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Generalizations of colorability and connectivity of graphs

Xiankun Zhang
West Virginia University

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Generalizations of Colorability and Connectivity of Graphs

by

Xiankun Zhang

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Abstract

Let $G = (V, E)$ be a graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions $f: E(G) \rightarrow A$. Denote by D an orientation of $E(G)$. Then G is A -colorable if and only if for every $f \in F(G, A)$ there exists an A -coloring $c: V(G) \rightarrow A$ such that for every $e = (x, y) \in E(G)$ (assumed to be directed from x to y), $c(x) - c(y) \neq f(e)$. We define the group chromatic number of $\chi_1(G)$ to be the minimum number m for which G is A -colorable for any Abelian group A of order $\geq m$ under the orientation D . Chapters 2 and 3 are mainly concerned with group chromatic number and some results are given.

The edge-integrity of a graph G is defined by $\min_{S \subseteq E(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ denotes the maximum order of a component of $G - S$. Let $I'(G)$ denote the edge-integrity of a graph G . We define a graph G to be I' -maximal if for every edge e in $\overline{G}$, the complement of graph G , $I'(G + e) > I'(G)$. In chapter 4, some results of I' -maximal graphs are established, the girth of a connected I' -maximal graph is given and lower and upper bounds on the size of I' -maximal connected graphs with given order and edge-integrity are investigated. Also, the I' -maximal trees and unicyclic graphs are completely characterized.

For any given edges e_1, e_2 in $E(G)$, a spanning trail of G with e_1 as the first edge and e_2 as the last edge is called a spanning (e_1, e_2) -trail. In chapter 5, we consider best possible degree conditions to assure the existence of these trails for every pair of edges e_1, e_2 in a 3-edge-connected graph G .

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Chapter 1

Introduction

1.1 Definitions

Graphs in this dissertation are simple, finite and loopless, unless otherwise stated. Undefined terms and notation follow from [7]. We use $H \subseteq G$ to denote the fact that H is a subgraph of G .

Three problems will be considered in this dissertation. They concern somewhat separate topics, but all of them involve some popular subjects in the graph theory.

First of all, we consider the problem called group coloring of graphs, which is another interesting variant which generalizes the classical vertex colorings of graphs.

Let $G = (V, E)$ be a graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions $f: E(G) \rightarrow A$. Denote by D an orientation of $E(G)$. Then G is A -colorable if and only if for every $f \in F(G, A)$ there exists an A -coloring $c: V(G) \rightarrow A$ such that for every $e = (x, y) \in E(G)$ (assumed to be directed from x to y), $c(x) - c(y) \neq f(e)$. If G is a graph, we define its **group chromatic number** $\chi_1(G)$ to be the minimum number m for which G is A -colorable for any Abelian group A of order $\geq m$ under the

orientation D . Chapter 2 is mainly devoted to group chromatic number and some results are given.

Secondly, we consider the **edge-integrity** of graphs. The edge-integrity of a graph G is given by $\min_{S \subseteq E(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ denotes the maximum order of a component of $G - S$. The edge-integrity of a graph attempts to measure the disruption caused by the removal of edges from the graph.

Let $I'(G)$ denote the edge-integrity of a graph G . We define a graph G to be I' -maximal if for every edge e in $\overline{G}$, the complement of graph G , $I'(G + e) > I'(G)$. In chapter 3, some results of I' -maximal graphs are established, the girth of a connected I' -maximal graph is given and lower and upper bounds on the size of I' -maximal connected graphs with given order and edge-integrity are investigated. Also, the I' -maximal trees and unicyclic graphs are completely characterized.

For any given edges e_1, e_2 in $E(G)$, a spanning trail of G with e_1 as the first edge and e_2 as the last edge is called a spanning (e_1, e_2) -trail. The problem to determine if a graph G has a spanning (e_1, e_2) -trail, for given $e_1, e_2 \in E(G)$, is NP-complete. In chapter 4, we consider best possible degree conditions to assure the existence of these trails for every pair of edges e_1, e_2 in a 3-edge-connected graph G .

1.2 Operations on graphs

Let G be a graph. The removal of an edge e involves simply deleting the edge e but leaving its end vertices, but the removal of a vertex v also implies deleting the vertex v and all edges incident on it. The operation of **contraction of an edge** involves not only removing it from the graph, but also identifying its two end vertices to form a new single one, all the other

edges incident on the two end vertices being retained. Let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. Figure 1.1 illustrates the operation of contracting the three edges e_1, e_2 and e_3 into one vertex v . For some applications of graph minors, it may be possible to avoid using multiple edges, for others multiple edges may be necessary.

Figure 1.1:

1.3 Method used

We describe the methods we use as follows:

- (a) The various possible cases that can arise are grouped together in categories, each of which has to be looked at separately.
- (b) We often consider the graphs with property P , but the deletion of any vertex (or edge, depending on context) destroys this property.

1.4 Conventions

If S is a finite set, we denote by $|S|$ the number of elements in S . We use $\lfloor x \rfloor$ for the largest integer not greater than x , and $\lceil x \rceil$ for the smallest integer not smaller than x . For example, $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$.

Let G be a simple graph. Let u and v be two different vertices of G . We denote by $G \cup \{uv\}$ the graph obtained from G by connecting u and v if they are not adjacent.

Let G be a graph and $S \subseteq V(G)$. We denote by $G[S]$ the subgraph of G induced by S and $G - S$ the subgraph of G obtained by deleting S .

Chapter 2

Group Colorability of Graphs

2.1 Introduction

Graphs in this chapter are simple, finite and loopless, unless otherwise stated. Undefined terms and notation follow from [7]. We use $H \subseteq G$ to denote the fact that H is a subgraph of G .

Let $G = (V, E)$ be a graph and A a non-trivial Abelian group, and let $F(G, A)$ denote the set of all functions $f: E(G) \rightarrow A$. Denote by D an orientation of $E(G)$. An oriented edge uv of G (assumed to be directed from u to v) is called an arc uv . The graph G under the orientation D is sometimes denoted by $D(G)$.

DEFINITION 1.1. For $f \in F(G, A)$, an (A, f) -**coloring** (**A -coloring**, if no confusion arises) of G under the orientation D is a function $c: V(G) \rightarrow A$ such that for every arc $e = uv \in E(G)$ $c(u) - c(v) \neq f(e)$.

DEFINITION 1.2. G is **A -colorable** under the orientation D if and only if for every $f \in F(G, A)$ there exists an A -coloring.

F. Jaeger, N. Linial, C. Payan, and M. Tarsi [22] proposed the definition

of group colorability of graphs as the dual concept of group connectivity of M , where M is a cographic matroid. Clearly, an A -colorable graph is $|A|$ -colorable (take $f = 0$) and A -colorability is the dual of local A -connectivity, in the same way that k -colorability is the dual of admitting a nowhere-zero k -flow.

DEFINITION 1.3. The **first group chromatic number** (simply called **group chromatic number**) of a graph G with an orientation D is defined to be the minimum m for which G is A -colorable for any group A of order $\geq m$ under the orientation D . The group chromatic number of graph G under the orientation D is simply denoted by $\chi_1(G)$.

DEFINITION 1.4. The **second group chromatic number** of a graph G is defined to be the minimum $|A|$ for which G is A -colorable under the orientation D . The second group chromatic number of graph G under the orientation D is simply denoted by $\chi_2(G)$.

Later in Lemma 2.1, we shall show that both $\chi_1(G)$ and $\chi_2(G)$ are independent of the orientation D .

By above definitions, for any graph G , $\chi_1(G) \geq \chi_2(G)$.

F. Jaeger, N. Linial, C. Payan, and M. Tarsi [22] proved the following result:

THEOREM 1.1 ([22], Proposition 4.2). Every simple planar graph is A -colorable for every Abelian group of order ≥ 6 .

Hence for every simple planar graph G , $\chi_1(G) \leq 6$.

Let $\chi(G)$ denote the **chromatic number** which is the minimum k for which G is k -colorable; if $\chi(G) = k$, G is said to be k -chromatic.

Since an A -colorable graph is $|A|$ -colorable, for any graph G , $\chi_2(G) \geq \chi(G)$.

Let us look at an example. Let C_4 be a cycle with 4 vertices. Then $\chi(C_4) = 2$. However, $\chi_1(C_4) \neq 2$ and $\chi_2(C_4) \neq 2$. (See Figure 2.1.)

Figure 2.1:

Let $f \in F(C_4, Z_2)$ and the orientation D be defined as Figure 2.1. Then C_4 has no (Z_2, f) -coloring c under D , since we can find an arc $e = v_i v_j$ such that $c(v_i) - c(v_j) = f(e)$ no matter what value $c(v_1)$ is.

