3.49) the following relations are obtained,

$$\begin{cases} \dot{\varepsilon}_{ij}^p = \dot{\lambda} \frac{\partial f(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)})}{\partial \sigma_{ij}} \\ \dot{a}^{(k)} = \dot{\lambda} \frac{\partial f(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)})}{\partial A^{(k)}}, \quad k = 1, 2, \dots, n \end{cases} \quad (3.54)$$

When several yield functions are used, an extension of the previous concept is applied. As seen in Eq. (3.41), the number of Lagrange multipliers is the same as the number of yield functions used. Thus, in general the stated relations in Eq. (3.53) can be generalized to

$$\begin{cases} \dot{\varepsilon}_{ij}^p = \sum_{i=1}^m \dot{\lambda}^{(i)} \frac{\partial f^{(i)}(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)})}{\partial \sigma_{ij}} \\ \dot{a}^{(k)} = \sum_{i=1}^m \dot{\lambda}^{(i)} \frac{\partial f^{(i)}(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)})}{\partial A^{(k)}} \end{cases} \quad (3.55)$$

where m is the total yield functions, $k = 1, 2, \dots, n$ is the total quantity of thermodynamic forces, and $m \leq n$. Each internal variable rate is related to one yield function, and one function can be related to several internal variables' rates.

3.5 Summary

In order to define a model the previous definitions are used in sequential steps. These steps could be condensed for the case of small deformations and neglecting temperature effects as:

1. Select the set of state variables.

$$(\varepsilon_{ij}, \varepsilon_{ij}^p, a^{(k)}) \quad , \quad k = 1, 2, \dots, n \quad (3.56)$$

2. Define the explicit form for the potentials, i.e., the specific Helmholtz (or Gibbs) free energy and the m-yield function(s).

$$\begin{cases} \psi = \psi(\varepsilon_{ij}, \varepsilon_{ij}^p, a^{(k)}) \\ f^{(i)} = f^{(i)}(\sigma_{ij}, A^{(k)}; \varepsilon_{ij}^p, a^{(k)}), \quad i = 1, 2, \dots, m \end{cases} \quad (3.57)$$

3. Apply the state equations:

$$\begin{cases} \sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}^e} \\ A^{(k)} = -\rho \frac{\partial \psi}{\partial a^{(k)}} \end{cases} \quad (3.58)$$

4. Apply the normality rule in order to evaluate the internal state variables evolution.

$$\begin{cases} \dot{\varepsilon}_{ij}^p = \sum_{i=1}^m \dot{\lambda}^{(i)} \frac{\partial f^{(i)}(\sigma_{ij}, A^{(k)})}{\partial \sigma_{ij}} \\ \dot{a}^{(k)} = \sum_{i=1}^m \dot{\lambda}^{(i)} \frac{\partial f^{(i)}(\sigma_{ij}, A^{(k)})}{\partial A^{(k)}} \end{cases} \quad (3.59)$$

5. Verify the homogeneity of each assumed yield function and the satisfaction of the Clausius-Duhem inequality,

$$\begin{cases} \sigma_{ij} \dot{\varepsilon}_{ij}^p \geq 0 \\ A^{(k)} \dot{a}^{(k)} \geq 0 \\ \dot{\lambda}^{(i)} > 0 \end{cases} \quad (3.60)$$

4. CONTINUUM DAMAGE MECHANICS MODELING

A model that takes into account both plasticity and damage evolution is desirable. This chapter contains an analysis of damage and plasticity as independent and dependent mechanisms. Also, a material parameter evaluation is done in order to conceptualize both failure mechanisms.

4.1 Damage

The basic definition of damage, as put forth by the early works of Kachanov (1958) and Kachanov (1986), and more recently by Krajcinovic (1996), Lemaître and Desmorat (2005), and Voyiadjis and Kattan (2005), expresses damage as a macroscopic measure. First a representative volume element (RVE) is considered, containing distributed cracks (expressed as micro or macro cracks, and voids) and loaded in only one direction (x_1 -direction) or uniaxial tension, as seen in Figure 4-1.

Figure 4-1 Uniaxial loading over a representative volume element (RVE)

When isotropic damage is considered, only one damage parameter relating the damaged and undamaged configurations, D , is used. This is defined as the ratio between the area

where only cracks and voids are considered over the whole RVE, A^D , and the initial (or damaged) area, A . After removing all cracks and voids from the damaged area, an effective undamaged area is obtained, $\bar{A}$, so

$$D = \frac{A^D}{A} = \frac{A - \bar{A}}{A} = 1 - \frac{\bar{A}}{A} \quad (4.1)$$

If all types of damage, including voids and cracks, are removed from the RVE, one fictitious undamaged (effective) configuration is obtained, Figure 4-2(b). Since both configurations are subjected to the same external load T , then

$$\sigma A = \bar{\sigma} \bar{A} \quad (4.2)$$

so the effective uniaxial stress $\bar{\sigma}$, as defined also by Odqvist and Hult (1961) and Rabotnov (1968), is derived as

$$\begin{aligned} \bar{\sigma} &= \frac{A}{\bar{A}} \sigma = \frac{A}{(1-D)A} \sigma \\ \bar{\sigma} &= \frac{1}{(1-D)} \sigma \end{aligned} \quad (4.3)$$

Figure 4-2 Removal of cracks and voids from the RVE

Also, it must be noted that the stress-strain relation for the damaged configuration is

$$\sigma = E \varepsilon \quad (4.4)$$

while for the effective undamaged configuration is

$$\bar{\sigma} = \bar{E} \bar{\varepsilon} \quad (4.5)$$

4.1.1 *Relations between damaged and effective configuration*

The relations between damage and effective configurations are not yet fully defined. In order to complete this relation some hypothesis need to be proposed. The following three hypotheses are defined explicitly.

4.1.1.1 Equivalent strain hypothesis

Lemaître and Chaboche (1978) and Sidoroff (1980) introduced the concept of equivalent strain. Here the strains for both configurations are assumed the same, $\varepsilon = \bar{\varepsilon}$. So, considering this equivalence and introducing it into Eqs. (4.4) and (4.5), also using Eq. (4.2) leads to a relation between the damaged and the effective stiffnesses as

$$\begin{aligned} \bar{\varepsilon} &= \varepsilon \\ \frac{\bar{\sigma}}{\bar{E}} &= \frac{\sigma}{E} \\ \frac{E}{\bar{E}} &= \frac{\sigma}{\bar{\sigma}} \\ \frac{E}{\bar{E}} &= (1 - D) \end{aligned} \quad (4.6)$$

4.1.1.2 Elastic strain-energy density hypothesis

Usually an energy approach is used in the literature in order to define the equivalence between the configurations. This approach states that the elastic strain energy is the same in both configurations, Cordebois and Sidoroff (1979) and Cordebois and Sidoroff (1982).

Under a threshold value defined at point A in Figure 4-3 damage is not developed, then there is no difference between both configuration. When the load is increased over the threshold value, damage will be produced and increased, i.e., in Figure 4-4(b) a damaged area is developed inside the RVE. If the load is continuously increased until point C, damage develops and the damaged area could be referred to the previous point as

$$A_C^D = A_B^D + \Delta A_{BC}^D \quad (4.7)$$

$$A_C^D = A_P^D = A_{OC}^D \quad (4.8)$$

If the loading is increased above the actual threshold value, C , damage is developed until an ultimate point is reached, U . At this point a critical damage value is obtained, where values beyond this cannot be obtained because failure has been reached.

Since the elastic strain energy is considered to be the same for both configurations, and taking into account that both configurations show linear behavior, the areas OAM , APQ , and $O'M'N'$ have the same value according to Figure 2-4. Moreover, based on the relationship between the effective stress and undamaged stress as defined in Eq. (4.3), and using the relations for damaged and undamaged configurations in Eqs (4.4) and (4.5), a relation between the stiffnesses in both configurations is obtained as

$$\begin{aligned}
\int_0^{\varepsilon} \sigma \, d\varepsilon &= \int_0^{\bar{\varepsilon}} \bar{\sigma} \, d\bar{\varepsilon} = \int_{\varepsilon_0}^{\varepsilon_0 + \bar{\varepsilon}} \bar{\sigma} \, d\bar{\varepsilon} \\
\frac{1}{2} \sigma \varepsilon &= \frac{1}{2} \bar{\sigma} \bar{\varepsilon} \\
\sigma \frac{\sigma}{E} &= \bar{\sigma} \frac{\bar{\sigma}}{\bar{E}} \\
\frac{E}{\bar{E}} &= \left(\frac{\sigma}{\bar{\sigma}} \right)^2 \\
\frac{E}{\bar{E}} &= (1 - D)^2
\end{aligned} \tag{4.9}$$

4.1.1.3 Li's hypothesis

An energy correlation hypothesis between a damaged macroscopic continuum and its sub-scale virgin or matrix material has been established by Li (2000). Based on the geometrical definitions of the damage parameter, thermodynamic principles and void growth model, Li defined two types of damage: the equivalent elastic strain energy and the other as function of the type of void. The first one was defined previously, while the second is defined as

$$\frac{E}{\bar{E}} = \frac{(1 - D)^2}{1 - F} \tag{4.10}$$

where F is defined as the volume damage density (analogous to area damage density, D) by Tvergaard (1981), Tvergaard (1990) as

$$F = \frac{V^D}{V} \tag{4.11}$$

Figure 4-5 Distributed voids in a representative cubic element

Two types of void are considered cubical and spherical shape, as seen in Figure 4-5. For the case of cubic void, the area and volume area density is then found to be

$$D = \frac{A^D}{A} = \frac{n^2 a^2}{L^2} \quad \text{and} \quad F = \frac{V^D}{V} = \frac{n^3 a^3}{L^3} \quad (4.12a,b)$$

where n is the number of cubic voids inside a cubic RVE of side L , and a is the side of the cubic void, Figure 4-6(a). The relation between the area and volume damage density is found combining Eqs. (4.12a) and (4.12b).

$$F = D^{3/2} \quad (4.13)$$

For the case of spherical void, the area and volume damage density are analogously stated as

$$D = \frac{A^D}{A} = \frac{n^2 \pi a^2}{4L^2} \quad \text{and} \quad F = \frac{V^D}{V} = \frac{n^3 \pi a^3}{6L^3} \quad (4.14a,b)$$

where n is the number of spherical voids inside a cubic RVE of side L , and d is the diameter of the spherical void, Figure 4-5(b). Combining Eqs. (4.14a) and (4.14b) the relation between the area and volume damage density is

$$F = \frac{4}{3\sqrt{\pi}} D^{3/2} = 0.752 D^{3/2} \quad (4.15)$$

Introducing Eqs. (4.13) and (4.15) into Eq. (4.10) a general relation between the damaged and effective stiffness as function only on the area damaged density, D , is

$$\frac{E}{\bar{E}} = \frac{(1-D)^2}{1-\alpha D^{3/2}} \quad (4.16)$$

where

$$\alpha = \begin{cases} 1, & \text{cubic void} \\ 0.752, & \text{spherical void} \end{cases}$$

Since D and F are statistical values among the RVE, Eq. (4.16) does not depend on a particular arrangement of the voids. Finally, a comparison between the stiffness degradation as a function of the damage parameter for various equivalent hypothesis are shown in Figure 4-6.

Finally, the damage variable has been defined as a macroscopic measure of the microscopic degradation of an RVE; then damage can be represented as:

- Specific voids and cracks surfaces,
- Specific voids and cracks volumes,
- Spacing between cracks and voids, or
- Geometry of voids.

Figure 4-6 Degradation of Young's modulus with respect to damage parameter D , for various equivalent hypotheses

4.1.2 Isotropic damage evolution

The evolution of the the damage variable when considered as an isotropic value is shown in Figure 4-7. It is noted that the evolution of this variable will start after a threshold value, S_{th} , has been reached. After point, A, two parameters will be developed: an isotropic damage and a plastic strain tensor. Both parameters not necessarily will be related or start their evolution from the same threshold value.

When the loading is reduced to zero after reaching B , the strain will not go to zero but to a plastic point O' . This value OO' is well known as the plastic strain tensor. When loading is between O' and B , damage or plastic evolution will not occur and a linear elastic behavior over $O'B$ is observed. Also, the relations in damaged and effective configurations, Eqs. (4.4) and (4.5), respectively, can be extended to

$$\sigma_{ij} = E_{ij} \varepsilon_{ij}^e \quad (4.17)$$

$$\bar{\sigma}_{ij} = \bar{E}_{ij} \bar{\varepsilon}_{ij}^e \quad (4.18)$$

Figure 4-7 Evolution of the damage and plastic isotropic variables

From this point forward the elastic strain energy density equivalence hypothesis will be used. That is the relations between the effective and damaged configurations are given

when triangles OMN and $O'PQ$ have the same value. When previous concepts are extended to this case Eqs. (2.3), (2.9), (2.17), and (2.18) relate the stress, elastic strain, and stiffness tensors in the effective and damaged configurations as

$$\sigma_{ij} = (1 - D) \bar{\sigma}_{ij} \quad (4.19)$$

$$\varepsilon_{ij}^e = \frac{1}{(1 - D)} \bar{\varepsilon}_{ij}^e \quad (4.20)$$

$$E_{ij} = (1 - D)^2 \bar{E}_{ij} \quad (4.21)$$

For small deformations the total strain tensor can be additively decomposed into its plastic and elastic components as defined below:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \quad (3.26)$$

Also the elastic strain can be decomposed into effective elastic strain and damaged elastic strain, as can be seen in Figure 4-7; and thus

$$\varepsilon_{ij}^e = \varepsilon_{ij}^{ee} + \varepsilon_{ij}^{ed} \quad (4.22)$$

where ε_{ij}^e is the total elastic strain recovered during unloading. Due to this additive decomposition and the assumption of an elastic strain energy equivalence, the effective and damaged elastic strains can be defined in terms of the total elastic strain, the effective elastic strain, and between each other, as also stated in Abu Al-Rub and Voyiadjis (2003), as

$$\varepsilon_{ij}^{ee} = \begin{cases} (1 - D) \bar{\varepsilon}_{ij}^e \\ (1 - D)^2 \varepsilon_{ij}^e \end{cases}, \quad \varepsilon_{ij}^{ed} = \begin{cases} \left[\frac{1 - (1 - D)^2}{(1 - D)} \right] \bar{\varepsilon}_{ij}^e \\ \left[\frac{1 - (1 - D)^2}{(1 - D)^2} \right] \varepsilon_{ij}^e \end{cases} \quad (4.23)$$

The ultimate value is related to an ultimate strength $S_{u_{ij}}$, ultimate strain $\hat{\varepsilon}_{ij}$, and ultimate plastic strain $\hat{\varepsilon}_{ij}^p$, which can be obtained experimentally. Knowing these values, the ultimate elastic strain $\hat{\varepsilon}_{ij}^e$ and the ultimate stiffness $\hat{E}_{ij}$ can be obtained. In order to define the damage parameter between 0 and 1, the critical damage value, D_{cr} , must be 1, then making a change in variables to include this unity critical value the relations between the effective and damage configurations can be redefined as,

$$E_{ij} = \begin{cases} (1-D)(\bar{E}_{ij} - \hat{E}_{ij}) + \hat{E}_{ij}, & \text{strain equivalence} \\ (1-D)(\bar{E}_{ij} - \hat{E}_{ij}) + \hat{E}_{ij}, & \text{equivalent elastic strain equivalence} \\ \frac{(1-D)^2}{1-\alpha D^{3/2}} (\bar{E}_{ij} - \hat{E}_{ij}) + \hat{E}_{ij}, & \text{cubic/spherical void} \end{cases} \quad (4.24)$$

where, $0 \leq D \leq 1$

4.1.3 Anisotropic damage

An anisotropic damage can be defined initially when unique damage is considered within a representative volume element, as seen in Figure 4-8. The projection of this volumetric damage is seen, when unloaded, over the sides of the RVE revealing the area damage density with respect to the principal directions ($i=1, 2, 3$). An extension of the isotropic damage presented in Eq. (4.1) could then be easily performed as

$$d_i = \frac{A_i^D}{A_i} \quad (\text{no sum over } i) \quad (4.25)$$

in order to define anisotropic damage. The damage here is defined in the three principal directions and its evolution in any of the three directions can occur freely, and is a function of the loading and material characteristics.

Figure 4-8 RVE with unique anisotropic damage

If the projected damaged areas are considered as ellipses, then the relations between the projected ellipse axes with respect to its correspondent side length is stated as,

$$\begin{cases} d_1 = \frac{\pi \phi_2 \phi_3}{4 L_2 L_3} \\ d_2 = \frac{\pi \phi_1 \phi_3}{4 L_1 L_3} \\ d_3 = \frac{\pi \phi_1 \phi_2}{4 L_1 L_2} \end{cases} \leftrightarrow \begin{cases} \frac{\phi_1}{L_1} = \sqrt{\frac{4 d_2 d_3}{\pi d_1}} \\ \frac{\phi_2}{L_2} = \sqrt{\frac{4 d_1 d_3}{\pi d_2}} \\ \frac{\phi_3}{L_3} = \sqrt{\frac{4 d_1 d_2}{\pi d_3}} \end{cases} \quad (4.26)$$

When an external load is applied beyond its threshold value, it is seen that the damage area in its correspondent cube's side will increase, see Figure 4-9. This increase in the area is related to an increase in its projected axes, $\Delta\phi_2$ and $\Delta\phi_3$. Also, this increase will increase the damaged area in the other two sides proportionally to the increase in the projected axis. Then an increase in d_1 will produce an increase in d_2 and d_3 proportionally to the increase of the projected axis.

Figure 4-9 Projected damage area dependencies

This approach is used extensively in the literature. Some preliminaries quantities such as the second-order identity tensor, also known as Kronecker delta δ_{ij} , and the symmetric fourth-order identity tensor are defined as

$$I_{ij} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (4.27)$$

$$I_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (4.28)$$

In the literature several expressions are defined to correspond to a fourth-order damage effective tensor $\mathbf{M}$. This tensor is a function of a second-order damage tensor, which in turn is function of the damaged vector $\mathbf{d}$, defined in Eq. (4.25). This effective damage tensor is used in order to extend the isotropic damage into anisotropic damage, so the relation between the damaged and effective configurations can be extended as

$$D \rightarrow d_i \rightarrow D_{ij}(d_i) \rightarrow M_{ijkl}(\delta_{ij} - D_{ij})$$

$$\begin{cases} \sigma_{ij} = (1 - D) \bar{\sigma}_{ij} \\ \varepsilon_{ij}^e = \frac{1}{(1 - D)} \bar{\varepsilon}_{ij}^e \\ E_{ij} = (1 - D)^2 \bar{E}_{ij} \end{cases} \leftrightarrow \begin{cases} \boldsymbol{\sigma} = \mathbf{M} : \bar{\boldsymbol{\sigma}} \\ \boldsymbol{\varepsilon}^e = \mathbf{M}^{-1} : \bar{\boldsymbol{\varepsilon}}^e \\ \mathbf{C} = \mathbf{M} : \bar{\mathbf{C}} : \mathbf{M}^T \end{cases} \quad (4.29)$$

First Cordebois and Sidorof (1979) and then Abu Al-Rub and Voyiadjis (2003) have used the elastic energy hypothesis define the effective fourth-order tensor and the relations between the damaged and effective configurations as,

$$d_i = \frac{A_i^D}{A_i} \text{ (no sum over } i)$$

$$D_{ij} = \sqrt{d_i d_j}$$

$$M_{ijkl} = \frac{1}{2} [(\delta_{ik} - D_{ik})\delta_{jl} + (\delta_{jl} - D_{jl})\delta_{ik}]$$

$$\begin{cases} \sigma_{ij} = M_{ijkl} \bar{\sigma}_{kl} \\ \varepsilon_{ij}^e = M_{ijkl}^{-1} \bar{\varepsilon}_{kl}^e \\ C_{ijkl} = M_{imjn} \bar{C}_{mnpq} M_{pkql} \end{cases} \quad (4.30)$$

where $\mathbf{C}$ and $\bar{\mathbf{C}}$ are the fourth-order damaged and effective stiffness tensors, respectively.

4.1.4 Anisotropic damage in unidirectional composite materials

Three mechanisms of failure are seen in composite materials, fiber breakage, inter-fiber failure, and delamination. The external loadings that induce these failures are seen in Figure 4-10, which have been defined by Papanikos *et al.* (2003).

(i) Fiber tension

(ii) Fiber compression

(a) Fiber Failure

(i) Matrix tension

(ii) Matrix compression

(iii) Fiber-matrix shear out

(b) Inter Fiber Failure

(i) Tension

(ii) Compression

(c) Delamination

Figure 4-10 Composite materials failure mechanisms

The three failure mechanisms are considered independent from each other. In other words the damage components in the three orthogonal directions, 1, 2, and 3, evolve without taking into account the damage evolution of the others. So, fiber breakage would not induce delamination nor inter-fiber failure.

When fiber reinforced plastics (FRP) is considered, the damage principal direction, 1, is coincident with the fiber direction. This allows consideration of fiber failure, interfiber failure and delamination to be coincident with the principal directions 1, 2, and 3, correspondingly. Also due to this damage visualization, the damage second-order tensor will be a diagonal tensor, defined as

$$D_{ij} = d_i \delta_{ij} \quad (\text{no sum over } i) \quad (4.31)$$

Barbero (2008) defines an integrity second-order tensor and then the fourth-order damage effective tensor and its inverse as

$$\begin{aligned} \Omega_{ij} &= \sqrt{I_{ij} - D_{ij}} = \sqrt{\delta_{ij} - d_i \delta_{ij}} = \sqrt{1 - d_i} \delta_{ij} = \Omega_i \delta_{ij} \\ M_{ijkl} &= \frac{1}{2} (\Omega_{ik} \Omega_{jl} + \Omega_{il} \Omega_{jk}) = \sqrt{1 - d_i} \sqrt{1 - d_j} I_{ijkl} = \Omega_i \Omega_j I_{ijkl} \\ M_{ijkl}^{-1} &= \frac{1}{\sqrt{1 - d_i} \sqrt{1 - d_j}} I_{ijkl} = \frac{1}{\Omega_i \Omega_j} I_{ijkl} \end{aligned} \quad (4.32)$$

where no sum over i or j is considered.

In this model the complete second-order damage tensor is considered. The previous analysis does not take into account the effect of shear. When shear damage is considered, the three principal damage values d_i are obtained after the eigenvalues of the D_{ij} are obtained. The principal directions of these principal values will coincide with the fiber and transverse directions when shear values are zero. Otherwise the damage ellipsoid will be rotated. Then,

$$\begin{aligned} d_k &= \text{eig}(D_{ij}) \\ D_{ij} &= \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \end{aligned} \quad (4.33)$$

From the previous statement, the integrity tensor is obtained using the energy equivalence and the symmetric fourth-order damage tensor is then defined as

$$\Omega_{ij} = \sqrt{I_{ij} - D_{ij}}$$

$$\begin{aligned}
M_{ijkl} &= \Omega_{ij} \Omega_{kl} I_{ijkl} \quad (\text{no sum over } i, j, k, l) \\
M_{ijkl}^{-1} &= \frac{1}{\Omega_{ij} \Omega_{kl}} I_{ijkl} \quad (\text{no sum over } i, j, k, l)
\end{aligned} \tag{4.34}$$

The relations between the effective and damaged configurations are developed explicitly and extensively in Appendix A.

Finally, it is stated that the damage variable is a macroscopic measure of the microscopic degradation of an RVE. Making an extension to the above explanation, damage can be represented as:

- Specific voids and cracks surfaces.
- Specific voids and cracks volumes.
- Spacing between cracks and voids.
- Geometry of voids.

But also the damage has a strong relation to the virgin stiffness modulus and the damaged one, as seen in Figure 4-3. This means that material's stiffness of the RVE will degrade following a damage function, $f(\phi)$.

4.1.5 *Damage quantification*

There are many ways to measure degradation in materials such as: direct measurements of the total crack areas lying on a surface, degradation of the elastic modulus, degradation of ultrasonic waves propagation speed, degradation of the microhardness, change in density, increase in electrical resistance, variation in the cyclic plastic response, change in creep properties, change in acoustic emission properties, remaining life, and cumulative hysteresis energy.

Damage processes corresponding to the degradation of microstructure is, in general, irreversible. During the cumulative damage process, the internal entropy production, which is a measure of disorder in the system, must increase according to the second law of thermodynamics. So, internal entropy production can also be used as a criterion for qualification of damage. This criterion is used to increase the concept of entropy into damage which is called as continuum damage mechanics.

4.2 **Plasticity**

The yield surface specifies the state of stress at which plastic flow initiates. This surface is generally defined as a function g , so $g=0$ indicates plastic evolution. Negative values

are given when an elastic behavior occurs, and nonnegative values are not admissible for rate-independent plasticity.

Figure 4-11 Yield surface definition

For a simple uniaxial case, as seen in Figure 4-11, the curve is used to define a plastic stress-strain curve. Observe that after an initial threshold value S_y , plastic evolution will occur, and below this value only elastic behavior is seen. This new curve is used to define the yield surface, as

$$g(\sigma, R) = \hat{g}(\sigma) - \hat{R}(R) \quad (4.35)$$

where $\hat{g}$ represents the state of positive value for the stress $\hat{g} = \sqrt{\sigma^2}$ in the case of uniaxial stress or an equivalent positive value of the state of stresses. The $\hat{R}$ represents the limit where plasticity occurs and this evolution is presented in Figure 4-11; so

$$\hat{R} = R + R_0 \quad (4.36)$$

Figure 4-12 Kuhn-Tucker and consistency conditions

A flow rule was specified in Eq. (3.59), which relates the internal state variable rate as a function of the partial derivative of a yield potential f with respect to its thermodynamic force. This rate and partial derivative are linearly related to a Lagrange multiplier. The multiplier and the yield surface g are restricted by the Kuhn-Tucker conditions, which are classic in the convex mathematical programming literature. The Kuhn-Tucker conditions states that

$$\begin{cases} \dot{\lambda} \geq 0 \\ g \leq 0 \\ \dot{\lambda} g = 0 \end{cases} \quad (4.37)$$

After differentiating $\dot{\lambda} g = 0$ and for the case of pure plastic evolution, $g=0$, a final condition is obtained. This is usually called consistency condition or persistency condition and defined as,

$$\dot{\lambda} \dot{g} = 0 \quad (4.38)$$

All conditions are represented in Figure 4-12, where elastics behavior and plastic evolution are clearly observed. When elastic behavior is expected, the value of the plastic surface should be a negative value and the multiplier should be zero, or no plastic flow will occur. If the multiplier is zero but the yield surface value is positive, there is a non possible state. So, the multiplier must increase and the yield value must go to zero in order to reach a plastic evolution state. At this point, the rate of change for the yield surface must be zero.

4.3 Damage - Plasticity

Until this point, general concepts have been defined and exposed. All these concepts will now be integrated into a method in order to produce a material model. In general the following steps are to be followed to completely define the model.

The definition of the state variables and the form of the specific Helmholtz (or Gibbs) free-energy potential constitutes a break point on the formulation, since they are the basis for the derivation of all constitutive equations.

- Step 1:* Select the state variables.
- Step 2:* Define the specific Helmholtz (or Gibbs) free-energy potential.
- Step 3:* Apply the state equations to evaluate the thermodynamic forces and its rates.
- Step 4:* Apply the normality rule to evaluate the state variables rates.
- Step 5:* Define the yield surfaces.
- Step 6:* Find the Jacobian after applying the Kuhn-Tucker and consistency conditions.

Step 7: Implement the iterative variables used to obtain the Lagrange multipliers numerically using the return mapping algorithm.

Since we are interested in a uniaxial fiber composite material model all the steps defined will be developed for this model. Also, a 3-D model will be considered. Appendix B expresses explicitly all the variables and definitions given in this section.

4.3.1 State variables selection

Plastic strain and damage state variables are considered. Damage will be taken as defined in Eqs. (4.25) through (4.30). An isotropic hardening plasticity p and a correspondent isotropic hardening damage δ are considered. The state variables are expressed as,

$$\psi(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij}, p, \delta) \quad (4.39)$$

4.3.2 Define the specific Helmholtz free-energy potential

The Helmholtz free-energy is divided into three different potentials. An elastic potential deduced from the strain energy density, a plastic-strain energy dissipation density, and a damage energy dissipation density. All three are considered independent to each other so the evolution of each state variable is clearly defined, so

$$\psi = \psi_e(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij}) + \psi_p(p) + \psi_d(\delta) \quad (4.40)$$

where,

$$\begin{cases} \psi_e = \psi_e(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij}) = \frac{1}{2\rho} (\varepsilon_{ij} - \varepsilon_{ij}^p) C_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^p) = \frac{1}{2\rho} \varepsilon_{ij}^e C_{ijkl} \varepsilon_{kl}^e \\ \psi_p = \psi_p(p) = \frac{1}{\rho} \sum_{i=1}^{n_p} \alpha_i^p \left(\beta_i^p e^{\left(-p/\beta_i^p\right)} + p \right) \\ \psi_d = \psi_d(\delta) = \frac{1}{\rho} \sum_{i=1}^{n_d} \alpha_i^d \beta_i^d e^{\left(-\delta/\beta_i^d\right)} \end{cases} \quad (4.41)$$

in which α_i^p , β_i^p , α_i^d , and β_i^d are material parameters, where the plastic and damage potentials are defined as n -terms Prony and exponential series respectively.

4.3.3 Thermodynamic forces and its rates

Applying the state equations expressed in Eq. (3.58) to the Helmholtz free-energy defined in Eqs. (4.40) and (4.41), the following set of equations are obtained,

$$\begin{aligned}
\sigma_{ij} &= \rho \frac{\partial \psi}{\partial \varepsilon_{ij}^e} = \rho \frac{\partial \psi_e}{\partial \varepsilon_{ij}^e} = \rho \frac{\partial}{\partial \varepsilon_{ij}^e} \left(\frac{1}{2\rho} \varepsilon_{kl}^e C_{klmn} \varepsilon_{mn}^e \right) = C_{ijkl} \varepsilon_{kl}^e = C_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^p) \\
Y_{ij} &= -\rho \frac{\partial \psi}{\partial D_{ij}} = -\rho \frac{\partial \psi_e}{\partial D_{ij}} = -\rho \frac{\partial}{\partial D_{ij}} \left(\frac{1}{2\rho} \varepsilon_{kl}^e C_{klmn} \varepsilon_{mn}^e \right) = -\frac{1}{2} \varepsilon_{kl}^e \frac{\partial C_{klmn}}{\partial D_{ij}} \varepsilon_{mn}^e \\
&= -\frac{1}{2} (\varepsilon_{kl} - \varepsilon_{kl}^p) \frac{\partial C_{klmn}}{\partial D_{ij}} (\varepsilon_{mn} - \varepsilon_{mn}^p) \\
R &= -\rho \frac{\partial \psi}{\partial p} = -\rho \frac{\partial \psi_p}{\partial p} = -\rho \frac{\partial}{\partial p} \left[\frac{1}{\rho} \sum_{i=1}^{n_p} \alpha_i^p \left(\beta_i^p e^{\left(-p/\beta_i^p\right)} + p \right) \right] = \sum_{i=1}^{n_p} \alpha_i^p \left(e^{\left(-p/\beta_i^p\right)} - 1 \right) \\
\gamma &= -\rho \frac{\partial \psi}{\partial \delta} = -\rho \frac{\partial \psi_d}{\partial \delta} = -\rho \frac{\partial}{\partial \delta} \left[\frac{1}{\rho} \sum_{i=1}^{n_d} \alpha_i^d \beta_i^d e^{\left(-\delta/\beta_i^d\right)} \right] = \sum_{i=1}^{n_d} \alpha_i^d e^{\left(-\delta/\beta_i^d\right)}
\end{aligned} \tag{4.42}$$

The rate of the thermodynamic forces are defined when the chain rule is used with respect to its variables:

$$\begin{aligned}
\dot{\sigma}_{ij} &= \frac{d\sigma_{ij}(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij})}{dt} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \dot{\varepsilon}_{kl}^p + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \dot{D}_{kl} \\
\dot{Y}_{ij} &= \frac{dY_{ij}(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij})}{dt} = \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \dot{\varepsilon}_{kl}^p + \frac{\partial Y_{ij}}{\partial D_{kl}} \dot{D}_{kl} \\
\dot{R} &= \frac{dR(p)}{dt} = \frac{\partial R}{\partial p} \dot{p} \\
\dot{\gamma} &= \frac{d\gamma(\delta)}{dt} = \frac{\partial \gamma}{\partial \delta} \dot{\delta}
\end{aligned} \tag{4.43}$$

Also,

$$\dot{\sigma}_{ij} = \frac{d}{dt} [C_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^p)] = C_{ijkl} \dot{\varepsilon}_{kl} - C_{ijkl} \dot{\varepsilon}_{kl}^p + \dot{C}_{ijkl} \varepsilon_{kl}^e \tag{4.44}$$

Then, equating Eqs (4.44) and (4.43a), the following relations are obtained,

$$\left\{ \begin{array}{l} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = C_{ijkl} \\ \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} = -C_{ijkl} \\ \frac{\partial \sigma_{ij}}{\partial D_{kl}} \dot{D}_{kl} = \dot{C}_{ijkl} \varepsilon_{kl}^e \end{array} \right. \quad (4.45)$$

Where the stress rate is finally reduced to,

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\varepsilon}_{kl} - C_{ijkl} \dot{\varepsilon}_{kl}^p + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \dot{D}_{kl} \quad (4.46)$$

4.3.4 State variables rates

Two yield potentials are defined, one related to the plastic evolution and other to the damage evolution. Similar to the case of the Helmholtz free-energy potential, both potentials develop independently from each other:

$$\left\{ \begin{array}{l} f_p(\sigma_{ij}, R) \\ f_d(Y_{ij}, \gamma) \end{array} \right. \quad (4.47)$$

Applying the normality rule to all internal state variables, its rates are then defined as

$$\begin{aligned} \dot{\varepsilon}_{ij}^p &= \dot{\lambda}^p \frac{\partial f_p(\sigma_{ij}, R)}{\partial \sigma_{ij}} + \dot{\lambda}^d \frac{\partial f_d(Y_{ij}, \gamma)}{\partial \sigma_{ij}} = \dot{\lambda}^p \frac{\partial f_p}{\partial \sigma_{ij}} \\ \dot{D}_{ij} &= \dot{\lambda}^p \frac{\partial f_p(\sigma_{ij}, R)}{\partial Y_{ij}} + \dot{\lambda}^d \frac{\partial f_d(Y_{ij}, \gamma)}{\partial Y_{ij}} = \dot{\lambda}^d \frac{\partial f_d}{\partial Y_{ij}} \\ \dot{\sigma} &= \dot{\lambda}^p \frac{\partial f_p(\sigma_{ij}, R)}{\partial \gamma} + \dot{\lambda}^d \frac{\partial f_d(Y_{ij}, \gamma)}{\partial \gamma} = \dot{\lambda}^d \frac{\partial f_d}{\partial \gamma} \\ \dot{p} &= \dot{\lambda}^p \frac{\partial f_p(\sigma_{ij}, R)}{\partial R} + \dot{\lambda}^d \frac{\partial f_d(Y_{ij}, \gamma)}{\partial R} = \dot{\lambda}^p \frac{\partial f_p}{\partial R} \end{aligned} \quad (4.48)$$

If both yield potential f and yield surface g are the same, the model is said to be associated otherwise it is called non-associated. The present model considers an associative behavior, needing the definition in detail of the surface yield function.

The stress rate can be redefined as a function of the Lagrange multipliers when Eqs. (4.48a) and (4.48b) are introduced into Eq. (4.46) as,

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\epsilon}_{kl} - C_{ijkl} \frac{\partial f_p}{\partial \sigma_{kl}} \dot{\lambda}^p + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{kl}} \dot{\lambda}^d \quad (4.49)$$

4.3.5 Yield surfaces

Two yield surfaces are defined, the plastic-strain g_p and damage g_d ,

$$\begin{cases} g_p(\sigma_{ij}, R) \\ g_d(Y_{ij}, \gamma) \end{cases} \quad (4.50)$$

The **plastic-strain surface** is defined as

$$g_p(\sigma_{ij}, R) = \hat{g}_p(\sigma_{ij}) - \hat{R}(R) \quad (4.51)$$

where

$$\begin{cases} \hat{g}_p = \frac{f_{ij} \sigma_{ij}}{2} + \sqrt{\left(\frac{f_{ij} \sigma_{ij}}{2}\right)^2 + \sigma_{ij} f_{ijkl} \sigma_{kl}} \\ \hat{R} = R(p) + R_0 \end{cases}$$

The yield-surface potential is based on the Tsai-Wu failure polynomial, which is an interacting failure criterion in which all stress components are used simultaneously to determine whether a failure at a material point has occurred or not. The tensorial expression of the three-dimensional form of the modified quadratic Tsai failure is used to define the equivalent state of stresses in the media, as define in Eq. (4.35). The values of the nonzero polynomial coefficients, f_{ij} and f_{ijkl} , are dependent on the allowable material ultimate strength values as given by

$$\begin{aligned} f_{11} = F_1 &= \frac{1}{F_{1t}} - \frac{1}{F_{1c}}; & f_{22} = F_2 &= \frac{1}{F_{2t}} - \frac{1}{F_{2c}}; & f_{33} = F_3 &= \frac{1}{F_{3t}} - \frac{1}{F_{3c}} \\ f_{1111} = F_{11} &= \frac{1}{F_{1t} F_{1c}}; & f_{2222} = F_{22} &= \frac{1}{F_{2t} F_{2c}}; & f_{3333} = F_{33} &= \frac{1}{F_{3t} F_{3c}} \\ f_{2233} = f_{3322} &= \frac{1}{2} F_{23} \cong \frac{1}{2} \frac{(-1)}{\sqrt{F_{2t} F_{2c} F_{3t} F_{3c}}} \\ f_{1133} = f_{3311} &= \frac{1}{2} F_{13} \cong \frac{1}{2} \frac{(-1)}{\sqrt{F_{1t} F_{1c} F_{3t} F_{3c}}} \\ f_{1122} = f_{2211} &= \frac{1}{2} F_{12} \cong \frac{1}{2} \frac{(-1)}{\sqrt{F_{1t} F_{1c} F_{2t} F_{2c}}} \\ f_{2323} = f_{3232} = f_{3223} = f_{2332} &= \frac{1}{4} F_{44} = \frac{1}{4} \frac{1}{F_4^2} \end{aligned}$$

$$\begin{aligned}
f_{1313} &= f_{1331} = f_{3113} = f_{3131} = \frac{1}{4} F_{55} = \frac{1}{4} \frac{1}{F_5^2} \\
f_{1212} &= f_{1221} = f_{2112} = f_{2121} = \frac{1}{4} F_{66} = \frac{1}{4} \frac{1}{F_6^2}
\end{aligned} \tag{4.52}$$

where,

- F_{1t} = Longitudinal tensile strength
- F_{1c} = Longitudinal compressive strength
- F_{2t} = Transverse tensile strength
- F_{2c} = Transverse compressive strength
- F_{3t} = Intralaminar tensile strength
- F_{3c} = Intralaminar compressive strength
- F_4 = Intralaminar shear strength
- F_5 = Intralaminar shear strength
- F_6 = Inplane shear strength

The **damage surface** is defined similar to the plastic-strain surface:

$$g_d(Y_{ij}, \gamma) = \hat{g}_d(Y_{ij}) - \hat{\gamma}(\gamma) \tag{4.53}$$

where

$$\begin{cases} \hat{g}_d = \sqrt{Y_{ij} h_{ijkl} Y_{kl}} \\ \hat{\gamma} = \gamma(\delta) + \gamma_0 \end{cases}$$

where Y^N and Y^S represent the thermodynamic forces due to normal and shear stresses respectively. The coefficients h_{ijkl}^N and h_{ijkl}^S are the tensorial expressions of the material properties as explained in detail in Section 5.1.2, where the nonzero terms are expressed as a function of the material strengths and the initial and ultimate stiffness,

$$\begin{aligned}
h_{1111} &= H_{11}^N = \begin{cases} \frac{\tilde{E}_{1t}^3}{\bar{E}_{11} F_{1t}^4}, & \sigma_{11} \geq 0 \\ \frac{\tilde{E}_{1c}^3}{\bar{E}_{11} F_{1c}^4}, & \sigma_{11} < 0 \end{cases} \\
h_{2222} &= H_{22}^N = \begin{cases} \frac{\tilde{E}_{2t}^3}{\bar{E}_{22} F_{2t}^4}, & \sigma_{22} \geq 0 \\ \frac{\tilde{E}_{2c}^3}{\bar{E}_{22} F_{2c}^4}, & \sigma_{22} < 0 \end{cases} \\
h_{3333} &= H_{33}^N = \begin{cases} \frac{\tilde{E}_{3t}^3}{\bar{E}_3 F_{3t}^4}, & \sigma_{33} \geq 0 \\ \frac{\tilde{E}_{3c}^3}{\bar{E}_3 F_{3c}^4}, & \sigma_{33} < 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
h_{1212} &= h_{1221} = h_{2112} = h_{2121} = \frac{1}{4} H_{12}^S = \frac{1}{4} \frac{\tilde{G}_{12}^3}{\overline{G}_{12} F_6^4} \\
h_{1313} &= h_{1331} = h_{3113} = h_{3131} = \frac{1}{4} H_{13}^S = \frac{1}{4} \frac{\tilde{G}_{13}^3}{\overline{G}_{13} F_5^4} \\
h_{2323} &= h_{2332} = h_{3223} = h_{3232} = \frac{1}{4} H_{23}^S = \frac{1}{4} \frac{\tilde{G}_{23}^3}{\overline{G}_{23} F_4^4}
\end{aligned} \tag{4.54}$$

The material properties H_i^S are those related to the shear components. While H_i^N are related to the principal directions and takes into account the sense of the strain direction, tension, or compression, and thus

$$H_i^N = \begin{cases} H_{it}^N, & \sigma_{ii} \geq 0 \\ H_{ic}^N, & \sigma_{ii} < 0 \end{cases} \text{ (no sum over } i) \tag{4.55}$$

4.3.6 Jacobian

The consistency conditions will be dependant on the type of evolution expected. These conditions can be divided into no plastic nor damage, only plastic, only damage, and both damage-plastic evolutions. For each case the Jacobian is defined.

4.3.6.1 No plasticity or damage

The Jacobian is well known and expressed as

$$\dot{\sigma}_{ij} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} \tag{4.56}$$

when no damage or plastic evolution is seen the Jacobian is the elastic stiffness, or

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = C_{ijkl}^e, \quad \begin{cases} \dot{\lambda}^p > 0 \\ \dot{\lambda}^d > 0 \end{cases} \tag{4.57}$$

4.3.6.2 Only plasticity

When no damage is present $\dot{\lambda}^d = 0$, and applying consistency conditions for plasticity potential only, we have

$$\dot{g}_p = \frac{d g_p(\sigma_{ij}, R)}{dt} = \frac{\partial g_p}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial g_p}{\partial R} \dot{R} = 0 \quad (4.58)$$

then, after mathematical manipulation the plastic multiplier is obtained as

$$\begin{aligned} \frac{\partial g_p}{\partial \sigma_{ij}} \left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \dot{\varepsilon}_{kl}^p + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \dot{D}_{kl} \right) + \frac{\partial g_p}{\partial R} \left(\frac{\partial R}{\partial p} \dot{p} \right) &= 0 \\ \frac{\partial g_p}{\partial \sigma_{ij}} \left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \dot{\lambda}^p \frac{\partial f_p}{\partial \sigma_{ij}} + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \dot{\lambda}^d \frac{\partial f_d}{\partial Y_{ij}} \right) + \frac{\partial g_p}{\partial R} \left(\frac{\partial R}{\partial p} \dot{\lambda}^p \frac{\partial f_p}{\partial R} \right) &= 0 \\ \left(\frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{ij}} + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial p} \frac{\partial f_p}{\partial R} \right) \dot{\lambda}^p &= - \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} \\ a_{11} \dot{\lambda}^p &= -b_{kl}^p \dot{\varepsilon}_{kl} \\ \dot{\lambda}^p &= - \frac{b_{kl}^p \dot{\varepsilon}_{kl}}{a_{11}} \\ \dot{\lambda}^p &= L_{kl}^{pp} \dot{\varepsilon}_{kl} \end{aligned} \quad (4.59)$$

Replacing into the stress rate given in Eq. (4.49)

$$\dot{\sigma}_{ij} = \left(C_{ijkl} - C_{ijmn} \frac{\partial f_p}{\partial \sigma_{mn}} L_{kl}^{pp} \right) \dot{\varepsilon}_{kl} \quad (4.60)$$

then the Jacobian or elasto-plasto tangent modulus is obtained for the case of only plastic evolution as,

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = C_{ijkl}^e + C_{ijkl}^{pp}, \quad \begin{cases} \dot{\lambda}^p > 0 \\ \dot{\lambda}^d = 0 \end{cases} \quad (4.61)$$

where,

$$\begin{cases} C_{ijkl}^e = C_{ijkl} \\ C_{ijkl}^{pp} = -C_{ijmn} \frac{\partial f_p}{\partial \sigma_{mn}} L_{kl}^{pp} \end{cases}$$

4.3.6.3 Only damage

When no plasticity is present $\dot{\lambda}^p = 0$, and applying the consistency conditions for only damage potential, we have

$$\dot{g}_d = \frac{dg_d(Y_{ij}, \gamma)}{dt} = \frac{\partial g_d}{\partial Y_{ij}} \dot{Y}_{ij} + \frac{\partial g_d}{\partial \gamma} \dot{\gamma} = 0 \quad (4.62)$$

Then, after mathematical manipulation the damage multiplier is obtained as

$$\begin{aligned} & \frac{\partial g_d}{\partial Y_{ij}} \left(\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \dot{\varepsilon}_{kl}^p + \frac{\partial Y_{ij}}{\partial D_{kl}} \dot{D}_{kl} \right) + \frac{\partial g_d}{\partial \gamma} \left(\frac{\partial \gamma}{\partial \delta} \dot{\delta} \right) \dot{\gamma} = 0 \\ & \frac{\partial g_d}{\partial Y_{ij}} \left(\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} + \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \dot{\lambda}^p \frac{\partial f_p}{\partial \sigma_{ij}} + \frac{\partial Y_{ij}}{\partial D_{kl}} \dot{\lambda}^d \frac{\partial f_d}{\partial Y_{ij}} \right) + \frac{\partial g_d}{\partial \gamma} \left(\frac{\partial \gamma}{\partial \delta} \dot{\lambda}^d \frac{\partial f_d}{\partial \gamma} \right) = 0 \\ & \left(\frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{ij}} \right) \dot{\lambda}^p + \left(\frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{ij}} + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma}{\partial \delta} \frac{\partial f_d}{\partial \gamma} \right) \dot{\lambda}^d = - \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} \\ & a_{22} \dot{\lambda}^d = -b_{kl}^d \dot{\varepsilon}_{kl} \\ & \dot{\lambda}^d = - \frac{b_{kl}^d \dot{\varepsilon}_{kl}}{a_{22}} \\ & \dot{\lambda}^d = L_{kl}^{dd} \dot{\varepsilon}_{kl} \end{aligned} \quad (4.63)$$

Substituting into the stress rate given in Eq. (4.49),

$$\dot{\sigma}_{ij} = \left(C_{ijkl} + \frac{\partial \sigma_{ij}}{\partial D_{mn}} \frac{\partial f_d}{\partial Y_{mn}} L_{kl}^{dd} \right) \dot{\varepsilon}_{kl} \quad (4.64)$$

and comparing it with the Jacobian definition given in Eq. (4.56), then the Jacobian or elasto-damage tangent modulus is obtained for the case of only damage evolution as,

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = C_{ijkl}^e + C_{ijkl}^{dd}, \quad \begin{cases} \dot{\lambda}^p = 0 \\ \dot{\lambda}^d > 0 \end{cases} \quad (4.65)$$

where,

$$\begin{cases} C_{ijkl}^e = C_{ijkl} \\ C_{ijkl}^{dd} = \frac{\partial \sigma_{ij}}{\partial D_{mn}} \frac{\partial f_d}{\partial Y_{mn}} L_{kl}^{dd} \end{cases}$$

4.3.6.4 Plasticity and damage

Applying the consistency conditions for both surface potentials, we have

$$\begin{cases} \dot{g}_p = \frac{\partial g_p}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial g_p}{\partial R} \dot{R} = 0 \\ \dot{g}_d = \frac{\partial g_d}{\partial Y_{ij}} \dot{Y}_{ij} + \frac{\partial g_d}{\partial \gamma} \dot{\gamma} = 0 \end{cases} \quad (4.66)$$

after using the previous relations, the consistency conditions can be expressed in a matrix form as,

$$\begin{bmatrix} \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{kl}} + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial p} \frac{\partial f_p}{\partial R} & \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{kl}} \\ \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{kl}} & \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{kl}} + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma}{\partial \delta} \frac{\partial f_d}{\partial \gamma} \end{bmatrix} \begin{Bmatrix} \dot{\lambda}^p \\ \dot{\lambda}^d \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} \\ -\frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \dot{\varepsilon}_{kl} \end{Bmatrix}$$

or,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \dot{\lambda}^p \\ \dot{\lambda}^d \end{Bmatrix} = \begin{Bmatrix} -b_{ij}^p \dot{\varepsilon}_{ij} \\ -b_{ij}^d \dot{\varepsilon}_{ij} \end{Bmatrix}$$

Inverting the matrix, a solution for the multipliers is obtained as,

$$\begin{Bmatrix} \dot{\lambda}^p \\ \dot{\lambda}^d \end{Bmatrix} = \frac{1}{a_{11} a_{22} - a_{12} a_{21}} \begin{Bmatrix} a_{12} b_{ij}^d - a_{22} b_{ij}^p \\ a_{21} b_{ij}^p - a_{11} b_{ij}^d \end{Bmatrix} \dot{\varepsilon}_{ij}$$

or, when expressed explicitly

$$\begin{cases} \dot{\lambda}^p = \frac{a_{12} b_{ij}^d - a_{22} b_{ij}^p}{a_{11} a_{22} - a_{12} a_{21}} \dot{\varepsilon}_{ij} = L_{ij}^p \dot{\varepsilon}_{ij} \\ \dot{\lambda}^d = \frac{a_{21} b_{ij}^p - a_{11} b_{ij}^d}{a_{11} a_{22} - a_{12} a_{21}} \dot{\varepsilon}_{ij} = L_{ij}^d \dot{\varepsilon}_{ij} \end{cases} \quad (4.67)$$

Replacing the previous relations into the stress rate,

$$\dot{\sigma}_{ij} = \left(C_{ijkl} - C_{ijmn} \frac{\partial f_p}{\partial \sigma_{mn}} L_{kl}^p + \frac{\partial \sigma_{ij}}{\partial D_{mn}} \frac{\partial f_d}{\partial Y_{mn}} L_{kl}^d \right) \dot{\varepsilon}_{kl} \quad (4.68)$$

then the Jacobian or elasto-plasto-damage tangent modulus is obtained for the case of plastic and damage evolution as,

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = C_{ijkl}^e + C_{ijkl}^p + C_{ijkl}^d, \quad \begin{cases} \dot{\lambda}^p > 0 \\ \dot{\lambda}^d > 0 \end{cases} \quad (4.69)$$

where,

$$\begin{cases} C_{ijkl}^e = C_{ijkl} \\ C_{ijkl}^{pp} = -C_{ijmn} \frac{\partial f_p}{\partial \sigma_{mn}} L_{kl}^p \\ C_{ijkl}^{pp} = \frac{\partial \sigma_{ij}}{\partial D_{mn}} \frac{\partial f_d}{\partial Y_{mn}} L_{kl}^d \end{cases}$$

4.3.7 Discretization

A plastic behavior is expected when the yield surface is positive $g>0$ or the state of stresses exceeds the yield boundary when the total strain has reached a new value. At this point the Lagrange multiplier is zero, point C' in Figure 4-13, but the equilibrium will be reached when the yield surface leads to zero, $g=0$ for the same total strain. At this point, C , the Lagrangian multiplier is bigger than zero.

Figure 4-13 Discretization of the Lagrange multiplier evolution

As it can be seen, the new value of the Lagrangian multiplier can be solved when the yield surface function is zero. Since this surface function can be obtained parametrically from the Lagrangian multiplier, then the root or zero of the function can be obtained using the Newton method, which is an iterative method where each step is defined as a function of the previous, as

$$\dot{\lambda}_k = \dot{\lambda}_{k-1} - \frac{g|_{k-1}}{\left. \frac{dg}{d\dot{\lambda}} \right|_{k-1}} \quad (4.70)$$

Since no variation in the total strain is expected for the actual time step $()^n$, by definition, the partial derivative of this total strain with respect either the plastic or damage multipliers are zero; then

$$\varepsilon_{ij} = (\varepsilon_{ij})^n = (\varepsilon_{ij})^{n-1} + (\Delta \varepsilon_{ij})^n \quad \Rightarrow \quad \frac{\partial \varepsilon_{ij}}{\partial \dot{\lambda}^p} = \frac{\partial \varepsilon_{ij}}{\partial \dot{\lambda}^d} = 0 \quad (4.71)$$

From the rate of the internal variables given in Eq. (4.48), the following relations are derived as,

$$\begin{aligned} \frac{\partial \varepsilon_{ij}^p}{\partial \dot{\lambda}^p} &= \frac{\partial f_p}{\partial \sigma_{ij}} \\ \frac{\partial D_{ij}}{\partial \dot{\lambda}^d} &= \frac{\partial f_d}{\partial Y_{ij}} \\ \frac{\partial \delta}{\partial \dot{\lambda}^d} &= \frac{\partial f_d}{\partial \gamma} \\ \frac{\partial p}{\partial \dot{\lambda}^p} &= \frac{\partial f_p}{\partial R} \end{aligned} \quad (4.72)$$

and also,

$$\frac{\partial \varepsilon_{ij}^p}{\partial \dot{\lambda}^d} = \frac{\partial D_{ij}}{\partial \dot{\lambda}^p} = \frac{\partial \delta}{\partial \dot{\lambda}^p} = \frac{\partial p}{\partial \dot{\lambda}^d} = 0 \quad (4.73)$$

The derivative of the plastic and damage surfaces with respect to the plastic and damage Lagrangian multipliers are obtained using the relations in Eqs. (4.71) to (4.73) and applying the chain rule, we get

$$\begin{aligned} \frac{dg_p}{d\dot{\lambda}^p} &= \frac{dg_p(\sigma_{ij}, R)}{d\dot{\lambda}^p} = \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \dot{\lambda}^p} + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial \dot{\lambda}^p} \\ &= \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij})}{\partial \dot{\lambda}^p} + \frac{\partial g_p}{\partial R} \frac{\partial R(p)}{\partial \dot{\lambda}^p} \\ &= \frac{\partial g_p}{\partial \sigma_{ij}} \left[\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \frac{\partial \varepsilon_{kl}}{\partial \dot{\lambda}^p} + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial \varepsilon_{kl}^p}{\partial \dot{\lambda}^p} + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial D_{kl}}{\partial \dot{\lambda}^p} \right] + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial p} \frac{\partial p}{\partial \dot{\lambda}^p} \\ &= \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{ij}} + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial p} \frac{\partial f_p}{\partial R} \end{aligned}$$

$$\frac{dg_p}{d\dot{\lambda}^p} = a_{11} \quad (4.74)$$

$$\begin{aligned} \frac{\partial g_p}{\partial \dot{\lambda}^d} &= \frac{\partial g_p(\sigma_{ij}, R)}{\partial \dot{\lambda}^d} = \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \dot{\lambda}^d} + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial \dot{\lambda}^d} \\ &= \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij})}{\partial \dot{\lambda}^d} + \frac{\partial g_p}{\partial R} \frac{\partial R(p)}{\partial \dot{\lambda}^d} \\ &= \frac{\partial g_p}{\partial \sigma_{ij}} \left[\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \frac{\partial \varepsilon_{kl}}{\partial \dot{\lambda}^d} + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial \varepsilon_{kl}^p}{\partial \dot{\lambda}^d} + \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial D_{kl}}{\partial \dot{\lambda}^d} \right] + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial p} \frac{\partial p}{\partial \dot{\lambda}^d} \\ &= \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{ij}} \end{aligned}$$

$$\frac{\partial g_p}{\partial \dot{\lambda}^d} = a_{12} \quad (4.75)$$

$$\begin{aligned} \frac{\partial g_d}{\partial \dot{\lambda}^d} &= \frac{\partial g_d(Y_{ij}, \gamma)}{\partial \dot{\lambda}^d} = \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \dot{\lambda}^d} + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma}{\partial \dot{\lambda}^d} \\ &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij})}{\partial \dot{\lambda}^d} + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma(\delta)}{\partial \dot{\lambda}^d} \\ &= \frac{\partial g_d}{\partial Y_{ij}} \left[\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \frac{\partial \varepsilon_{kl}}{\partial \dot{\lambda}^d} + \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial \varepsilon_{kl}^p}{\partial \dot{\lambda}^d} + \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial D_{kl}}{\partial \dot{\lambda}^d} \right] + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma}{\partial \delta} \frac{\partial \delta}{\partial \dot{\lambda}^d} \\ &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{ij}} + \frac{\partial g_d}{\partial \gamma} \frac{\partial R}{\partial p} \frac{\partial f_d}{\partial \gamma} \end{aligned}$$

$$\frac{\partial g_d}{\partial \dot{\lambda}^d} = a_{22} \quad (4.76)$$

$$\begin{aligned} \frac{\partial g_d}{\partial \dot{\lambda}^p} &= \frac{\partial g_d(Y_{ij}, \gamma)}{\partial \dot{\lambda}^p} = \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \dot{\lambda}^p} + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma}{\partial \dot{\lambda}^p} \\ &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}(\varepsilon_{ij}, \varepsilon_{ij}^p, D_{ij})}{\partial \dot{\lambda}^p} + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma(\delta)}{\partial \dot{\lambda}^p} \\ &= \frac{\partial g_d}{\partial Y_{ij}} \left[\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} \frac{\partial \varepsilon_{kl}}{\partial \dot{\lambda}^p} + \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial \varepsilon_{kl}^p}{\partial \dot{\lambda}^p} + \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial D_{kl}}{\partial \dot{\lambda}^p} \right] + \frac{\partial g_d}{\partial \gamma} \frac{\partial \gamma}{\partial \delta} \frac{\partial \delta}{\partial \dot{\lambda}^p} \\ &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{kl}} \end{aligned}$$

$$\frac{\partial g_d}{\partial \dot{\lambda}^p} = a_{21} \quad (4.77)$$

When an associated formulation is considered, the discretization of the internal variables is obtained from the rate of internal variables as,

$$\begin{aligned}
\dot{\varepsilon}_{ij}^p &= \dot{\lambda}^p \frac{\partial f_p}{\partial \sigma_{ij}} \quad \Rightarrow \quad (\varepsilon_{ij}^p)_k^n = (\varepsilon_{ij}^p)_k^{n-1} + \dot{\lambda}_k^p \left(\frac{\partial f_p}{\partial \sigma_{ij}} \right)_k^n = (\varepsilon_{ij}^p)_k^{n-1} + \dot{\lambda}_k^p \left(\frac{\partial g_p}{\partial \sigma_{ij}} \right)_k^n \\
\dot{p} &= \dot{\lambda}^p \frac{\partial f_p}{\partial R} \quad \Rightarrow \quad (p)_k^n = (p)_k^{n-1} + \dot{\lambda}_k^p \left(\frac{\partial f_p}{\partial R} \right)_k^n = (p)_k^{n-1} - \dot{\lambda}_k^p \\
\dot{D}_{ij} &= \dot{\lambda}^d \frac{\partial f_d}{\partial Y_{ij}} \quad \Rightarrow \quad (D_{ij})_k^n = (D_{ij})_k^{n-1} + \dot{\lambda}_k^d \left(\frac{\partial f_d}{\partial Y_{ij}} \right)_k^n = (D_{ij})_k^{n-1} + \dot{\lambda}_k^d \left(\frac{\partial g_d}{\partial Y_{ij}} \right)_k^n \\
\dot{\delta} &= \dot{\lambda}^d \frac{\partial f_d}{\partial \gamma} \quad \Rightarrow \quad (\delta)_k^n = (\delta)_k^{n-1} + \dot{\lambda}_k^d \left(\frac{\partial f_d}{\partial \gamma} \right)_k^n = (\delta)_k^{n-1} - \dot{\lambda}_k^d
\end{aligned} \tag{4.78}$$

The previous formulations are used to define the evolution of the discrete Lagrange multipliers which will converge to values that make true consistency conditions or cause yield or damage functions become zero. Different approaches are taken into account depending on the type of evolution considered.

4.3.7.1 Only plasticity

For the case of only plastic evolution, the Newton Rhapson iteration method is defined as,

$$\dot{\lambda}_k^p = \dot{\lambda}_{k-1}^p - \frac{g_p|_{k-1}}{\left. \frac{dg_p}{d\lambda^p} \right|_{k-1}} \tag{4.79}$$

and finally the discretization of the Lagrange multiplier evolution is then obtained as,

$$\lambda_k^p = \lambda_{k-1}^p - \frac{(g_p)_{k-1}^n}{(a_{11})_{k-1}^n} \tag{4.80}$$

4.3.7.2 Only damage

For the case of only damage evolution, the Newton-Rhapson iteration method is defined as,

$$\dot{\lambda}_k^d = \dot{\lambda}_{k-1}^d - \frac{g_d|_{k-1}}{\left. \frac{dg_d}{d\dot{\lambda}^d} \right|_{k-1}} \quad (4.81)$$

and its discretization of the Lagrange damage multiplier evolution is obtained as,

$$\dot{\lambda}_k^d = \dot{\lambda}_{k-1}^d - \frac{(g_d)_{k-1}^n}{(a_{22})_{k-1}^n} \quad (4.82)$$

4.3.7.3 Plasticity and damage

For the case of both plastic and damage evolution, the Newton Rhapsion iteration method is defined using a Jacobian matrix defined as,

$$\begin{bmatrix} \frac{\partial g_p}{\partial \dot{\lambda}^p} & \frac{\partial g_p}{\partial \dot{\lambda}^d} \\ \frac{\partial g_d}{\partial \dot{\lambda}^p} & \frac{\partial g_d}{\partial \dot{\lambda}^d} \end{bmatrix}_{k-1} \begin{Bmatrix} \dot{\lambda}_k^p - \dot{\lambda}_{k-1}^p \\ \dot{\lambda}_k^d - \dot{\lambda}_{k-1}^d \end{Bmatrix} = - \begin{Bmatrix} g_p \\ g_d \end{Bmatrix}_{k-1} \quad (4.83)$$

and the discretization of both plastic and damage Lagrange multipliers are obtained as,

$$\begin{bmatrix} (a_{11})_{k-1}^n & (a_{12})_{k-1}^n \\ (a_{21})_{k-1}^n & (a_{22})_{k-1}^n \end{bmatrix} \begin{Bmatrix} \dot{\lambda}_k^p - \dot{\lambda}_{k-1}^p \\ \dot{\lambda}_k^d - \dot{\lambda}_{k-1}^d \end{Bmatrix} = - \begin{Bmatrix} (g_p)_{k-1}^n \\ (g_d)_{k-1}^n \end{Bmatrix}$$

then,

$$\begin{cases} \dot{\lambda}_k^p = \dot{\lambda}_{k-1}^p - \frac{(a_{22})_{k-1}^n (g_p)_{k-1}^n - (a_{12})_{k-1}^n (g_d)_{k-1}^n}{(a_{11})_{k-1}^n (a_{22})_{k-1}^n - (a_{12})_{k-1}^n (a_{21})_{k-1}^n} \\ \dot{\lambda}_k^d = \dot{\lambda}_{k-1}^d - \frac{(a_{11})_{k-1}^n (g_d)_{k-1}^n - (a_{21})_{k-1}^n (g_p)_{k-1}^n}{(a_{11})_{k-1}^n (a_{22})_{k-1}^n - (a_{12})_{k-1}^n (a_{21})_{k-1}^n} \end{cases} \quad (4.84)$$

4.4 Material Parameters Evaluation

The material parameters used to define the plastic yield surface are well known as the strength failure values. These plastic parameters are defined in Eq. (4.52) and included in

the Tsai-Wu failure coefficients. No parameters are available in the literature for the coefficients of the damage surface, as expressed in Eq. (4.54).

The coefficients that define the plastic or damage evolution, expressed in Eqs. (4.42c) and (4.42d) respectively, for the case of Prony series are not known. The proposed method to find these coefficients starts with an experimental inplane shear stress strain data, where load and unload have been performed to the probe, as seen in Figure 4-14.

Figure 4-14 Stress-strain load/unload available data

Important information is obtained directly from this stress-strain curve, where the information will depend on the type of loading done onto the specimen, tension, compression, shear and the applied load direction with respect to the material coordinates 1, 2 or 3. The initial or undamaged stiffness $\bar{E}_{ij}$, the final, ultimate, or critical stiffness $\tilde{E}_{ij}$, the ultimate strength F_{ij} , the limit linear strength S_{ij}^{Linear} , the final, ultimate, or critical total strain $\tilde{\epsilon}_{ij}$, and the critical, ultimate, or final plastic strain $\tilde{\epsilon}_{ij}^p$. A fitted stress-strain curve can also be evaluated.

When the strain vs. plastic strain is plotted using the available data, the strain damage threshold can be obtained after the points are fitted. Where this fitted function intersects the strain axis the strain threshold is defined. In Figure 4-15(a) this fitted curve is represented as a straight line, but it could also be concave or convex curve.

In an analogous manner the strain damage threshold can be found. The main difference is that the strain threshold is defined when the fitted curve intersects the undamaged stiffness. These strain threshold values can now be plotted in the stress-strain curve, as seen in Figure 4-16. Using the fitted stress-strain curve obtained for Figure 4-14, strength threshold values are obtained for plastic and damage behaviors.

Figure 4-15 Plastic and damage strain threshold

It is important to notice that not all the threshold values for plasticity and damage will be the same. Experimental data suggests usually plasticity has bigger values, depending on the material evaluated. In few cases the damage threshold values are bigger.

Figure 4-16 Plastic and damage stress threshold

Considering an associative behavior and noting that the Lagrangian multiplier is the same for the isotropic plastic evolution and plastic strain ratios, a relation of these rates can be expressed as

$$\begin{cases} \dot{p} = \dot{\lambda}^p \frac{\partial f_p}{\partial R} \\ \dot{\varepsilon}_{ij}^p = \dot{\lambda}^p \frac{\partial f_p}{\partial \sigma_{ij}} \end{cases} \Rightarrow \frac{\partial g_p}{\partial \sigma_{ij}} \dot{p} = -\dot{\varepsilon}_{ij}^p \quad (4.85)$$

Since the partial derivate of the plastic surface w.r.t. the stress is independent of the plastic strain and the isotropic plastic evolution and since at the beginning of the integration time $t = 0$, $\varepsilon_{ij}^p = 0$ and $p = 0$, then after performing the integration of Eq. (4.85), a relation between these two internal variables can be expressed as

$$\frac{\partial g_p}{\partial \sigma_{ij}} p = -\varepsilon_{ij}^p \quad (4.86)$$

Starting with a plastic stress-strain plot of the available data, a plot of the isotropic plastic evolution stress can be done using Eq. (4.86), as seen in Figure 4-17. In this plot the plastic surface is define for $g_p = 0$ with boundaries at $p = 0$ and $p = \tilde{p}$, so

$$g_p = \hat{g}_p - \hat{R} = \hat{g}_p - (R + R_0) = 0 \quad \Rightarrow \quad \hat{g}_p(\sigma_{ij}) - [R(p) + R_0] = 0 \quad (4.87)$$

Figure 4-17 Plastic surface fitting given available data

Finally, after fitting the data and using the boundary conditions, all material properties and Prony coefficients can be obtained for plastic evolution.

A similar procedure is used when damage is analyzed. First a relation between damage D_{ij} and isotropic hardening damage evolution δ is obtained as

$$\frac{\partial g_d}{\partial Y_{ij}} \delta = -D_{ij} \quad (4.88)$$

The damage surface g_d is obtained given the thermodynamic damage force Y_{ij} , Eq. (4.53), which are function of the state of stresses. Also from Eq. (4.15b) a relation between the

damaged stiffness and the total strain was defined; from this relation and from Eq. (4.88) a relation between the total stiffness with the isotropic hardening damage evolution can be extended. Then a plot of the available data can be obtained as seen in Figure 4-18. The damage surface is obtained for $g_d = 0$, so

$$g_d = \hat{g}_d - \hat{\gamma} = \hat{g}_p - (\gamma + \gamma_0) = 0 \quad \Rightarrow \quad \hat{g}_d(Y_{ij}(\sigma_{ij})) - [\gamma(\delta) + \gamma_0] = 0 \quad (4.89)$$

Figure 4-18 Damage surface fitting given available data

With the use of the boundary conditions seen in Figure 4-18(b) and fitting the curve for $g_d = 0$, the damage Prony coefficients and material damage properties can be obtained when comparing it with Eq. (4.89).

A complete plasticity-damage model has been developed in detail. This model can be implemented in a finite element code, which can be used to accurately predict failure in directional fiber composites, which is the main objective of this section.

This tool will alleviate the uncertainty of future designs, increasing the accuracy of new developments.