DEFINITION 1.5. Let G be a graph, $H \subseteq G$ and A a non-trivial Abelian group. Then (G, H) is said to be **A -extensible** if for any $f \in F(G, A)$ and for any $(A, f|_{E(H)})$ -coloring c' (if exists), there is an (A, f) -coloring c of G such that $c|_{V(H)} = c'$. G is said to be **strong A -colorable** if for any subgraph H of G , (G, H) is A -extensible.

By definition, (G, H) is A -extensible if and only if for any $f \in F(G, A)$, any A -coloring of H for $f|_{E(H)}$ can be extended to G for f .

2.2 Elementary Properties

LEMMA 2.1. Let D be an orientation of $E(G)$ and E_0 be a subset of $E(G)$. Let D' be the orientation of $E(G)$ obtained from D by reversing the direction of every arc in E_0 . Assume that A is a non-trivial Abelian group. If G is A -colorable under the orientation D , then G is also A -colorable under the orientation D' .

Proof. Let $f' \in F(G, A)$. We consider the ordered pair (D, f) , where f is defined as follows:

$$f(e) = \begin{cases} f'(e), & \text{if } e \notin E_0 \\ -f'(e), & \text{if } e \in E_0. \end{cases} \quad (2.1)$$

Since G is A -colorable under the orientation D , by the definition 1.1, there exists a function $c: V(G) \rightarrow A$ such that for every arc $e = xy \in E[D(G)]$, $c(x) - c(y) \neq f(e)$. If $e \notin E_0$, then $e \in E[D'(G)]$ and $c(x) - c(y) \neq f(e) = f'(e)$; if $e \in E_0$, then $yx \in E[D'(G)]$ and $c(x) - c(y) \neq f(e)$, namely, $c(y) - c(x) \neq -f(e) = f'(e)$. Hence, G is A -colorable under the orientation D' . $\square$

By Lemma 2.1, it is easy to see that

THEOREM 2.1. Let G be a graph and D be an orientation of $E(G)$. Then, for any Abelian group A , graph G is A -colorable under the orientation D if and only if G is A -colorable under every orientation of $E(G)$.

THEOREM 2.2. Let A be an Abelian group. Then graph G is A -colorable if and only if each block of G is A -colorable.

Proof. If G is A -colorable, then every subgraph of G is also A -colorable, and so each block of G is A -colorable.

It clearly suffices to prove the converse for connected graphs with two

blocks. Let G be a connected graph with two blocks G_1 and G_2 and assume that G_1 and G_2 are A -colorable. Let v_0 be the cut vertex of G . Then $v_0 \in V(G_1) \cap V(G_2)$.

For any $f \in F(G, A)$, we can get two functions $f|_{E(G_1)} \in F(G_1, A)$ and $f|_{E(G_2)} \in F(G_2, A)$. Let $f_1 = f|_{E(G_1)}$ and $f_2 = f|_{E(G_2)}$. Since G_1 and G_2 are A -colorable, there exist an A -coloring $c_1 : V(G_1) \rightarrow A$ for $f_1 \in F(G_1, A)$ and an A -coloring $c_2 : V(G_2) \rightarrow A$ for $f_2 \in F(G_2, A)$. Let $c'_2 : V(G_2) \rightarrow A$ defined by

$$c'_2(v) = c_2(v) - c_2(v_0) + c_1(v_0)$$

for each $v \in V(G_2)$. Obviously, c'_2 is an A -coloring of G_2 for f_2 . Define $c : V(G) \rightarrow A$ as follows:

$$c(v) = \begin{cases} c_1(v), & \text{if } v \in V(G_1) \\ c'_2(v), & \text{if } v \in V(G_2). \end{cases} \quad (2.2)$$

It is easy to see that c is an A -coloring of G for $f \in F(G, A)$. $\square$

THEOREM 2.3. Let A be an Abelian group and $H \subseteq G$. If (G, H) be A -extensible and H is A -colorable, then G is A -colorable.

Proof. For any $f \in F(G, A)$, since H is A -colorable, there is an A -coloring $c_1 : V(H) \rightarrow A$ for $f|_{E(H)}$. Since (G, H) is A -extensible, for f , c_1 can be extended to G such that $c|_{V(H)} = c_1$. Then G is A -colorable. $\square$

THEOREM 2.4. Let A be an Abelian group and $H_2 \subseteq H_1 \subseteq G$. If (G, H_1) and (H_1, H_2) are A -extensible, then (G, H_2) is also A -extensible.

Proof. For any $f \in F(G, A)$, let $f_1 = f|_{E(H_1)}$ and $f_2 = f|_{E(H_2)}$. Since (H_1, H_2) is A -extensible, any A -coloring c_1 of H_2 for f_2 can be extended to an A -coloring c'_1 of H_1 for f_1 such that $c'_1|_{V(H_2)} = c_1$. Since (G, H_1) is A -extensible, A -coloring c'_1 of H_1 for f_1 can be extended to an A -coloring c of G for f such that $c|_{V(H_1)} = c'_1$ and $c|_{V(H_2)} = c'_1|_{V(H_2)} = c_1$. Hence (G, H_2) is

A -extensible. $\square$

let A and A' be two Abelian groups and let $\varphi : A \rightarrow A'$ be a homomorphism. Then $\text{im}(\varphi)$, the image of A under φ , is a subgroup of A' .

THEOREM 2.5. Let $\varphi : A \rightarrow A'$ be a homomorphism and G be a graph. If G is $\text{im}(\varphi)$ -colorable, then G is also A -colorable.

Proof. Let $f \in F(G, A)$. Then $\varphi f \in F(G, A')$. Since G is $\text{im}(\varphi)$ -colorable, there exists an $\text{im}(\varphi)$ -coloring $c' : V(G) \rightarrow \text{im}(\varphi)$ such that for every arc $e = xy$ of G , $c'(x) - c'(y) \neq \varphi f(e)$. Define $c : V(G) \rightarrow A$ as follows: for $v \in V(G)$, let $c(v) = a \in A$ such that $\varphi(a) = c'(v)$. For every arc $e = xy \in E(G)$, it is easy to see that $c(x) - c(y) \neq f(e)$. Otherwise $\varphi(c(x) - c(y)) = \varphi f(e)$, namely, $c'(x) - c'(y) = \varphi f(e)$, a contradiction. $\square$

COROLLARY 2.1. Let G be a graph. If φ be a homomorphism of A onto A' and G is A' -colorable, then G is also A -colorable.

Let N be a normal subgroup of A . The function $\pi : A \rightarrow A/N$ (A/N is call the quotient group of A) defined by $\pi(a) = aN$ is a homomorphism of A onto A/N . By Corollary 2.1, we have the following result.

COROLLARY 2.2. Let A be a Abelian group and N be any subgroup of A . If graph G is A/N -colorable, then G is also A -colorable.

Suppose that A and A' are two finite cyclic groups with orders m and n , respectively. Then there exists a homomorphism of A onto A' if and only if $n|m$.

COROLLARY 2.3. For any graph G , if G is Z_n -colorable, then G is also Z_{kn} -colorable, where $k \in \mathbb{Z}^+$.

In Section 4, we will show that for any graph G , $\chi_1(G) \leq \Delta(G) + 1 \leq |V(G)|$, which implies that any graph G is A -colorable, where $\Delta(G)$ is the maximum degree of G and A is an Abelian group with order $\geq \Delta(G) + 1$. Here we propose:

PROBLEM 2.1. Prove or disprove: Let A be an Abelian group. If graph G is A -colorable, then for any integer k ($|A| < k \leq \Delta(G)$), there exists an Abelian group A' such that $|A'| = k$ and G is also A' -colorable.

More generally:

PROBLEM 2.2. For any graph G , Is $\chi_1(G) = \chi_2(G)$?

2.3 Characterization of Z_2 -colorable Graphs

LEMMA 3.1. If G is Z_2 -colorable, then G is a forest.

Proof. We prove it by contradiction. Assume that $c = v_0v_1 \cdots v_kv_0$ is a directed cycle of G . Let $e_i = v_iv_{i+1}$ ($i = 0, 1, \dots, k-1$) and $e_k = v_kv_0$. Let $f \in F(G, Z_2)$ be defined as follows.

(i). If k is odd, then

$$f(e) = \begin{cases} 1, & \text{if } e = e_k, \\ 0, & \text{otherwise.} \end{cases}$$

(ii). If k is even, let $f(e) = 0$ for any $e \in E(G)$.

We only consider the case when k is odd. The other case is similar.