4.5 Yield Surface Material Parameters

4.5.1 Plastic-strain surface parameters

Barbero, E.J. and Lonetti, P. (2002) defined the plastic-strain surface as the Tsai-Wu polynomial criterion:

$$\hat{g}_p = \sigma_{ij} f_{ijkl} \sigma_{kl} + f_{ij} \sigma_{ij} \quad (4.90a)$$

or taking its square root as,

$$\hat{g}_p = \sqrt{\sigma_{ij} f_{ijkl} \sigma_{kl} + f_{ij} \sigma_{ij}} \quad (4.90b)$$

where the coefficients are defined as

$$\begin{cases} f_{ij} \sigma_{ij} = F_1 \sigma_{11} + F_2 \sigma_{22} + F_3 \sigma_{33} \\ \sigma_{ij} f_{ijkl} \sigma_{kl} = F_{11} \sigma_{11}^2 + F_{22} \sigma_{22}^2 + F_{33} \sigma_{33}^2 + F_{12} \sigma_{11} \sigma_{22} + F_{13} \sigma_{11} \sigma_{33} + F_{23} \sigma_{22} \sigma_{33} + \\ F_{66} \sigma_{12}^2 + F_{55} \sigma_{13}^2 + F_{44} \sigma_{23}^2 \end{cases}$$

where,

$$\begin{aligned} F_1 &= \frac{1}{F_{1t}} - \frac{1}{F_{1c}}; & F_2 &= \frac{1}{F_{2t}} - \frac{1}{F_{2c}}; & F_3 &= \frac{1}{F_{3t}} - \frac{1}{F_{3c}} & F_{11} &= \frac{1}{F_{1t} F_{1c}}; \\ F_{22} &= \frac{1}{F_{2t} F_{2c}}; & F_{33} &= \frac{1}{F_{3t} F_{3c}} & F_{44} &= \frac{1}{F_4^2}; & F_{55} &= \frac{1}{F_5^2}; & F_{66} &= \frac{1}{F_6^2} \\ F_{23} &= -\frac{1}{\sqrt{F_{2t} F_{2c} F_{3t} F_{3c}}}; & F_{13} &= -\frac{1}{\sqrt{F_{1t} F_{1c} F_{3t} F_{3c}}}; & F_{12} &= -\frac{1}{\sqrt{F_{1t} F_{1c} F_{2t} F_{2c}}} \end{aligned}$$

and,

F_{1t} : Longitudinal tensile strength
 F_{1c} : Longitudinal compressive strength
 F_{2t} : Transverse tensile strength
 F_{2c} : Transverse compressive strength
 F_{3t} : Intralaminar tensile strength
 F_{3c} : Intralaminar compressive strength
 F_4 : Intralaminar shear strength
 F_5 : Intralaminar shear strength
 F_6 : Inplane shear strength

The evolution of this polynomial failure index is plotted in the longitudinal and transversal directions, as seen in Figure 4-19. Here can be observed that there is a zone where the failure index is negative. Also the evolution from zero to one is observed as a proportionally bigger ellipse with respect to the initial zero-value ellipse.

Due to this negative value, the polynomial expression of the failure index or the plastic surface definition cannot be supported or accepted. A more developed definition given by Liu and Tsai (1998) expresses a quadratic failure criterion, defined as,

$$\frac{1}{\hat{g}_p^2} \sigma_{ij} f_{ijkl} \sigma_{kl} + \frac{1}{\hat{g}_p} f_{ij} \sigma_{ij} - 1 = 0 \quad (4.91)$$

Figure 4-19 σ_{11} - σ_{22} Tsai-Wu polynomial failure index evolution

Rewriting the quadratic Equation (4.91)

$$\frac{1}{\hat{g}_p^2} a + \frac{1}{\hat{g}_p} b - 1 = 0 \quad (4.92)$$

then the failure index, equivalent stress, or plastic-strain function is obtained as

$$\hat{g}_p = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + a} \quad (4.93)$$

where

$$\begin{cases} a = \sigma_{ij} f_{ijkl} \sigma_{kl} \\ b = f_{ij} \sigma_{ij} \end{cases}$$

and the coefficients are the same as those expressed in Eq. (4.90) for the Tsai-Wu polynomial index failure.

Figure 4-20 σ_{11} - σ_{22} Tsai-Liu quadratic failure index evolution

The definition given in Eq. (4.93) is always positive as seen in Figure 4-20. Also, when the failure index is one, the three definitions given in Eqs. (4.90a), (4.90b), and (4.93) are the same.

The coefficient a defined in Eq. (4.93) in its expanded form is developed as,

$$\begin{aligned}
 a &= \sigma_{ij} f_{ijkl} \sigma_{kl} \\
 &= F_{11} \sigma_{11}^2 + F_{22} \sigma_{22}^2 + F_{33} \sigma_{33}^2 + F_{12} \sigma_{11} \sigma_{22} + F_{13} \sigma_{11} \sigma_{33} + F_{23} \sigma_{22} \sigma_{33} \\
 &\quad + F_{66} \sigma_{12}^2 + F_{55} \sigma_{13}^2 + F_{44} \sigma_{23}^2 \\
 &= \left[\frac{1}{2} F_{11} \sigma_{11}^2 - \sqrt{F_{11} F_{22}} \sigma_{11} \sigma_{22} + \frac{1}{2} F_{22} \sigma_{22}^2 \right] + \left[\frac{1}{2} F_{11} \sigma_{11}^2 - \sqrt{F_{11} F_{33}} \sigma_{11} \sigma_{33} + \frac{1}{2} F_{33} \sigma_{33}^2 \right] \\
 &\quad + \left[\frac{1}{2} F_{22} \sigma_{22}^2 - \sqrt{F_{22} F_{33}} \sigma_{22} \sigma_{33} + \frac{1}{2} F_{33} \sigma_{33}^2 \right] + F_{66} \sigma_{12}^2 + F_{55} \sigma_{13}^2 + F_{44} \sigma_{23}^2 \\
 &= \frac{1}{2} \left(\sqrt{F_{11}} \sigma_{11} - \sqrt{F_{22}} \sigma_{22} \right)^2 + \frac{1}{2} \left(\sqrt{F_{11}} \sigma_{11} - \sqrt{F_{33}} \sigma_{33} \right)^2 + \frac{1}{2} \left(\sqrt{F_{22}} \sigma_{22} - \sqrt{F_{33}} \sigma_{33} \right)^2 + \\
 &\quad + F_{66} \sigma_{12}^2 + F_{55} \sigma_{13}^2 + F_{44} \sigma_{23}^2
 \end{aligned} \tag{4.94}$$

where it can be seen that the coefficient a is always positive or

$$a \geq 0 \quad (4.95)$$

also,

$$\sqrt{\left(\frac{b}{2}\right)^2 + a} \geq \sqrt{\left(\frac{b}{2}\right)^2} = \left|\frac{b}{2}\right| \geq \frac{b}{2}$$

finally, can be concluded that the plastic-strain function is always positive or,

$$\hat{g}_p = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + a} \geq 0 \quad (4.96)$$

An equivalence between the Tsai-Liu quadratic index failure and the von Mises distortion energy theory for isotropic materials can be stated when

$$\begin{cases} F_{11} = F_{22} = F_{33} = \frac{1}{S_y^2} \\ F_{44} = F_{55} = F_{66} = \frac{1}{\tau_y^2} \\ \tau_y = \frac{S_y}{\sqrt{3}} \end{cases} \quad (4.97)$$

where S_y and τ_y are the normal strength and shear strength respectively. Then when introducing Eq. (4.97) into Eq. (4.94), the coefficient a is obtained as

$$\begin{aligned} a &= \frac{1}{2} \frac{1}{S_y^2} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2 + (\sigma_{22} - \sigma_{33})^2] + \frac{1}{\tau_y^2} (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \\ &= \frac{1}{S_y^2} \left\{ \frac{1}{2} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2 + (\sigma_{22} - \sigma_{33})^2] + 3 (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2) \right\} \\ &= \left(\frac{\sigma_{eq}}{S_y} \right)^2 \end{aligned}$$

and since $b=0$, the quadratic index failure is then reduced to

$$\hat{g}_p = \sqrt{a} = \sqrt{\left(\frac{\sigma_{eq}}{S_y}\right)^2} = \frac{\sigma_{eq}}{S_y} \quad (4.98)$$

which is the same definition given for the von Mises distortion-energy failure index.

With this surface definition, failure conditions are satisfied when the loading is applied since no negative values are obtained for any state of stresses. The failure condition is expressed in Figure 4-17 and defined as,

$$\hat{g}_p(F_{ij}) = 1 \quad (4.99)$$

and the failure conditions stated explicitly are those for uniaxial tension and uniaxial compression in each direction and shear loading are stated as,

$$\begin{aligned} \hat{g}_p(\sigma_{11} = F_{1t}, \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{11} = -F_{1c}, \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{22} = F_{2t}, \sigma_{11} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{22} = -F_{2c}, \sigma_{11} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{33} = F_{3t}, \sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{33} = -F_{3c}, \sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{12} = F_6, \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{13} = F_5, \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = 0) &= 1 \\ \hat{g}_p(\sigma_{23} = F_4, \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = 0) &= 1 \end{aligned} \quad (4.100)$$

It must be noted that this failure condition is also reached when a combination of loads is taken into account.

4.5.2 Damage surface parameters

The surface damage parameters are related to the evolution of microvoids into macrocracks as shown in Figure 2-1. An equivalence between fracture mechanics and damage evolution will be discussed and presented in this section. These conclusions will allow us to define the shape of the surface damage function and its related coefficients.

Eshelby (1957) stated that when an infinite, isotropic, homogeneous, linear elastic solid or matrix, and initially stress free, is submitted to an internal ellipsoidal eigenstrain inclusion, it will produce displacement fields through the matrix, Figure 4-21. Stress-free transformation is the name given by Eshelby (1957) to the generic terminology eigenstrains, which can also be named as initial-, plastic-, inelastic-, plastic transformation-, thermal-strain, etc. [Qu and Cherkaoui (2006)]. These eigenstrains are produced when the ellipsoidal region, centered at the origin with semiaxes a_1 , a_2 , and a_3 , is loaded by a uniform stress σ_{ij}^∞ at infinity.

After Faivre (1971) analyzed Eshelby's ellipsoidal inclusion concept for anisotropic solids, Laws (1977) and Dvorak *et al.* (1985) defined some energy interactions for matrix

cracking in composites laminates. They stated that the crack can be modeled first by considering an elliptical cylinder ($a_3 \rightarrow \infty$) and then analyzing a flat ellipsoidal ($a_1 > a_2 \gg a_3$) which is the simplest way to define a crack. They stated that the energy released by the introduction of an elliptical shape crack that lies in the 1-3 plane and only grows in the x_3 - axis for an orthotropic material is defined as,

$$W_{a_3} = \frac{1}{2} \pi a_3^2 (\Lambda_{22} \sigma_{22}^2 + \Lambda_{66} \sigma_{12}^2 + \Lambda_{44} \sigma_{23}^2) \quad (4.101)$$

where

$$\begin{cases} \Lambda_{22} = \Lambda_{44} = 2 \left(\frac{1}{E_2} - \frac{\nu_{12}^2}{E_1} \right) \\ \Lambda_{66} = \frac{1}{G_{12}} \end{cases}$$

Figure 4-21 Eshelby's ellipsoidal inclusion, Eshelby (1957)

Extensively in the literature, Barbero (2008), Camanho (2001), Davila (2005), Dvorak and Laws (1987), Hanh and Johannesson (1984), Hashin (1980), Lapczyk and Hurtado (2007), Lonetti *et al.* (2003), Lucioni (2003), Maimí *et al.* (2007), Pinho (2005), Puck (2002), Schuecker and Pettermann (2006), Turon (2007), and others expressed that the failure in the longitudinal direction of the fiber is due to fiber breakage. Matrix cracks are developed inside the matrix that grows through the thickness and longitudinally, as seen in Figure 4-22(a).

The matrix cracks lies in the 1-3 plane running in the direction of the fibers. After Dvorak and Laws (1987) defined an elliptical shape crack, as seen in Figure 4-22(b), they stated that the crack will grow in the transverse direction (through the thickness). Making use of the early definition given by Irwin (1956) related to the energy release rate that measures the energy available in order to an increment in crack length a , or

$$G = \frac{1}{2} \frac{\partial W}{\partial a} \quad (4.102)$$

(a) 3-D inplane crack.

(b) Inplane elliptical crack.

Figure 4-22 Inplane crack

Dvorak and Laws (1987) defined the energy release rate of a crack growing only in the transverse direction, Figure 4-23(a), by

$$G_{a_3} = \frac{1}{2} \frac{\partial W_{a_3}}{\partial a_3} = \frac{1}{2} \pi a_3 (\Lambda_{22} \sigma_{22}^2 + \Lambda_{66} \sigma_{12}^2 + \Lambda_{44} \sigma_{23}^2) \quad (4.103)$$

Also, Dvorak and Laws (1987) defined the energy released rate in the longitudinal direction, Figure 4-23(b), as

$$G_{a_1} = \frac{1}{a_3} \int_0^{a_3} G_{a_3} da_3 = \frac{G_{a_3}}{2a_3} \quad (4.104)$$

then

$$G_{a_1} = \frac{1}{4} \pi a_3 (\Lambda_{22} \sigma_{22}^2 + \Lambda_{66} \sigma_{12}^2 + \Lambda_{44} \sigma_{23}^2) \quad (4.105)$$

(a) Transversal crack growth.

(b) Longitudinal crack growth.

Figure 4-23 Intralaminar crack directional growth

Since the energy release rate in the longitudinal direction is less than the transverse one, the following analysis will consider only the energy at the longitudinal direction. The known definitions of the crack modes, Anderson (1995), are used to see the crack mode evolution of the crack in the longitudinal direction.

From Eq. (4.105) it can be observed that the crack mode evolution is defined as seen in Figure 4-24. It follows that the energy release rate can be separated into the three modes, as

$$G_{a_1} = G_{I-a_1} + G_{II-a_1} + G_{III-a_1} \quad (4.106)$$

The failure criterion is accepted extensively at the literature when the energy release at each mode reaches a toughness value, also mode III is neglected in inplane applications. Hahn and Hohannesson (1993) and (1994) made a correlation between the energy release rate and the critical stress intensity factor. From here, it is stated that a mixed-mode fracture criterion could be defined as,

$$\begin{cases} \frac{G_I}{G_{Ic}} + \frac{G_{II}}{G_{IIc}} = 1 \\ (1-g)\sqrt{\frac{G_I}{G_{Ic}}} + g\left(\frac{G_I}{G_{Ic}}\right) + \left(\frac{G_{II}}{G_{IIc}}\right) = 1 \end{cases} \quad (4.107)$$

where

$$g = \frac{G_{Ic}}{G_{IIc}}$$

Figure 4-24 Crack modes definition, Anderson (1995).

The fracture toughness is evaluated when the ply will fail. This is analyzed individually for each crack mode and obtained toughness as

$$\begin{cases} G_{Ic} = \frac{\pi a_3}{4} \Lambda_{22} F_{2t}^2 \\ G_{IIc} = \frac{\pi a_3}{4} \Lambda_{66} F_6^2 \end{cases} \quad (4.108)$$

Finally introducing Eqs. (4.105) and (4.108) into Eq. (4.107), the failure criterion is derived as

$$\begin{cases} \frac{\sigma_{22}^2}{F_{2t}^2} + \frac{\sigma_{12}^2}{F_6^2} = 1 \\ (1-g) \frac{\sigma_{22}}{F_{2t}} + g \frac{\sigma_{22}^2}{F_{2t}^2} + \frac{\sigma_{12}^2}{F_6^2} = 1 \end{cases} \quad (4.109)$$

The definition of the thermodynamic damage force, can be restated in terms of stress, damage, and undamaged compliance, Eq. (B.17), in the normal direction as

$$\begin{cases} Y_{11} = \frac{1}{(1-D_{11})^2} \left(\frac{\bar{S}_{11}}{(1-D_{11})} \sigma_{11}^2 + \frac{\bar{S}_{12}}{(1-D_{22})} \sigma_{11} \sigma_{22} + \frac{\bar{S}_{13}}{(1-D_{33})} \sigma_{11} \sigma_{33} \right) \\ Y_{22} = \frac{1}{(1-D_{22})^2} \left(\frac{\bar{S}_{12}}{(1-D_{11})} \sigma_{11} \sigma_{22} + \frac{\bar{S}_{22}}{(1-D_{22})} \sigma_{22}^2 + \frac{\bar{S}_{23}}{(1-D_{33})} \sigma_{22} \sigma_{33} \right) \\ Y_{33} = \frac{1}{(1-D_{33})^2} \left(\frac{\bar{S}_{13}}{(1-D_{11})} \sigma_{11} \sigma_{33} + \frac{\bar{S}_{23}}{(1-D_{22})} \sigma_{22} \sigma_{33} + \frac{\bar{S}_{33}}{(1-D_{33})} \sigma_{33}^2 \right) \end{cases} \quad (4.110)$$

and in the shear direction as

$$\begin{cases} Y_{12} = \frac{\bar{S}_{66}}{(1-D_{12})^3} \sigma_{12}^2 \\ Y_{13} = \frac{\bar{S}_{66}}{(1-D_{13})^3} \sigma_{13}^2 \\ Y_{23} = \frac{\bar{S}_{66}}{(1-D_{23})^3} \sigma_{23}^2 \end{cases} \quad (4.111)$$

A comparison among Eqs. (4.105), (4.106), (4.110), and (4.111) leads us to observe an analogy between crack mode and the normal or shear components of the thermodynamic damage forces. Here, mode I is referred to the normal component, while the shear could be seen either by modes II or III. But since, as stated previously, mode III is considered negligible for inplane applications, only mode II will be considered.

The damage surface has been defined as a polynomial in Eq. (4.53), where the normal and shear thermodynamic damage forces are assumed to work in conjunction with each other. This dependency can also be seen when the Tsai-Wu polynomial is used for the previous plastic surface evolution analysis. Employing the concept given by Barbero and Lonetti (2001) and Barbero (2008) the damage surface evolution can be expressed as a function of the damage force components, then the damage surface can be expressed explicitly as

$$\begin{aligned} \hat{g}_d = & \left\{ H_1^N (Y_1^N)^2 + H_2^N (Y_2^N)^2 + H_3^N (Y_3^N)^2 + H_{12}^N Y_1^N Y_2^N + H_{13}^N Y_1^N Y_3^N + H_{23}^N Y_2^N Y_3^N \right. \\ & \left. + H_{12}^S (Y_{12}^S)^2 + H_{13}^S (Y_{13}^S)^2 + H_{23}^S (Y_{23}^S)^2 + H_{12-13}^S Y_{12}^S Y_{13}^S + H_{12-23}^S Y_{12}^S Y_{23}^S + H_{13-23}^S Y_{13}^S Y_{23}^S \right\}^{1/2} \end{aligned} \quad (4.112)$$

In the fiber direction, damage is only considered along the fiber, so damage associated with shear is not seen. Also, when analyzing the transverse direction, damage associated with the inplane shear σ_{23} is not seen. From the well known and extensively documented concepts found in the literature, no coupling among the shear components is seen. Then, some terms can be neglected in the polynomial expression, and so

$$H_{12-13}^S = H_{12-23}^S = H_{13-23}^S = 0 \quad (4.113)$$

When failure is expected, the damage surface value is one. This condition is seen in Figure 4-17 and will be used to define the damage-surface coefficients. This is done considering failures only due to normal or shear forces.

For the case of failure in the fiber direction due to normal tension forces, the stresses should be $\sigma_{11} = F_{1t}$ and $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$, from which the damage forces can then be obtained as

$$\begin{cases} \sigma_{11} = F_{1t} > 0 \\ d_1 = \tilde{d}_{1t} \end{cases} \Rightarrow \begin{cases} Y_1^N = \frac{\bar{S}_{11} F_{1t}^2}{(1 - \tilde{d}_{1t})^3} \\ Y_2^N = Y_3^N = Y_2^{S_{12}} = Y_3^{S_{13}} = Y_3^{S_{23}} = 0 \end{cases} \quad (4.114)$$

Under this uniaxial tension failure the damage surface value is the unity, or,

$$\hat{g}_d = \sqrt{H_{1t}^N \left(\frac{\bar{S}_{11} F_{1t}^2}{(1 - \tilde{d}_{1t})^3} \right)^2} = 1 \quad (4.115)$$

Knowing that

$$\begin{cases} \bar{S}_{11} = \frac{1}{\bar{E}_{11}} \\ E_{11} = (1-d_1)^2 \bar{E}_{11} \Rightarrow \tilde{E}_{1t} = (1-\tilde{d}_{1t})^2 \bar{E}_{11} \end{cases} \quad (4.116)$$

the coefficient referred to the normal tension in the fiber direction is obtained as

$$H_{1t}^N = \frac{\tilde{E}_{11}^3}{\bar{E}_{11} F_{1t}^4} \quad (4.117)$$

An extension of the previous development to tension and compression in the other directions gives us directly that

$$H_1^N = \begin{cases} \frac{\tilde{E}_{1t}^3}{\bar{E}_{11} F_{1t}^4}, & \sigma_{11} \geq 0 \\ \frac{\tilde{E}_{1c}^3}{\bar{E}_{11} F_{1c}^4}, & \sigma_{11} < 0 \end{cases}, H_2^N = \begin{cases} \frac{\tilde{E}_{2t}^3}{\bar{E}_{22} F_{2t}^4}, & \sigma_{22} \geq 0 \\ \frac{\tilde{E}_{2c}^3}{\bar{E}_{22} F_{2c}^4}, & \sigma_{22} < 0 \end{cases}, H_3^N = \begin{cases} \frac{\tilde{E}_{3t}^3}{\bar{E}_3 F_{3t}^4}, & \sigma_{33} \geq 0 \\ \frac{\tilde{E}_{3c}^3}{\bar{E}_3 F_{3c}^4}, & \sigma_{33} < 0 \end{cases} \quad (4.118)$$

When an elliptical shape is considered in the case of normal inplane thermodynamic damage forces, as seen in Figure 4-25, the coefficient of the term $Y_1^N Y_2^N$ is zero. If the ellipse is rotated so the intercepts with the axes remain the same, then the coefficient is a negative square root of the multiplication of both intercepts. This definition is also used to define the von Mises equivalent stress. For the case of orthotropic materials the Tsai-Wu failure criteria is the extension of the von Mises equivalent stress, where the tension and compression strengths are different. In this case, only the positive values are considered, where the tension and compression cases are seen separately, working only in the positive or first quadrant.

An extension of this concept for normal-related coefficients could be stated as

$$\begin{cases} H_{12}^N = -\sqrt{H_1^N H_2^N} \\ H_{13}^N = -\sqrt{H_1^N H_3^N} \\ H_{23}^N = -\sqrt{H_2^N H_3^N} \end{cases} \quad (4.119)$$

on the other hand, since the damage evolution in the longitudinal direction, or fiber breakage, is not influenced by the intralaminar or interlaminar matrix damage evolution, its correspondent coefficients are zero,

$$H_{12}^N = H_{13}^N = 0 \quad (4.120)$$

Also, with the intention to define an independent evolution of delamination and intralaminar matrix evolution, its coefficient is also considered zero,

$$H_{23}^N = 0 \quad (4.121)$$

Figure 4-25 Elliptical shape of the normal inplane thermodynamic damage forces

Similar to the previous normal case, in the case of inplane failure, the stresses should be $\sigma_{12} = F_6$ and $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$. At this point the inplane shear modulus has a critical value, when

$$\begin{cases} \sigma_{12} = F_6 \\ G_{12} = \tilde{G}_{12} \end{cases} \quad (4.122)$$

Thakkar and Pandey (2006), among others, state that the inplane shear damage is related to the damages in the fiber and transverse direction. Then, the shear modulus is related to the undamaged one by the relation

$$G_{12} = (1 - d_1)(1 - d_2) \bar{G}_{12}$$

or

$$(1 - d_1)(1 - d_2) = \frac{G_{12}}{\bar{G}_{12}} \quad (4.123)$$

Figure 4-26 shows the form of the relation of the damages d_1 and d_2 when the inplane shear force acts alone. If damage in the fiber direction is not relevant ($d_1 = 0$), then the

damage will only be produced in the transverse direction for a critical value of $\tilde{d}_{2-12}$. Analogously, when only damage in the fiber direction is considered, a critical damage value $\tilde{d}_{1-12}$ is reached. From Eq. (4.123), both critical damages have the same value of

$$\tilde{d}_{1-12} = \tilde{d}_{2-12} = 1 - \frac{\tilde{G}_{12}}{G_{12}} \quad (4.124)$$

These critical damages are different from which critical values where only tension or compression loads are considered, $\tilde{D}_{11} \neq \tilde{d}_{1-12}$ and $\tilde{D}_{22} \neq \tilde{d}_{2-12}$. Then two different critical values must be considered in this case in order to define fiber and shear failure or transverse and shear failure. So, the relationship among damages as stated by Thakkar and Pandey (2006) as $(1 - D_{12})^2 = (1 - d_1)(1 - d_2)$ will not be applied in the development of this model. Moreover, damage in the transversal direction is considered to be irrelevant if compared to damage in the longitudinal direction, Barbero (2008).

Figure 4-26 Inplane shear damage evolution

From the previous discussion, we will prefer to have an independent inplane shear damage variable, which relates the shear modulus to its undamaged value as

$$G_{12} = (1 - D_{12})^2 \bar{G}_{12}$$

Then, damage in the inplane shear direction has the following relation when the critical failure point is reached:

$$\tilde{D}_{12} = 1 - \sqrt{\frac{\tilde{G}_{12}}{G_{12}}} \quad (4.125)$$

For this inplane shear damage evolution assumption, the ultimate shear damage forces can be obtained as

$$\begin{cases} \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \\ \sigma_{12} = F_6 \\ D_{12} = \tilde{D}_{12} \end{cases} \Rightarrow \begin{cases} Y_{12}^S = \frac{\bar{S}_{66} F_6^2}{(1 - \tilde{D}_{12})^3} \\ Y_1^N = Y_2^N = Y_3^N = Y_{13}^S = Y_{23}^S = 0 \end{cases} \quad (4.126)$$

and introducing Eqs. (4.125) and (4.126) into the surface damage function Eq. (4.112), it can be reduced to

$$\hat{g}_d = \sqrt{H_{12}^S \left[\left(\frac{\bar{G}_{12}}{\tilde{G}_{12}} \right)^{3/2} \frac{F_6^2}{\bar{G}_{12}} \right]^2} = 1 \quad (4.127)$$

and,

$$H_{12}^S = \frac{\tilde{G}_{12}^3}{\bar{G}_{12} F_6^4} \quad (4.128)$$

By extension the other shear coefficients are defined similarly as

$$\begin{cases} H_{12}^S = \frac{\tilde{G}_{12}^3}{\bar{G}_{12} F_6^4} \\ H_{13}^S = \frac{\tilde{G}_{13}^3}{\bar{G}_{13} F_5^4} \\ H_{23}^S = \frac{\tilde{G}_{23}^3}{\bar{G}_{23} F_4^4} \end{cases} \quad (4.129)$$

Finally, the damage surface expressed in Eq. (4.112) is redefined as

$$\hat{g}_d = \sqrt{H_1^N (Y_1^N)^2 + H_2^N (Y_2^N)^2 + H_3^N (Y_3^N)^2 + H_{12}^S (Y_{12}^S)^2 + H_{13}^S (Y_{13}^S)^2 + H_{23}^S (Y_{23}^S)^2} \quad (4.130)$$

where its coefficients are given in Eqs. (4.118) and (4.129).

5. MODEL SETUP

The first step in finding some preliminary results is to identify the parameters used to define the model. These parameters are obtained with the help of the Matlab[®] program *eps12_Sig12.m*, and the functions *DataFun.m*, *FitFun.m*, *FitFun2P.m*, and *fitfunLINEAR.m*.

After identifying these parameters some preliminary results are obtained using the Newton Rhapsion nonlinear analysis. The explicit variables obtained in Appendix B and the flow chart defined in Appendix C are implemented in the Matlab[®] *Newton.m* program and several subroutines or functions.

5.1 Isotropic Evolution Parameters

The isotropic plastic-strain and damage evolution parameters will be defined in this section. The analysis is based upon the loading and unloading stress-strain data found by Ladavez and Dantec (1992) for the carbon fiber composite material T300/914, as seen in Figure 5-1. The methodology exposed in Section 4.4 will be performed. The data used is the inplane shear. Tensile or compressive loading along the fiber or transversal directions does not clearly show this evolution of the strain, until suddenly a failure is seen, as fragile failure.

Figure 5-1 Initial experimental data for inplane shear ϵ_{12} - σ_{12} , T300/914 Ladaveze and Dantec (1992)

Several functions will be obtained using fitting techniques with Matlab[®]. These functions are Prony series, Exponential series, or polynomials developed as Matlab functions, as *FitFun.m* that fits *n*-order Prony and Exponential series. This user defined function accepts the data to be fitted and internally defines the initial coefficients and fits this curve, considering the lower value of the sum of the squares error, minimizing this.

The first step in this methodology is to fit the stress-strain curve for the inplane shear. Table 5-1 shows the exponential and Prony series of fourth-orders series or below from data obtained by Ladavez and Dantec (1992), for. It can be seen that for fourth-order series the error goes up a little with respect to third-order. This is caused because of the sensibility of the minimizing method that Matlab uses.

Table 5-1 Fitting of inplane shear experimental data.

<i>Function Fitting Type</i>	<i>n</i>	<i>i</i>	α_i	β_i	<i>Sum Square Error</i>
Exponential series: $y = \sum_{i=1}^n \alpha_i e^{\left(-x/\beta_i\right)}$	1	1	35.5654	-31.2465×10^{-3}	5148.8844
	2	1	-64.9186	4.6778×10^{-3}	8.0887
		2	65.2130	-106.1265×10^{-3}	
	3	1	-1899.1173	73.8736×10^{-3}	2.4647
		2	-50.6700	3.6670×10^{-3}	
		3	1949.3863	81.9264×10^{-3}	
	4	1	-49.4352	3.4847×10^{-3}	2.0766
		2	-142.8674	42.3754×10^{-3}	
		3	1.0400	0.0511×10^{-3}	
		4	191.2610	150.8430×10^{-3}	
Prony series: $y = \sum_{i=1}^n \alpha_i \left(e^{\left(-x/\beta_i\right)} - 1 \right)$	1	1	-85.0324	6.8498×10^{-3}	134.5801
	2	1	-50.0647	3.7448×10^{-3}	2.8982
		2	-53.6042	26.5155×10^{-3}	
	3	1	19.0032	1.6572×10^{-3}	1.9698
		2	-60.9410	2.6579×10^{-3}	
	4	3	-56.3934	19.3514×10^{-3}	2.9065
		1	-49.9981	3.7425×10^{-3}	
		2	0.0065	3.1244×10^{-3}	
		3	-10.1979	446.7659×10^{-3}	
		4	-52.0827	25.9062×10^{-3}	

The best fit function in this case is the third-order Prony series, as seen in Figure 5-2 which is the visualization of Table 5-1. The inplane shear stress is then defined as a function of the inplane shear strain as,

$$\sigma_{12} = 19.0032 \left(e^{\left(\frac{\varepsilon_{12}}{1.6572 \times 10^{-3}} \right)} - 1 \right) - 60.9410 \left(e^{\left(\frac{\varepsilon_{12}}{2.6579 \times 10^{-3}} \right)} - 1 \right) - 56.3934 \left(e^{\left(\frac{\varepsilon_{12}}{19.3514 \times 10^{-3}} \right)} - 1 \right) , [\text{MPa}] \quad (5.1)$$

Figure 5-2 Fitting of inplane shear stress - strain

The ultimate inplane shear strain is obtained from the data given by Ladeveze and Dantec (1992). This value is related to the inplane shear strength using the fitted relation in Eq. (5.1), then

$$\begin{cases} \tilde{\varepsilon}_{12} = 33.16 \times 10^{-3} \\ F_6 = 88.1677, \text{ MPa} \end{cases} \quad (5.2)$$

Figure 5-3 Fitting of inplane-shear load/unload stress-strain values

The next step is to find the points that define the load-unload conditions. These points are found first from the data given by Ladeveze and Dantec (1992) and shown as triangles in Figure 5-3. Each pair of load-unload data is intercepted with the fitted inplane shear stress-strain curve, Eq. (5.42), and with the ordinate axis which gives zero stress value. Forward analysis will be made using this new set of load-unload data, which intercepts the fitted curve, in order to minimize errors in the following steps.

5.1.1 Elastic linear limit

The next step is to find the limit of the linear elastic behavior of the material. Four different criteria are considered in this analysis which can be seen in Figure 5-4, where the goal is to define how many points are needed to define the value of the initial stiffness.

Figure 5-4 Methods to obtain the linear initial stiffness

The first method evaluates the stiffness in the last two points of the analyzed data series. The second obtains the stiffness considering the last point. The stiffness is also obtained, fitting a line that passes through the center of axis considering all the initial n -points. Finally, a fourth analysis is made fitting a line that does not necessarily pass through the center of axis.

From the data given by Ladeveze and Dantec (1992) and using the criteria seen in Figure 5-4, a set of data is obtained in Table 5-2. It is observed that after the fifth point the stiffness reduces its value.

Table 5-2 Initial linear shear stiffness as a function of the first n -data points, G_{12} [MPa]

Method	n -th data point			
	2	3	4	5
(a) Line last 2 $n-1, n$ -Data points	6769.19	6436.93 (6588.06)	6088.78 (6421.63)	4682.40 (5986.83)
(b) Line last n -Data point	6769.19	6609.84 (6689.52)	6421.66 (6600.23)	5986.84 (6446.88)
(c) Fit $y=mx$	6769.19	6643.96 (6706.58)	6497.64 (6636.93)	6222.21 (6533.25)
(d) Fit $y=mx+b$	6769.19	6612.27 (6690.73)	6419.88 (6600.45)	6034.53 (6458.97)

() indicates the mean of the actual and previous values.

The variation in the stiffness before point 5 is also seen in Figure 5-5, which contains plots of data obtained in Table 5-2. So, considering only the first four data points, the average value of the inplane shear stiffness, $\bar{G}_{12}$ is approximately 6420 MPa. With this value and considering that the initial linear elastic behavior passes through the center of axis, an interception of this and the plotted strain-stress curve, Eq. (5.1), will define the limit of the linear elastic behavior. Two additional values are then defined, the linear strain value and the linear strength value:

$$\begin{cases} \bar{G}_{12_{Linear}} = 6419.88, & \text{MPa} \\ \varepsilon_{12_{Linear}} = 1.4479 \times 10^{-3} \\ S_{12_{Linear}} = 18.5913, & \text{MPa} \end{cases} \quad (5.3)$$

Figure 5-5 Initial linear shear stiffness as a function of the first n -data points

5.1.2 Isotropic hardening plasticity parameters

From the pair of load-unload data obtained, as seen in Figure 5-3, the inplane-shear-strain and plastic-shear-strain data can be extracted. These data points are used to fit them in order to find a relation of the plastic strain as a function of the total strain.

The fitted data is represented as polynomial and exponential series functions. Table 5-3 presents the coefficients for these fittings, while in Figure 5-6 the third-order polynomial and second-order exponential series are plotted. Fitting using Prony series can not be used because this series always passes through the axis center and this is not possible in this analysis. In this table the best fitted function is the third-order polynomial.

Table 5-3 Fitting of inplane shear total strain–shear plastic strain data

<i>Function Fitting Type</i>	<i>n</i>	<i>i</i>	α_i	β_i	<i>Sum Square Error</i>
Exponential series: $y = \sum_{i=1}^n \alpha_i e^{\left(-x/\beta_i\right)}$	1	1	1.7754×10^{-3}	-13.5639×10^{-3}	10.08×10^{-6}
		2	-13.9868×10^{-3}	$69828.9094 \times 10^{-3}$	0.60×10^{-6}
	2	2	12.6000×10^{-3}	-33.4705×10^{-3}	
		3	0.4791×10^{-3}	55.4747×10^{-3}	
	3	2	-9.7850×10^{-3}	83.1794×10^{-3}	0.67×10^{-6}
		3	7.8580×10^{-3}	-27.4828×10^{-3}	
	4	1	-31.5109×10^{-3}	20.4725×10^{-3}	
		2	99.8644×10^{-3}	20.6072×10^{-3}	
		3	-143.8734×10^{-3}	44.8264×10^{-3}	0.02×10^{-6}
		4	75.7551×10^{-3}	2494.5904×10^{-3}	
Polynomial $y = \sum_{i=0}^n \alpha_i x^n$	1	1	67.0813×10^{-2}		5.19×10^{-6}
		0	-33.3456×10^{-4}		
	2	2	10.4744		
		1	28.5606×10^{-2}		0.38×10^{-6}
	3	0	-10.1857×10^{-4}		
		3	-3.5579×10^2		
		2	30.153.2		
		1	-0.8893×10^{-2}		0.04×10^{-6}
	4	0	-0.4384×10^{-4}		
		4	2.0631×10^4		
		3	-18.6462×10^2		
		2	66.3125		0.00×10^{-6}
		1	32.3596×10^{-2}		
		0	6.8590×10^{-4}		

The third-order polynomial function that best fits the inplane shear and plastic shear function is then expressed as

$$\varepsilon_{12}^p = -3.5579 \times 10^2 \varepsilon_{12}^3 + 30.1532 \varepsilon_{12}^2 - 0.8893 \times 10^{-2} \varepsilon_{12} - 0.4384 \times 10^{-4} \quad (5.4)$$

From Eq. (5.4) some important values like the plastic threshold strain, plastic threshold strength, plastic threshold stiffness, and ultimate shear plastic strain are obtained, whose values are,

$$\begin{cases} \varepsilon_{12Th-p} = 1.3747 \times 10^{-3} \\ S_{12Th-p} = 17.7641 \text{ , MPa} \\ \bar{G}_{12Th-p} = \frac{S_{12Th-p}}{2 \varepsilon_{12Th-p}} = 6460.9252 \text{ , MPa} \\ \tilde{\varepsilon}_{12}^p = 19.8446 \times 10^{-3} \end{cases} \quad (5.5)$$

Figure 5-6 Fitting of inplane shear total strain–shear plastic strain data

Considering only an inplane shear loading, $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ and $\sigma_{12} \neq 0$, Eq. (4.93) is reduced to

$$\hat{g}_p = \sqrt{\frac{1}{F_6^2} \sigma_{12}^2} = \frac{\sigma_{12}}{F_6} \quad (5.6)$$

and from the definition of the plastic surface given in Eq. (4.87), it can be noted that

$$\hat{g}_p = \frac{\sigma_{12}}{F_6} = R + R_0 \quad (5.7)$$

From the condition given in Figure 4-17, for $p = 0$ then $R = 0$ and $\sigma_{12}|_{p=0} = S_{12_{Th-p}}$. The initial plastic surface value can be evaluated when the given conditions are applied into Eq. (5.7), and so

$$R_0 = \frac{S_{12_{Th-p}}}{F_6} \quad (5.8)$$

Since the initial plastic surface value R_0 can be obtained from known data, then a relation between the strength and the plastic surface can be obtained directly using Eq. (5.48). At this point the plastic surface and the plastic strain are functions of the total strain when Eqs. (5.42), (5.45), (5.48), and Eq. (5.49) are used respectively.

Table 5-4 Fitting of isotropic hardening plasticity internal variable and thermodynamic isotropic hardening plasticity force

<i>Function Fitting Type</i>	<i>n</i>	<i>i</i>	α_i^p	β_i^p	<i>Sum Square Error</i>
Prony series: $R = \sum_{i=1}^n \alpha_i^p \left(e^{\left(-\frac{p}{\beta_i^p} \right)} - 1 \right)$	1	1	-0.7177	-0.1495	0.1726
	2	1	-0.3801	-0.0349	0.0038
		2	-0.4333	-0.6254	
	3	1	-0.1783	-0.0140	0.0001
		2	-0.2827	-0.0911	
		3	-0.4005	-0.9664	
	4	1	-0.0564	-0.0005	0.0011
		2	-0.2469	-2.9625	
		3	-0.3353	-0.0471	
		4	-0.2991	-0.4932	
Prony series, forced to pass through $(\tilde{p}, 1 - R_0)$: $R = \sum_{i=1}^n \alpha_i^p \left(e^{\left(-\frac{p}{\beta_i^p} \right)} - 1 \right)$	1	1	-0.7989	-0.2252	0.3049
	2	1	-0.3960	-0.0374	0.0047
		2	-0.4431	-0.7321	
	3	1	-0.186597	-0.0146632	0.0002
		2	-0.283851	-0.0971623	
		3	-0.405007	-1.05341	
	4	1	-0.0254	-0.0014	0.0001
		2	-0.4001	-1.1422	
		3	-0.2067	-0.0221	
		4	-0.2528	-0.1186	

The partial derivative of the plastic surface w.r.t. the stress for the case of inplane shear loading can be obtained as

$$\frac{\partial g_p}{\partial \sigma_{12}} = \frac{1}{2 \sqrt{\left(\frac{b}{2}\right)^2 + a}} 2 F_{66} \sigma_{12} = \frac{F_{66} \sigma_{12}}{\sqrt{F_{66} \sigma_{12}^2}} = \sqrt{F_{66}} = \frac{1}{F_6} \quad (5.9)$$

Applying Eq. (5.9) into Eq. (4.86), a relation between the isotropic hardening plasticity and the plastic strain is obtained as

$$p = -\frac{\varepsilon_{12}^p}{\frac{\partial g_p}{\partial \sigma_{12}}} = -F_6 \varepsilon_{12}^p \quad (5.10)$$

From an initial-threshold plastic strain to an ultimate strain, a set of points can be discretized. This set of points can be seen in Figure 5-7 as dots and obtained since R and p are functions of the total shear strain, as stated in Eqs. (5.1) and (5.4), respectively, or

$$\begin{cases} R = \frac{\sigma_{12} - S_{12m-p}}{F_6} \\ p = -F_6 \varepsilon_{12}^p \end{cases} \quad (5.11)$$

These points are to be fitted as Prony series using the definition given in Eq. (4.42),

$$R = \sum_{i=1}^{n_p} \alpha_i^p \left(e^{\left(-\frac{p}{\beta_i^p} \right)} - 1 \right) \quad (5.12)$$

It is known that when fitting, the fitted curve will not pass necessarily through all initial points. In this case, it is desirable that the fitted curve pass through the origin or center of axis and the ultimate point defined by $(\tilde{p}, 1 - R_0)$. Because of the Prony series definition, it will always pass through the origin. In order to assure it passes through the ultimate point, a second fitting is performed modifying the Prony series function so that one coefficient is not fitted, but is a function of this point and the other coefficients. The Matlab function *Fitting_02.m* is used, having the initial fitting parameters as input data and the point of interest.

Figure 5-7 Fitting of isotropic hardening plasticity internal variable and thermodynamic isotropic hardening plasticity force

Table 5-4 shows the coefficients for the first curve fit and then the one that is forced to pass through the ultimate defined point. A third-order Prony series has the least error and this is chosen for the isotropic hardening plasticity parameters:

$$n^p = 3, \quad \begin{cases} \alpha_1^p = -0.186597 \\ \alpha_2^p = -0.283851 \\ \alpha_3^p = -0.405007 \end{cases}, \quad \begin{cases} \beta_1^p = -1.46632 \times 10^{-2} \\ \beta_2^p = -9.71623 \times 10^{-2} \\ \beta_3^p = -1.05341 \end{cases} \quad (5.13)$$

or,

$$R = -0.186597 \left(e^{\left(\frac{-p}{-1.46632 \times 10^{-2}} \right)} - 1 \right) - 0.283851 \left(e^{\left(\frac{-p}{-9.71623 \times 10^{-2}} \right)} - 1 \right) - 0.405007 \left(e^{\left(\frac{-p}{-1.05341} \right)} - 1 \right) \quad (5.14)$$

5.1.3 Isotropic hardening damage parameters

As in the case of the previous section, the evaluation of the isotropic hardening damage parameters starts from the pair of load/unload data obtained as seen in Figure 5-3. First, the inplane shear stiffness is obtained from this load/unload data using the well known relation between the inplane stress and elastic strain,

$$G_{12} = \frac{\sigma_{12}}{2 \varepsilon_{12}^e} = \frac{\sigma_{12}}{2 (\varepsilon_{12} - \varepsilon_{12}^p)} \quad (5.15)$$

Table 5-5 Fitting of inplane shear strain-shear stiffness data

<i>Function Fitting Type</i>	<i>n</i>	<i>i</i>	α_i	β_i	<i>Sum Square Error</i>
Exponential series: $y = \sum_{i=1}^n \alpha_i e^{\left(-x/\beta_i\right)}$	1	1	5798.4694	48.5312×10^{-3}	807851.99
	2	1	4307.8025	4.4334×10^{-3}	1113.70
		2	3933.6693	188.8677×10^{-3}	
	3	1	7007.7022	1.7326×10^{-3}	0.00
		2	2722.7794	15.1926×10^{-3}	
		3	2645.5134	-260.8653×10^{-3}	
	4	1	5316.6397	2.1792×10^{-3}	0.00
		2	-54.2271	0.2978×10^{-3}	
		3	2964.2484	19.5465×10^{-3}	
		4	2188.3211	-141.2267×10^{-3}	

The load/unload data is used to plot the strain vs. the inplane shear stiffness as seen in Figure 5-8 in dots. This data is then curve fitted using an exponential series. Again, it is not possible to fit using Prony series because this series always passes through the origin. The fitted coefficients are shown in Table 5-5 up to a fourth-order series, and plotted in Figure 5-8.

Figure 5-8 Fitting of inplane shear strain-shear stiffness data

The best fitted curve is the third-order exponential series and then the inplane shear stiffness can be expressed as a function of the total inplane shear strain as,

$$G_{12} = 7007.7022 e^{\left(\frac{\varepsilon_{12}}{1.7326 \times 10^{-3}}\right)} + 2722.7794 e^{\left(\frac{\varepsilon_{12}}{15.1926 \times 10^{-3}}\right)} + 2645.5134 e^{\left(\frac{\varepsilon_{12}}{260.8653 \times 10^{-3}}\right)}, \text{ MPa} \quad (16)$$

Also, the inplane damage threshold strain, damage threshold inplane strength and ultimate shear stiffness can be obtained whose values are:

$$\begin{cases} \varepsilon_{12_{Th-d}} = 2.6666 \times 10^{-3} \\ S_{12_{Th-d}} = 30.6538 \text{ , MPa} \\ \tilde{G}_{12} = 3311.0814 \text{ , MPa} \end{cases} \quad (5.17)$$

From values given in Eqs. (5.3), (5.5), and (5.17), Figure 5-9 can be plotted. Here the elastic limit, the plastic threshold, and damage threshold are plotted. It can be seen that since the elastic limit and the plastic threshold are almost the same, both points will coincide in one. This condition will give us three zones.

The first zone is a linear-elastic one, a pure-plastic the second zone, and then a plastic-damage zone. In the second zone where a pure plastic behavior is expected, the stiffness remains the same and equal to the undamaged initial stiffness.

Figure 5-9 Plastic and damage stress threshold

Eq. (4.125) can be rewritten for a general inplane shear damage and shear stiffness evolution in time as

$$1 - D_{12} = \sqrt{\frac{G_{12}}{\tilde{G}_{12}}} \quad (5.18)$$

Also when considering only an inplane shear loading, ($\sigma_{12} \neq 0$ and $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$), and Eq. (5.18), the thermodynamic damage forces are reduced to

$$\begin{cases} Y_{12}^S = \frac{\bar{S}_{66}}{(1-D_{12})^3} \sigma_{12}^2 = \frac{\sigma_{12}^2}{(1-D_{12})^3 \bar{G}_{12}} = \sqrt{\frac{\bar{G}_{12}}{G_{12}^3}} \sigma_{12}^2 \\ Y_1^N = Y_2^N = Y_3^N = Y_{13}^S = Y_{23}^S = 0 \end{cases} \quad (5.19)$$

Then substituting Eqs. (4.128) and (5.19) into Eq. (4.130), the damage yield surface can be expressed as

$$\hat{g}_d = \sqrt{H_{12}^S (Y_{12}^S)^2} = \sqrt{\frac{\tilde{G}_{12}^3}{\bar{G}_{12} F_6^4} \left(\sqrt{\frac{\bar{G}_{12}}{G_{12}^3}} \sigma_{12}^2 \right)^2} = \sqrt{\frac{\tilde{G}_{12}^3}{G_{12}^3} \frac{\sigma_{12}^2}{F_6^2}} \quad (5.20)$$

Table 5-6 Fitting of isotropic hardening damage internal variable and thermodynamic isotropic hardening damage force using Prony series

<i>Function Fitting Type</i>	<i>n</i>	<i>i</i>	α_i	β_i	<i>Sum Square Error</i>
Prony series: $y = \sum_{i=1}^n \alpha_i \left(e^{\left(-\frac{x}{\beta_i} \right)} - 1 \right)$	1	1	-5.4125 x 10 ⁷	-8.6158 x 10 ⁷	0.9335
	2	1	0.0030 x 10 ⁸	-0.5995 x 10 ⁸	0.9335
	2	2	-1.1947 x 10 ⁸	-1.8858 x 10 ⁸	
		1	-95.3571	-978.9690	0.0781
	3	2	8.9154	-1.6726	
		3	-280.0128	-56.9120	
	4	1	0.1551	-0.0899	0.4966
		2	-1.9804	-323.6296	
		3	-8.9319	-12.9386	
		4	-3.0077	-15.3191	
Prony series, forced to pass through $(\tilde{d}_{2,1} - \gamma_0)$: $y = \sum_{i=1}^n \alpha_i \left(e^{\left(-\frac{x}{\beta_i} \right)} - 1 \right)$	1	1	-8.8917 x 10 ⁷	-8.6697 x 10 ⁷	3.6972
	2	1	0.0032 x 10 ⁸	-0.6032 x 10 ⁸	3.6972
	2	2	-1.9440 x 10 ⁸	-1.8858 x 10 ⁸	
		1	0.0150 x 10 ⁸	-2.6936 x 10 ⁶	0.1549
	3	2	0.0001 x 10 ⁸	-0.0001 x 10 ⁶	
		3	-2.2425 x 10 ⁸	-1.2749 x 10 ⁶	
	4	1	0.4498	-0.2395	0.0030
		2	1.3582	0.6315	
		3	-20.0738	-1.4148	
		4	153.3982	-10.6340	

From the definition of the damage surface given in Eq. (4.89),

$$\hat{g}_d = \sqrt{\frac{\tilde{G}_{12}^3}{G_{12}^3} \frac{\sigma_{12}^2}{F_6^2}} = \gamma + \gamma_0 \quad (5.21)$$

From Figure 4-18 when $\delta = 0$ then $\gamma = 0$, $G_{12} = \bar{G}_{12}$, and $\sigma_{12} = S_{12Th-d}$. Then the initial thermodynamic isotropic hardening damage force γ_0 can be evaluated as

$$\gamma_0 = \sqrt{\frac{\tilde{G}_{12}^3}{G_{12}^3} \frac{S_{12Th-d}^2}{F_6^2}} \quad (5.22)$$

After substituting into Eq. (5.21), the damage surface is derived as,

$$\gamma = \frac{\tilde{G}_{12}^{3/2}}{F_6^2} \left(\frac{\sigma_{12}^2}{G_{12}^{3/2}} - \frac{S_{12Th-d}^2}{\bar{G}_{12}^{3/2}} \right) \quad (5.23)$$

Figure 5-10 Plot of isotropic hardening damage internal variable and thermodynamic isotropic hardening damage force using Prony series

The partial derivative of the damage surface with respect to the thermodynamic damage force in the transverse direction is obtained as

$$\frac{\partial g_d}{\partial Y_{12}} = \frac{H_{12}^S Y_{12}^S}{\sqrt{H_{12}^S (Y_{12}^S)^2}} = \sqrt{H_{12}^S} \quad (5.24)$$

From Eq. (4.88), the isotropic damage evolution is given as a function of the stiffness in the inplane shear direction using Eqs. (4.129), (5.18), and (5.24) as

$$\delta = -\frac{D_{12}}{\frac{\partial g_d}{\partial Y_{12}}} = -\frac{D_{12}}{\sqrt{H_{12}^s}} = -F_6^2 \sqrt{\frac{\tilde{G}_{12}}{\tilde{G}_{12}^3}} \left(1 - \sqrt{\frac{G_{12}}{\tilde{G}_{12}}}\right) \quad (5.25)$$

and from Eqs. (5.22) and (5.25) the isotropic hardening damage internal variable and the thermodynamic isotropic hardening damage force relation can be fitted since both are functions of the inplane total shear. This discretization can be done from an initial total strain threshold damage to the ultimate strain defined in Eqs. (5.2) and (5.17) respectively. Both relations have been stated as,

$$\begin{cases} \gamma = \frac{\tilde{G}_{12}^{3/2}}{F_6^2} \left(\frac{\sigma_{12}^2}{\tilde{G}_{12}^{3/2}} - \frac{S_{12,th-d}^2}{\tilde{G}_{12}^{3/2}} \right) \\ \delta = -F_6^2 \frac{\tilde{G}_{12}^{1/2}}{\tilde{G}_{12}^{3/2}} \left(1 - \sqrt{\frac{G_{12}}{\tilde{G}_{12}}} \right) \end{cases} \quad (5.26)$$

and when fitted using Prony series of the type,

$$\gamma = \sum_{i=1}^{n_d} \alpha_i^d \left(e^{\left(-\delta/\beta_i^d\right)} - 1 \right) \quad (5.27)$$

then Table 5-6 and Figure 5-7 are obtained.

From the data seen in Table 5-6 and Figure 5-10 it can be concluded that Prony series can not be used with the shape given. The fitting process shows a considerable instability of the procedure. So, the Prony series will not be used for curve fitting.

When exponential series are used to fit this relation, the functions are restated as,

$$\begin{cases} \gamma = \frac{\tilde{G}_{12}^{3/2}}{F_6^2} \frac{\sigma_{12}^2}{\tilde{G}_{12}^{3/2}} \\ \delta = -F_6^2 \frac{\tilde{G}_{12}^{1/2}}{\tilde{G}_{12}^{3/2}} \left(1 - \sqrt{\frac{G_{12}}{\tilde{G}_{12}}} \right) \end{cases} \quad (5.28)$$

where the exponential series is given as

$$\hat{\gamma} = \sum_{i=1}^{n_d} \alpha_i^d e^{\left(-\delta/\beta_i^d\right)} \quad (5.29)$$

Table 5-7 Fitting of isotropic hardening damage internal variable and thermodynamic isotropic hardening damage force using exponential series

<i>Function Fitting Type</i>	<i>n</i>	<i>i</i>	α_i^d	β_i^d	<i>Sum Square Error</i>
Exponential series: $\hat{\gamma} = \sum_{i=1}^n \alpha_i^d e^{\left(-\frac{\delta}{\beta_i^d}\right)}$	1	1	0.0403	0.2923	0.0013
	2	1	0.0064	-5.1441 x 10 ⁵	0.0009
		2	0.0377	0.0000 x 10 ⁵	
	3	1	-85.6561	-0.9529	0.0009
		2	85.6621	-0.9529	
		3	0.0390	0.2890	
	4	1	0.0653 x 10 ⁷	-0.0797 x 10 ⁸	0.7545
		2	4.0579 x 10 ⁷	-2.1607 x 10 ⁸	
		3	-2.9961 x 10 ⁷	-0.2654 x 10 ⁸	
		4	-1.1270 x 10 ⁷	-7.9760 x 10 ⁸	
Exponential series, forced to pass through (0, γ_0) and ($\tilde{\delta}$, 1): $\hat{\gamma} = \sum_{i=1}^n \alpha_i^d e^{\left(-\frac{\delta}{\beta_i^d}\right)}$	1	1	0.0443	0.2991	0.0175
	2	1	0.0096	-1.7518 x 10 ¹²	0.0030
		2	0.0348	0.0000 x 10 ¹²	
	3	1	-85.6579	-0.9538	0.0026
		2	85.6710	-0.9544	
		3	0.0312	0.2706	
	4	1	0.0654 x 10 ⁷	-0.0800 x 10 ⁸	4.9505
		2	4.0605 x 10 ⁷	-2.2606 x 10 ⁸	
		3	-2.9886 x 10 ⁷	-0.2652 x 10 ⁸	
		4	-1.1373 x 10 ⁷	-0.7106 x 10 ⁸	

After discretizing the functions in 80 points, Table 5-7 and Figure 5-11 are obtained. A third-order exponential series represents the damage surface best, whose parameters are

$$n^d = 3, \quad \begin{cases} \alpha_1^d = -85.6579 \\ \alpha_2^d = 85.6710 \\ \alpha_3^d = 0.031245 \end{cases}, \quad \begin{cases} \beta_1^d = -0.953836 \\ \beta_2^d = -0.954361 \\ \beta_3^d = 0.270605 \end{cases} \quad (5.30)$$

or

$$\hat{\gamma} = -85.6579 e^{\left(\frac{-\delta}{-0.953836}\right)} + 85.6710 e^{\left(\frac{-\delta}{-0.954361}\right)} + 0.031245 e^{\left(\frac{-\delta}{0.270605}\right)} \quad (5.31)$$

Figure 5-11 Plot of isotropic hardening damage internal variable and thermodynamic isotropic hardening damage force using exponential series

5.2 Preliminary Results

The material properties needed to define the plastic and damage coefficients are obtained from Ladeveze and Dantec (1992) and Barbero (2008). These properties are listed in Tables 5-8, 5-9, and 5-10 which present the undamaged elastic, strength, and ultimate elastic properties for T300/914 unidirectional carbon fiber composite.

The plastic and damage evolution is simulated in a user-defined Matlab program *Newton.m*. This program uses the nonlinear evolution approach expressed as Newton Raphson, Chen and Han (1988). It makes use of a user-defined function *subUserMat.m*, which has the same input and output variables as those used to define material subroutines in ANSYS. These inputs and outputs are expressed in Figure 5-12.

Figure 5-12 Input/output usermat variables

Table 5-8 Undamaged elastic properties for T300/914 unidirectional carbon fiber composite

$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$	$\bar{G}_{12}$	$\bar{G}_{13}$	$\bar{G}_{23}$	$\bar{\nu}_{12}$	$\bar{\nu}_{13}$	$\bar{\nu}_{23}^{(1)}$
[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]			
142000	10300	10300	6461	6461	3710	0.21	0.21	0.39

⁽¹⁾ obtained using Eq. (5.73)

Poisson's ratio ν_{23} is obtained as a function of the other 5 parameters when a transversely-isotropic material is considered, then

$$\nu_{23} = \frac{E_2}{2G_{23}} - 1 = 0.3881 \quad (5.32)$$

Table 5-9 Strength properties for T300/914 unidirectional carbon fiber composite

F_{1t}	F_{1c}	F_{2t}	F_{2c}	F_{3t}	F_{3c}	F_6	F_5	F_4
[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]
1830	1096	57	57	57	57	88	88	78

The ultimate shear elastic stiffness is not found in the literature. For this case, an approximation will be considered as

$$\tilde{G}_{23} = \frac{\tilde{G}_{12}}{\bar{G}_{12}} \bar{G}_{23} \quad (5.33)$$

Table 5-10 Ultimate elastic properties for T300/914 unidirectional carbon fiber composite

$\tilde{E}_{1t}$	$\tilde{E}_{1c}$	$\tilde{E}_{2t}$	$\tilde{E}_{2c}$	$\tilde{E}_{3t}$	$\tilde{E}_{3c}$	$\tilde{G}_{12}$	$\tilde{G}_{13}$	$\tilde{G}_{23}$
[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]
110941	112250	2375	2375	2375	2375	3311	3311	1901

Considering an element subjected to an increasing inplane shear stress, its response of the model can be visualized in Figure 5-13. Four different material behaviors are considered, purely elastic, purely plastic, purely damage and mixed plastic/damage.

Figure 5-13 Preliminary results for inplane shear stress-strain considering linear, plastic, damage and plastic/damage behaviors.

These results are compared with the experimental load unload inplane shear evolution given by Ladeveze and Dantec (1992). It can be seen that the present results are very close to the experimental ones.

5.3 Correction of Hardening Damage Parameters

The coefficients obtained in Eq. (4.118) are true when the condition seen in Figure 4-18(b) is fulfilled. Since the thermodynamic, isotropic hardening force has been obtained for the case of inplane shear evolution, a correction must be performed for the other evolutions. This is the fiber tension and compression, transverse tension and compression, etc. coefficients must be corrected.

The condition referred to in Figure 4-18(b) is

$$\begin{cases} \hat{g}_d(Y_{ij}(F_{ij})) = 1 \\ \hat{\gamma}(\tilde{\delta}) = \gamma(\tilde{\delta}) + \gamma_0 = 1 \end{cases} \quad (5.34)$$

For a more general case, where a single isotropic hardening is considered, at the critical maximum point the value will no longer be unity. What will be always true is that both the isotropic hardening and the damage will be same for this ultimate point, or,

$$\hat{g}_d(Y_{ij}(F_{ij})) = \hat{\gamma}(\tilde{\delta}_{ij}) \quad (5.35)$$

From Eq. (4.88) a relation between damage and isotropic damage for the case of critical, or ultimate, fiber tension and can be redefined as

$$\tilde{\delta}_{1t} = - \frac{\tilde{D}_{1t}}{\frac{\partial g_d}{\partial Y_{11}}} \quad (5.36)$$

where the partial derivative of the damage surface with respect to the fiber tension thermodynamic force, from Eq. (B.29), is

$$\frac{\partial g_d}{\partial Y_{11}} = \sqrt{H_1^N} \quad (5.37)$$

After replacing Eq. (5.37) into Eq. (5.36), the coefficient is obtained by solving the nonlinear system of equations at Eqs. (5.35) and (5.36). The same procedure is used to obtain the other tension or compression coefficients:

$$\begin{cases} \tilde{\delta}_{1t} = -\frac{\tilde{D}_{1t}}{\sqrt{H_1^N}} \\ \sqrt{H_{1t}^N} \frac{F_{1t}^2}{(1-\tilde{D}_{1t})^3 \bar{E}_1} = \hat{\gamma}(\tilde{\delta}_{1t}) \end{cases}, \quad \sigma_{11} \geq 0 \\
\begin{cases} \tilde{\delta}_{1c} = -\frac{\tilde{D}_{1c}}{\sqrt{H_1^N}} \\ \sqrt{H_{1c}^N} \frac{F_{1c}^2}{(1-\tilde{D}_{1c})^3 \bar{E}_1} = \hat{\gamma}(\tilde{\delta}_{1c}) \end{cases}, \quad \sigma_{11} < 0 \\
\begin{cases} \tilde{\delta}_{2t} = -\frac{\tilde{D}_{2t}}{\sqrt{H_2^N}} \\ \sqrt{H_{2t}^N} \frac{F_{2t}^2}{(1-\tilde{D}_{2t})^3 \bar{E}_2} = \hat{\gamma}(\tilde{\delta}_{2t}) \end{cases}, \quad \sigma_{22} \geq 0 \\
\begin{cases} \tilde{\delta}_{2c} = -\frac{\tilde{D}_{2c}}{\sqrt{H_2^N}} \\ \sqrt{H_{2c}^N} \frac{F_{2c}^2}{(1-\tilde{D}_{2c})^3 \bar{E}_2} = \hat{\gamma}(\tilde{\delta}_{2c}) \end{cases}, \quad \sigma_{22} < 0 \\
\begin{cases} \tilde{\delta}_{3t} = -\frac{\tilde{D}_{3t}}{\sqrt{H_3^N}} \\ \sqrt{H_{3t}^N} \frac{F_{3t}^2}{(1-\tilde{D}_{3t})^3 \bar{E}_3} = \hat{\gamma}(\tilde{\delta}_{3t}) \end{cases}, \quad \sigma_{33} \geq 0 \\
\begin{cases} \tilde{\delta}_{3c} = -\frac{\tilde{D}_{3c}}{\sqrt{H_3^N}} \\ \sqrt{H_{3c}^N} \frac{F_{3c}^2}{(1-\tilde{D}_{3c})^3 \bar{E}_3} = \hat{\gamma}(\tilde{\delta}_{3c}) \end{cases}, \quad \sigma_{33} < 0
\end{cases} \tag{5.38}$$

Using the material properties given in Tables 5-8, 5-9, and 5-10 and the shear coefficients stated at Eq. (4.129), the damage surface coefficients can be redefined. Table 6-1 shows these redefined values, which are going to be used as input material properties in the ANSYS code.

Table 5-11 Damage surface coefficients and critical values for T300/914 unidirectional carbon fiber composite

H_{1t}	H_{1c}	H_{2t}	H_{3c}	H_{2t}	H_{3c}	H_{12}	H_{13}	H_{23}
0.008643	0.012078		0.251552			0.093144		0.081015
$\tilde{D}_{1t}$	$\tilde{D}_{1c}$	$\tilde{D}_{2t}$	$\tilde{D}_{2t}$	$\tilde{D}_{2t}$	$\tilde{D}_{2t}$	$\tilde{D}_{12}$	$\tilde{D}_3$	$\tilde{D}_{23}$
0.1161	0.1109		0.5			0.2841		0.2842
$\tilde{\delta}_{1t}$	$\tilde{\delta}_{1c}$	$\tilde{\delta}_{2t}$	$\tilde{\delta}_{2t}$	$\tilde{\delta}_{2t}$	$\tilde{\delta}_{2t}$	$\tilde{\delta}_{12}$	$\tilde{\delta}_3$	$\tilde{\delta}_{23}$
-1.2488	-1.0091		-0.9969			-0.9310		-0.9984

5.4 General Procedure

The general procedure to obtain the isotropic plastic-strain and damage evolution parameters is summarized in the following steps.

- Step 1:* Use a third-order Prony series to fit the stress-strain shear curve from inplane shear experimental data.
- Step 2:* Obtain the strength inplane and ultimate shear strain.
- Step 3:* Obtain the inplane shear stress and strain elastic limit and the undamaged stiffness.
- Step 4:* Use a third-order polynomial function to fit the inplane shear plastic strain as a function of the total inplane shear strain.
- Step 5:* Use a third-order Prony series to fit the thermodynamic isotropic hardening plasticity as a function of the isotropic hardening plasticity internal variable. The data used to fit the variables is obtained after knowing that both variables are functions of the total inplane shear strain. So, using a set of discretized total inplane shear values, both variables can be obtained and used in data fitting.

These **isotropic hardening plasticity parameters** are used as material properties and treated as input data when the user-defined function is implemented.

- Step 6:* Use a third-order exponential series to fit the stiffness inplane shear as a function of the total inplane shear.
- Step 7:* Use a third-order exponential series to fit the thermodynamic isotropic hardening damage force as a function of the isotropic hardening damage internal variable. As in the case of Step 5, also these two variables can be defined as a function of the total inplane shear.

These **isotropic hardening damage parameters** are used as material properties and used as input data when the user-defined function is implemented.

The material properties that are needed to define a T300/914 unidirectional carbon fiber composite are then obtained from:

Undamaged elastic properties	Table 5-8
Strength properties	Table 5-9
Isotropic hardening plasticity parameters	Eq. (5.14)
Isotropic hardening damage parameters	Eq. (5.31)
Damage surface coefficients	Table 5-11

6. RESULTS

The model described in previous chapters is implemented in ANSYS through a user-defined Fortran subroutine called *usermat3d.f*. Several cases are analyzed, starting with an inplane shear then tension in the fiber direction and tension in the transverse direction. A tension case for a $[\pm 45^\circ]_{2S}$ laminate is compared with data given by Ladeveze and Dantec (1992). Finally a four-point-bending analysis is made, where quantitative damage values are obtained for the critical mid-section.

6.1 Inplane Shear

The first task is to verify that the input data used to define the parameters and coefficients match with those obtained after compiling the user-defined subroutine *usermat3d.f*. In order to do this, a unique 8-node cube is subjected to an inplane shear loading. This model can be observed in Figure 6-1, where the maximum shear strength was reached.

Figure 6-1 Inplane shear stress model using nonlinear Ansys

The evolution of the inplane shear stress as a function of the inplane shear strain is plotted in ANSYS and shown in Figure 6-2. The evolution of the inplane plastic strain and inplane damage internal variables are also plotted in this graph.

Figure 6-2 Inplane shear loading graph, with plastic strain and damage evolution

Data obtained from this analysis allows us to compare with the initial experimental data. This comparison is plotted in Figure 6-3, where it can be seen that the evolution of both the inplane shear variable and the inplane damage internal variables agree excellently with the experimental ones. These initial values have been seen in Figures 5-14 and 5-16 for inplane plastic and inplane damage or stiffness evolution, respectively.

(a) Inplane shear internal variable

(b) Inplane damage internal variable

Figure 6-3 Comparison of evolution of internal shear variables with those obtained with the initial experimental data

Figure 6-4 Comparison of the total inplane shear stress-strain evolution with respect to the initial experimental data

Also a comparison is made in Figure 6-4 for the total evolution of the inplane shear stress-strain until failure with respect to the initial experimental data. The initial inplane shear stress-strain evolution is from Figure 5-10.

Figure 6-5 Load-unload inplane shear loading graph, with plastic strain and damage evolution in Ansys

Finally a load/unload inplane shear analysis is performed with this model. The results obtained are plotted in ANSYS and seen in Figure 6-5. Here the evolution of the inplane stress and the internal parameters, inplane plastic shear and damage, are also plotted. It is seen that during the loading and unloading in the linear elastic regions, no internal

damage or plastic strain is seen, as expected. Figure 6-6 makes a plotted comparison of the initial load-unload initial inplane shear experimental data, Figure 5-9, with respect to the data obtained from the nonlinear analysis using the present material model.

Figure 6-6 Comparison of the load/unload total inplane shear stress-strain evolution with respect to the initial experimental data

Since all the initial data has been referred to the load/unload inplane shear experimental data, the obtained results must correlate. This validates the usability of the model.

6.2 Fiber Tension

Using the modified damage surface coefficients, Figure 5-11, a nonlinear fiber tension analysis is carried out in ANSYS. Figure 6-7 shows the model used in ANSYS, considering eight cubic solid model elements of 1-mm per side, completely restrained at the center node of the whole model. The material properties are the one obtained for T300/914.

Figure 6-7 Fiber tension stress model using nonlinear Ansys

First a load analysis until failure is done. The stress-strain curve is plotted in ANSYS, where the evolution of the fiber plastic strain and damage are also plotted. These plots are seen in Figure 6-8.

Figure 6-8 Fiber tension ANSYS loading graph with plastic strain and damage evolution

In order to verify the data, several cases are analyzed for fiber tension. Linear, only plasticity, only damage, and plastic-damage material behaviors are considered. For the critical or ultimate point the critical values stated in Table 6-1 are obtained. Also it can be seen that for this fiber tension case, the overall behavior is governed by damage rather than plasticity. These behaviors are shown in Figure 6-9.

In the case of shear, plasticity is the phenomena that govern essentially the behavior of the matrix.

Figure 6-9 Fiber tension loading graph for linear, only plastic, only damage, plastic-damage behaviorS

In order to verify the model in ANSYS, a case where the fiber is rotated 90° and tensioned in the y-direction is also done. The values as expected are the same. Finally, a load/unload case is performed until failure for the case of fiber tension, Figure 6-10. Values of tension damage and plastic strain remained constant in the zones of linear behavior, as expected.

Figure 6-10 Load-unload fiber tension loading graph, with plastic strain and damage evolution in ANSYS

6.3 Transverse Tension

Similarly to the previous case of fiber tension, an analysis of transverse tension is also performed. Figure 6-11 shows the stress-strain behavior for the cases of linear, plastic only, damage only, and plastic-damage evolutions.