Assume that for function f , there exists an A -coloring $c : V(G) \rightarrow Z_2$ such that for every arc $e = xy \in E(G)$, $c(x) - c(y) \neq f(e)$. If $c(v_0) = 1$, then $c(v_1) = 0, c(v_2) = 1, \dots, c(v_k) = 0$; if $c(v_0) = 0$, then $c(v_1) = 1, c(v_2) = 0, \dots, c(v_k) = 1$. Thus $c(v_k) - c(v_0) = 1 = f(e)$, a contradiction.

Hence, if G is Z_2 -colorable, then G is acyclic. $\square$

Since every non-empty subgraph of a forest has a vertex of degree less than 2, we have the following lemma.

LEMMA 3.2. For any forest G , $\chi_1(G) = 2$.

By Lemmas 3.1 and 3.2, we have:

THEOREM 3.1. Let G be a graph. Then $\chi_1(G) = 2$ if and only if G is a forest.

Further we have:

THEOREM 3.2. Let G be a forest and $H \subseteq G$. Then (G, H) is Z_2 -extensible if and only if any two components of H belong to two different components of G .

Proof. Without loss of generality, we may assume that G is a tree. We need to prove that (G, H) is Z_2 -extensible if and only if H is a connected subgraph of G .

If H is connected, let u_0v_0 be an arc of G such that $u_0 \in E(H)$ and $v_0 \notin E(H)$. For any $f \in F(G, A)$, any Z_2 -coloring c' of H for $f|_{E(H)}$ is easily extended to a Z_2 -coloring c_1 of subgraph $H_1 = H \cup \{u_0v_0\}$ by a simple extension: let $c_1(v) = c'(v)$ if $v \in V(H)$ and $c_1(v_0) = a \neq c'(u_0) - f(u_0v_0)$. Hence any Z_2 -coloring of H for $f|_{E(G)}$ can be extended to a Z_2 -coloring of G for f by $|V(G)| - |V(H)|$ simple extensions, and so (G, H) is Z_2 -extensible.

If H is not connected, we may assume that H has two components H_1 and H_2 . Let $v_0v_1 \cdots v_k$ be a directed path of G such that $v_0 \in V(H_1)$, $v_k \in V(H_2)$ and $v_i \notin V(H)$ ($3 \leq i \leq k-1$). Let $e_i = v_iv_{i+1}$ ($i = 0, 1, \dots, k-1$) and $f \in F(G, Z_2)$ be defined as follows: For any $e \in E(G)$, let $f(e) = 0$ if

$e = e_{k-1}$ and 1 otherwise. Let c_1 be a Z_2 -coloring of H for $f|_{E(H)}$ such that $c_1(v) = 1$ for every $v \in V(H)$. It is easy to see that c_1 can not be extended to G for F , and so (G, H) is not Z_2 -extensible. $\square$

THEOREM 3.3. Let A be an Abelian group of order ≥ 3 and G be a forest. Then G is strong A -colorable.

Proof. We need to prove that for any subgraph H of G , (G, H) is A -extensible.

We may, without loss of generality, assume that G is a tree and prove it by induction on $\omega(H)$, the number of components of subgraph H .

From the proof of above theorem, we easily know that the theorem holds when $\omega(H) = 1$. Let k be a positive integer and assume that the theorem is valid when $\omega(H) \leq k$. Suppose, now, that H has $k + 1$ components. Choose two components H_1 and H_2 of H such that there exists a directed path $P = v_0 v_1 \cdots v_k$ with $v_0 \in H_1$, $v_k \in H_2$ and $v_i \notin V(H)$ ($1 \leq i \leq k - 1$). For any $f \in F(G, A)$, let $c_1 : V(H) \rightarrow A$ be an A -coloring of H for $f|_{E(H)}$. Define $c : V(H \cup P) \rightarrow A$ as follows: Let $c(v) = c_1(v)$ if $v \in V(H)$, $c(v_i) = a_i \in A - \{c(v_{i-1}) + f(v_{i-1}v_i)\}$ ($1 \leq i \leq k - 2$) and $c(v_{k-1}) = a_{k-1} \in A - \{c(v_{k-2}) - f(v_{k-2}v_{k-1}), c(v_k) + f(v_{k-1}v_k)\}$. Then c is an A -coloring of $H \cup P$ for $f|_{E(H \cup P)}$ and is an extension of c_1 , namely, $(H \cup P, H)$ is A -extensible. Now, $\omega(H \cup P) = k$, and by the induction hypothesis, $(G, H \cup P)$ is A -extensible. By Theorem 2.4, (G, H) is A -extensible.

Thus (G, H) is A -extensible for all subgraphs H of G . $\square$

2.4 The Analogue of Brooks' Theorem

Denote the maximum degree of graph G by $\Delta(G)$. The following theorem is the well-known theorem of Brooks which relates the chromatic number of a graph to its maximum degree.

THEOREM 4.1 (Brooks [8]). For any connected graph G ,

$$\chi(G) \leq \Delta(G) + 1$$

with equality if and only if either $\Delta(G) = 2$ and G is an odd cycle; or $\Delta(G) \geq 3$ and G is complete.

For group chromatic number of graph G , we can get the following analogue to Brooks' Theorem.

THEOREM 4.2. For any connected simple graph G ,

$$\chi_1(G) \leq \Delta(G) + 1$$

with equality if and only if either $\Delta(G) = 2$ and G is a cycle; or $\Delta(G) \geq 3$ and G is complete.

The proof of Theorem 4.2 needs the following lemmas.

LEMMA 4.1. Let G be a graph and $V(G)$ can be linearly ordered as $v_1, v_2, \dots, v_n$ such that $d_{G_i}(v_i) \leq k$ ($i = 1, 2, \dots, n$), where G_i is a subgraph $G[\{v_1, v_2, \dots, v_i\}]$ of G . Then for any Abelian group A of order $\geq k + 1$, (G_{i+1}, G_i) , where $i = 1, 2, \dots, n - 1$, is A -extensible and so G is A -colorable.

Proof. Let D be an orientation of $E(G_{i+1})$ such that every $e = v_{j_1} v_{j_2} \in E(G_{i+1})$ is directed from v_{j_1} to v_{j_2} if $j_1 > j_2$ and from v_{j_2} to v_{j_1} otherwise. For any $f \in F(G_{i+1}, A)$ and any A -coloring c_1 of G_i for $f|_{E(G_i)}$, we define an A -coloring $c : V(G_{i+1}) \rightarrow A$ as follows: Assume that $v_{i_1} v_{i+1}, v_{i_2} v_{i+1}, \dots, v_{i_r} v_{i+1}$ are all edges joining v_{i+1} ($0 \leq r \leq k$) in G_{i+1} and let $c(v) = c_1(v)$ if $v \in V(G_i)$, $c(v_{i+1}) = a'$ such that $a' \in A' = A - \{c(v_{i_p}) + f(v_{i_p} v_{i+1}) | p = 1, 2, \dots, r\}$. Since $|A| \geq k + 1$, $A' \neq \emptyset$. Hence (G_{i+1}, G_i) is A -extensible, where $i = 1, 2, \dots, n - 1$.

Since G_1 is A -colorable, by Theorem 2.3 and 2.4, G is A -colorable. $\square$

By Lemma 4.1, we have the following lemma, which is similar to the result on chromatic number due to G. Szekeres and H. S. Wilf [34].

LEMMA 4.2. Let $k = \max_H \delta(H)$, where the maximum is taken over all induced subgraphs of G and $\delta(H)$ is the minimum degree of subgraph H . Then $\chi_1(G) \leq k + 1$.

Proof. Let $|V(G)| = n$ and v_n be a vertex of degree at most k . Put $H_{n-1} = G - \{v_n\}$. By assumption H_{n-1} has a vertex, say v_{n-1} , of degree at most k . Put $H_{n-2} = G - \{v_n, v_{n-1}\}$. Continuing in this way we enumerate all the vertices of G . Hence we get a sequence $v_1, v_2, \dots, v_n$ such that each v_j is joined to at most k vertices preceding it. By above Lemma, this lemma holds. $\square$

An immediate corollary is given below.

COROLLARY 4.1. For any graph G , $\chi_1(G) \leq \Delta(G) + 1$.

Since every nontrivial simple graph without subdivision of K_4 has a vertex of degree at most 2, by Theorem 3.1 and Lemma 4.2, we have the following result.

COROLLARY 4.2. Let G be a nontrivial simple graph without subdivision of K_4 . Then $\chi_1(G) \leq 3$, where equality holds if and only if G has a cycle.