Figure 6-11 Fiber tension loading graph for linear, plastic only, damage only, plastic-damage behavior

When the transverse strength is reached, similar critical values are obtained, as expected, from Tables 5-10 and 6-1. This gives us the confidence that the model is proper. Also, since transverse tension is related to matrix tension, plastic behavior dominates. Also, to verify the model a 90° rotation was analyzed, where the loading was done in the x -direction. It is seen that the same values are obtained.

Figure 6-12 Load-unload transverse tension graph, with plastic strain and damage evolution in ANSYS

A load/unload analysis for transverse tension is made. The plot of the plastic-damage stress-strain evolution is seen in Figure 6-12. It is seen here that the plastic strain and shear damage internal variables remain constant in the linear behavior of the material.

6.4 $[\pm 45]_{2s}$ Tension in the x -direction

In the previous models only loading in a single direction has been considered. The next model is a $[\pm 45]_{2s}$ laminate made of carbon fiber composite material T300/914 is tensioned in the x -direction. A square cell with periodic boundaries is analyzed in ANSYS. This model is described in Barbero (2008).

This model is compared with the data given by Ladavez and Dantec (1992) who made load/unload tension in the x -direction experiments for a $[\pm 45]_{2s}$ laminate made of carbon fiber composite material T300/914. Figure 6-13 shows this comparison.

Figure 6-13 Comparison of the load/unload tension in the x -direction with data obtained from Ladavez and Dantec (1992) for a $[\pm 45]_{2s}$ laminate

Since a $[\pm 45]_{2s}$ laminate is tensioned in the global x -direction, same values through the thickness are obtained. Then all nodes will have the same stress-strain evolution curve.

The values obtained from the model are similar to those obtained experimentally, as seen in Figure 6-13. Both graphs are not exactly the same. One of the possible explanations of this is the fact that the isotropic hardening evolution has been obtained from the inplane

shear curve and the coefficients have been modified accordingly. If individual isotropic hardening evolutions could be obtained for each loading condition, maybe the accuracy would be better.

Figure 6-14 Damage evolution for a $[\pm 45]_{2s}$ laminate load/unload tension in the x -direction

One of the main purposes of this dissertation is the quantification of damage. In Figure 6-14 damages in the inplane fiber, transverse, and shear directions are obtained. From this quantified damage evolution, it is seen that shear damage is the dominant failure mechanism. The value obtained for the ultimate $\gamma = -0.956$, which is close to the ultimate or critical value $\gamma_{12-cr} = -0.9310$.

Also, the ultimate damage value for inplane shear is $D_{12} = 0.2917$, which is close to the critical inplane shear value of $D_{12-cr} = 0.2842$. The other damage values are small when compared to the inplane shear damage.

6.5 $[0/90]_s$ 4-point Bending Beam

Since the model has been validated as consistent for several cases, a 4-point bending beam application is analyzed. A 20-mm long, 12-mm width, and 1-mm thick beam is modeled considering a $[0/90]_s$ laminate with all layers of equal thickness, material made of carbon fiber composite material T300/914. The loads are applied 4 mm from the ends.

Figure 6-15 $[0/90]_s$ 4-point bending model using nonlinear Ansys

The model of the beam considers two symmetries, one longitudinal and other transverse. A simple supported fixture is considered and the loading is assumed as distributed force through the width, as seen in Figure 6-15.

Since a 4-point bending beam is analyzed, it is known that the middle zone is subjected to pure bending where the shear forces are zero. This means that at the middle span all internal stresses are the same through the length. If the beam were longer but keeping the load at the same positions with respect to the ends, the shown stresses and displacements would be the same.

The analysis is restricted to the middle span. Stresses in global coordinates at the center of the beam are seen in Figure 6-16. Since pure bending is applied at this section, compression above the neutral axis and tension below are seen, as expected this can be seen in Figure 6-16(a), where tension in the fiber at the 0° layer supports all the tensile and compressive loads generated by bending.

Stresses in the global x -direction are seen in Figure 6-16(a). The compressive fiber strength is 1096 Mpa, while the compressive transverse strength is 57 Mpa. The maximum stresses in fiber compression and transverse compression are recorded as -1243.81 Mpa and -46.3353 Mpa for a maximum applied load of 210 N at the upper node 0° above the neutral axis. Then compression fiber failure will occur first.

Due to the fiber geometry, delamination is also a mechanism that is seen. In Figure 6-16(e) the global yz -stress at the 90° fiber has increasing values at the upper layer. The local or material coordinate for this layer is the 13-stress. Due to the geometry, this value is not significant, but what this applications intends to see if all failure mechanisms can be seen.

(a) Global σ_x

(b) Global σ_y

(c) Global σ_z

(d) Global σ_{xy}

(e) Global σ_{xz}

(f) Global σ_{yz}

Figure 6-16 Global stresses for a $[0/90]_s$ 4-point bending model using nonlinear ANSYS

Figure 6-17 Damage evolution for a $[\pm 45]_{2s}$ laminate load-unload tension in the x -direction

For several points through the thickness a graph that shows all six damage evolutions are presented in Figure 6-17. Damages are related to their critical values. From these graphs, it is seen that failure is expected at the upper node above the neutral axis. This failure zone corresponds to that seen at the x -stress graph for transverse compression.

For $u_z = -2.18$ mm the fiber compression stress at the upper node of the 0° layer above the neutral axis is the same as its fiber compression strength. Also, at this point a value of the transverse stress is -12.92 MPa. Then, according to the Tsai-Wu failure criterion, failure will occur for higher values in the fiber and transverse directions. This is seen in Figure 6-17, where a value of 1 is reached for a relative value of the D_{11}/D_{1c} corresponding to a vertical displacement of -2.2 mm.

Quantitative values of stresses, strains, displacements, plastic strains and damages are obtained through the evolution of the loading, which is one the main objectives of this work.

7. SUMMARY AND CONCLUSIONS

A newer 3-D phenomenological model has been developed for the analysis of fiber reinforced composites.

As a first stage, classical thermodynamic and continuum mechanics principals are used to build all the thermechanics conditions that govern a general model for an n -set of internal variables. A first limitation imposed to this model has been related to small displacements, where the total strain is the addition plastic and elastic strains. Also this model is limited for local dissipation, where no heat conduction is included.

A selected failure mechanism involving plastic strain and damage evolution has been chosen. These two mechanisms are intended to work independently. The term “damage” is used only in the context of failure mechanisms associated with fracture, which are commonly associated with degradation in stiffness. Plasticity and damage have been thoroughly defined in order to state clearly its behavior.

Definitions of isotropic and anisotropic damage in unidirectional composite materials have been explained. When damage in unidirectional composites is seen, fiber, inter-fiber and intra-fiber failure mechanisms have been stated. Damage in the three principal directions are related to the projected principal diameters of an assumed ellipsoidal void.

Since internal damage has values in the three principal directions and also the three associated shear values, the whole damage tensor is used in this model. This complete tensor is related to the theoretical ellipsoid void when its eigenvalues are obtained. So the three eigenvalues states the dimensions of the three principal diameters of this ellipsoid and the correspondent eigenvector will define the direction of these diameters. So, when damage shear is seen, this ellipsoid void will be rotated accordingly.

Three specific Helmholtz free energy potentials have been stated: linear elastic, plastic, and damage. All three are considered independent from each other and the total potential is the addition of these three. The internal state variables for the elastic potential, deduced from the strain energy density, are the total strain, plastic strain, and damage. An isotropic hardening plasticity and isotropic damage hardening are used as internal variables for the plastic and damage potentials.

Using the thermomechanic principles defined previously, all the needed thermodynamic forces, thermodynamic force rates, state variables rates, yield surfaces have been developed analytically first and then obtained explicitly. A deep and strict compressed

vector representation of second- and fourth-order tensors has also been developed and used to define explicitly all used variables.

The plastic strain surface is defined on the basis of the Tsai-Wu failure criterion. Here the polynomial approach used by others, proved to be inconsistent for small stress values in the fiber and transverse tension directions. A quadratic formulation is used for consistency. The parameters are the same as those used in the Tsai-Wu polynomial.

The damage surface is defined based on the energy-release crack growth. Here, the fracture toughness in modes I, II, and III are compared to the thermodynamic damage forces. the damage evolution is related to typical failure mechanisms in composite materials such as fiber, inter-fiber, and intra-fiber fracture.

Data obtained from inter-fiber shear load/unload experimental results are used to define the plastic and damage anisotropic associative evolution. The plastic and damage thresholds are obtained by using a nonlinear extrapolation. The thermodynamic isotropic hardening plasticity force defined as a function of the isotropic plasticity internal variable is defined as a third-order Prony series. Its coefficients are obtained by fitting the appropriate extracted data from the inplane shear load/unload experiments.

Similarly, the isotropic hardening damage force expressed as a function of the isotropic damage internal variable is defined as a third-order exponential series. Also, its coefficients are obtained by fitting the appropriate extracted data from the inplane shear load/unload experiments.

From the undamaged material properties, strength properties, and ultimate or critical material properties, the plastic strain surface and damage surface coefficients have been obtained. Since the thermodynamic isotropic hardening force is obtained for the case of inplane shear evolution, a correction has been performed in the other evolutions in order to assure that failure will occur when the strength values are reached.

The model is then implemented in Matlab in order to verify for inconsistencies. An initial shear analysis shows that the model gives same results as those for shear loading.

Finally, this model is implemented as a new, user-defined material complied in the commercial finite element analysis software ANSYS. An inplane shear evolution model until failure shows the same results as those used as input values for the definition of all coefficients.

Then, tensions in the fiber and transverse directions until failure are analyzed. Final stress values are the same as the input material strengths, as expected. Published experimental data from load unload tension tests of a $[\pm 45^\circ]_{2S}$ composite laminate is used to validate the model.

Finally, the new material model is implemented in ANSYS to predict the durability of a composite beam subjected to four-point bending, where the evolution of fiber, inter-

laminar, and intra-laminar types of damage are quantified. This damage quantification is the closure of this work, since this damage values allow us to evaluate the dimensions and orientation of the crack after its eigenvector and eigenvalues are obtained.

7.1 Contributions to the Current State-of the Art

This work shows a comprehensive effort in modeling damage, where the thermomechanics basics are clearly stated for a general case and also for the specific case of small displacements.

The definition of the specific Helmholtz free-energy potential is the biggest challenge when new models are defined. In this case since a separation of effects are considered, where linear elastic, plastic, and damage potentials are used. The plastic and damage potentials are related to the thermodynamic isotropic hardening plasticity force and thermodynamic isotropic damage force. From the analysis done, it is stated that the isotropic plastic function can be defined using a Prony series, while the isotropic damage function using exponential series.

The plastic surface is defined using the Tsai-Wu quadratic formulation. It is stated that the use of Tsai-Wu polynomial formulation has some inconsistencies for small stress values in the fiber and transverse directions.

The damage surface is defined using fracture mechanics concepts and taken as a function of the square of the damage forces in all directions. These coefficients are defined after a correction factor is applied so the critical or ultimate point matches with the initial input material properties.

The coefficients of the isotropic plastic and damage functions are obtained after curve fitting techniques are used extracting information from the inplane shear load/unload experimental data. Threshold values in the plastic and damage evolution are found. For different threshold values plastic only or damage only behavior can be simulated after linear elastic and before plastic-damage behaviors.

For the numerical implementation, a return mapping algorithm is used. The use of Newton-Rhapson iteration for two variables allows us to reduce the number of iterations. Also the final Jacobian or elasto-plasto-damage tangent modulus is obtained for the case of plastic and damage evolution. Finally, the use of tensorial analysis in all the development steps is employed. All variables are defined explicitly after compressed representations of variables are transformed into second- and fourth-order tensors.

8. RECOMMENDATIONS

The present research lands itself to several possible avenues of extension and some of them are listed below.

- Modify the code so linear elastic, nonlinear elastic, plastic only or damage only, and plastic damage behavior can be modeled when different values in the threshold plastic and plastic values and elastic limit are used.
- Instead of a small-deformation theory or infinitesimal-strain theory, use a large-strain theory, finite-strain theory or large deformations.
- Consider heat conduction.
- Consider gradients as internal variables, in order to model the effects of impacts.
- Use the present material model in conjunction with shell elements.
- Use the present material model with zero-thickness elements, in order to evaluate more accurately intra-laminar failure or delamination.
- Modify the model in order to predict fatigue.
- Perform experiments to:
 - obtain inplane stress-strain load/unload data of other unidirectional composites.
 - obtain the other material properties of unidirectional composites.
 - obtain from other loading/unloading and laminate stacking sequence data so validation of the model can be increased.
 - obtain data from bidirectional composites.
 - obtain data from impact and fatigue experiments so validation for impact and fatigue models can be done.

- Measure the generated voids so that a comparison can be made with the ones obtained from the model, and this will allow to predict other failure or durability predictions.

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Appendix A **COMPRESSED REPRESENTATION OF SECOND- AND FOURTH ORDER TENSORS**

A.1 Compressed vector representation of a symmetric second-order tensor

A second-order tensor

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad i, j = 1, 2, 3 \quad (\text{A.1a})$$

is defined symmetric if the tensor components remain invariable when its scripts are switched, $T_{ij}=T_{ji}$, and it can be expressed with 6 terms but not 9 due to this symmetry.

$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & T_{22} & T_{23} \\ \text{symm} & & T_{33} \end{bmatrix} \quad (\text{A.1b})$$

A vector (6x1 matrix) representation of this symmetric second-order tensor is possible, where an hypothetical non uniqueness solution is possible, as shown in the literature. The derivation of this representation is done by means of contravariant and covariant coordinates in the vector domain, as expressed and done by Helnwein (2001), Nadeau and Ferrari (1998), and Mieke (1997).

The simplest way to express this representation is with a direct relation. This relation is also expressed as contravariant coordinates and defined as

$$\begin{Bmatrix} T^1 \\ T^2 \\ T^3 \\ T^4 \\ T^5 \\ T^6 \end{Bmatrix} = \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{Bmatrix} \quad (\text{A.2a})$$

or

$$\{T^a\} = \{T_{ij}\} \quad a = 1, 2, \dots, 6 \quad (\text{A.2b})$$

where the script transformation between the tensorial expression i, j and the vector expression a is defined as

$$a = \begin{cases} i & , i = j \\ 9 - (i + j) & , i \neq j \end{cases} \quad (\text{A.3})$$

Its corresponding covariant coordinate transformation, which is also called its conjugate or associate transformation, are expressed then as

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ 2T_{23} \\ 2T_{13} \\ 2T_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{Bmatrix} \quad (\text{A.4a})$$

or

$$\{T_a\} = [R] \{T_{ij}\} \quad (\text{A.4b})$$

where a factor of “2” is seen in the last three terms. In matrix manipulation this factor can be replaced with the so called *Reuter* matrix R , Barbero (2008).

The difference between the contravariant and variant coordinates is proved by means of the unique vector invariant, $I_{vector} = T^a T_a$, and the equivalence with the first and second invariants, $I_1 = T_{ii} = \text{tr} \mathbf{T}$ and $I_2 = \frac{1}{2}(T_{ii} T_{jj} - T_{ij} T_{ij}) = \frac{1}{2}(\text{tr}^2 \mathbf{T} - \text{tr} \mathbf{T}^2)$ respectively, of a symmetric second-order tensor. A combined invariant can be obtained from the first two as $I_{matrix} = T_{ij} T_{ij}$, so the invariance of the tensor and its vector representation are expressed as

$$Inv = T^a T_a = T_{ij} T_{ij} \quad (\text{A.5})$$

No difference between the contravariant and covariant coordinates can be obtained when a normalization of the tensorial basis is done from its ortho-normal basis. This normalized transformation is expressed as

$$\{\hat{T}_a\} = \{\hat{T}^a\} = [\hat{R}] \{T_{ij}\} \quad (\text{A.6a})$$

or

$$\begin{Bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \\ \hat{T}_4 \\ \hat{T}_5 \\ \hat{T}_6 \end{Bmatrix} = \begin{Bmatrix} \hat{T}^1 \\ \hat{T}^2 \\ \hat{T}^3 \\ \hat{T}^4 \\ \hat{T}^5 \\ \hat{T}^6 \end{Bmatrix} = \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ \sqrt{2} T_{23} \\ \sqrt{2} T_{13} \\ \sqrt{2} T_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{Bmatrix} \quad (\text{A.6b})$$

where $\hat{R}$ represents the normalized Reuter matrix. This normalized transformation also validated by the invariance expressed in Eq. (A.5). The use of one of these transformations, contravariant, covariant, or normalized, must be adequate to its physical meaning as expressed for the cases of stresses and strains.

A.2 Compressed matrix representation of a symmetric fourth-order tensor

In this analysis, the tensor symmetry, defined as its invariance under permutation of its arguments, is taken into account to define the best physical transformation. Two tensor symmetries are defined for the case of a fourth-order tensor, minor and major symmetries. Minor symmetry refers to identical values of tensor components when adjacent scripts are permuted, as expressed in Eq. (A.7). While major symmetry refers when adjacent pairs of scripts are permutes, as seen in Eq. (A.8).

$$T_{ijkl} = T_{jikl} = T_{ijlk} = T_{jilk} \quad (\text{minor symmetry}) \quad (\text{A.7})$$

$$T_{ijkl} = T_{klij} \quad (\text{major symmetry}) \quad (\text{A.8})$$

If symmetry is not taken into account a fourth-order tensor is defined completely with 81 elements, Eq. (A.9a), while 36 elements are necessary to define a fourth-order tensor with minor symmetry, Eq. (A.9b). When both minor and major symmetries are considered, only 21 elements define completely a fourth-order tensor, as seen explicitly in Eq. (A.9c).

$$T_{ijkl} = \begin{bmatrix} \begin{bmatrix} T_{1111} & T_{1112} & T_{1113} \\ T_{1121} & T_{1122} & T_{1123} \\ T_{1131} & T_{1132} & T_{1133} \end{bmatrix}_{11kl} & \begin{bmatrix} T_{1211} & T_{1212} & T_{1213} \\ T_{1221} & T_{1222} & T_{1223} \\ T_{1231} & T_{1232} & T_{1233} \end{bmatrix}_{12kl} & \begin{bmatrix} T_{1311} & T_{1312} & T_{1313} \\ T_{1321} & T_{1322} & T_{1323} \\ T_{1331} & T_{1332} & T_{1333} \end{bmatrix}_{13kl} \\ \begin{bmatrix} T_{2111} & T_{2112} & T_{2113} \\ T_{2121} & T_{2122} & T_{2123} \\ T_{2131} & T_{2132} & T_{2133} \end{bmatrix}_{21kl} & \begin{bmatrix} T_{2211} & T_{2212} & T_{2213} \\ T_{2221} & T_{2222} & T_{2223} \\ T_{2231} & T_{2232} & T_{2233} \end{bmatrix}_{22kl} & \begin{bmatrix} T_{2311} & T_{2312} & T_{2313} \\ T_{2321} & T_{2322} & T_{2323} \\ T_{2331} & T_{2332} & T_{2333} \end{bmatrix}_{23kl} \\ \begin{bmatrix} T_{3111} & T_{3112} & T_{3113} \\ T_{3121} & T_{3122} & T_{3123} \\ T_{3131} & T_{3132} & T_{3133} \end{bmatrix}_{31kl} & \begin{bmatrix} T_{3211} & T_{3212} & T_{3213} \\ T_{3221} & T_{3222} & T_{3223} \\ T_{3231} & T_{3232} & T_{3233} \end{bmatrix}_{32kl} & \begin{bmatrix} T_{3311} & T_{3312} & T_{3313} \\ T_{3321} & T_{3322} & T_{3323} \\ T_{3331} & T_{3332} & T_{3333} \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.9a})$$

$$T_{ijkl} = \begin{bmatrix} \begin{bmatrix} T_{1111} & T_{1112} & T_{1113} \\ \text{minor symm} & T_{1122} & T_{1123} \\ & & T_{1133} \end{bmatrix}_{11kl} & \begin{bmatrix} T_{1211} & T_{1212} & T_{1213} \\ \text{minor symm} & T_{1222} & T_{1223} \\ & & T_{1233} \end{bmatrix}_{12kl} & \begin{bmatrix} T_{1311} & T_{1312} & T_{1313} \\ \text{minor symm} & T_{1322} & T_{1323} \\ & & T_{1333} \end{bmatrix}_{13kl} \\ & \begin{bmatrix} T_{2211} & T_{2212} & T_{2213} \\ \text{minor symm} & T_{2222} & T_{2223} \\ & & T_{2233} \end{bmatrix}_{22kl} & \begin{bmatrix} T_{2311} & T_{2312} & T_{2313} \\ \text{minor symm} & T_{2322} & T_{2323} \\ & & T_{2333} \end{bmatrix}_{23kl} \\ & & \begin{bmatrix} T_{3311} & T_{3312} & T_{3313} \\ \text{minor symm} & T_{3322} & T_{3323} \\ & & T_{3333} \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.9b})$$

$$T_{ijkl} = \begin{bmatrix} \begin{bmatrix} T_{1111} & T_{1112} & T_{1113} \\ \text{minor symm} & T_{1122} & T_{1123} \\ & & T_{1133} \end{bmatrix}_{11kl} & \begin{bmatrix} \text{MAJOR SYMM} & T_{1212} & \text{MAJOR SYMM} \\ \text{minor symm} & \text{MAJOR SYMM} & \text{MAJOR SYMM} \\ & & \text{MAJOR SYMM} \end{bmatrix}_{12kl} & \begin{bmatrix} \text{MAJOR SYMM} & T_{1312} & T_{1313} \\ \text{minor symm} & \text{MAJOR SYMM} & \text{MAJOR SYMM} \\ & & \text{MAJOR SYMM} \end{bmatrix}_{13kl} \\ & \begin{bmatrix} \text{MAJOR SYMM} & T_{2212} & T_{2213} \\ & T_{2222} & T_{2223} \\ \text{minor symm} & & T_{2233} \end{bmatrix}_{22kl} & \begin{bmatrix} \text{MAJOR SYMM} & T_{2312} & T_{2313} \\ \text{minor symm} & \text{MAJOR SYMM} & T_{2323} \\ & & \text{MAJOR SYMM} \end{bmatrix}_{23kl} \\ & & \begin{bmatrix} \text{MAJOR SYMM} & T_{3312} & T_{3313} \\ \text{minor symm} & \text{MAJOR SYMM} & T_{3323} \\ & & T_{3333} \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.9c})$$

The 6x6 matrix representation of a fourth-order tensor is discussed by Helnwein (2001), Nadeau and Ferrari (1998), and Miehe (1997). They define several cases of matrix representation for the cases of contravariant, mixed-variant, and covariant coordinates, which are explicitly defined as follows:

– Contravariant coordinates,

$$\begin{bmatrix} T^{11} & T^{12} & T^{13} & T^{14} & T^{15} & T^{16} \\ T^{21} & T^{22} & T^{23} & T^{24} & T^{25} & T^{26} \\ T^{31} & T^{32} & T^{33} & T^{34} & T^{35} & T^{36} \\ T^{41} & T^{42} & T^{43} & T^{44} & T^{45} & T^{46} \\ T^{51} & T^{52} & T^{53} & T^{54} & T^{55} & T^{56} \\ T^{61} & T^{62} & T^{63} & T^{64} & T^{65} & T^{66} \end{bmatrix} = \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} & T_{1123} & T_{1113} & T_{1112} \\ T_{2211} & T_{2222} & T_{2233} & T_{2223} & T_{2213} & T_{2212} \\ T_{3311} & T_{3322} & T_{3333} & T_{3323} & T_{3313} & T_{3312} \\ T_{2311} & T_{2322} & T_{2333} & T_{2323} & T_{2313} & T_{2312} \\ T_{1311} & T_{1322} & T_{1333} & T_{1323} & T_{1313} & T_{1312} \\ T_{1211} & T_{1222} & T_{1233} & T_{1223} & T_{1213} & T_{1212} \end{bmatrix} \quad (\text{A.10a})$$

or

$$[T^{ab}] = [T_{ijkl}] \quad (\text{A.10b})$$

– Mixed-Variant coordinates,

$$\begin{bmatrix} T_{.1}^1 & T_{.2}^1 & T_{.3}^1 & T_{.4}^1 & T_{.5}^1 & T_{.6}^1 \\ T_{.1}^2 & T_{.2}^2 & T_{.3}^2 & T_{.4}^2 & T_{.5}^2 & T_{.6}^2 \\ T_{.1}^3 & T_{.2}^3 & T_{.3}^3 & T_{.4}^3 & T_{.5}^3 & T_{.6}^3 \\ T_{.1}^4 & T_{.2}^4 & T_{.3}^4 & T_{.4}^4 & T_{.5}^4 & T_{.6}^4 \\ T_{.1}^5 & T_{.2}^5 & T_{.3}^5 & T_{.4}^5 & T_{.5}^5 & T_{.6}^5 \\ T_{.1}^6 & T_{.2}^6 & T_{.3}^6 & T_{.4}^6 & T_{.5}^6 & T_{.6}^6 \end{bmatrix} = \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} & 2T_{1123} & 2T_{1113} & 2T_{1112} \\ T_{2211} & T_{2222} & T_{2233} & 2T_{2223} & 2T_{2213} & 2T_{2212} \\ T_{3311} & T_{3322} & T_{3333} & 2T_{3323} & 2T_{3313} & 2T_{3312} \\ T_{2311} & T_{2322} & T_{2333} & 2T_{2323} & 2T_{2313} & 2T_{2312} \\ T_{1311} & T_{1322} & T_{1333} & 2T_{1323} & 2T_{1313} & 2T_{1312} \\ T_{1211} & T_{1222} & T_{1233} & 2T_{1223} & 2T_{1213} & 2T_{1212} \end{bmatrix} \quad (\text{A.11a})$$

or

$$[T_{.b}^a] = [T_{ijkl}][R] \quad (\text{A.11b})$$

and,

$$\begin{bmatrix} T_1^{.1} & T_1^{.2} & T_1^{.3} & T_1^{.4} & T_1^{.5} & T_1^{.6} \\ T_2^{.1} & T_2^{.2} & T_2^{.3} & T_2^{.4} & T_2^{.5} & T_2^{.6} \\ T_3^{.1} & T_3^{.2} & T_3^{.3} & T_3^{.4} & T_3^{.5} & T_3^{.6} \\ T_4^{.1} & T_4^{.2} & T_4^{.3} & T_4^{.4} & T_4^{.5} & T_4^{.6} \\ T_5^{.1} & T_5^{.2} & T_5^{.3} & T_5^{.4} & T_5^{.5} & T_5^{.6} \\ T_6^{.1} & T_6^{.2} & T_6^{.3} & T_6^{.4} & T_6^{.5} & T_6^{.6} \end{bmatrix} = \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} & T_{1123} & T_{1113} & T_{1112} \\ T_{2211} & T_{2222} & T_{2233} & T_{2223} & T_{2213} & T_{2212} \\ T_{3311} & T_{3322} & T_{3333} & T_{3323} & T_{3313} & T_{3312} \\ 2T_{2311} & 2T_{2322} & 2T_{2333} & 2T_{2323} & 2T_{2313} & 2T_{2312} \\ 2T_{1311} & 2T_{1322} & 2T_{1333} & 2T_{1323} & 2T_{1313} & 2T_{1312} \\ 2T_{1211} & 2T_{1222} & 2T_{1233} & 2T_{1223} & 2T_{1213} & 2T_{1212} \end{bmatrix} \quad (\text{A.12a})$$

or

$$[T_a^{.b}] = [R][T_{ijkl}] \quad (\text{A.12b})$$

– Covariant coordinates,

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} = \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} & 2T_{1123} & 2T_{1113} & 2T_{1112} \\ T_{2211} & T_{2222} & T_{2233} & 2T_{2223} & 2T_{2213} & 2T_{2212} \\ T_{3311} & T_{3322} & T_{3333} & 2T_{3323} & 2T_{3313} & 2T_{3312} \\ 2T_{2311} & 2T_{2322} & 2T_{2333} & 4T_{2323} & 4T_{2313} & 4T_{2312} \\ 2T_{1311} & 2T_{1322} & 2T_{1333} & 4T_{1323} & 4T_{1313} & 4T_{1312} \\ 2T_{1211} & 2T_{1222} & 2T_{1233} & 4T_{1223} & 4T_{1213} & 4T_{1212} \end{bmatrix} \quad (\text{A.13a})$$

or

$$[T_{ab}] = [R][T_{ijkl}][R] \quad (\text{A.13b})$$

It should be noticed that only tensors with at least minor symmetry can be expressed in compressed notation as a 6x6 matrix. When a tensor has minor symmetry, it can be transformed into matrix representation according to Eqs. (A.11) to (A.12), or by mixed-variant coordinates respectively. While if major and minor symmetries are observed,

contravariant and covariant coordinate transformations are possible and defined in Eqs. (A.10) and (A.13) respectively.

As in the case of the second-order tensor, also a normalized coordinate transformation is possible

$$\begin{aligned}
 \begin{bmatrix} \hat{T}^{11} & \hat{T}^{12} & \hat{T}^{13} & \hat{T}^{14} & \hat{T}^{15} & \hat{T}^{16} \\ \hat{T}^{21} & \hat{T}^{22} & \hat{T}^{23} & \hat{T}^{24} & \hat{T}^{25} & \hat{T}^{26} \\ \hat{T}^{31} & \hat{T}^{32} & \hat{T}^{33} & \hat{T}^{34} & \hat{T}^{35} & \hat{T}^{36} \\ \hat{T}^{41} & \hat{T}^{42} & \hat{T}^{43} & \hat{T}^{44} & \hat{T}^{45} & \hat{T}^{46} \\ \hat{T}^{51} & \hat{T}^{52} & \hat{T}^{53} & \hat{T}^{54} & \hat{T}^{55} & \hat{T}^{56} \\ \hat{T}^{61} & \hat{T}^{62} & \hat{T}^{63} & \hat{T}^{64} & \hat{T}^{65} & \hat{T}^{66} \end{bmatrix} &= \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} & \hat{T}_{13} & \hat{T}_{14} & \hat{T}_{15} & \hat{T}_{16} \\ \hat{T}_{21} & \hat{T}_{22} & \hat{T}_{23} & \hat{T}_{24} & \hat{T}_{25} & \hat{T}_{26} \\ \hat{T}_{31} & \hat{T}_{32} & \hat{T}_{33} & \hat{T}_{34} & \hat{T}_{35} & \hat{T}_{36} \\ \hat{T}_{41} & \hat{T}_{42} & \hat{T}_{43} & \hat{T}_{44} & \hat{T}_{45} & \hat{T}_{46} \\ \hat{T}_{51} & \hat{T}_{52} & \hat{T}_{53} & \hat{T}_{54} & \hat{T}_{55} & \hat{T}_{56} \\ \hat{T}_{61} & \hat{T}_{62} & \hat{T}_{63} & \hat{T}_{64} & \hat{T}_{65} & \hat{T}_{66} \end{bmatrix} \\
 &= \begin{bmatrix} T_{1111} & T_{1122} & T_{1133} & \sqrt{2}T_{1123} & \sqrt{2}T_{1113} & \sqrt{2}T_{1112} \\ T_{2211} & T_{2222} & T_{2233} & \sqrt{2}T_{2223} & \sqrt{2}T_{2213} & \sqrt{2}T_{2212} \\ T_{3311} & T_{3322} & T_{3333} & \sqrt{2}T_{3323} & \sqrt{2}T_{3313} & \sqrt{2}T_{3312} \\ \sqrt{2}T_{2311} & \sqrt{2}T_{2322} & \sqrt{2}T_{2333} & 2T_{2323} & 2T_{2313} & 2T_{2312} \\ \sqrt{2}T_{1311} & \sqrt{2}T_{1322} & \sqrt{2}T_{1333} & 2T_{1323} & 2T_{1313} & 2T_{1312} \\ \sqrt{2}T_{1211} & \sqrt{2}T_{1222} & \sqrt{2}T_{1233} & 2T_{1223} & 2T_{1213} & 2T_{1212} \end{bmatrix} \quad (\text{A.14a})
 \end{aligned}$$

or,

$$[\hat{T}^{ab}] = [\hat{T}_{ab}] = [\hat{R}][T_{ijkl}][\hat{R}] \quad (\text{A.14b})$$

A special case of this normalized coordinate transformation is seen when the compressed matrix representation is a diagonal matrix. This case is observed when the only nonzero terms are T_{1111} , T_{2222} , T_{3333} , T_{2323} , T_{1313} , T_{1212} , and the equivalences of the last three due to symmetry. Then the compressed representation of this symmetric fourth-order tensor can be expressed explicitly as

$$\begin{aligned}
 \begin{bmatrix} \tilde{T}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{T}_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{T}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{T}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{T}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{T}_{66} \end{bmatrix} &= \begin{bmatrix} T_{1111} & 0 & 0 & 0 & 0 & 0 \\ 0 & T_{2222} & 0 & 0 & 0 & 0 \\ 0 & 0 & T_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2T_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2T_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2T_{1212} \end{bmatrix} \\
 &\quad (\text{A.15a})
 \end{aligned}$$

or,

$$[\tilde{T}^{ab}] = [\tilde{T}_{ab}] = [R][T_{ijkl}] \quad (\text{A.15b})$$

A.3 Stress and strain tensor

The contravariant or covariant nature of tensor depends on its physical meaning. It is common to see that force (external and internal) and stress (Cauchy, Kirchhoff, first and second-Piola Kirchhoff, etc) are referred to as contravariant components, while displacement and strain (Cauchy, Green Lagrange, Almansi-Euler, Henkey, etc) as covariant, Yokoseki and Aoki (2004). Since stress is defined as contravariant coordinates, its vectorial representation is given in Eq. (A.2) as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}, \text{ or } \quad \{\tau_a\} = \{\sigma_{ij}\} = \{\sigma_a\} \quad (\text{A.16a})$$

or in tensorial notation

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \quad (\text{A.16b})$$

The engineering notation of strain is given as the basis of all strength of materials concepts. This notation is usually represented as a vector and when compared with compressed tensorial notation the known *Voigt* notation is used, as given in Eq. (A.4)

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{Bmatrix} = \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ 2\epsilon_4 \\ 2\epsilon_5 \\ 2\epsilon_6 \end{Bmatrix}, \text{ or } \quad \{\epsilon_a\} = [R]\{\epsilon_{ij}\} = [R]\{\epsilon_a\} \quad (\text{A.17a})$$

where ϵ_a is the engineering strain and ϵ_{ij} is the tensorial strain. When expressed in tensorial notation, this relation is expressed as

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_6 & \epsilon_5 \\ \epsilon_6 & \epsilon_2 & \epsilon_4 \\ \epsilon_5 & \epsilon_4 & \epsilon_3 \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \epsilon_2 & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \epsilon_3 \end{bmatrix} \quad (\text{A.17b})$$

For the case of a linear stress-strain relation, the strain energy density is defined by Mase and Mase (1999) as,

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \quad (\text{A.18})$$

being this scalar an invariant or the value must be the same when using compressed or tensorial notation. This is true when Eq. (A.16) and (A.17) are used independently using engineering or tensorial strains. Another property which also preserves this property is the *Kelvin* notation. Here both strain and stress are compressed as normalized coordinates, Eq. (A.6), where a mathematical equivalence is seen but a physical meaning is not clear. This is clarified in the next section, where the stiffness and compliance are defined.

A.4 Compressed fourth-order identity tensor

For any second or fourth-order tensor, $\mathbf{t}$ or $\mathbf{T}$ respectively, a fourth-order identity tensor is defined so that,

$$\begin{aligned} \mathbf{I} : \mathbf{t} &= \mathbf{t} \\ \mathbf{I} : \mathbf{T} &= \mathbf{T} \end{aligned} \quad (\text{A.19})$$

The simplest way to define this identity tensor may be as $I_{ijkl} = \delta_{ij}\delta_{kl}$, but this definition does not satisfy Eq. (19). Other possible definitions are expressed in Eqs. (20) and (21), both satisfy the identity condition, but without minor symmetry.

$$\mathbf{I}^{(1)} = I_{ijkl}^{(1)} = \delta_{ik}\delta_{jl} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.20})$$

$$\mathbf{I}^{(2)} = I_{ijkl}^{(2)} = \delta_{il} \delta_{jk} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.21})$$

Both, (20) and (21) satisfies the identity property given in Eq. (19), without minor symmetry. A symmetric fourth-order identity tensor is then defined combining the previous definitions as

$$\mathbf{I}^s = I_{ijkl}^s = \frac{1}{2}(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.22})$$

A compressed matrix representation of this symmetric identity tensor must be done independently if contravariant or covariant coordinates are considered. Also it is seen that only diagonal components in the compressed matrix notation are nonzero. So, normalized coordinates must be considered as in Eq. (A.15),

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \times 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \times 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \times 0.5 \end{bmatrix}$$

then

$$[I_{ab}] = [R][I_{ijkl}] \quad (\text{A.23})$$

As expected, an identity 6x6 matrix is obtained after compressing the identity fourth-order tensor, because only with this identity matrix the identity condition expressed in Eq. (A.19) is accomplished. In order to simplify the notation, I and not I^* will be used when referred to this symmetric fourth-order identity tensor.

A.5 Relation between undamaged stiffness and compliance compressed matrix and fourth-order tensor

The material compliance is defined in engineering notation as a relation between the stress and strain in the linear domain as

$$\{\epsilon_a\} = [\bar{S}_{ab}] \{\tau_a\} \quad (\text{A.24a})$$

or in expanded notation

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & 0 \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \quad (\text{A.24b})$$

Knowing that the stiffness is defined as the inverse of the compliance, $C=S^{-1}$, then the stress is also defined as a function of the stiffness and the strain as

$$\{\tau_a\} = [\bar{C}_{ab}] \{\epsilon_a\} = [\bar{S}_{ab}]^{-1} \{\epsilon_a\} \quad (\text{A.25a})$$

or in expanded notation

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \quad (\text{A.25b})$$

This stress definition is invariant according to any representation. In particular, when tensor notation is used the only possibility for the stiffness representation in compressed matrix notation is in contravariant coordinates. When stress is expressed in contravariant and strain in covariant coordinates, then

$$\boldsymbol{\sigma} = \bar{\mathbf{C}} : \boldsymbol{\varepsilon} \quad \rightarrow \quad \sigma_{ij} = \bar{C}_{ijkl} \varepsilon_{kl} \quad (\text{A.26a})$$

or

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{1111} & \bar{C}_{1122} & \bar{C}_{1133} & 0 & 0 & 0 \\ \bar{C}_{1122} & \bar{C}_{2222} & \bar{C}_{2223} & 0 & 0 & 0 \\ \bar{C}_{1133} & \bar{C}_{2223} & \bar{C}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \quad (\text{A.26b})$$

Since Eqs. (A.25b) and (A.26b) are the same, then the symmetric fourth-order stiffness tensor expressed in terms of the compressed matrix components is expressed as

$$\bar{C}_{ijkl} = \begin{bmatrix} \begin{bmatrix} \bar{C}_{11} & 0 & 0 \\ 0 & \bar{C}_{12} & 0 \\ 0 & 0 & \bar{C}_{13} \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & \bar{C}_{66} & 0 \\ \bar{C}_{66} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & \bar{C}_{55} \\ 0 & 0 & 0 \\ \bar{C}_{55} & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & \bar{C}_{66} & 0 \\ \bar{C}_{66} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} \bar{C}_{12} & 0 & 0 \\ 0 & \bar{C}_{22} & 0 \\ 0 & 0 & \bar{C}_{23} \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{C}_{44} \\ 0 & \bar{C}_{44} & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & \bar{C}_{55} \\ 0 & 0 & 0 \\ \bar{C}_{55} & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{C}_{44} \\ 0 & \bar{C}_{44} & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} \bar{C}_{13} & 0 & 0 \\ 0 & \bar{C}_{23} & 0 \\ 0 & 0 & \bar{C}_{33} \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.27})$$

and its correspondent symmetric fourth-order compliance tensor expressed in terms of its covariant compressed matrix components as

$$\bar{S}_{ijkl} = \begin{bmatrix} \begin{bmatrix} \bar{S}_{11} & 0 & 0 \\ 0 & \bar{S}_{12} & 0 \\ 0 & 0 & \bar{S}_{13} \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & 0.25\bar{S}_{66} & 0 \\ 0.25\bar{S}_{66} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & 0.25\bar{S}_{55} \\ 0 & 0 & 0 \\ 0.25\bar{S}_{55} & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & 0.25\bar{S}_{66} & 0 \\ 0.25\bar{S}_{66} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} \bar{S}_{12} & 0 & 0 \\ 0 & \bar{S}_{22} & 0 \\ 0 & 0 & \bar{S}_{23} \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.25\bar{S}_{44} \\ 0 & 0.25\bar{S}_{44} & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & 0.25\bar{S}_{55} \\ 0 & 0 & 0 \\ 0.25\bar{S}_{55} & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.25\bar{S}_{44} \\ 0 & 0.25\bar{S}_{44} & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} \bar{S}_{13} & 0 & 0 \\ 0 & \bar{S}_{23} & 0 \\ 0 & 0 & \bar{S}_{33} \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.28})$$

The mathematical definition of the inverse of a fourth-order tensor $\mathbf{T}$ is the tensor $\mathbf{T}^{-1}$ such that,

$$\mathbf{T} : \mathbf{T}^{-1} = \mathbf{I} \quad \rightarrow \quad T_{ijpq} T_{pqkl}^{-1} = I_{ijkl} \quad (\text{A.30})$$

where $\mathbf{I}$ is the symmetric fourth-order identity tensor defined in Eq. (A.22). This definition is satisfied when the stiffness and its inverse, the compliance, is evaluated using Eqs. (A.25), (A.27) and (A.28) when compressed matrix notation and tensorial notation are used, then

$$[\bar{\mathbf{C}}_{ab}][\bar{\mathbf{S}}_{ab}] = [\bar{\mathbf{C}}_{ab}][\bar{\mathbf{C}}_{ab}]^{-1} = [I_{ab}] \quad (\text{A.31a})$$

$$\bar{\mathbf{C}} : \bar{\mathbf{S}} = \mathbf{I} \quad \rightarrow \quad \bar{C}_{ijpq} \bar{S}_{pqkl} = I_{ijkl} \quad (\text{A.31b})$$

This definition is also satisfied when stress, strain, stiffness, and compliance are reduced as normalized coordinates, so

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2} \sigma_{23} \\ \sqrt{2} \sigma_{13} \\ \sqrt{2} \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{1111} & \bar{C}_{1122} & \bar{C}_{1133} & 0 & 0 & 0 \\ \bar{C}_{1122} & \bar{C}_{2222} & \bar{C}_{2233} & 0 & 0 & 0 \\ \bar{C}_{1133} & \bar{C}_{2233} & \bar{C}_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\bar{C}_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\bar{C}_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\bar{C}_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2} \varepsilon_{23} \\ \sqrt{2} \varepsilon_{13} \\ \sqrt{2} \varepsilon_{12} \end{Bmatrix} \quad (\text{A.32})$$

also known as *Kelvin* or *Mandel* notation. While mathematically the above definition is correct, when trying to understand its physics a conflict is apparent. In this notation it is seen that the engineering and tensorial stress notation are the same. This is not true, because when the linearized Lagrangian strain is defined, Masse and Masse (1999), a relation between the off-diagonals of ε_{ij} and the engineering shear strain components γ_{ij} are defined as

$$\gamma_{ij} = 2\varepsilon_{ij} \quad (i \neq j) \quad (\text{A.33})$$

and not the relation seen in Eq. (A.32), but similar to that expressed in Eq. (A.26b).

A.6 Effective damage tensor

A second-order identity tensor, also known as Kronecker delta, is defined as

$$I_{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{A.34})$$

used to define the second-order *damage tensor*

$$D_{ij} = d_i \delta_{ij} \quad (\text{no sum on } i) \quad (\text{A.35a})$$

or in expanded notation as

$$D_{ij} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad (\text{A.35b})$$

where d_i are the eigenvalues of $\mathbf{D}$. The *integrity tensor* is related to this damage tensor, so using energy equivalence it is defined as,

$$\Omega_{ij} = \sqrt{I_{ij} - D_{ij}} = \sqrt{\delta_{ij} - d_i \delta_{ij}} = \sqrt{1 - d_i} \delta_{ij} \quad (\text{no sum on } i) \quad (\text{A.36a})$$

or when expressed explicitly as

$$\Omega_{ij} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} = \begin{bmatrix} \sqrt{1-d_1} & 0 & 0 \\ 0 & \sqrt{1-d_2} & 0 \\ 0 & 0 & \sqrt{1-d_3} \end{bmatrix} = \begin{bmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_2 & 0 \\ 0 & 0 & \Omega_3 \end{bmatrix} \quad (\text{A.36b})$$

A symmetric fourth-order *effective damage tensor* is introduced by Barbero (2008) and defined as

$$\begin{aligned} M_{ijkl} &= \frac{1}{2} (\Omega_{ik} \Omega_{jl} + \Omega_{il} \Omega_{jk}) \\ &= \frac{1}{2} (\sqrt{1-d_i} \delta_{ik} \sqrt{1-d_j} \delta_{jl} + \sqrt{1-d_i} \delta_{il} \sqrt{1-d_j} \delta_{jk}) \\ &= \sqrt{1-d_i} \sqrt{1-d_j} \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (\text{no sum on } i, j) \\ &= \sqrt{1-d_i} \sqrt{1-d_j} I_{ijkl} \\ &= \Omega_i \Omega_j I_{ijkl} \end{aligned} \quad (\text{A.37a})$$

or

$$M_{ijkl} = \begin{bmatrix} \begin{bmatrix} \Omega_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & \frac{1}{2}\Omega_1\Omega_2 & 0 \\ \frac{1}{2}\Omega_1\Omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & \frac{1}{2}\Omega_1\Omega_3 \\ 0 & 0 & 0 \\ \frac{1}{2}\Omega_1\Omega_3 & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & \frac{1}{2}\Omega_1\Omega_2 & 0 \\ \frac{1}{2}\Omega_1\Omega_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Omega_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Omega_2\Omega_3 \\ 0 & \frac{1}{2}\Omega_2\Omega_3 & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2}\Omega_1\Omega_3 \\ 0 & 0 & 0 \\ \frac{1}{2}\Omega_1\Omega_3 & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Omega_2\Omega_3 \\ 0 & \frac{1}{2}\Omega_2\Omega_3 & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Omega_3^2 \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.37b})$$

In order to obtain the inverse of the damage effect tensor, the tensor properties $\mathbf{A}:\mathbf{A}^{-1}=\mathbf{I}$ and $\mathbf{A}:\mathbf{I}=\mathbf{I}:\mathbf{A}=\mathbf{A}$, Eqs. (A.30) and (A.19) respectively, are used, then

$$\begin{aligned} M_{ijmn} M_{mnkl}^{-1} &= I_{ijkl} \\ \sqrt{1-d_i} \sqrt{1-d_j} I_{ijmn} M_{mnkl}^{-1} &= I_{ijkl} \\ M_{ijkl}^{-1} &= \frac{1}{\sqrt{1-d_i} \sqrt{1-d_j}} I_{ijkl} \quad (\text{no sum on } i, j) \\ &= \frac{1}{\Omega_i \Omega_j} I_{ijkl} \end{aligned} \quad (\text{A.38a})$$

or

$$M_{ijkl}^{-1} = \begin{bmatrix} \begin{bmatrix} \Omega_1^{-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & \frac{1}{2}\Omega_1^{-1}\Omega_2^{-1} & 0 \\ \frac{1}{2}\Omega_1^{-1}\Omega_2^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & \frac{1}{2}\Omega_1^{-1}\Omega_3^{-1} \\ 0 & 0 & 0 \\ \frac{1}{2}\Omega_1^{-1}\Omega_3^{-1} & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & \frac{1}{2}\Omega_1^{-1}\Omega_2^{-1} & 0 \\ \frac{1}{2}\Omega_1^{-1}\Omega_2^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Omega_2^{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Omega_2^{-1}\Omega_3^{-1} \\ 0 & \frac{1}{2}\Omega_2^{-1}\Omega_3^{-1} & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & \frac{1}{2}\Omega_1^{-1}\Omega_3^{-1} \\ 0 & 0 & 0 \\ \frac{1}{2}\Omega_1^{-1}\Omega_3^{-1} & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Omega_2^{-1}\Omega_3^{-1} \\ 0 & \frac{1}{2}\Omega_2^{-1}\Omega_3^{-1} & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Omega_3^{-2} \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{A.38b})$$

The effective damage tensor and its inverse are define as functions of the symmetric fourth-order identity tensor. This identity tensor has been reduced to a matrix notation using normalized coordinates. Then this effective damage tensor and its inverse can be reduced into matrix notation using normalized coordinates. This compressed matrix notation are expressed explicitly in Eqs. (A.39) and (A.40) when effective and effective inverse damage tensors are reduced. This reduction is performed using Eq. (A.23) from its definition given in Eq. (A.38).

$$[M_{ab}] = \begin{bmatrix} \Omega_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_2^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_2\Omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_1\Omega_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_1\Omega_2 \end{bmatrix} \quad (\text{A.39})$$

$$[M_{ab}]^{-1} = \begin{bmatrix} \Omega_1^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_2^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_3^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_2^{-1}\Omega_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_1^{-1}\Omega_3^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_1^{-1}\Omega_2^{-1} \end{bmatrix} \quad (\text{A.40})$$

A.7 Stress - elastic strain relations

Stress-elastic strain relations considering effective and damaged configurations are expressed explicitly for tensor and compressed matrix or vector manipulation. Since the relation is unique, if tensor or compressed tools are used, same results are obtained for all cases as seen next. For all cases, first is expressed the tensorial manipulation and then the result obtained for compressed matrix manipulation. These calculations are made in the *Compressed_Tensor.m* Matlab[®] program.

– Effective stress,

$$\underline{\bar{\sigma}} = \underline{\mathbf{M}}^{-1} : \underline{\sigma} \quad \rightarrow \quad \bar{\sigma}_{ij} = M_{ijkl}^{-1} \sigma_{kl} \quad \leftrightarrow \quad \{\bar{\sigma}_a\} = [M_{ab}]^{-1} \{\sigma_a\}$$

$$\bar{\sigma}_{ij} = \begin{bmatrix} \Omega_1^{-2} \sigma_{11} & \Omega_1^{-1} \Omega_2^{-1} \sigma_{12} & \Omega_1^{-1} \Omega_3^{-1} \sigma_{13} \\ & \Omega_2^{-2} \sigma_{22} & \Omega_2^{-1} \Omega_3^{-1} \sigma_{23} \\ \text{symm} & & \Omega_3^{-2} \sigma_{33} \end{bmatrix} \quad (\text{A.41a})$$

$$\underline{\bar{\sigma}} = \underline{\bar{\mathbf{C}}} : \underline{\bar{\epsilon}}^e \quad \rightarrow \quad \bar{\sigma}_{ij} = \bar{C}_{ijkl} \bar{\epsilon}_{kl}^e \quad \leftrightarrow \quad \{\bar{\sigma}_a\} = [\bar{C}_{ab}][R]\{\bar{\epsilon}_a^e\}$$

$$\bar{\sigma}_{ij} = \begin{bmatrix} \bar{C}_{11} \bar{\epsilon}_{11}^e + \bar{C}_{12} \bar{\epsilon}_{22}^e + \bar{C}_{13} \bar{\epsilon}_{33}^e & 2\bar{C}_{66} \bar{\epsilon}_{12}^e & 2\bar{C}_{55} \bar{\epsilon}_{13}^e \\ & \bar{C}_{12} \bar{\epsilon}_{11}^e + \bar{C}_{22} \bar{\epsilon}_{22}^e + \bar{C}_{23} \bar{\epsilon}_{33}^e & 2\bar{C}_{44} \bar{\epsilon}_{23}^e \\ \text{symm} & & \bar{C}_{13} \bar{\epsilon}_{11}^e + \bar{C}_{23} \bar{\epsilon}_{22}^e + \bar{C}_{33} \bar{\epsilon}_{33}^e \end{bmatrix} \quad (\text{A.41b})$$

$$\underline{\bar{\sigma}} = \underline{\bar{\mathbf{C}}} : \underline{\mathbf{M}} : \underline{\epsilon}^e \quad \rightarrow \quad \bar{\sigma}_{ij} = \bar{C}_{ijkl} M_{klpq} \epsilon_{pq}^e \quad \leftrightarrow \quad \{\bar{\sigma}_a\} = [\bar{C}_{ab}][M_{ab}][R]\{\epsilon_a^e\}$$

$$\bar{\sigma}_{ij} = \begin{bmatrix} \bar{C}_{11} \Omega_1^2 \epsilon_{11}^e + \bar{C}_{12} \Omega_2^2 \epsilon_{22}^e + \bar{C}_{13} \Omega_3^2 \epsilon_{33}^e & 2\bar{C}_{66} \Omega_1 \Omega_2 \epsilon_{12}^e & 2\bar{C}_{55} \Omega_1 \Omega_3 \epsilon_{13}^e \\ & \bar{C}_{12} \Omega_1^2 \epsilon_{11}^e + \bar{C}_{22} \Omega_2^2 \epsilon_{22}^e + \bar{C}_{23} \Omega_3^2 \epsilon_{33}^e & 2\bar{C}_{44} \Omega_2 \Omega_3 \epsilon_{23}^e \\ \text{symm} & & \bar{C}_{13} \Omega_1^2 \epsilon_{11}^e + \bar{C}_{23} \Omega_2^2 \epsilon_{22}^e + \bar{C}_{33} \Omega_3^2 \epsilon_{33}^e \end{bmatrix} \quad (\text{A.41c})$$

$$\{\bar{\sigma}_a\} = \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \Omega_1^{-2} \sigma_1 \\ \Omega_2^{-2} \sigma_2 \\ \Omega_3^{-2} \sigma_3 \\ \Omega_2^{-1} \Omega_3^{-1} \sigma_4 \\ \Omega_1^{-1} \Omega_3^{-1} \sigma_5 \\ \Omega_1^{-1} \Omega_2^{-1} \sigma_6 \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} \bar{\epsilon}_1^e + \bar{C}_{12} \bar{\epsilon}_2^e + \bar{C}_{13} \bar{\epsilon}_3^e \\ \bar{C}_{12} \bar{\epsilon}_1^e + \bar{C}_{22} \bar{\epsilon}_2^e + \bar{C}_{23} \bar{\epsilon}_3^e \\ \bar{C}_{13} \bar{\epsilon}_1^e + \bar{C}_{23} \bar{\epsilon}_2^e + \bar{C}_{33} \bar{\epsilon}_3^e \\ 2\bar{C}_{44} \bar{\epsilon}_4^e \\ 2\bar{C}_{55} \bar{\epsilon}_5^e \\ 2\bar{C}_{66} \bar{\epsilon}_6^e \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} \Omega_1^2 \epsilon_1^e + \bar{C}_{12} \Omega_2^2 \epsilon_2^e + \bar{C}_{13} \Omega_3^2 \epsilon_3^e \\ \bar{C}_{12} \Omega_1^2 \epsilon_1^e + \bar{C}_{22} \Omega_2^2 \epsilon_2^e + \bar{C}_{23} \Omega_3^2 \epsilon_3^e \\ \bar{C}_{13} \Omega_1^2 \epsilon_1^e + \bar{C}_{23} \Omega_2^2 \epsilon_2^e + \bar{C}_{33} \Omega_3^2 \epsilon_3^e \\ 2\bar{C}_{44} \Omega_2 \Omega_3 \epsilon_4^e \\ 2\bar{C}_{55} \Omega_1 \Omega_3 \epsilon_5^e \\ 2\bar{C}_{66} \Omega_1 \Omega_2 \epsilon_6^e \end{bmatrix} \quad (\text{A.41d})$$

– Effective elastic strain,

$$\underline{\bar{\boldsymbol{\varepsilon}}^e} = \underline{\mathbf{M} : \boldsymbol{\varepsilon}^e} \quad \rightarrow \quad \bar{\varepsilon}_{ij}^e = M_{ijkl} \varepsilon_{kl}^e \quad \leftrightarrow \quad \begin{aligned} [R] \{ \bar{\boldsymbol{\varepsilon}}_a^e \} &= [M_{ab}] [R] \{ \boldsymbol{\varepsilon}_a^e \} \\ \{ \bar{\boldsymbol{\varepsilon}}_a^e \} &= [R]^{-1} [M_{ab}] [R] \{ \boldsymbol{\varepsilon}_a^e \} \end{aligned}$$

$$\bar{\varepsilon}_{ij}^e = \begin{bmatrix} \Omega_1^2 \varepsilon_{11}^e & \Omega_1 \Omega_2 \varepsilon_{12}^e & \Omega_1 \Omega_3 \varepsilon_{13}^e \\ & \Omega_2^2 \varepsilon_{22}^e & \Omega_2 \Omega_3 \varepsilon_{23}^e \\ \text{symm} & & \Omega_3^2 \varepsilon_{33}^e \end{bmatrix} \quad (\text{A.42a})$$

$$\underline{\bar{\boldsymbol{\varepsilon}}^e} = \underline{\bar{\mathbf{S}} : \bar{\boldsymbol{\sigma}}^e} \quad \rightarrow \quad \bar{\varepsilon}_{ij}^e = \bar{S}_{ijkl} \bar{\sigma}_{kl}^e \quad \leftrightarrow \quad \begin{aligned} [R] \{ \bar{\boldsymbol{\varepsilon}}_a^e \} &= [\bar{S}_{ab}] \{ \bar{\boldsymbol{\sigma}}_a^e \} \\ \{ \bar{\boldsymbol{\varepsilon}}_a^e \} &= [R]^{-1} [\bar{S}_{ab}] \{ \bar{\boldsymbol{\sigma}}_a^e \} \end{aligned}$$

$$\bar{\varepsilon}_{ij}^e = \begin{bmatrix} \bar{S}_{11} \bar{\sigma}_{11} + \bar{S}_{12} \bar{\sigma}_{22} + \bar{S}_{13} \bar{\sigma}_{33} & \frac{1}{2} \bar{S}_{66} \bar{\sigma}_{12} & \frac{1}{2} \bar{S}_{55} \bar{\sigma}_{13} \\ & \bar{S}_{12} \bar{\sigma}_{11} + \bar{S}_{22} \bar{\sigma}_{22} + \bar{S}_{23} \bar{\sigma}_{33} & \frac{1}{2} \bar{S}_{44} \bar{\sigma}_{23} \\ \text{symm} & & \bar{S}_{13} \bar{\sigma}_{11} + \bar{S}_{23} \bar{\sigma}_{22} + \bar{S}_{33} \bar{\sigma}_{33} \end{bmatrix} \quad (\text{A.42b})$$

$$\underline{\bar{\boldsymbol{\varepsilon}}^e} = \underline{\bar{\mathbf{S}} : \mathbf{M}^{-1} : \boldsymbol{\sigma}} \quad \rightarrow \quad \bar{\varepsilon}_{ij}^e = \bar{S}_{ijkl} M_{klpq}^{-1} \sigma_{pq} \quad \leftrightarrow \quad \begin{aligned} [R] \{ \bar{\boldsymbol{\varepsilon}}_a^e \} &= [\bar{S}_{ab}] [M_{ab}]^{-1} \{ \boldsymbol{\sigma}_a \} \\ \{ \bar{\boldsymbol{\varepsilon}}_a^e \} &= [R]^{-1} [\bar{S}_{ab}] [M_{ab}]^{-1} \{ \boldsymbol{\sigma}_a \} \end{aligned}$$

$$\bar{\varepsilon}_{ij}^e = \begin{bmatrix} \bar{S}_{11} \Omega_1^{-2} \sigma_{11} + \bar{S}_{12} \Omega_2^{-2} \sigma_{22} + \bar{S}_{13} \Omega_3^{-2} \sigma_{33} & \frac{1}{2} \bar{S}_{66} \Omega_1^{-1} \Omega_2^{-1} \sigma_{12} & \frac{1}{2} \bar{S}_{55} \Omega_1^{-1} \Omega_3^{-1} \sigma_{13} \\ & \bar{S}_{12} \Omega_1^{-2} \sigma_{11} + \bar{S}_{22} \Omega_2^{-2} \sigma_{22} + \bar{S}_{23} \Omega_3^{-2} \sigma_{33} & \frac{1}{2} \bar{S}_{44} \Omega_2^{-1} \Omega_3^{-1} \sigma_{23} \\ \text{symm} & & \bar{S}_{13} \Omega_1^{-2} \sigma_{11} + \bar{S}_{23} \Omega_2^{-2} \sigma_{22} + \bar{S}_{33} \Omega_3^{-2} \sigma_{33} \end{bmatrix} \quad (\text{A.42c})$$

$$\{ \bar{\boldsymbol{\varepsilon}}_a \} = \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix} = \begin{bmatrix} \Omega_1^2 \varepsilon_1^e \\ \Omega_2^2 \varepsilon_2^e \\ \Omega_3^2 \varepsilon_3^e \\ \Omega_2 \Omega_3 \varepsilon_4^e \\ \Omega_1 \Omega_3 \varepsilon_5^e \\ \Omega_1 \Omega_2 \varepsilon_6^e \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} \bar{\sigma}_1 + \bar{S}_{12} \bar{\sigma}_2 + \bar{S}_{13} \bar{\sigma}_3 \\ \bar{S}_{12} \bar{\sigma}_1 + \bar{S}_{22} \bar{\sigma}_2 + \bar{S}_{23} \bar{\sigma}_3 \\ \bar{S}_{13} \bar{\sigma}_1 + \bar{S}_{23} \bar{\sigma}_2 + \bar{S}_{33} \bar{\sigma}_3 \\ \frac{1}{2} \bar{S}_{44} \bar{\sigma}_4 \\ \frac{1}{2} \bar{S}_{55} \bar{\sigma}_5 \\ \frac{1}{2} \bar{S}_{66} \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} \Omega_1^{-2} \sigma_1 + \bar{S}_{12} \Omega_2^{-2} \sigma_2 + \bar{S}_{13} \Omega_3^{-2} \sigma_3 \\ \bar{S}_{12} \Omega_1^{-2} \sigma_1 + \bar{S}_{22} \Omega_2^{-2} \sigma_2 + \bar{S}_{23} \Omega_3^{-2} \sigma_3 \\ \bar{S}_{13} \Omega_1^{-2} \sigma_1 + \bar{S}_{23} \Omega_2^{-2} \sigma_2 + \bar{S}_{33} \Omega_3^{-2} \sigma_3 \\ \frac{1}{2} \bar{S}_{44} \Omega_2^{-1} \Omega_3^{-1} \sigma_4 \\ \frac{1}{2} \bar{S}_{55} \Omega_1^{-1} \Omega_3^{-1} \sigma_5 \\ \frac{1}{2} \bar{S}_{66} \Omega_1^{-1} \Omega_2^{-1} \sigma_6 \end{bmatrix} \quad (\text{A.42d})$$

– Damaged stress,

$$\underline{\boldsymbol{\sigma}} = \mathbf{M} : \bar{\boldsymbol{\sigma}} \quad \rightarrow \quad \sigma_{ij} = M_{ijkl} \bar{\sigma}_{kl} \quad \leftrightarrow \quad \{\sigma_a\} = [M_{ab}] \{\bar{\sigma}_a\}$$

$$\sigma_{ij} = \begin{bmatrix} \Omega_1^2 \bar{\sigma}_{11} & \Omega_1 \Omega_2 \bar{\sigma}_{12} & \Omega_1 \Omega_3 \bar{\sigma}_{13} \\ & \Omega_2^2 \bar{\sigma}_{22} & \Omega_2 \Omega_3 \bar{\sigma}_{23} \\ \text{symm} & & \Omega_3^2 \bar{\sigma}_{33} \end{bmatrix} \quad (\text{A.43a})$$

$$\underline{\boldsymbol{\sigma}} = \mathbf{C} : \boldsymbol{\varepsilon}^e \quad \rightarrow \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}^e \quad \leftrightarrow \quad \{\sigma_a\} = [C_{ab}] [R] \{\varepsilon_a^e\}$$

$$\sigma_{ij} = \begin{bmatrix} \bar{C}_{11} \Omega_1^4 \varepsilon_{11}^e + \bar{C}_{12} \Omega_1^2 \Omega_2^2 \varepsilon_{22}^e + \bar{C}_{13} \Omega_1^2 \Omega_3^2 \varepsilon_{33}^e & 2 \bar{C}_{66} \Omega_1^2 \Omega_2^2 \varepsilon_{12}^e & 2 \bar{C}_{55} \Omega_1^2 \Omega_3^2 \varepsilon_{13}^e \\ & \bar{C}_{12} \Omega_1^2 \Omega_2^2 \varepsilon_{11}^e + \bar{C}_{22} \Omega_2^4 \varepsilon_{22}^e + \bar{C}_{23} \Omega_2^2 \Omega_3^2 \varepsilon_{33}^e & 2 \bar{C}_{44} \Omega_2^2 \Omega_3^2 \varepsilon_{23}^e \\ \text{symm} & & \bar{C}_{13} \Omega_1^2 \Omega_3^2 \varepsilon_{11}^e + \bar{C}_{23} \Omega_2^2 \Omega_3^2 \varepsilon_{22}^e + \bar{C}_{33} \Omega_3^4 \varepsilon_{33}^e \end{bmatrix} \quad (\text{A.43b})$$

$$\underline{\boldsymbol{\sigma}} = \mathbf{M} : \bar{\mathbf{C}} : \bar{\boldsymbol{\varepsilon}}^e \quad \rightarrow \quad \sigma_{ij} = M_{ijkl} C_{klpq} \bar{\varepsilon}_{pq}^e \quad \leftrightarrow \quad \{\sigma_a\} = [M_{ab}] [\bar{C}_{ab}] [R] \{\bar{\varepsilon}_a^e\}$$

$$\sigma_{ij} = \begin{bmatrix} \Omega_1^2 (\bar{C}_{11} \bar{\varepsilon}_{11}^e + \bar{C}_{12} \bar{\varepsilon}_{22}^e + \bar{C}_{13} \bar{\varepsilon}_{33}^e) & 2 \bar{C}_{66} \Omega_1 \Omega_2 \bar{\varepsilon}_{12}^e & 2 \bar{C}_{55} \Omega_1 \Omega_3 \bar{\varepsilon}_{13}^e \\ & \Omega_1^2 (\bar{C}_{12} \bar{\varepsilon}_{11}^e + \bar{C}_{22} \bar{\varepsilon}_{22}^e + \bar{C}_{23} \bar{\varepsilon}_{33}^e) & 2 \bar{C}_{44} \Omega_2 \Omega_3 \bar{\varepsilon}_{23}^e \\ \text{symm} & & \Omega_3^2 (\bar{C}_{13} \bar{\varepsilon}_{11}^e + \bar{C}_{23} \bar{\varepsilon}_{22}^e + \bar{C}_{33} \bar{\varepsilon}_{33}^e) \end{bmatrix} \quad (\text{A.43c})$$

$$\{\sigma_a\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \Omega_1^2 \sigma_1 \\ \Omega_2^2 \sigma_2 \\ \Omega_3^2 \sigma_3 \\ \Omega_2 \Omega_3 \sigma_4 \\ \Omega_1 \Omega_3 \sigma_5 \\ \Omega_1 \Omega_2 \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \bar{C}_{11} \Omega_1^4 \varepsilon_{11}^e + \bar{C}_{12} \Omega_1^2 \Omega_2^2 \varepsilon_{22}^e + \bar{C}_{13} \Omega_1^2 \Omega_3^2 \varepsilon_{33}^e \\ \bar{C}_{12} \Omega_1^2 \Omega_2^2 \varepsilon_{11}^e + \bar{C}_{22} \Omega_2^4 \varepsilon_{22}^e + \bar{C}_{23} \Omega_2^2 \Omega_3^2 \varepsilon_{33}^e \\ \bar{C}_{13} \Omega_1^2 \Omega_3^2 \varepsilon_{11}^e + \bar{C}_{23} \Omega_2^2 \Omega_3^2 \varepsilon_{22}^e + \bar{C}_{33} \Omega_3^4 \varepsilon_{33}^e \\ 2 \bar{C}_{44} \Omega_2^2 \Omega_3^2 \varepsilon_{23}^e \\ 2 \bar{C}_{55} \Omega_1^2 \Omega_3^2 \varepsilon_{13}^e \\ 2 \bar{C}_{66} \Omega_1^2 \Omega_2^2 \varepsilon_{12}^e \end{Bmatrix} = \begin{Bmatrix} \Omega_1^2 (\bar{C}_{11} \bar{\varepsilon}_1^e + \bar{C}_{12} \bar{\varepsilon}_2^e + \bar{C}_{13} \bar{\varepsilon}_3^e) \\ \Omega_1^2 (\bar{C}_{12} \bar{\varepsilon}_1^e + \bar{C}_{22} \bar{\varepsilon}_2^e + \bar{C}_{23} \bar{\varepsilon}_3^e) \\ \Omega_3^2 (\bar{C}_{13} \bar{\varepsilon}_1^e + \bar{C}_{23} \bar{\varepsilon}_2^e + \bar{C}_{33} \bar{\varepsilon}_3^e) \\ 2 \bar{C}_{44} \Omega_2 \Omega_3 \bar{\varepsilon}_4^e \\ 2 \bar{C}_{55} \Omega_1 \Omega_3 \bar{\varepsilon}_5^e \\ 2 \bar{C}_{66} \Omega_1 \Omega_2 \bar{\varepsilon}_6^e \end{Bmatrix} \quad (\text{A.43d})$$