COROLLARY 4.3. Let K_n be a complete graph with n vertices. then $\chi_1(K_n) = n$.

Proof. $n = \chi(K_n) \leq \chi_1(K_n) \leq \Delta(K_n) + 1 = n$. $\square$

By modifying the proof of Brooks' Theorem in [6], we obtain the following:

Proof of Theorem 4.2. If G is connected and not regular of degree $\Delta(G)$, then $\max_{H \subset G} \delta(H) \leq \Delta(G) - 1$ and so $\chi_1(G) \leq \Delta(G)$. Without loss of generality, let G be 2-connected and $\Delta(G)$ -regular. If G is a complete graph, then $\chi_1(G) = |V(G)| = \Delta(G) + 1$.

If $\Delta(G) = 2$, then G is a cycle and so $\mu_1(G) = 3 = \Delta(G) + 1$. If G is 3-connected and G is not complete, then there are three vertices v_1, v_2 and v_n ($n = |V(G)|$) in G such that $v_1v_n, v_2v_n \in E(G)$ and $v_1v_2 \notin E(G)$. If G is 2-connected, let $\{v_n, v'\}$ be a cut set of G . Then there are two vertices v_1 and v_2 belonging to different endblocks of $G - v_n$. Now, we arrange the vertices of $G - \{v_1, v_2\}$ in nonincreasing order of their distance from v_n , say $v_3, v_4, \dots, v_n$. Then the sequence $v_1, v_2, \dots, v_n$ is such that each vertex other than v_n is adjacent to at least one vertex following it, namely each vertex other than v_n is joined to at most $\Delta(G) - 1$ vertices preceding it.

Let D be an orientation of $E(G)$ such that every $e = v_i v_j \in E(G)$ is directed from v_i to v_j if $i > j$ and from v_j to v_i otherwise. For any $f \in F(G, A)$ ($|A| \geq \Delta(G)$), we define an A -coloring $c : V(G) \rightarrow A$ as follows: Assign $a_1 \in A$ to $c(v_1)$ and $a_2 \in A$ to $c(v_2)$ such that $a_1 + f(v_1v_n) = a_2 + f(v_2v_n)$; for v_j ($3 \leq j \leq n$), let $v_{i_1}v_j, v_{i_2}v_j, \dots, v_{i_r}v_j \in E(G)$ ($r \leq \Delta(G) - 1$ if $j < n$) are all edges joining v_j and with $i_p < j$ ($p = 1, 2, \dots, r$) and assign a_j to $c(v_j)$ such that $a_j \in A_j = A - \{c(v_{i_p}) + f(v_{i_p}v_j) | p = 1, 2, \dots, r\}$. If $j < n$, then $r \leq \Delta(G) - 1$ and so $A_j \neq \emptyset$; if $j = n$, since $a_1 + f(v_1v_n) = a_2 + f(v_2v_n)$, $A_n \neq \emptyset$.

Hence for every $f \in F(G, A)$ ($|A| \geq \Delta(G)$), there exists an A -coloring.

□

2.5 Group Chromatic Numbers of a Graph and its Chromatic Number

Following the definition of $\chi_1(G)$ and $\chi(G)$, we know that for any graph, $\chi_1(G) \geq \chi(G)$. In this section, we present a result that there exists a graph G such that $\chi_1(G) - \chi(G)$ may be arbitrarily large.

Let's prove the following Theorem first.

THEOREM 5.1. Let $K_{m,n}$ be a complete bipartite graph with $n \geq m^m$. Then $\chi_1(K_{m,n}) = m + 1$.

Proof. Assume that $K_{m,n}$ has partition (X, Y) with $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. Let A be an Abelian group of order $\leq m$ and D be an orientation of $E(K_{m,n})$ such that every $e = x_i y_j \in E(K_{m,n})$ is directed from y_j to x_i .

Denote the set of all functions $c : V(K_{m,n}) \rightarrow A$ by $C(K_{m,n}, A)$. For every function $c \in C(K_{m,n}, A)$, we can get a function $c|_X : X \rightarrow A$. Let $C(X, A) = \{c|_X \mid c \in C(K_{m,n}, A)\}$. Since $|A| \leq m$, $|C(X, A)| = m^{|A|} \leq m^m$. Assume that $C(X, A) = \{c_1, c_2, \dots, c_r\}$ ($r = m^{|A|}$). Now we define $f_l \in F(K_{m,n}, A)$ ($l = 1, 2, \dots, r$) as follows: If $j \neq l$, let $f_l(x_i y_j) = 0$ for every i , and otherwise let $f_l(x_i y_l) = a_{li} \in A$ such that $\{c_l(x_i) + a_{li} \mid i = 1, 2, \dots, m\} = A$. Let $f = \sum_{l=1}^r f_l$. Then for any function $c : V(K_{m,n}) \rightarrow A$, there exists at least one arc $e = y_j x_i \in E(K_{m,n})$ such that $c(y_j) - c(x_i) = f(e)$. Hence $\chi_1(K_{m,n}) \geq m + 1$.

On the other hand, by Lemma 4.2, $\chi_1(K_{m,n}) \leq m + 1$.

Therefore $\chi_1(K_{m,n}) = m + 1$. $\square$

THEOREM 5.2. For any positive integers m and k , there exists a graph G such that $\chi(G) = m$ and $\chi_1(G) = m + k$.

Proof. Let G be a graph with $(2m + k) + (m + k)^{m+k} - 1$ vertices formed from a complete subgraph K_m with m vertices and a complete bipartite subgraph K_{r_1, r_2} with $r_1 = m + k$ and $r_2 = (m + k)^{m+k}$ such that

$$|V(K_m) \cap V(K_{r_1, r_2})| = 1.$$

Obviously $\chi(G) = m$, and by Theorem 5.1 we easily know that $\chi_1(G) = m + k$. $\square$

2.6 Group Chromatic Numbers of a Graph and of its Complement

Let G^c be the complement of a graph G . E. A. Nordhaus and J. W. Gaddum ([31], 1956) proved the follow theorem:

THEOREM 6.1. If G is a graph of order n , then $2\sqrt{n} \leq \chi(G) + \chi(G^c) \leq n + 1$, and $n \leq \chi(G)\chi(G^c) \leq ((n + 1)/2)^2$.

In this section, we present the following result on the group chromatic numbers of a graph and of its complement which generalizes Theorem 6.1.

THEOREM 6.2. If G is a graph of order n , then $2\sqrt{n} \leq \chi(G) + \chi(G^c) \leq \chi_1(G) + \chi_1(G^c) \leq n + 1$, and $n \leq \chi(G)\chi(G^c) \leq \chi_1(G)\chi_1(G^c) \leq ((n + 1)/2)^2$.

Before giving the proof of Theorem 6.2, let us prove the following three lemmas.

By a simple argument which is similar to the proof of Lemma 4.1, we have the following lemma.

LEMMA 6.1. Let G be a graph and $d_G(v_0) \leq k$. Then for any Abelian group A with $|A| \geq k+1$, $(G, G-v_0)$ is A -extensible, and so G is A -colorable if and only if $G-v_0$ is A -colorable.

LEMMA 6.2. Let G be a simple graph with $\chi_1(G) = k$. Then G has at least k vertices of degree $\geq k$.

Proof. By lemma 6.1, we may assume that each vertex of G has degree at least k . Since $\chi_1(G) = k$, $|V(G)| \geq k$. Hence, G has at least k vertices of degree $\geq k$. $\square$

LEMMA 6.3. If $d_1 \geq d_2 \geq \dots \geq d_n$ is the degree sequence of G , then $\chi_1(G) \leq \max \min \{d_i + 1, i\}$.

Proof. Suppose that $\chi_1(G) = k$. By lemma 2, we have $\chi_1(G) = k = \min \{d_k + 1, k\} \leq \max \min \{d_i + 1, i\}$. $\square$

Here is the proof of Theorem 6.2.

Proof. Since, for any graph G , $\chi_1(G) \geq \chi(G)$, we only need to show that $\chi_1(G) + \chi_1(G^c) \leq n + 1$, and $\chi_1(G)\chi_1(G^c) \leq ((n+1)/2)^2$.

Let $d_1 \geq d_2 \geq \dots \geq d_n$ be the degree sequence of G , and $d'_1 \geq d'_2 \geq \dots \geq d'_n$ be the degree sequence of G^c . By lemma 6.3, there is two integers p and q such that $\chi_1(G) \leq \min\{d_p + 1, p\}$ and $\chi_1(G^c) \leq \min\{d'_q + 1, q\}$. We consider the following cases.

Case 1. $q \geq n - p + 1$.