– Elastic damaged strain,

$$\underline{\boldsymbol{\varepsilon}}^e = \mathbf{M}^{-1} : \bar{\boldsymbol{\varepsilon}}^e \quad \rightarrow \quad \varepsilon_{ij}^e = M_{ijkl}^{-1} \bar{\varepsilon}_{kl}^e \quad \leftrightarrow \quad \begin{aligned} [R] \{\varepsilon_a^e\} &= [M_{ab}]^{-1} [R] \{\bar{\varepsilon}_a^e\} \\ \{\varepsilon_a^e\} &= [R]^{-1} [M_{ab}]^{-1} [R] \{\bar{\varepsilon}_a^e\} \end{aligned}$$

$$\varepsilon_{ij} = \begin{bmatrix} \Omega_1^{-2} \bar{\varepsilon}_{11} & \Omega_1^{-1} \Omega_2^{-1} \bar{\varepsilon}_{12} & \Omega_1^{-1} \Omega_3^{-1} \bar{\varepsilon}_{13} \\ & \Omega_2^{-2} \bar{\varepsilon}_{22} & \Omega_2^{-1} \Omega_3^{-1} \bar{\varepsilon}_{23} \\ \text{symm} & & \Omega_3^{-2} \bar{\varepsilon}_{33} \end{bmatrix} \quad (\text{A.44a})$$

$$\underline{\boldsymbol{\varepsilon}}^e = \mathbf{S} : \boldsymbol{\sigma} \quad \rightarrow \quad \varepsilon_{ij}^e = S_{ijkl} \sigma_{kl} \quad \leftrightarrow \quad \begin{aligned} [R] \{ \boldsymbol{\varepsilon}_a^e \} &= [S_{ab}] \{ \boldsymbol{\sigma}_a \} \\ \{ \boldsymbol{\varepsilon}_a^e \} &= [R]^{-1} [S_{ab}] \{ \boldsymbol{\sigma}_a \} \end{aligned}$$

$$\varepsilon_{ij}^e = \begin{bmatrix} \bar{S}_{11} \Omega_1^{-4} \sigma_{11} + \bar{S}_{12} \Omega_1^{-2} \Omega_2^{-2} \sigma_{22} & \frac{1}{2} \bar{S}_{66} \Omega_1^{-2} \Omega_2^{-2} \sigma_{12} & \frac{1}{2} \bar{S}_{55} \Omega_1^{-2} \Omega_3^{-2} \sigma_{13} \\ & \bar{S}_{12} \Omega_1^{-2} \Omega_2^{-2} \sigma_{11} + \bar{S}_{22} \Omega_2^{-4} \sigma_{22} & \frac{1}{2} \bar{S}_{44} \Omega_2^{-2} \Omega_3^{-2} \sigma_{23} \\ & & \bar{S}_{13} \Omega_1^{-2} \Omega_3^{-2} \sigma_{11} + \bar{S}_{23} \Omega_2^{-2} \Omega_3^{-2} \sigma_{22} \\ & & & + \bar{S}_{33} \Omega_3^{-4} \sigma_{33} \\ \text{symm} & & & \end{bmatrix} \quad (\text{A.44b})$$

$$\underline{\boldsymbol{\varepsilon}}^e = \mathbf{M}^{-1} : \bar{\mathbf{S}} : \bar{\boldsymbol{\sigma}} \quad \rightarrow \quad \varepsilon_{ij}^e = M_{ijkl}^{-1} \bar{S}_{klpq} \bar{\sigma}_{pq} \quad \leftrightarrow \quad \begin{aligned} [R] \{ \boldsymbol{\varepsilon}_a^e \} &= [M_{ab}]^{-1} [\bar{S}_{ab}] \{ \bar{\boldsymbol{\sigma}}_a \} \\ \{ \boldsymbol{\varepsilon}_a^e \} &= [R]^{-1} [M_{ab}]^{-1} [\bar{S}_{ab}] \{ \bar{\boldsymbol{\sigma}}_a \} \end{aligned}$$

$$\varepsilon_{ij}^e = \begin{bmatrix} \Omega_1^{-2} (\bar{S}_{11} \bar{\sigma}_{11} + \bar{S}_{12} \bar{\sigma}_{22} + \bar{S}_{13} \bar{\sigma}_{33}) & \frac{1}{2} \bar{S}_{66} \Omega_1^{-1} \Omega_2^{-1} \bar{\sigma}_{12} & \frac{1}{2} \bar{S}_{55} \Omega_1^{-1} \Omega_3^{-1} \bar{\sigma}_{13} \\ & \Omega_2^{-2} (\bar{S}_{12} \bar{\sigma}_{11} + \bar{S}_{22} \bar{\sigma}_{22} + \bar{S}_{23} \bar{\sigma}_{33}) & \frac{1}{2} \bar{S}_{44} \Omega_2^{-1} \Omega_3^{-1} \bar{\sigma}_{23} \\ & & \Omega_3^{-2} (\bar{S}_{13} \bar{\sigma}_{11} + \bar{S}_{23} \bar{\sigma}_{22} + \bar{S}_{33} \bar{\sigma}_{33}) \\ \text{symm} & & \end{bmatrix} \quad (\text{A.44c})$$

$$\{ \boldsymbol{\varepsilon}_a^e \} = \begin{Bmatrix} \varepsilon_1^e \\ \varepsilon_2^e \\ \varepsilon_3^e \\ \varepsilon_4^e \\ \varepsilon_5^e \\ \varepsilon_6^e \end{Bmatrix} = \begin{Bmatrix} \Omega_1^{-2} \bar{\varepsilon}_1 \\ \Omega_2^{-2} \bar{\varepsilon}_2 \\ \Omega_3^{-2} \bar{\varepsilon}_3 \\ \Omega_2^{-1} \Omega_3^{-1} \bar{\varepsilon}_4 \\ \Omega_1^{-1} \Omega_3^{-1} \bar{\varepsilon}_5 \\ \Omega_1^{-1} \Omega_2^{-1} \bar{\varepsilon}_6 \end{Bmatrix} = \begin{Bmatrix} \bar{S}_{11} \Omega_1^{-4} \sigma_1 + \bar{S}_{12} \Omega_1^{-2} \Omega_2^{-2} \sigma_2 + \bar{S}_{13} \Omega_1^{-2} \Omega_3^{-2} \sigma_3 \\ \bar{S}_{12} \Omega_1^{-2} \Omega_2^{-2} \sigma_1 + \bar{S}_{22} \Omega_2^{-4} \sigma_2 + \bar{S}_{23} \Omega_2^{-2} \Omega_3^{-2} \sigma_3 \\ \bar{S}_{13} \Omega_1^{-2} \Omega_3^{-2} \sigma_1 + \bar{S}_{23} \Omega_2^{-2} \Omega_3^{-2} \sigma_2 + \bar{S}_{33} \Omega_3^{-4} \sigma_3 \\ \frac{1}{2} \bar{S}_{44} \Omega_2^{-2} \Omega_3^{-2} \sigma_4 \\ \frac{1}{2} \bar{S}_{55} \Omega_1^{-2} \Omega_3^{-2} \sigma_5 \\ \frac{1}{2} \bar{S}_{66} \Omega_1^{-2} \Omega_2^{-2} \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \Omega_1^{-2} (\bar{S}_{11} \bar{\sigma}_{11} + \bar{S}_{12} \bar{\sigma}_{22} + \bar{S}_{13} \bar{\sigma}_{33}) \\ \Omega_2^{-2} (\bar{S}_{12} \bar{\sigma}_{11} + \bar{S}_{22} \bar{\sigma}_{22} + \bar{S}_{23} \bar{\sigma}_{33}) \\ \Omega_3^{-2} (\bar{S}_{13} \bar{\sigma}_{11} + \bar{S}_{23} \bar{\sigma}_{22} + \bar{S}_{33} \bar{\sigma}_{33}) \\ \frac{1}{2} \bar{S}_{44} \Omega_2^{-1} \Omega_3^{-1} \bar{\sigma}_{23} \\ \frac{1}{2} \bar{S}_{55} \Omega_1^{-1} \Omega_3^{-1} \bar{\sigma}_{13} \\ \frac{1}{2} \bar{S}_{66} \Omega_1^{-1} \Omega_2^{-1} \bar{\sigma}_{12} \end{Bmatrix} \quad (\text{A.44d})$$

Appendix B EXPLICIT VARIABLE DEFINITION

The variables defined in the first chapters are now defined explicitly as scalars or tensors as it corresponds. For all cases each variable is defined first as it is defined mathematically. Then is defined as a function of all its variables, this definition allows us to define the input variables in order to define this variable completely. In some cases this variable definition is supported in previously defined variables. Finally the explicit definition is expressed. For all cases i, j, k , and $l=1,2,3$ and $\alpha, \beta=1,\dots,6$.

The variables definition where found using Matlab[®] and the programs *ExplicitVar_01.m* and *ExplicitVar_02.m* and the functions *CompTensor.m* and *TensorProd.m*.

As stated before, an additive decomposition of the strain into an elastic and plastic is considered due to small deformations initial assumptions. This was stated in Eq. (1.23) and shown here the elastic strain in terms of the plastic and plastic components. This definition is considered in the definitions stated in this Appendix as

$$\varepsilon_{kl}^e = \varepsilon_{kl}^e(\varepsilon_{ij}, \varepsilon_{ij}^p) = \varepsilon_{ij} - \varepsilon_{ij}^p \quad (1.23)$$

B.1 Damage tensor

The general second-order *damage tensor* is stated for a complete set of internal damage variables in all directions, normal and shear. This second-order damage tensor is defined as,

$$D_{ij} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \quad (B.1)$$

As seen in Eq. (A.36a), the integrity tensor is obtained using the energy equivalence and stated as

$$\Omega_{ij} = \sqrt{1 - D_{ij}} \quad (B.2a)$$

or when expressed explicitly as

$$\Omega_{ij} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} = \begin{bmatrix} \sqrt{1-D_{11}} & \sqrt{1-D_{12}} & \sqrt{1-D_{13}} \\ \sqrt{1-D_{12}} & \sqrt{1-D_{22}} & \sqrt{1-D_{23}} \\ \sqrt{1-D_{13}} & \sqrt{1-D_{23}} & \sqrt{1-D_{33}} \end{bmatrix} \quad (\text{B.2b})$$

By extension of the definition given in Eq. (A.37a), the symmetric fourth-order damage tensor is then defined as,

$$M_{ijkl} = \Omega_{ij} \Omega_{kl} I_{ijkl} \quad (\text{no sum on } i, j, k, l) \quad (\text{B.3a})$$

$$M_{ijkl}^{-1} = \frac{1}{\Omega_{ij} \Omega_{kl}} I_{ijkl} \quad (\text{no sum on } i, j, k, l) \quad (\text{B.3b})$$

and when used Eqs. (B.1) and (B.2), the explicit definition of the fourth-order damage tensor is then stated as,

$$M_{ijkl} = \begin{bmatrix} \begin{bmatrix} (1-D_{11}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{11kl} & \begin{bmatrix} 0 & \frac{(1-D_{12})}{2} & 0 \\ \frac{(1-D_{12})}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl} & \begin{bmatrix} 0 & 0 & \frac{(1-D_{13})}{2} \\ 0 & 0 & 0 \\ \frac{(1-D_{13})}{2} & 0 & 0 \end{bmatrix}_{13kl} \\ \begin{bmatrix} 0 & \frac{(1-D_{12})}{2} & 0 \\ \frac{(1-D_{12})}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{21kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-D_{22}) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{22kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{(1-D_{23})}{2} \\ 0 & \frac{(1-D_{23})}{2} & 0 \end{bmatrix}_{23kl} \\ \begin{bmatrix} 0 & 0 & \frac{(1-D_{13})}{2} \\ 0 & 0 & 0 \\ \frac{(1-D_{13})}{2} & 0 & 0 \end{bmatrix}_{31kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{(1-D_{23})}{2} \\ 0 & \frac{(1-D_{23})}{2} & 0 \end{bmatrix}_{32kl} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1-D_{33}) \end{bmatrix}_{33kl} \end{bmatrix} \quad (\text{B.4})$$

B.2 Undamaged and damaged stiffness

For orthotropic materials, its compliance matrix is well defined when all its material properties are known as,

$$\begin{aligned}\bar{S}_{\alpha\beta} &= \bar{S}_{\alpha\beta}(\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{G}_{12}, \bar{G}_{13}, \bar{G}_{23}, \nu_{12}, \nu_{13}, \nu_{23}) \\ &= \begin{bmatrix} 1/\bar{E}_1 & -\nu_{12}/\bar{E}_1 & -\nu_{12}/\bar{E}_1 & 0 & 0 & 0 \\ -\nu_{12}/\bar{E}_1 & 1/\bar{E}_2 & -\nu_{23}/\bar{E}_2 & 0 & 0 & 0 \\ -\nu_{12}/\bar{E}_1 & -\nu_{23}/\bar{E}_2 & 1/\bar{E}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\bar{G}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\bar{G}_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\bar{G}_{12} \end{bmatrix}\end{aligned}\quad (\text{B.4})$$

when the compliance is inverted, the stiffness matrix is obtained as

$$\begin{aligned}\bar{C}_{\alpha\beta} &= \bar{C}_{\alpha\beta}(\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{G}_{12}, \bar{G}_{13}, \bar{G}_{23}, \nu_{12}, \nu_{13}, \nu_{23}) = \bar{S}_{\alpha\beta}^{-1} \\ &= \begin{bmatrix} \frac{(\bar{E}_2 - \nu_{23}^2 \bar{E}_3) \bar{E}_1^2}{\Delta^{(2)}} & \frac{(\nu_{12} \bar{E}_2 + \nu_{13} \nu_{23} \bar{E}_3) \bar{E}_1 \bar{E}_2}{\Delta^{(1)}} & \frac{(\nu_{13} + \nu_{12} \nu_{23}) \bar{E}_1 \bar{E}_2 \bar{E}_3}{\Delta^{(1)}} & 0 & 0 & 0 \\ \frac{(\nu_{12} \bar{E}_2 + \nu_{13} \nu_{23} \bar{E}_3) \bar{E}_1 \bar{E}_2}{\Delta^{(1)}} & \frac{(\bar{E}_1 - \nu_{13}^2 \bar{E}_3) \bar{E}_2^2}{\Delta^{(1)}} & \frac{(\nu_{23} \bar{E}_1 + \nu_{12} \nu_{13} \bar{E}_2) \bar{E}_2 \bar{E}_3}{\Delta^{(1)}} & 0 & 0 & 0 \\ \frac{(\nu_{13} + \nu_{12} \nu_{23}) \bar{E}_1 \bar{E}_2 \bar{E}_3}{\Delta^{(1)}} & \frac{(\nu_{23} \bar{E}_1 + \nu_{12} \nu_{13} \bar{E}_2) \bar{E}_2 \bar{E}_3}{\Delta^{(1)}} & \frac{(\bar{E}_1 - \nu_{12}^2 \bar{E}_2) \bar{E}_2 \bar{E}_3}{\Delta^{(1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{G}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{G}_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{G}_{12} \end{bmatrix}\end{aligned}\quad (\text{B.5})$$

where,

$$\Delta^{(1)} = \bar{E}_1 \bar{E}_2 - \nu_{12}^2 \bar{E}_2^2 - \nu_{13}^2 \bar{E}_2 \bar{E}_3 - \nu_{23}^2 \bar{E}_1 \bar{E}_3 - 2\nu_{12} \nu_{13} \nu_{23} \bar{E}_2 \bar{E}_3$$

For the case of transversely isotropic materials, its corresponding stiffness is reduced from the previous considering $\bar{E}_2 = \bar{E}_3$, $\bar{G}_{12} = \bar{G}_{13}$, and $\nu_{12} = \nu_{13}$, so

$$\begin{aligned}\bar{C}_{\alpha\beta} &= \bar{C}_{\alpha\beta}(\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{G}_{12}, \bar{G}_{13}, \bar{G}_{23}, \nu_{12}, \nu_{13}, \nu_{23}) \\ &= \begin{bmatrix} \frac{(1 - \nu_{23}^2) \bar{E}_1^2}{\Delta_1^{(2)}} & \frac{\nu_{12} \bar{E}_1 \bar{E}_2}{\Delta_1^{(2)}} & \frac{\nu_{12} \bar{E}_1 \bar{E}_2}{\Delta_1^{(2)}} & 0 & 0 & 0 \\ \frac{\nu_{12} \bar{E}_1 \bar{E}_2}{\Delta_1^{(2)}} & \frac{(\bar{E}_1 - \nu_{12}^2 \bar{E}_2) \bar{E}_2}{\Delta_2^{(2)}} & \frac{(\nu_{23} \bar{E}_1 + \nu_{12}^2 \bar{E}_2) \bar{E}_2}{\Delta_2^{(2)}} & 0 & 0 & 0 \\ \frac{\nu_{12} \bar{E}_1 \bar{E}_2}{\Delta_1^{(2)}} & \frac{(\nu_{23} \bar{E}_1 + \nu_{12}^2 \bar{E}_2) \bar{E}_2}{\Delta_2^{(2)}} & \frac{(\bar{E}_1 - \nu_{12}^2 \bar{E}_2) \bar{E}_2}{\Delta_2^{(2)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{G}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{G}_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{G}_{12} \end{bmatrix}\end{aligned}\quad (\text{B.6})$$

where,

$$\begin{aligned}\Delta_1^{(2)} &= \bar{E}_1 - \nu_{23} \bar{E}_1 - 2\nu_{12}^2 \bar{E}_2 \\ \Delta_2^{(2)} &= (1 + \nu_{23}) \Delta_1^{(2)}\end{aligned}$$

The undamaged stiffness tensor is defined in Appendix A and here is defined as a function of the damage internal variables and the effective stiffness matrix coefficients, When defined as stated, a general case of the material definition as isotropic, transversely isotropic, anisotropic or other could be used. When transformed to a 4th-order tensor, the damaged stiffness tensor is expressed as,

$$C_{ijkl} = C_{ijkl}(D_{ij}, \bar{C}_{\alpha\beta}) = M_{ijmn} \bar{C}_{mnpq} M_{pqkl} = \begin{bmatrix} \Delta_{11kl}^{(3)} & \Delta_{12kl}^{(3)} & \Delta_{13kl}^{(3)} \\ \Delta_{21kl}^{(3)} & \Delta_{22kl}^{(3)} & \Delta_{23kl}^{(3)} \\ \Delta_{31kl}^{(3)} & \Delta_{32kl}^{(3)} & \Delta_{33kl}^{(3)} \end{bmatrix} \quad (\text{B.7})$$

where,

$$\begin{aligned}\Delta_{11kl}^{(3)} &= \begin{bmatrix} (1-D_{11})^2 \bar{C}_{11} & 0 & 0 \\ 0 & (1-D_{11})(1-D_{22}) \bar{C}_{12} & 0 \\ 0 & 0 & (1-D_{11})(1-D_{33}) \bar{C}_{13} \end{bmatrix}_{11kl} \\ \Delta_{22kl}^{(3)} &= \begin{bmatrix} (1-D_{11})(1-D_{22}) \bar{C}_{12} & 0 & 0 \\ 0 & (1-D_{22})^2 \bar{C}_{22} & 0 \\ 0 & 0 & (1-D_{22})(1-D_{33}) \bar{C}_{23} \end{bmatrix}_{22kl} \\ \Delta_{33kl}^{(3)} &= \begin{bmatrix} (1-D_{11})(1-D_{33}) \bar{C}_{13} & 0 & 0 \\ 0 & (1-D_{22})(1-D_{33}) \bar{C}_{23} & 0 \\ 0 & 0 & (1-D_{33})^2 \bar{C}_{33} \end{bmatrix}_{33kl} \\ \Delta_{12kl}^{(3)} = \Delta_{21kl}^{(3)} &= \begin{bmatrix} 0 & (1-D_{12})^2 \bar{C}_{66} & 0 \\ (1-D_{12})^2 \bar{C}_{66} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl=21kl} \\ \Delta_{13kl}^{(3)} = \Delta_{31kl}^{(3)} &= \begin{bmatrix} 0 & 0 & (1-D_{13})^2 \bar{C}_{55} \\ 0 & 0 & 0 \\ (1-D_{13})^2 \bar{C}_{55} & 0 & 0 \end{bmatrix}_{13kl=31kl} \\ \Delta_{23kl}^{(3)} = \Delta_{32kl}^{(3)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (1-D_{23})^2 \bar{C}_{44} \\ 0 & (1-D_{23})^2 \bar{C}_{44} & 0 \end{bmatrix}_{23kl=32kl}\end{aligned}$$

The compressed representation of the damaged stiffness is stated as,

$$C_{\alpha\beta} = C_{\alpha\beta}(D_{ij}, \bar{C}_{\alpha\beta})$$

$$= \begin{bmatrix} (1-D_{11})^2 \bar{C}_{11} & (1-D_{11})(1-D_{22}) \bar{C}_{12} & (1-D_{11})(1-D_{33}) \bar{C}_{13} & 0 & 0 & 0 \\ (1-D_{11})(1-D_{22}) \bar{C}_{12} & (1-D_{22})^2 \bar{C}_{22} & (1-D_{22})(1-D_{33}) \bar{C}_{23} & 0 & 0 & 0 \\ (1-D_{11})(1-D_{33}) \bar{C}_{13} & (1-D_{22})(1-D_{33}) \bar{C}_{23} & (1-D_{33})^2 \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-D_{23})^2 \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-D_{13})^2 \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-D_{12})^2 \bar{C}_{66} \end{bmatrix}$$

(B.8)

B.3 Damaged stress

The damaged stress is defined in Eq. (4.30) and expressed as

$$\sigma_{ij} = \sigma_{ij}(D_{ij}, \bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) = C_{ijkl} \varepsilon_{kl}^e$$

$$= \begin{bmatrix} (1-D_{11})^2 \bar{C}_{11} \varepsilon_{11}^e + & & \\ (1-D_{11})(1-D_{22}) \bar{C}_{12} \varepsilon_{22}^e + & 2(1-D_{12})^2 \bar{C}_{66} \varepsilon_{12}^e & 2(1-D_{13})^2 \bar{C}_{55} \varepsilon_{13}^e \\ (1-D_{11})(1-D_{33}) \bar{C}_{13} \varepsilon_{33}^e & & \\ \hline & (1-D_{11})(1-D_{22}) \bar{C}_{12} \varepsilon_{11}^e + & \\ 2(1-D_{12})^2 \bar{C}_{66} \varepsilon_{12}^e & (1-D_{22})^2 \bar{C}_{22} \varepsilon_{22}^e + & 2(1-D_{23})^2 \bar{C}_{44} \varepsilon_{23}^e \\ & (1-D_{22})(1-D_{33}) \bar{C}_{23} \varepsilon_{33}^e & \\ \hline & & \\ 2(1-D_{13})^2 \bar{C}_{55} \varepsilon_{13}^e & 2(1-D_{23})^2 \bar{C}_{44} \varepsilon_{23}^e & (1-D_{11})(1-D_{33}) \bar{C}_{13} \varepsilon_{11}^e + \\ & & (1-D_{22})(1-D_{33}) \bar{C}_{23} \varepsilon_{22}^e + \\ & & (1-D_{33})^2 \bar{C}_{33} \varepsilon_{33}^e \end{bmatrix} \quad (B.9)$$

The partial derivatives of the damaged stress with respect the strain and the plastic strain are

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^e} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^e} \Rightarrow \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^e} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^e} (C_{ijkl}) = C_{ijkl} \quad (B.10)$$

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} = -\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^e} \Rightarrow \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} (C_{ijkl}) = -C_{ijkl} \quad (B.11)$$

The partial derivative of the damaged stress with respect damage is

$$\frac{\partial \sigma_{ij}}{\partial D_{kl}} = \frac{\partial \sigma_{ij}}{\partial D_{kl}} (D_{ij}, \bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) = \begin{bmatrix} \Delta_{11kl}^{(4)} & \Delta_{12kl}^{(4)} & \Delta_{13kl}^{(4)} \\ \Delta_{21kl}^{(4)} & \Delta_{22kl}^{(4)} & \Delta_{23kl}^{(4)} \\ \Delta_{31kl}^{(4)} & \Delta_{32kl}^{(4)} & \Delta_{33kl}^{(4)} \end{bmatrix} \quad (B.12)$$

where,

$$\begin{aligned}
\Delta_{11kl}^{(4)} &= \begin{bmatrix} -2(1-D_{11})\bar{C}_{11}\varepsilon_{11}^e - (1-D_{22})\bar{C}_{12}\varepsilon_{22}^e - (1-D_{33})\bar{C}_{13}\varepsilon_{33}^e & 0 & 0 \\ 0 & -(1-D_{11})\bar{C}_{12}\varepsilon_{22}^e & 0 \\ 0 & 0 & -(1-D_{11})\bar{C}_{13}\varepsilon_{33}^e \end{bmatrix}_{11kl} \\
\Delta_{22kl}^{(4)} &= \begin{bmatrix} -(1-D_{22})\bar{C}_{12}\varepsilon_{11}^e & 0 & 0 \\ 0 & -(1-D_{11})\bar{C}_{12}\varepsilon_{11}^e - 2(1-D_{22})\bar{C}_{22}\varepsilon_{22}^e - (1-D_{33})\bar{C}_{23}\varepsilon_{33}^e & 0 \\ 0 & 0 & -(1-D_{22})\bar{C}_{23}\varepsilon_{33}^e \end{bmatrix}_{22kl} \\
\Delta_{33kl}^{(4)} &= \begin{bmatrix} -(1-D_{33})\bar{C}_{13}\varepsilon_{11}^e & 0 & 0 \\ 0 & -(1-D_{33})\bar{C}_{23}\varepsilon_{22}^e & 0 \\ 0 & 0 & -(1-D_{11})\bar{C}_{13}\varepsilon_{11}^e - (1-D_{22})\bar{C}_{23}\varepsilon_{22}^e - 2(1-D_{33})\bar{C}_{33}\varepsilon_{33}^e \end{bmatrix}_{33kl} \\
\Delta_{12kl}^{(4)} = \Delta_{21kl}^{(4)} &= \begin{bmatrix} 0 & -4(1-D_{12})\bar{C}_{66}\varepsilon_{12}^e & 0 \\ -4(1-D_{12})\bar{C}_{66}\varepsilon_{12}^e & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl=21kl} \\
\Delta_{13kl}^{(4)} = \Delta_{31kl}^{(4)} &= \begin{bmatrix} 0 & 0 & -4(1-D_{13})\bar{C}_{55}\varepsilon_{13}^e \\ 0 & 0 & 0 \\ -4(1-D_{13})\bar{C}_{55}\varepsilon_{13}^e & 0 & 0 \end{bmatrix}_{13kl=31kl} \\
\Delta_{23kl}^{(4)} = \Delta_{32kl}^{(4)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4(1-D_{23})\bar{C}_{44}\varepsilon_{23}^e \\ 0 & -4(1-D_{23})\bar{C}_{44}\varepsilon_{23}^e & 0 \end{bmatrix}_{23kl=32kl}
\end{aligned}$$

B.4 Thermodynamic isotropic hardening plasticity force

The thermodynamic isotropic hardening plasticity force and its partial derivative with respect isotropic hardening are

$$R = R(n_p, \alpha_i^p, \beta_i^p, p) = \sum_{i=1}^{n_p} \alpha_i^p \left(e^{\left(-\frac{p}{\beta_i^p} \right)} - 1 \right) \quad (\text{B.10})$$

$$\frac{\partial R}{\partial p} = \frac{\partial R}{\partial p}(n_p, \alpha_i^p, \beta_i^p, p) = \sum_{i=1}^{n_p} -\frac{\alpha_i^p}{\beta_i^p} e^{\left(-\frac{p}{\beta_i^p} \right)} \quad (\text{B.11})$$

B.5 Thermodynamic damage force

The thermodynamic damage force can be defined as the addition of a normal and shear diagonal components as

$$\begin{aligned}
Y_{ij} &= Y_{ij}(D_{ij}, \bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) = Y_{ij}^N(D_{ij}, \bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) + Y_{ij}^S(D_{ij}, \bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) \\
&= -\frac{1}{2}(\varepsilon_{kl}^e - \varepsilon_{kl}^p) \frac{\partial C_{klmn}}{\partial D_{ij}} (\varepsilon_{mn}^e - \varepsilon_{mn}^p) = -\frac{1}{2} \varepsilon_{kl}^e \frac{\partial C_{klmn}}{\partial D_{ij}} \varepsilon_{kl}^e \\
&= \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12} & Y_{22} & Y_{23} \\ Y_{13} & Y_{23} & Y_{33} \end{bmatrix} = \begin{bmatrix} Y_{11}^N & 0 & 0 \\ 0 & Y_{22}^N & 0 \\ 0 & 0 & Y_{33}^N \end{bmatrix} + \begin{bmatrix} 0 & Y_{12}^S & Y_{13}^S \\ Y_{12}^S & 0 & Y_{23}^S \\ Y_{13}^S & Y_{23}^S & 0 \end{bmatrix}
\end{aligned} \tag{B.12}$$

where,

$$\begin{cases} Y_{11} = Y_{11}^N = (1 - D_{11}) \bar{C}_{11} (\varepsilon_{11}^e)^2 + (1 - D_{22}) \bar{C}_{12} \varepsilon_{11}^e \varepsilon_{22}^e + (1 - D_{33}) \bar{C}_{13} \varepsilon_{11}^e \varepsilon_{33}^e \\ Y_{22} = Y_{22}^N = (1 - D_{11}) \bar{C}_{12} \varepsilon_{11}^e \varepsilon_{22}^e + (1 - D_{22}) \bar{C}_{22} (\varepsilon_{22}^e)^2 + (1 - D_{33}) \bar{C}_{23} \varepsilon_{22}^e \varepsilon_{33}^e \\ Y_{33} = Y_{33}^N = (1 - D_{11}) \bar{C}_{13} \varepsilon_{11}^e \varepsilon_{33}^e + (1 - D_{22}) \bar{C}_{23} \varepsilon_{22}^e \varepsilon_{33}^e + (1 - D_{33}) \bar{C}_{33} (\varepsilon_{33}^e)^2 \end{cases}$$

and,

$$\begin{cases} Y_{12} = Y_{12}^S = 4(1 - D_{12}) \bar{C}_{66} (\varepsilon_{12}^e)^2 \\ Y_{13} = Y_{13}^S = 4(1 - D_{13}) \bar{C}_{55} (\varepsilon_{13}^e)^2 \\ Y_{23} = Y_{23}^S = 4(1 - D_{23}) \bar{C}_{44} (\varepsilon_{23}^e)^2 \end{cases}$$

The partial derivative of the thermodynamic damage force with respect to damage is

$$\frac{\partial Y_{ij}}{\partial D_{kl}} = \frac{\partial Y_{ij}}{\partial D_{kl}}(\bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) = \begin{bmatrix} \Delta_{11kl}^{(5)} & \Delta_{12kl}^{(5)} & \Delta_{13kl}^{(5)} \\ \Delta_{21kl}^{(5)} & \Delta_{22kl}^{(5)} & \Delta_{23kl}^{(5)} \\ \Delta_{31kl}^{(5)} & \Delta_{32kl}^{(5)} & \Delta_{33kl}^{(5)} \end{bmatrix} \tag{B.13}$$

where,

$$\begin{aligned}
\Delta_{11kl}^{(5)} &= \begin{bmatrix} -\bar{C}_{11} (\varepsilon_{11}^e)^2 & 0 & 0 \\ 0 & -\bar{C}_{12} \varepsilon_{11}^e \varepsilon_{22}^e & 0 \\ 0 & 0 & -\bar{C}_{13} \varepsilon_{11}^e \varepsilon_{33}^e \end{bmatrix}_{11kl} \\
\Delta_{22kl}^{(5)} &= \begin{bmatrix} -\bar{C}_{12} \varepsilon_{11}^e \varepsilon_{22}^e & 0 & 0 \\ 0 & -\bar{C}_{22} (\varepsilon_{22}^e)^2 & 0 \\ 0 & 0 & -\bar{C}_{23} \varepsilon_{22}^e \varepsilon_{33}^e \end{bmatrix}_{22kl} \\
\Delta_{33kl}^{(5)} &= \begin{bmatrix} -\bar{C}_{13} \varepsilon_{11}^e \varepsilon_{33}^e & 0 & 0 \\ 0 & -\bar{C}_{23} \varepsilon_{22}^e \varepsilon_{33}^e & 0 \\ 0 & 0 & -\bar{C}_{33} (\varepsilon_{33}^e)^2 \end{bmatrix}_{33kl}
\end{aligned}$$

$$\begin{aligned}
\Delta_{12kl}^{(5)} = \Delta_{21kl}^{(5)} &= \begin{bmatrix} 0 & -4\bar{C}_{66}(\varepsilon_{12}^e)^2 & 0 \\ -4\bar{C}_{66}(\varepsilon_{12}^e)^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl=21kl} \\
\Delta_{13kl}^{(5)} = \Delta_{31kl}^{(5)} &= \begin{bmatrix} 0 & 0 & -4\bar{C}_{55}(\varepsilon_{13}^e)^2 \\ 0 & 0 & 0 \\ -4\bar{C}_{55}(\varepsilon_{13}^e)^2 & 0 & 0 \end{bmatrix}_{13kl=31kl} \\
\Delta_{23kl}^{(5)} = \Delta_{32kl}^{(5)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4\bar{C}_{44}(\varepsilon_{23}^e)^2 \\ 0 & -4\bar{C}_{44}(\varepsilon_{23}^e)^2 & 0 \end{bmatrix}_{23kl=32kl}
\end{aligned}$$

The partial derivative of the thermodynamic damage force with respect to the elastic strain is,

$$\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} = \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} (D_{ij}, \bar{C}_{\alpha\beta}, \varepsilon_{ij}^e) = \begin{bmatrix} \Delta_{11kl}^{(6)} & \Delta_{12kl}^{(6)} & \Delta_{13kl}^{(6)} \\ \Delta_{21kl}^{(6)} & \Delta_{22kl}^{(6)} & \Delta_{23kl}^{(6)} \\ \Delta_{31kl}^{(6)} & \Delta_{32kl}^{(6)} & \Delta_{33kl}^{(6)} \end{bmatrix} \quad (\text{B.14})$$

where,

$$\begin{aligned}
\Delta_{11kl}^{(6)} &= \begin{bmatrix} 2(1-D_{11})\bar{C}_{11}\varepsilon_{11}^e + (1-D_{22})\bar{C}_{12}\varepsilon_{22}^e + (1-D_{33})\bar{C}_{13}\varepsilon_{33}^e & 0 & 0 \\ 0 & (1-D_{22})\bar{C}_{12}\varepsilon_{11}^e & 0 \\ 0 & 0 & (1-D_{33})\bar{C}_{13}\varepsilon_{11}^e \end{bmatrix}_{11kl} \\
\Delta_{22kl}^{(6)} &= \begin{bmatrix} (1-D_{11})\bar{C}_{12}\varepsilon_{22}^e & 0 & 0 \\ 0 & (1-D_{11})\bar{C}_{12}\varepsilon_{11}^e + 2(1-D_{22})\bar{C}_{22}\varepsilon_{22}^e + (1-D_{33})\bar{C}_{23}\varepsilon_{33}^e & 0 \\ 0 & 0 & (1-D_{33})\bar{C}_{23}\varepsilon_{22}^e \end{bmatrix}_{22kl} \\
\Delta_{33kl}^{(6)} &= \begin{bmatrix} (1-D_{11})\bar{C}_{13}\varepsilon_{33}^e & 0 & 0 \\ 0 & (1-D_{22})\bar{C}_{23}\varepsilon_{33}^e & 0 \\ 0 & 0 & (1-D_{11})\bar{C}_{13}\varepsilon_{11}^e + (1-D_{22})\bar{C}_{23}\varepsilon_{22}^e + 2(1-D_{33})\bar{C}_{33}\varepsilon_{33}^e \end{bmatrix}_{33kl} \\
\Delta_{12kl}^{(6)} = \Delta_{21kl}^{(6)} &= \begin{bmatrix} 0 & 8(1-D_{12})\bar{C}_{66}\varepsilon_{12}^e & 0 \\ 8(1-D_{12})\bar{C}_{66}\varepsilon_{12}^e & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{12kl=21kl} \\
\Delta_{13kl}^{(6)} = \Delta_{31kl}^{(6)} &= \begin{bmatrix} 0 & 0 & 8(1-D_{13})\bar{C}_{55}\varepsilon_{13}^e \\ 0 & 0 & 0 \\ 8(1-D_{13})\bar{C}_{55}\varepsilon_{13}^e & 0 & 0 \end{bmatrix}_{13kl=31kl}
\end{aligned}$$

$$\Delta_{23kl}^{(6)} = \Delta_{32kl}^{(6)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 8(1-D_{23})\bar{C}_{44}\varepsilon_{23}^e \\ 0 & 8(1-D_{23})\bar{C}_{44}\varepsilon_{23}^e & 0 \end{bmatrix}_{23kl=32kl}$$

Considering small deformations and additive decomposition of the strain in elastic and plastic components, the partial derivatives of the thermodynamic damage force with respect to the total and plastic strain are

$$\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} \quad (\text{B.15})$$

$$\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} = -\frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} \quad (\text{B.16})$$

Also, the thermodynamic damage force can be expressed as a function of the effective compliance and the stress as

$$\begin{cases} Y_{11} = \frac{1}{(1-D_{11})^2} \left(\frac{\bar{S}_{11}}{(1-D_{11})} \sigma_{11}^2 + \frac{\bar{S}_{12}}{(1-D_{22})} \sigma_{11} \sigma_{22} + \frac{\bar{S}_{13}}{(1-D_{33})} \sigma_{11} \sigma_{33} \right) \\ Y_{22} = \frac{1}{(1-D_{22})^2} \left(\frac{\bar{S}_{12}}{(1-D_{11})} \sigma_{11} \sigma_{22} + \frac{\bar{S}_{22}}{(1-D_{22})} \sigma_{22}^2 + \frac{\bar{S}_{23}}{(1-D_{33})} \sigma_{22} \sigma_{33} \right) \\ Y_{33} = \frac{1}{(1-D_{33})^2} \left(\frac{\bar{S}_{13}}{(1-D_{11})} \sigma_{11} \sigma_{33} + \frac{\bar{S}_{23}}{(1-D_{22})} \sigma_{22} \sigma_{33} + \frac{\bar{S}_{33}}{(1-D_{33})} \sigma_{33}^2 \right) \\ Y_{12} = \frac{\bar{S}_{66}}{(1-D_{12})^3} \sigma_{12}^2 \\ Y_{13} = \frac{\bar{S}_{66}}{(1-D_{13})^3} \sigma_{13}^2 \\ Y_{23} = \frac{\bar{S}_{66}}{(1-D_{23})^3} \sigma_{23}^2 \end{cases} \quad (\text{B.17})$$

B.6 Thermodynamic isotropic hardening damage force

The thermodynamic isotropic hardening damage force and its partial derivative with respect to the isotropic damage are

$$\gamma = \gamma(n_d, \alpha_i^d, \beta_i^d, \delta) = \sum_{i=1}^{n_d} \alpha_i^d e^{\left(-\delta/\beta_i^d\right)} \quad (\text{B.18})$$

$$\frac{\partial \gamma}{\partial \delta} = \frac{\partial \gamma}{\partial \delta} (n_d, \alpha_i^d, \beta_i^d, \delta) = \sum_{i=1}^{n_d} -\frac{\alpha_i^d}{\beta_i^d} e^{\left(-\delta/\beta_i^d\right)} \quad (\text{B.19})$$

B.7 Plastic surface

The plastic surface is defined explicitly as

$$g_p = g_p(F_i, F_{\alpha\beta}, n_p, \alpha_i^p, \beta_i^p, R_0, \sigma_{ij}, p) = \hat{g}_p - \hat{R} \quad (\text{B.20})$$

where

$$\begin{aligned} \hat{g}_p &= \hat{g}_p(F_i, F_{\alpha\beta}, \sigma_{ij}) \\ &= \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + a} \end{aligned}$$

$$\begin{cases} a = F_{11} \sigma_{11}^2 + F_{22} \sigma_{22}^2 + F_{33} \sigma_{33}^2 + F_{12} \sigma_{11} \sigma_{22} + F_{13} \sigma_{11} \sigma_{33} + F_{23} \sigma_{22} \sigma_{33} \\ \quad + F_{66} \sigma_{12}^2 + F_{55} \sigma_{13}^2 + F_{44} \sigma_{23}^2 \\ b = F_1 \sigma_{11} + F_2 \sigma_{22} + F_3 \sigma_{33} \end{cases}$$

$$\begin{cases} F_1 = \frac{1}{F_{1t}} - \frac{1}{F_{1c}} & F_2 = \frac{1}{F_{2t}} - \frac{1}{F_{2c}} & F_3 = \frac{1}{F_{3t}} - \frac{1}{F_{3c}} \\ F_{11} = \frac{1}{F_{1t} F_{1c}} & F_{22} = \frac{1}{F_{2t} F_{2c}} & F_{33} = \frac{1}{F_{3t} F_{3c}} \\ F_{44} = \frac{1}{F_4^2} & F_{55} = \frac{1}{F_5^2} & F_{66} = \frac{1}{F_6^2} \\ F_{23} = -\frac{1}{\sqrt{F_{2t} F_{2c} F_{3t} F_{3c}}} & F_{13} = -\frac{1}{\sqrt{F_{1t} F_{1c} F_{3t} F_{3c}}} & F_{12} = -\frac{1}{\sqrt{F_{1t} F_{1c} F_{2t} F_{2c}}} \end{cases}$$

and

$$\begin{aligned} \hat{R} &= \hat{R}(R, R_0) = R(n_p, \alpha_i^p, \beta_i^p, p) + R_0 \\ &= \sum_{i=1}^{n_p} \alpha_i^p \left(e^{\left(-p/\beta_i^p\right)} - 1 \right) + R_0 \end{aligned}$$

The partial derivatives of the plastic surface with respect to the strain and isotropic hardening plasticity force are

$$\frac{\partial g_p}{\partial \sigma_{ij}} = \frac{\partial g_p}{\partial \sigma_{ij}}(F_i, F_{\alpha\beta}, \sigma_{ij}) = \frac{\partial \hat{g}_p}{\partial \sigma_{ij}}$$

$$= \left(\frac{\hat{g}_p}{\hat{g}_p^2 + a} \right) \begin{bmatrix} F_1 \hat{g}_p + 2F_{11} \sigma_{11} + F_{12} \sigma_{22} + F_{13} \sigma_{33} & 2F_{66} \sigma_{12} & 2F_{55} \sigma_{13} \\ 2F_{66} \sigma_{12} & F_2 \hat{g}_p + 2F_{22} \sigma_{22} + F_{12} \sigma_{11} + F_{23} \sigma_{33} & 2F_{44} \sigma_{23} \\ 2F_{55} \sigma_{13} & 2F_{44} \sigma_{23} & F_3 \hat{g}_p + 2F_{33} \sigma_{33} + F_{13} \sigma_{11} + F_{23} \sigma_{22} \end{bmatrix} \quad (\text{B.21})$$

where also the multiplier can be obtained as,

$$\frac{\hat{g}_p}{\hat{g}_p^2 + a} = \frac{1}{2 \sqrt{\left(\frac{b}{2}\right)^2 + a}} \quad (\text{B.22})$$

The partial derivative of the plastic surface with respect to the thermodynamic isotropic hardening plastic force is

$$\frac{\partial g_p}{\partial R} = -\frac{\partial \hat{R}}{\partial R} = -1 \quad (\text{B.23})$$

B.8 Damage surface

The damage surface is defined explicitly as

$$g_d = g_d(H_i^N, H_{ij}^S, n_d, \alpha_i^d, \beta_i^d, \gamma_0, Y_i^N, Y_i^S, \delta) = \hat{g}_d - \hat{\gamma} \quad (\text{B.24})$$

where

$$\begin{aligned} \hat{g}_d &= \hat{g}_d(H_i^N, H_{ij}^S, Y_i^N, Y_i^S) \\ &= \sqrt{H_1^N (Y_1^N)^2 + H_2^N (Y_2^N)^2 + H_3^N (Y_3^N)^2 + H_{12}^S (Y_{12}^S)^2 + H_{13}^S (Y_{13}^S)^2 + H_{23}^S (Y_{23}^S)^2} \end{aligned}$$

and

$$\begin{aligned} Y_1^N &= \frac{1}{(1-D_{11})^2} \left(\frac{1}{(1-D_{11}) \bar{E}_1} \sigma_{11}^2 - \frac{\nu_{12}}{(1-D_{22}) \bar{E}_1} \sigma_{11} \sigma_{22} - \frac{\nu_{13}}{(1-D_{33}) \bar{E}_1} \sigma_{11} \sigma_{33} \right) \\ Y_2^N &= \frac{1}{(1-D_{22})^2} \left(\frac{1}{(1-D_{22}) \bar{E}_2} \sigma_{22}^2 - \frac{\nu_{12}}{(1-D_{11}) \bar{E}_1} \sigma_{11} \sigma_{22} - \frac{\nu_{23}}{(1-D_{33}) \bar{E}_2} \sigma_{22} \sigma_{33} \right) \end{aligned}$$

$$\begin{aligned}
Y_3^N &= \frac{1}{(1-D_{33})^2} \left(\frac{1}{(1-D_{33})\bar{E}_3} \sigma_{33}^2 - \frac{\nu_{12}}{(1-D_{11})\bar{E}_1} \sigma_{11} \sigma_{33} - \frac{\nu_{23}}{(1-D_{22})\bar{E}_2} \sigma_{22} \sigma_{33} \right) \\
Y_{12}^S &= \frac{1}{(1-D_{12})^3 \bar{G}_{12}} \sigma_{12}^2 \\
Y_{13}^S &= \frac{1}{(1-D_{13})^3 \bar{G}_{13}} \sigma_{13}^2 \\
Y_{23}^S &= \frac{1}{(1-D_{23})^3 \bar{G}_{23}} \sigma_{23}^2
\end{aligned}$$

finally

$$\begin{aligned}
\hat{\gamma} &= \hat{\gamma}(n_d, \alpha_i^d, \beta_i^d, \delta) \\
&= \sum_{i=1}^{n_d} \alpha_i^d e^{\left(-\frac{\delta}{\beta_i^d}\right)}
\end{aligned}$$

The material normal coefficients H_i^N are dependent on the type of loading in tension or compression. So, for each of the principal directions each coefficient is defined for tension or compression as,

$$\begin{cases} \tilde{\delta}_{1t} = -\frac{\tilde{D}_{1t}}{\sqrt{H_1^N}}, \sigma_{11} \geq 0 \\ \sqrt{H_{1t}^N} \frac{F_{1t}^2}{(1-\tilde{D}_{1t})^3 \bar{E}_1} = \hat{\gamma}(\tilde{\delta}_{1t}) \end{cases}
\begin{cases} \tilde{\delta}_{1c} = -\frac{\tilde{D}_{1c}}{\sqrt{H_1^N}}, \sigma_{11} < 0 \\ \sqrt{H_{1c}^N} \frac{F_{1c}^2}{(1-\tilde{D}_{1c})^3 \bar{E}_1} = \hat{\gamma}(\tilde{\delta}_{1c}) \end{cases}$$

$$\begin{cases} \tilde{\delta}_{2t} = -\frac{\tilde{D}_{2t}}{\sqrt{H_2^N}}, \sigma_{22} \geq 0 \\ \sqrt{H_{2t}^N} \frac{F_{2t}^2}{(1-\tilde{D}_{2t})^3 \bar{E}_2} = \hat{\gamma}(\tilde{\delta}_{2t}) \end{cases}
\begin{cases} \tilde{\delta}_{2c} = -\frac{\tilde{D}_{2c}}{\sqrt{H_2^N}}, \sigma_{22} < 0 \\ \sqrt{H_{2c}^N} \frac{F_{2c}^2}{(1-\tilde{D}_{2c})^3 \bar{E}_2} = \hat{\gamma}(\tilde{\delta}_{2c}) \end{cases}$$

$$\left\{ \begin{array}{l} \tilde{\delta}_{3t} = -\frac{\tilde{D}_{3t}}{\sqrt{H_3^N}}, \sigma_{33} \geq 0 \\ \sqrt{H_{3t}^N} \frac{F_{2t}^2}{(1-\tilde{D}_{3t})^3 \bar{E}_3} = \hat{\gamma}(\tilde{\delta}_{3t}) \end{array} \right. \quad \left\{ \begin{array}{l} \tilde{\delta}_{3c} = -\frac{\tilde{D}_{3c}}{\sqrt{H_3^N}}, \sigma_{33} < 0 \\ \sqrt{H_{3c}^N} \frac{F_{2c}^2}{(1-\tilde{D}_{3c})^3 \bar{E}_3} = \hat{\gamma}(\tilde{\delta}_{3c}) \end{array} \right. \quad (\text{B.25})$$

or

$$H_i^N = \begin{cases} H_{it}^N, & \text{if } \sigma_{ii} \geq 0 \\ H_{ic}^N, & \text{if } \sigma_{ii} < 0 \end{cases} \quad (\text{no sum on } i) \quad (\text{B.26})$$

The shear coefficients are defined as,

$$H_{12}^S = \frac{\tilde{G}_{12}^3}{\bar{G}_{12} F_6^4}; \quad H_{13}^S = \frac{\tilde{G}_{13}^3}{\bar{G}_{13} F_5^4}; \quad H_{23}^S = \frac{\tilde{G}_{23}^3}{\bar{G}_{23} F_4^4} \quad (\text{B.27})$$

Since the total thermodynamic damage force is the addition of its normal and shear components, then the partial derivative of these components with respect to the total damage force is

$$Y_{ij} = Y_{ij}^N + Y_{ij}^S \quad \Rightarrow \quad \begin{cases} \frac{\partial Y_{ij}^N}{\partial Y_{ij}} = 1 \\ \frac{\partial Y_{ij}^S}{\partial Y_{ij}} = 1 \end{cases} \quad (\text{B.28})$$

Applying Eqs. (B.24) and (B.28), the partial derivative of the damage surface with respect to the thermodynamic damage force is then

$$\begin{aligned} \frac{\partial g_d}{\partial Y_{ij}} &= \frac{\partial g_d}{\partial Y_{ij}}(Y_i^N, Y_i^S, H_i^N, H_{ij}^S) = \frac{\partial \hat{g}_d}{\partial Y_{ij}^N} \frac{\partial Y_{ij}^N}{\partial Y_{ij}} + \frac{\partial \hat{g}_d}{\partial Y_{ij}^S} \frac{\partial Y_{ij}^S}{\partial Y_{ij}} = \frac{\partial \hat{g}_d}{\partial Y_{ij}^N} + \frac{\partial \hat{g}_d}{\partial Y_{ij}^S} \\ &= \frac{1}{\hat{g}_d} \begin{bmatrix} H_1^N Y_1^N & H_{12}^S Y_{12}^S & H_{13}^S Y_{13}^S \\ H_{12}^S Y_{12}^S & H_2^N Y_2^N & H_{23}^S Y_{23}^S \\ H_{13}^S Y_{13}^S & H_{23}^S Y_{23}^S & H_3^N Y_3^N \end{bmatrix} \end{aligned} \quad (\text{B.29})$$

The partial derivative of the damage surface with respect to the thermodynamic isotropic hardening damage force is

$$\frac{\partial g_d}{\partial \gamma} = -\frac{\partial \hat{\gamma}}{\partial \gamma} = -1 \quad (\text{B.30})$$

B.9 Coefficients a_{ij} , b_{ij}^p and b_{ij}^d

The proposed model considers an associated relation between the yield potential and the yield surface, or

$$\begin{cases} f_p = g_p \\ f_d = g_d \end{cases} \quad (\text{B.31})$$

Considering this associated model, the coefficients needed to define the Jacobian matrix are defined as

$$\begin{aligned} -) \quad a_{11} &= \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{kl}} + \frac{\partial g_p}{\partial R} \frac{\partial R}{\partial p} \frac{\partial f_p}{\partial R} = -\frac{\partial g_p}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial g_p}{\partial \sigma_{kl}} + \frac{\partial R}{\partial p} \\ a_{11} &= a_{11} \left(D_{ij}, \bar{C}_{\alpha\beta}, \frac{\partial g_p}{\partial \sigma_{ij}}, \frac{\partial R}{\partial p} \right) \\ &= -(1-D_{11})^2 \bar{C}_{11} \left(\frac{\partial g_p}{\partial \sigma_{11}} \right)^2 - (1-D_{22})^2 \bar{C}_{22} \left(\frac{\partial g_p}{\partial \sigma_{22}} \right)^2 - (1-D_{33})^2 \bar{C}_{33} \left(\frac{\partial g_p}{\partial \sigma_{33}} \right)^2 \\ &\quad - 2(1-D_{11})(1-D_{22}) \bar{C}_{12} \frac{\partial g_p}{\partial \sigma_{11}} \frac{\partial g_p}{\partial \sigma_{22}} \\ &\quad - 2(1-D_{11})(1-D_{33}) \bar{C}_{13} \frac{\partial g_p}{\partial \sigma_{11}} \frac{\partial g_p}{\partial \sigma_{33}} \\ &\quad - 2(1-D_{22})(1-D_{33}) \bar{C}_{23} \frac{\partial g_p}{\partial \sigma_{22}} \frac{\partial g_p}{\partial \sigma_{33}} \\ &\quad - 4(1-D_{12})^2 \bar{C}_{66} \left(\frac{\partial g_p}{\partial \sigma_{12}} \right)^2 - 4(1-D_{13})^2 \bar{C}_{55} \left(\frac{\partial g_p}{\partial \sigma_{13}} \right)^2 - 4(1-D_{23})^2 \bar{C}_{44} \left(\frac{\partial g_p}{\partial \sigma_{23}} \right)^2 \\ &\quad + \frac{\partial R}{\partial p} \end{aligned} \quad (\text{B.32})$$

$$\begin{aligned}
-) \quad a_{12} &= \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{kl}} = \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial D_{kl}} \frac{\partial g_d}{\partial Y_{kl}} \\
a_{12} &= a_{12} \left(\frac{\partial g_p}{\partial \sigma_{ij}}, \frac{\partial \sigma_{ij}}{\partial D_{kl}}, \frac{\partial g_d}{\partial Y_{kl}} \right) \\
a_{12} &= \frac{\partial g_p}{\partial \sigma_{11}} \left(\frac{\partial \sigma_{11}}{\partial D_{11}} \frac{\partial g_d}{\partial Y_{11}} + \frac{\partial \sigma_{11}}{\partial D_{22}} \frac{\partial g_d}{\partial Y_{22}} + \frac{\partial \sigma_{11}}{\partial D_{33}} \frac{\partial g_d}{\partial Y_{33}} \right) \\
&+ \frac{\partial g_p}{\partial \sigma_{22}} \left(\frac{\partial \sigma_{22}}{\partial D_{11}} \frac{\partial g_d}{\partial Y_{11}} + \frac{\partial \sigma_{22}}{\partial D_{22}} \frac{\partial g_d}{\partial Y_{22}} + \frac{\partial \sigma_{22}}{\partial D_{33}} \frac{\partial g_d}{\partial Y_{33}} \right) \\
&+ \frac{\partial g_p}{\partial \sigma_{33}} \left(\frac{\partial \sigma_{33}}{\partial D_{11}} \frac{\partial g_d}{\partial Y_{11}} + \frac{\partial \sigma_{33}}{\partial D_{22}} \frac{\partial g_d}{\partial Y_{22}} + \frac{\partial \sigma_{33}}{\partial D_{33}} \frac{\partial g_d}{\partial Y_{33}} \right) \\
&+ 4 \frac{\partial g_p}{\partial \sigma_{12}} \frac{\partial \sigma_{12}}{\partial D_{12}} \frac{\partial g_d}{\partial Y_{12}} + 4 \frac{\partial g_p}{\partial \sigma_{13}} \frac{\partial \sigma_{13}}{\partial D_{13}} \frac{\partial g_d}{\partial Y_{13}} + 4 \frac{\partial g_p}{\partial \sigma_{23}} \frac{\partial \sigma_{23}}{\partial D_{23}} \frac{\partial g_d}{\partial Y_{23}}
\end{aligned} \tag{B.33}$$

$$\begin{aligned}
-) \quad a_{21} &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^p} \frac{\partial f_p}{\partial \sigma_{kl}} = - \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} \frac{\partial f_p}{\partial \sigma_{kl}} \\
a_{21} &= a_{21} \left(\frac{\partial g_d}{\partial Y_{ij}}, \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e}, \frac{\partial g_p}{\partial \sigma_{ij}} \right) = - \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} \frac{\partial g_p}{\partial \sigma_{kl}} \\
&= - \frac{\partial g_d}{\partial Y_{11}} \left(\frac{\partial Y_{11}}{\partial \varepsilon_{11}^e} \frac{\partial g_p}{\partial \sigma_{11}} + \frac{\partial Y_{11}}{\partial \varepsilon_{22}^e} \frac{\partial g_p}{\partial \sigma_{22}} + \frac{\partial Y_{11}}{\partial \varepsilon_{33}^e} \frac{\partial g_p}{\partial \sigma_{33}} \right) \\
&- \frac{\partial g_d}{\partial Y_{22}} \left(\frac{\partial Y_{22}}{\partial \varepsilon_{11}^e} \frac{\partial g_p}{\partial \sigma_{11}} + \frac{\partial Y_{22}}{\partial \varepsilon_{22}^e} \frac{\partial g_p}{\partial \sigma_{22}} + \frac{\partial Y_{22}}{\partial \varepsilon_{33}^e} \frac{\partial g_p}{\partial \sigma_{33}} \right) \\
&- \frac{\partial g_d}{\partial Y_{33}} \left(\frac{\partial Y_{33}}{\partial \varepsilon_{11}^e} \frac{\partial g_p}{\partial \sigma_{11}} + \frac{\partial Y_{33}}{\partial \varepsilon_{22}^e} \frac{\partial g_p}{\partial \sigma_{22}} + \frac{\partial Y_{33}}{\partial \varepsilon_{33}^e} \frac{\partial g_p}{\partial \sigma_{33}} \right) \\
&- 4 \frac{\partial g_d}{\partial Y_{12}} \frac{\partial Y_{12}}{\partial \varepsilon_{12}^e} \frac{\partial g_p}{\partial \sigma_{12}} - 4 \frac{\partial g_d}{\partial Y_{13}} \frac{\partial Y_{13}}{\partial \varepsilon_{13}^e} \frac{\partial g_p}{\partial \sigma_{13}} - 4 \frac{\partial g_d}{\partial Y_{23}} \frac{\partial Y_{23}}{\partial \varepsilon_{23}^e} \frac{\partial g_p}{\partial \sigma_{23}}
\end{aligned} \tag{B.34}$$

$$\begin{aligned}
-) \quad a_{22} &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial f_d}{\partial Y_{kl}} + \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial \gamma}{\partial \delta} \frac{\partial f_d}{\partial \gamma} = \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial D_{kl}} \frac{\partial g_d}{\partial Y_{kl}} + \frac{\partial \gamma}{\partial \delta} \\
a_{22} &= a_{22} \left(\frac{\partial g_d}{\partial Y_{ij}}, \frac{\partial Y_{ij}}{\partial D_{kl}}, \frac{\partial \gamma}{\partial \delta} \right)
\end{aligned}$$

$$\begin{aligned}
a_{22} = & \left(\frac{\partial g_d}{\partial Y_{11}} \right)^2 \frac{\partial Y_{11}}{\partial D_{11}} + \left(\frac{\partial g_d}{\partial Y_{22}} \right)^2 \frac{\partial Y_{22}}{\partial D_{22}} + \left(\frac{\partial g_d}{\partial Y_{33}} \right)^2 \frac{\partial Y_{33}}{\partial D_{33}} \\
& + 2 \frac{\partial g_d}{\partial Y_{11}} \frac{\partial g_d}{\partial Y_{22}} \frac{\partial Y_{11}}{\partial D_{22}} + 2 \frac{\partial g_d}{\partial Y_{11}} \frac{\partial g_d}{\partial Y_{33}} \frac{\partial Y_{11}}{\partial D_{33}} + 2 \frac{\partial g_d}{\partial Y_{22}} \frac{\partial g_d}{\partial Y_{33}} \frac{\partial Y_{22}}{\partial D_{33}} \\
& + 4 \left(\frac{\partial g_d}{\partial Y_{12}} \right)^2 \frac{\partial Y_{12}}{\partial D_{12}} + 4 \left(\frac{\partial g_d}{\partial Y_{13}} \right)^2 \frac{\partial Y_{13}}{\partial D_{13}} + 4 \left(\frac{\partial g_d}{\partial Y_{23}} \right)^2 \frac{\partial Y_{23}}{\partial D_{23}} \\
& + \frac{\partial \gamma}{\partial \delta}
\end{aligned} \tag{B.35}$$

$$-) \quad b_{ij}^p = \frac{\partial g_p}{\partial \sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}}$$

$$\begin{aligned}
b_{ij}^p &= b_{ij}^p \left(D_{ij}, \bar{C}_{\alpha\beta}, \frac{\partial g_p}{\partial \sigma_{ij}} \right) \\
&= \begin{bmatrix}
(1-D_{11})^2 \bar{C}_{11} \frac{\partial g_p}{\partial \sigma_{11}} + & & \\
(1-D_{11})(1-D_{22}) \bar{C}_{12} \frac{\partial g_p}{\partial \sigma_{22}} + & 2(1-D_{12})^2 \bar{C}_{66} \frac{\partial g_p}{\partial \sigma_{12}} & 2(1-D_{13})^2 \bar{C}_{55} \frac{\partial g_p}{\partial \sigma_{13}} \\
(1-D_{11})(1-D_{33}) \bar{C}_{13} \frac{\partial g_p}{\partial \sigma_{33}} & & \\
\hline
2(1-D_{12})^2 \bar{C}_{66} \frac{\partial g_p}{\partial \sigma_{12}} & (1-D_{11})(1-D_{22}) \bar{C}_{12} \frac{\partial g_p}{\partial \sigma_{11}} + & \\
& (1-D_{22})^2 \bar{C}_{22} \frac{\partial g_p}{\partial \sigma_{22}} + & 2(1-D_{23})^2 \bar{C}_{44} \frac{\partial g_p}{\partial \sigma_{23}} \\
& (1-D_{22})(1-D_{33}) \bar{C}_{23} \frac{\partial g_p}{\partial \sigma_{33}} & \\
\hline
2(1-D_{13})^2 \bar{C}_{55} \frac{\partial g_p}{\partial \sigma_{13}} & 2(1-D_{23})^2 \bar{C}_{44} \frac{\partial g_p}{\partial \sigma_{23}} & (1-D_{11})(1-D_{33}) \bar{C}_{13} \frac{\partial g_p}{\partial \sigma_{11}} + \\
& & (1-D_{22})(1-D_{33}) \bar{C}_{23} \frac{\partial g_p}{\partial \sigma_{22}} + \\
& & (1-D_{33})^2 \bar{C}_{33} \frac{\partial g_p}{\partial \sigma_{33}}
\end{bmatrix}
\end{aligned} \tag{B.36}$$

$$\begin{aligned}
-) \quad b_{ij}^d &= \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial g_d}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} \\
b_{ij}^d &= b_{ij}^p \left(\frac{\partial g_d}{\partial Y_{ij}}, \frac{\partial Y_{ij}}{\partial \varepsilon_{kl}^e} \right) \\
&= \begin{bmatrix} \frac{\partial g_d}{\partial Y_{11}} \frac{\partial Y_{11}}{\partial \varepsilon_{11}^e} + & 2 \frac{\partial g_d}{\partial Y_{12}} \frac{\partial Y_{12}}{\partial \varepsilon_{12}^e} & 2 \frac{\partial g_d}{\partial Y_{13}} \frac{\partial Y_{13}}{\partial \varepsilon_{13}^e} \\ \frac{\partial g_d}{\partial Y_{22}} \frac{\partial Y_{22}}{\partial \varepsilon_{11}^e} + \frac{\partial g_d}{\partial Y_{33}} \frac{\partial Y_{33}}{\partial \varepsilon_{11}^e} & \frac{\partial g_d}{\partial Y_{11}} \frac{\partial Y_{11}}{\partial \varepsilon_{22}^e} + & 2 \frac{\partial g_d}{\partial Y_{23}} \frac{\partial Y_{23}}{\partial \varepsilon_{23}^e} \\ 2 \frac{\partial g_d}{\partial Y_{12}} \frac{\partial Y_{12}}{\partial \varepsilon_{12}^e} & \frac{\partial g_d}{\partial Y_{22}} \frac{\partial Y_{22}}{\partial \varepsilon_{22}^e} + \frac{\partial g_d}{\partial Y_{33}} \frac{\partial Y_{33}}{\partial \varepsilon_{22}^e} & \\ 2 \frac{\partial g_d}{\partial Y_{13}} \frac{\partial Y_{13}}{\partial \varepsilon_{13}^e} & 2 \frac{\partial g_d}{\partial Y_{23}} \frac{\partial Y_{23}}{\partial \varepsilon_{23}^e} & \frac{\partial g_d}{\partial Y_{11}} \frac{\partial Y_{11}}{\partial \varepsilon_{33}^e} + \\ & & \frac{\partial g_d}{\partial Y_{22}} \frac{\partial Y_{22}}{\partial \varepsilon_{33}^e} + \frac{\partial g_d}{\partial Y_{33}} \frac{\partial Y_{33}}{\partial \varepsilon_{33}^e} \end{bmatrix} \quad (B.37)
\end{aligned}$$

B.10 C_{ijkl}^p

The plastic material Jacobian tensor is defined as,

$$\begin{aligned}
\left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \right)^p &= -C_{ijmn} \frac{\partial f_p}{\partial \sigma_{mn}} L_{kl}^p = -C_{ijmn} \frac{\partial g_p}{\partial \sigma_{mn}} L_{kl}^p \\
&= \left(\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \right)^p \left(D_{ij}, \bar{C}_{\alpha\beta}, \frac{\partial g_p}{\partial \sigma_{ij}}, L_{ij}^p \right) = \begin{bmatrix} \Delta_{11kl}^{(8)} & \Delta_{12kl}^{(8)} & \Delta_{13kl}^{(8)} \\ \Delta_{21kl}^{(8)} & \Delta_{22kl}^{(8)} & \Delta_{23kl}^{(8)} \\ \Delta_{31kl}^{(8)} & \Delta_{32kl}^{(8)} & \Delta_{33kl}^{(8)} \end{bmatrix} \quad (B.38)
\end{aligned}$$

where,

$$\begin{aligned}
\Delta_{11kl}^{(8)} &= -(1-D_{11}) \left((1-D_{11}) \bar{C}_{11} \frac{\partial g_p}{\partial \sigma_{11}} + (1-D_{22}) \bar{C}_{12} \frac{\partial g_p}{\partial \sigma_{22}} + (1-D_{33}) \bar{C}_{13} \frac{\partial g_p}{\partial \sigma_{33}} \right) \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{11kl} \\
\Delta_{22kl}^{(8)} &= -(1-D_{22}) \left((1-D_{11}) C_{12} \frac{\partial g_p}{\partial \sigma_{11}} + (1-D_{22}) C_{22} \frac{\partial g_p}{\partial \sigma_{22}} + (1-D_{33}) C_{23} \frac{\partial g_p}{\partial \sigma_{33}} \right) \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{22kl}
\end{aligned}$$

$$\Delta_{33kl}^{(8)} = -(1-D_{33}) \left((1-D_{11})C_{13} \frac{\partial g_p}{\partial \sigma_{11}} + (1-D_{22})C_{23} \frac{\partial g_p}{\partial \sigma_{22}} + (1-D_{33})C_{33} \frac{\partial g_p}{\partial \sigma_{33}} \right) \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{33kl}$$

$$\Delta_{12kl}^{(8)} = \Delta_{21kl}^{(8)} = -2(1-D_{12})^2 C_{66} \frac{\partial g_p}{\partial \sigma_{12}} \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{12kl=21kl}$$

$$\Delta_{13kl}^{(8)} = \Delta_{31kl}^{(8)} = -2(1-D_{13})^2 C_{55} \frac{\partial g_p}{\partial \sigma_{13}} \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{13kl=31kl}$$

$$\Delta_{23kl}^{(8)} = \Delta_{32kl}^{(8)} = -2(1-D_{23})^2 C_{44} \frac{\partial g_p}{\partial \sigma_{23}} \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{23kl=32kl}$$

The transformation of this 4th-order tensor into compressed matrix notation is performed according to Eq. (A.10a) defined in Appendix A for stiffness notations.

B.11 C_{ijkl}^d

The damage material Jacobian tensor is defined as,

$$\begin{aligned} \frac{\partial \sigma_{ij}^d}{\partial \varepsilon_{kl}} &= \frac{\partial \sigma_{ij}}{\partial D_{mn}} \frac{\partial f_d}{\partial Y_{mn}} L_{kl}^d = \frac{\partial \sigma_{ij}}{\partial D_{mn}} \frac{\partial g_d}{\partial Y_{mn}} L_{kl}^d \\ &= \frac{\partial \sigma_{ij}^d}{\partial \varepsilon_{kl}} \left(\frac{\partial \sigma_{ij}}{\partial D_{kl}}, \frac{\partial g_d}{\partial Y_{ij}}, L_{ij}^d \right) = \begin{bmatrix} \Delta_{11kl}^{(9)} & \Delta_{12kl}^{(9)} & \Delta_{13kl}^{(9)} \\ \Delta_{21kl}^{(9)} & \Delta_{22kl}^{(9)} & \Delta_{23kl}^{(9)} \\ \Delta_{31kl}^{(9)} & \Delta_{32kl}^{(9)} & \Delta_{33kl}^{(9)} \end{bmatrix} \end{aligned} \quad (B.39)$$

where,

$$\Delta_{11kl}^{(9)} = \left(\frac{\partial \sigma_{11}}{\partial D_{11}} \frac{\partial g_d}{\partial Y_{11}} + \frac{\partial \sigma_{11}}{\partial D_{22}} \frac{\partial g_d}{\partial Y_{22}} + \frac{\partial \sigma_{11}}{\partial D_{33}} \frac{\partial g_d}{\partial Y_{33}} \right) \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{11kl}$$

$$\Delta_{22kl}^{(9)} = \left(\frac{\partial \sigma_{22}}{\partial D_{11}} \frac{\partial g_d}{\partial Y_{11}} + \frac{\partial \sigma_{22}}{\partial D_{22}} \frac{\partial g_d}{\partial Y_{22}} + \frac{\partial \sigma_{22}}{\partial D_{33}} \frac{\partial g_d}{\partial Y_{33}} \right) \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{22kl}$$

$$\Delta_{33kl}^{(9)} = \left(\frac{\partial \sigma_{33}}{\partial D_{11}} \frac{\partial g_d}{\partial Y_{11}} + \frac{\partial \sigma_{33}}{\partial D_{22}} \frac{\partial g_d}{\partial Y_{22}} + \frac{\partial \sigma_{33}}{\partial D_{33}} \frac{\partial g_d}{\partial Y_{33}} \right) \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{33kl}$$

$$\Delta_{12kl}^{(9)} = \Delta_{21kl}^{(9)} = 2 \frac{\partial \sigma_{12}}{\partial D_{12}} \frac{\partial g_d}{\partial Y_{12}} \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{12kl=21kl}$$

$$\Delta_{13kl}^{(9)} = \Delta_{31kl}^{(9)} = 2 \frac{\partial \sigma_{13}}{\partial D_{13}} \frac{\partial g_d}{\partial Y_{13}} \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{13kl=31kl}$$

$$\Delta_{23kl}^{(9)} = \Delta_{32kl}^{(9)} = 2 \frac{\partial \sigma_{23}}{\partial D_{23}} \frac{\partial g_d}{\partial Y_{23}} \begin{bmatrix} L_{11}^p & L_{12}^p & L_{13}^p \\ L_{12}^p & L_{22}^p & L_{23}^p \\ L_{13}^p & L_{23}^p & L_{33}^p \end{bmatrix}_{23kl=32kl}$$

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