Then $n - 1 = d_p + d'_{n-p+1} \geq d_p + d'_q \geq (\chi_1(G) - 1) + (\chi_1(G^c) - 1)$, and so $\chi_1(G) + \chi_1(G^c) \leq n + 1$. Also $\chi_1(G)\chi_1(G^c) \leq (d_p + 1)(d'_q + 1) = d_p d'_q + d_p + d'_q + 1 \leq d_p d'_q + n \leq d_p d'_{n-p+1} + n \leq ((n-1)/2)^2 + n = ((n+1)/2)^2$.

Case 2. $q \leq n - p + 1$.

Then, since $\chi_1(G) \leq p$ and $\chi_1(G^c) \leq q$, we have $n + 1 \geq p + (n - p + 1) \geq p + q \geq \chi_1(G) + \chi_1(G^c)$, and also $\chi_1(G)\chi_1(G^c) \leq pq \leq p(n - p + 1) = pn - p^2 + p \leq ((n + 1)/2)^2 - (p - (n + 1)/2)^2 \leq ((n + 1)/2)^2$. $\square$

Obviously, for any graph G with order n , if $\chi(G) + \chi(G^c) = n + 1$, then $\chi_1(G) + \chi_1(G^c) = n + 1$; if $\chi(G)\chi(G^c) = ((n + 1)/2)^2$, then $\chi_1(G)\chi_1(G^c) = ((n + 1)/2)^2$. For any $n \geq 6$, we define a graph G with n vertices as follows: $G = K_{n/2, n/2}$ if n is even; $G = K_{k, k} \cup v_0$, where $k = (n - 1)/2$ and v_0 is a isolated vertex. It is easy to check that $\chi(G) + \chi(G^c) < \chi_1(G) + \chi_1(G^c) < n + 1$ and $\chi(G)\chi(G^c) < \chi_1(G)\chi_1(G^c) < ((n + 1)/2)^2$.

2.7 The Group Chromatic Number of a Bipartite Graph

For the complete bipartite graph $K_{m, n}$, if m or n is one, then $K_{m, n}$ is a tree and so its group chromatic number equals to two. In this section, we will consider the complete bipartite graph $K_{m, n}$ with m and $n > 1$. Let $K_{m, n}$ have two partite sets U with m vertices and V with n vertices. By Lemma 2.1, we may assume that each edge uv is directed from v to u , where $v \in V$ and $u \in U$.

THEOREM 7.1. If $\min\{m, n\} = 2$, then $\chi_1(K_{m, n}) = 3$.

Proof. By Lemma 4.2, we have that $\chi_1(K_{m, n}) \leq \max_H \delta(H) + 1$, where the maximum is taken over all induced subgraphs of $K_{m, n}$. Hence $\chi_1(K_{m, n}) \leq 3$. On the other hand, since $\min\{m, n\} = 2$, and so by Theorem 3.1, $\chi_1(K_{m, n}) > 2$. Therefore, $\chi_1(K_{m, n}) = 3$. $\square$

THEOREM 7.2. $\chi_1(K_{3,n}) = 4$, where $n \geq 6$.

Proof. By Lemma 4.2, it is easy to see that $\chi_1(K_{3,n}) \leq 4$. Hence, we need to show that there exists a function $f \in F(K_{3,n}, Z_3)$, where Z_3 a non-trivial Abelian group with order 3, such that there is not any Z_3 -coloring of $K_{m,n}$ for f .

Obviously, we need to consider only for the graph $K_{3,6}$ with partite sets $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. We define a function $f \in F(K_{3,6}, Z_3)$ as follows (Figure 2.2): $f(v_1u_2) = 1$, $f(v_1u_3) = 2$, $f(v_3u_3) = 1$, $f(v_4u_2) = 1$, $f(v_4u_3) = 1$, $f(v_5u_2) = 1$, $f(v_6u_2) = 2$, and $f(vu) = 0$ for any other edge vu .

Figure 2.2: $K_{3,6}$

We can see that

- (a). if $(c(u_1), c(u_2), c(u_3)) = (0, 0, 0)$, $(1, 1, 1)$ or $(2, 2, 2)$, then $\{f(u_1v_1) + c(u_1), f(u_2v_1) + c(u_2), f(u_3v_1) + c(u_3)\} = \{0, 1, 2\}$;

- (b). if $(c(u_1), c(u_2), c(u_3)) = (1, 2, 0), (1, 0, 2), (2, 1, 0), (2, 0, 1), (0, 1, 2)$ or $(0, 2, 1)$, then $\{f(u_1v_2) + c(u_1), f(u_2v_2) + c(u_2), f(u_3v_2) + c(u_3)\} = \{0, 1, 2\}$;
- (c). if $(c(u_1), c(u_2), c(u_3)) = (2, 0, 0), (0, 2, 0), (0, 1, 1), (1, 0, 1), (1, 2, 2)$ or $(2, 1, 2)$, then $\{f(u_1v_3) + c(u_1), f(u_2v_3) + c(u_2), f(u_3v_3) + c(u_3)\} = \{0, 1, 2\}$;
- (d). if $(c(u_1), c(u_2), c(u_3)) = (0, 0, 1), (0, 1, 0), (1, 2, 1), (1, 1, 2), (2, 2, 0)$ or $(2, 0, 2)$, then $\{f(u_1v_4) + c(u_1), f(u_2v_4) + c(u_2), f(u_3v_4) + c(u_3)\} = \{0, 1, 2\}$;
- (e). if $(c(u_1), c(u_2), c(u_3)) = (0, 0, 2), (1, 1, 0)$ or $(2, 2, 1)$, then $\{f(u_1v_5) + c(u_1), f(u_2v_5) + c(u_2), f(u_3v_5) + c(u_3)\} = \{0, 1, 2\}$;
- (f). if $(c(u_1), c(u_2), c(u_3)) = (1, 0, 0), (2, 1, 1)$ or $(0, 2, 2)$, then $\{f(u_1v_6) + c(u_1), f(u_2v_6) + c(u_2), f(u_3v_6) + c(u_3)\} = \{0, 1, 2\}$;

Hence, no matter which element we assign to u_1, u_2 and u_3 , we can find a vertex v_i such that $\{f(u_1v_i) + c(u_1), f(u_2v_i) + c(u_2), f(u_3v_i) + c(u_3)\} = \{0, 1, 2\}$ and so a proper value of $c(v_i)$ can not be found. Hence, there is no any Z_3 -coloring of $K_{3,6}$ for the function f . $\square$

LEMMA 7.1. Suppose that c is an (A, f) -coloring of G . Then for any $a \in A$ and $\sigma \in \text{Aut}(A)$, we have (1) $c + a$ is an (A, f) -coloring of G . (2) σc is (A, f) -coloring of G .

The proof of this lemma is skipped since it is pretty straightforward.

LEMMA 7.2. $\chi_1(K_{4,4}) = 4$.

Proof. By Theorem 4.2, $\chi_1(K_{4,4}) \leq 4$. Hence, it suffices to show that $K_{4,4}$ is not Z_3 -colorable. Suppose that $K_{4,4}$ has partite sets $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$. We define a function $f \in F(K_{4,4}, Z_3)$ as follows: $f(v_1u_2) = 2, f(v_1u_3) = 1, f(v_2u_1) = 1, f(v_2u_3) = 2, f(v_3u_1) = 2, f(v_3u_4) = 1$ and $f(vu) = 0$ for any other edge vu , as shown in Figure 2.3.

Suppose that $K_{4,4}$ has a Z_3 -coloring $c: V(K_{4,4}) \rightarrow Z_3$ for f . By lemma 7.1, we may assume that $c(u_1) = 0$ and $c(v_1) = 1$. Therefore, a contradiction

Figure 2.3: $K_{4,4}$

arises if one of the following holds:

- (i) $\{c(u_1), c(u_2), c(u_3)\} = Z_3$, or
- (ii) $\{c(v_1), c(v_2), c(v_3)\} = Z_3$, or
- (iii) $Z_3 - \{c(u_1), c(u_2), c(u_3)\} = Z_3 - \{c(v_1), c(v_2), c(v_3)\} \neq \emptyset$.

Note that when (i) holds, no color is available for $c(v_4)$; when (ii) holds, no color is available for $c(u_4)$; when (iii) holds, since $c(u_3) \neq 0$ and $c(v_2) \neq 1$, each sides of (iii) has exactly one element. On the other hand, since $f(v_4u_4) = 0$, no color is available for $c(u_4)$ and $c(v_4)$ if $c(u_4) \neq c(v_4)$. If $c(u_2) = 0$, then $c(v_2) = 2$ and $c(v_3) = 0$. Hence (ii) holds. Hence, we may assume that $c(u_2) = 1$ and so $c(v_2) \in \{0, 2\}$. To avoid (i), $c(u_3) = 1$ and so $c(v_2) \neq 0$ which implies $c(v_2) = 2$. Since $c(v_3) \neq 2$ and $c(v_3) \neq c(u_3) + 0 = 1$, $c(v_3)$ must be 0 and so (ii) holds. $\square$

LEMMA 7.3. $\chi_1(K_{3,4}) = 3$.

Proof. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4\}$ be the partite sets of $K_{3,4}$. For a given function $f \in F(K_{3,4}, Z_3)$ and a vertex v_i , there are at most $3! (=6)$ coloring possibilities $c(u_1), c(u_2)$ and $c(u_3)$ such that $\{c(u_1) + f(u_1v_i), c(u_2) + f(u_2v_i), c(u_3) + f(u_3v_i)\} = Z_3$. Since $|V| = 4$, there are at most 24 coloring possibilities for $c(u_1), c(u_2)$ and $c(u_3)$ such that there is not a coloring for this given f . However, there are $3^3 = 27$ coloring possibilities for the vertex set U . Therefore, we can find a coloring for $c(u_1), c(u_2)$ and $c(u_3)$ such that $\{c(u_1) + f(u_1v_i), c(u_2) + f(u_2v_i), c(u_3) + f(u_3v_i)\} \neq Z_3$ for each $i (= 1, 2, 3 \text{ or } 4)$. $\square$

LEMMA 7.4. $\chi_1(K_{3,5}) = 3$.

Proof. Let $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4, v_5\}$ be the partite sets of $K_{3,5}$. For a given function $f \in F(K_{3,5}, Z_3)$ and a vertex v_i , if we color vertices u_1, u_2, u_3 by $c(u_1), c(u_2)$ and $c(u_3)$ such that $\{c(u_1) + f(u_1v_i), c(u_2) +$

$f(u_2v_i), c(u_3) + f(u_3v_i)\} = Z_3$, then we say the coloring $\{c(u_1), c(u_2), c(u_3)\}$ is prohibited by v_i . At each v_i , there are $3!=6$ prohibited colorings for a given f . We check each situation and see that they can be only one of the following nine cases:

$$(7.4.1). \{(0, 0, 1), (0, 1, 0), (1, 1, 2), (1, 2, 1), (2, 0, 2), (2, 2, 0)\}$$

$$(7.4.2). \{(0, 0, 0), (0, 1, 2), (1, 1, 1), (1, 2, 0), (2, 0, 1), (2, 2, 2)\}$$

$$(7.4.3). \{(0, 0, 2), (0, 1, 1), (1, 1, 0), (1, 2, 2), (2, 0, 0), (2, 2, 1)\}$$

$$(7.4.4). \{(0, 0, 0), (0, 2, 1), (1, 0, 2), (1, 1, 1), (2, 1, 0), (2, 2, 2)\}$$

$$(7.4.5). \{(0, 0, 2), (0, 2, 0), (1, 0, 1), (1, 1, 0), (2, 1, 2), (2, 2, 1)\}$$

$$(7.4.6). \{(0, 0, 1), (0, 2, 2), (1, 0, 0), (1, 1, 2), (2, 1, 1), (2, 2, 0)\}$$

$$(7.4.7). \{(0, 1, 1), (0, 2, 0), (1, 0, 1), (1, 2, 2), (2, 0, 0), (2, 1, 2)\}$$

$$(7.4.8). \{(0, 1, 0), (0, 2, 2), (1, 0, 0), (1, 2, 1), (2, 0, 2), (2, 1, 1)\}$$

$$(7.4.9). \{(0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 2, 0), (2, 0, 1), (2, 1, 0)\}$$

From these cases, we see that at most 24 colorings of vertices u_1, u_2, u_3 are prohibited by 5 vertices v_1, v_2, v_3, v_4, v_5 for any given function $f \in F(K_{3,5}, Z_3)$. Hence, we find a coloring of u_1, u_2, u_3 which no vertex in V can control. Therefore, $K_{3,5}$ is Z_3 -colorable. $\square$

By above lemmas and theorems, we can conclude that:

THEOREM 7.3. Let $K_{m,n}$ be a complete bipartite graph with m and $n \geq 2$. Then $\chi_1(K_{m,n}) = 3$ if and only if one of the following holds:

- (1) $\min\{m, n\} = 2$ or
- (2) $(m, n) = (3, 4), (4, 3), (3, 5)$ or $(5, 3)$.

By using the same idea of the proof of Lemma 7.3, we can similarly show the following theorem.

THEOREM 7.4. $\chi_1(K_{4,n}) = 4$ if $4 \leq n \leq 10$.

Proof. By Lemma 4.2, $\chi_1(K_{4,n}) \leq 5$, and by Lemma 7.2, we know that $\chi_1(K_{4,n}) \geq 4$ if $n \geq 4$. Let A_4 be an Abelian with order 4. We show that $K_{4,n}$ is A_4 -colorable if $4 \leq n \leq 10$. Let $U = \{u_1, u_2, u_3, u_4\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be the partite sets of $K_{4,n}$. For a given function $f \in F(K_{4,n}, A_4)$ and a vertex v_i , if we color vertices u_1, u_2, u_3, u_4 by $c(u_1), c(u_2), c(u_3)$ and $c(u_4)$ such that $\{c(u_1) + f(u_1v_i), c(u_2) + f(u_2v_i), c(u_3) + f(u_3v_i), c(u_4) + f(u_4v_i)\} = A_4$, then we say the coloring $\{c(u_1), c(u_2), c(u_3), c(u_4)\}$ is prohibited by v_i . We see that there are 24 colorings of vertices u_1, u_2, u_3, u_4 which are prohibited by a vertex v_i for a given f , where $1 \leq i \leq n$. However, there are total 4^4 colorings for the vertex set U . If $4 \leq n \leq 10$, then $4^4 > 24n$, and so we find a coloring of u_1, u_2, u_3, u_4 which is not prohibited by any vertex in V . Hence $K_{4,n}$ is A_4 -colorable if $4 \leq n \leq 10$. $\square$

2.8 χ_2 -Maximal Graphs

In Section 2, we once proposed the following problem:

Prove or disprove: Let A be an Abelian group. If simple graph G is A -colorable, then for any integer k ($|A| < k \leq \Delta(G)$), there exists an Abelian group A' such that $|A'| = k$ and G is also A' -colorable.

One way of tackling this problem may be through the study of χ_2 -maximal graphs.

DEFINITION 8.1. A graph G is said to be χ_2 -**maximal** if $\chi_2(G+e) > \chi_2(G)$ for any edge e of G^c , the complement of G .

This notion is similar to the notion of critically with respect to vertex coloring which was introduced by Dirac [18] in 1952.

By the definition, if graph G is χ_2 -maximal, then $\chi_2(G) < |V(G)|$.

We shall now study the basic properties of χ_2 -maximal graphs.

THEOREM 8.1. Any graph with $\chi_2(G) = k < |V(G)|$ is a subgraph of χ_2 -maximal graph G' with $\chi_2(G') = k$.

Proof. If G is not χ_2 -maximal graph, then there is an edge e_1 in G^c such that $\chi_2(G + e_1) = \chi_2(G)$. If $G + e_1$ is not χ_2 -maximal graph, then again there is an edge e_2 in G^c such that $\chi_2(G + e_1 + e_2) = \chi_2(G)$, etc. Eventually, this process locates a χ_2 -maximal graph G' with $\chi_2(G') = k$. $\square$

Here we propose the following problem.

PROBLEM 8.1. Prove or disprove: If G is a χ_2 -maximal graph with $\chi_2(G) = k < |V(G)|$, then $\chi_2(G + e) = \chi_2(G) + 1$ for any e of G^c .

More specifically:

PROBLEM 8.2. Prove or disprove: If G is a χ_2 -maximal graph with $\chi_2(G) = k < |V(G)|$, then there is an edge $e \in G^c$ such that $\chi_2(G + e) = \chi_2(G) + 1$.

If Problem 8.2 can be proved, then we easily show Problem 2.1 in Section 2.

Similarly, we propose other notions of minimality and criticality with respect to group chromatic number $\chi_2(G)$. A graph G is said to be χ_2 -**minimal** if $\chi_2(G - e) < \chi_2(G)$ for any edge e of G or χ_2 -**critical** if $\chi_2(G - v) < \chi_2(G)$ for any vertex v of G .

By simple argument which is similar to the proof of Lemma 4.1, we have

the following Lemma.

LEMMA 8.1. Let G be a graph and $d_G(v_0) \leq k$. Then for any Abelian group A with $|A| \geq k + 1$, $(G, G - v_0)$ is A -extensible.

THEOREM 8.2. If G is χ_2 -maximal graph with $\chi_2(G) = k$, then $d_G(v) \geq k - 1$ for all $v \in V(G)$.

Proof. Assume that $d_G(v_0) \leq k - 2$. Since $\chi_2(G) = k$, there is a group A of order $|A| = k$ such that $G - v_0$ is A -colorable. Since $|V(G)| > k$, $|V(G - v_0)| \geq k$, and so there exists at least an edge $e \in E(G^c)$ joining v_0 .

Consider graph $G + e$. Since $d_{G+e}(v_0) \leq k - 1$, by Lemma 8.1, $(G + e, G - v_0)$ is A -extensible. Since $G - v_0$ is A -colorable, by Theorem 2.3, $G + e$ is A -colorable, and so $\chi_2(G + e) = |A| = k$, a contradiction. $\square$

COROLLARY 8.1. For any χ_2 -maximal graph G ,

$$\chi_2(G) \leq 1 + p(G),$$

where $p(G)$ is the length of a longest path in G .

Proof. We may assume that $\chi_2(G) = k \geq 2$. By Theorem 8.2, $\delta(G) \geq k - 1$, where $\delta(G)$ is the minimum degree of G . Then G contains a path of length $k - 1$ (see [14], Theorem 3.6). Hence $p(G) \geq k - 1 = \chi_2(G) - 1$, namely, $\chi_2(G) \leq 1 + p(G)$. $\square$

THEOREM 8.3. Any χ_2 -maximal graph is connected.

Proof. Assume that $\chi_2(G) = k$ and G has two components G_1 and G_2 . Let G' be the graph obtained from G_1 and G_2 by adding a new edge e joining a vertex of G_1 to a vertex of G_2 . Since $\chi_2(G) = k$, there exists an Abelian group A of order k such that G is A -colorable. Therefore, G_1 and G_2 are

A -colorable. By Theorem 2.2, G' is A -colorable and so $\chi_2(G + e) = k$. $\square$

By Theorems 3.1 and 8.3, an immediate Corollary is given below.

COROLLARY 8.2. Graph G is a χ_2 -maximal graph with $\chi_2(G) = 2$ if and only if G is a tree.

THEOREM 8.4. Let G be a χ_2 -maximal graph with $\chi_2(G) = k \geq 3$. Then G is 2-edge-connected.

Proof. Assume the theorem is false. By Theorem 8.3, there exists a bridge $e = u_0v_0$ in G . Let G_1 and G_2 be two subgraphs obtained from G by splitting vertex u_0 into $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$.

Since $\chi_2(G) = k$, there is an Abelian group A of order k such that G is A -colorable, and so G_1 and G_2 are A -colorable. Since G is χ_2 -maximal and $k \geq 3$, by Theorem 8.2, $d_G(v_0) \geq 2$. Then there is an edge $e' \in E(G_2^c)$ joining u_2 .

Since $d_{G_2+e'}(u_2) = 2$ and $G_2 - u_2$ is A -colorable, by Lemma 8.1 and Theorem 2.3, $G_2 + e'$ is A -colorable. By Theorem 2.2, $G + e'$ is A -colorable, a contradiction. $\square$

Chapter 3

Group Coloring on a Surface

3.1 Introduction

The coloring problem of graphs on a surface S is a very popular topic of graph theory. Naturally, the most famous problem in this area is the Four Color Theorem. In this chapter, we will investigate the group coloring problem on a surface. Before doing that, let us look at some interesting results on vertex coloring and list coloring problems.

3.1.1 Map Coloring

Given a graph G , the minimal number $g \geq 0$ such that G can be embedded in an orientable surface of genus g is called the **genus** of G . Euler's polyhedron theorem states that a polyhedron of genus g with n vertices, e edges and f faces satisfies

$$n - e + f = 2 - 2g. \tag{3.1}$$

An immediate consequence of this is that if G is embeddable in a surface of genus g then

$$e \leq 3n + 6(g - 1). \quad (3.2)$$

Heawood showed in 1890 that the maximum number $h(g)$ of colors necessary to color this graph (the Chromatic Number) is

$$h(g) = \left\lfloor \frac{1}{2}(7 + \sqrt{48g + 1}) \right\rfloor.$$

For any orientable surface other than the Sphere (or equivalently, the Plane) and any nonorientable surface other than the Klein Bottle, $h(g)$ was shown to be not merely a maximum, but the actual number needed (Ringel and Youngs [32]) in 1968.

When the Four-Color Theorem was proven in 1976, the Heawood Formula was shown to hold also for all orientable and nonorientable surfaces with the exception of the Klein Bottle. For this case (which has Euler Characteristic 1, and therefore can be considered to have $g = 1/2$), the actual number of colors h needed is six-one less than $h(g) = 7$ (Franklin [20] 1934; Saaty [33] 1986).

surface	g	h(g)	h
Klein Bottle	1	7	6
Möbius Strip	$\frac{1}{2}$	6	6
Plane	0	4	4
Projective Plane	$\frac{1}{2}$	6	6
Sphere	0	4	4
Torus	1	7	7

Table 3.1:

3.1.2 Choosability in Graphs

The concepts of list coloring and choosability were introduced by Vizing [36] and independently by Erdős et al. [19]. A list assignment of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. An L -coloring is a function $\lambda : V(G) \rightarrow \cup_v L(v)$ such that $\lambda(v) \in L(v)$ for every $v \in V(G)$ and $\lambda(u) \neq \lambda(v)$ whenever u, v are adjacent vertices of G . If G admits an L -coloring then it is L -colorable. A graph G is m -choosable if, for every list assignment L with $|L(v)| \geq m$ for all $v \in V$, there exists an L -coloring of G .

Vizing [36] raised the question whether every planar graph is 5-choosable. Erdős et al. [19] conjectured that every planar graph is 5-choosable, but not necessarily 4-choosable. In 1993, Voigt [37] described planar graphs that are not 4-choosable. In 1994, Thomassen [35] showed that all planar graphs are 5-choosable. Recently, Škrekovski [38] extended Thomassen's result, by showing that all K_5 -minor-free graphs are 5-choosable.

3.2 Heawood Theorem on Group Coloring

A graph G is said to be χ_1 -critical if $\chi_1(G-v) < \chi_1(G)$ for any vertex v of G .

LEMMA 2.1. If G is χ_1 -critical graph with $\chi_1(G) = k$, then $d_G(v) \geq k - 1$ for all $v \in V(G)$.

Proof. By contradiction. Assume that G has a vertex v_0 with $d_G(v_0) \leq k - 2$. If $\chi_1(G - v_0) = k - 1$, then $\chi_1(G) = k - 1$. Since $\chi_1(G) = k$ and G is χ_1 -critical, $\chi_1(G - v_0) = k - 1$ and so we get a contradiction. $\square$

Let S be an orientable surface with genus $g > 0$. Suppose that S contains a graph with $\chi_1(G) = k$. Then, deleting some vertices of G if necessary, we may assume that G is χ_1 -critical. By above Lemma, $d_G(v) \geq k - 1$ for all

$v \in V(G)$. By equation 3.2, we have

$$\frac{1}{2}k - 1|V(G)| \leq |E(G)| \leq 3|V(G)| + 6(g - 1).$$

Since $|V(G)| \geq k$,

$$k(k - 7)/2 \leq 6(g - 1),$$

and so

$$k \leq \left\lfloor \frac{1}{2}(7 + \sqrt{48g + 1}) \right\rfloor.$$

Hence we show that

$$\chi_1(G) \leq \left\lfloor \frac{1}{2}(7 + \sqrt{48g + 1}) \right\rfloor.$$

On the other hand, Ringel and Youngs [32] showed that the largest integer p such that the complete graph K_p can be embedded in S with genus g is

$$p = \left\lfloor \frac{1}{2}(7 + \sqrt{48g + 1}) \right\rfloor.$$

Since $\chi_1(K_p) = p$, there exists a graph with genus g and group chromatic number equal to $\left\lfloor \frac{1}{2}(7 + \sqrt{48g + 1}) \right\rfloor$.

Therefore, Heawood Map-Coloring Theorem still holds for group coloring. Denoted by $\chi_1(S_g)$ the maximum group chromatic number among all graphs which can be embedded on S , an orientable surface with genus g . Then we have:

THEOREM 2.1. For every positive integer g ,

$$\chi_1(S_g) = \left\lfloor \frac{1}{2}(7 + \sqrt{48g + 1}) \right\rfloor.$$

□

3.3 $\chi_1(G) \leq 5$ for Every Planar Graph G

The Four Color Theorem claims that every planar graph is 4-colorable. Thomassen [35] showed that every planar graph is 5-choosable and Voigt [37] showed there exist planar graphs that are not 4-choosable. In this section, we prove that, for every planar graph G , $\chi_1(G) \leq 5$.

If G is a plane graph which has no loops or multiple edges and which consists of an exterior cycle $C : v_1 v_2 \cdots v_p v_1$, and vertices and edges inside C such that each bounded face is bounded by a triangle, then we say that G is a **near-triangulation**. A cycle C in a plane graph G is called **separating** if both the interior and exterior of C contain vertices of G .

THEOREM 3.1. Assume that G is a near-triangulation with an orientation D and an exterior cycle $C : v_1 v_2 \cdots v_p v_1$. Let A be an Abelian group with $|A| \geq 5$ and $f \in F(G, A)$. Let $a_1, a_2 \in A$ such that $a_2 - a_1 \neq f(v_2 v_1)$, where $v_2 v_1 \in E(D(G))$, and $A_3, A_4, \dots, A_p$ are subsets of A such that $|A_i| = 2$ for $3 \leq i \leq p$. Then there is an (A, f) -coloring $c : V(G) \rightarrow A$ such that $c(v_1) = a_1$, $c(v_2) = a_2$ and $c(v_i) \notin A_i$, $3 \leq i \leq p$.

Proof. We proceed by induction on the number $|V(G)|$ of vertices. Assume first that $|V(G)| = 3$ and $C = K_3$.

Since $|A| \geq 5$ and $|A_3| = 2$, there is an element $a_3 \in A - \{a_1 + f(v_3 v_1), a_2 + f(v_3 v_2)\} \cup A_3$, and so $c : V(G) \rightarrow A$ given by $c(v_i) = a_i$ ($1 \leq i \leq 3$) is the

desired extension. We proceed to the induction step.

If C has a chord $v_i v_j$ (see Figure 3.1), where $2 \leq i \leq j-2 \leq p-1$ ($v_{p+1} = v_1$), then we apply the induction hypothesis to G_1 with the exterior cycle $v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_1$ and then to G_2 with the exterior $v_j v_i v_{i+1} \cdots v_{j-1} v_j$.

Figure 3.1:

Now we assume that C has no chord. Let $v_1, u_1, u_2, \dots, u_m, v_p$ be the neighbors of v_p in that clockwise order around v_p (see Figure 3.2).

For the graph $G - v_p$, we define:

$$b_1, b_2 \in A - \{f(v_p v_1) + a_1\} \text{ and } b_1 \neq b_2$$

$$A'_i = \{f(u_i v_p) + b_1, f(u_i v_p) + b_2\}$$

Then $|A'_i| = 2$ for $i = 1, 2, \dots, m$.

By the inductive hypothesis, there exists an A -coloring $c_1 : V(G - v_p) \rightarrow A$ such that

Figure 3.2:

$$c_1(v_1) = a_1 \text{ and } c_1(v_2) = a_2,$$

$$c(v_i) \notin A_i, 3 \leq i \leq p-1, \text{ and}$$

$$c(u_j) \notin A'_j, 1 \leq j \leq m.$$

Since $\{b_1, b_2\} - \{c_1(v_{p-1}) - f(v_{p-1}v_p)\} \neq \emptyset$, we may assume that $b_1 \in \{b_1, b_2\} - \{c_1(v_{p-1}) - f(v_{p-1}v_p)\}$. We obtain an A -coloring c for G by defining:

$$c(x) = \begin{cases} c_1(x) & \text{if } x \neq v_p, \\ b_1 & \text{if } x = v_p. \end{cases}$$

□

An easy consequence of Theorem 3.1 is following result.

COROLLARY 3.1. $\chi_1(G) \leq 5$ for every planar graph G .

3.4 Group Coloring of K_5 -minor-free Graphs

In this section, we extend the result of last section by showing that all K_5 -minor-free graphs are A -colorable for all Abelian group A with $|A| \geq 5$. A graph H is a minor of G if we can obtain H from a subgraph H' of G by contracting some of its edges, and G is H -minor-free if H is not a minor of G . H' is also called a contraction image of H . We may assume that there is no multiple edges after each contraction in this section. The proof is based on Theorem 3.1 and the following two theorems.

THEOREM 4.1 (Wagner [39]). Graph G is planar if and only if G has no K_5 -minor or $K_{3,3}$ -minor.

THEOREM 4.2 (Halin [21]). Every 4-connected non-planar graph contains K_5 as a minor.

THEOREM 4.3. Let G be a connected K_5 -minor-free graph and let A be an Abelian group with $|A| \geq 5$. Suppose that H is a subgraph of G isomorphic to K_2 or K_3 . Then (G, H) is A -extensible.

Proof. We consider the following two cases.

Case 1: G is a planar graph.

We assume that G is a plane graph. Let $f \in F(G, A)$ and c_0 be an $(A, f|_{E(H)})$ -coloring.

If $H \cong K_2$, we may assume that G is a near-triangulation and H is an edge of the exterior cycle. By Theorem 3.1, (G, H) is A -extensible.

If $H \cong K_3$ and is not a separating cycle, we may assume that G is a near-triangulation, $C : v_1 v_2 \cdots v_{p-1} w$ is the exterior cycle and $V(H) = \{v_1, u_1, w\}$. Let $v_1, u_1, u_2, \cdots, u_m, v_{p-1}$ be the neighbors of w in that clockwise order

around w (see Figure 3.3).

Figure 3.3:

Consider graph $G - w$. By Theorem 3.1, we can extend $(A, f|_{E(v_1 u_1)})$ -coloring c_0 to an $(A, f|_{E(G-w)})$ -coloring c_1 from $\{v_1, u_1\}$ to $G - w$ such that $c(u_i) \neq f(u_i w) + c(w)$, $2 \leq i \leq m$, and $c(v_{p-1}) \neq f(v_{p-1} m) + c(w)$. Then we obtain an (A, f) -coloring c of G as follows:

$$c(v) = \begin{cases} c_0(v), & \text{if } v = u_1, v_1 \text{ or } w, \\ c_1(v), & \text{otherwise} \end{cases} \quad (3.3)$$

Suppose that $H \cong K_3$ and is a separating cycle. Let $G_1 = \text{Int}(H)$ and $G_2 = \text{Out}(H)$, where $\text{Int}(H)$ is the subgraph of all vertices and edges inside or on H and $\text{Out}(H)$ is the subgraph of all vertices and edges outside or on H . By the same reasoning as above, c_0 can be extended from H to G_1 and from H to G_2 .

Case 2: G is not planar.

Since G is K_5 -minor-free, by Theorem 4.2, the connectivity of G , $\kappa(G)$, ≤ 3 . We proceed by induction on $|V(G)|$, the number of vertices. Let $f \in F(G, A)$ and c_0 be an $(A, f|_{E(H)})$ -coloring.

If $|V(G)| = 6$, since G is not planar and is K_5 -minor-free, by Theorem 4.1, G must contains $K_{3,3}$ as a subgraph. Hence, the number of vertices with degree 5 is at most 3 (see Figure 3.4). Otherwise, G contains K_5 as a

Figure 3.4:

subgraph. If $H \cong K_3$ and G has three vertices of degree 3, then H contains at least one vertex with degree 3 in G . Now, we can arrange the vertices of G in this order: $v_1 v_2 \cdots v_6$ such that $d_G(v_6) \leq 4$, $V(H) = \{v_1, v_2\}$ if $H \cong K_2$ and $V(H) = \{v_1, v_2, v_3\}$ if $H \cong K_3$. Let G_i be a subgraph $G[\{v_1, v_2, \dots, v_i\}]$ of G . Then $d_{G_i}(v_i) \leq 4$ ($i = 1, 2, \dots, 6$). Since $|A| \geq 5$, by Lemma 4.1 of Chapter 2, c_0 can be extended from H to G .

We proceed to the induction step.

Let G is connected and $\kappa(G) = t$ and $T = \{x_1, \dots, x_t\}$ be a minimum cut set of G , where $1 \leq t \leq 3$.

Note that, as the inductive hypothesis we assume that, for each not-planar

K_5 -minor-free graph with vertex number $< |V(G)|$, the theorem holds.

Let G_1 and G_2 be induced subgraphs of G such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = T$ and $V(G_i) - V(G_{3-i}) \neq \emptyset$, where $i = 1$ or 2 . Assume that $H \subseteq G_1$. For $i = 1$ and 2 , let $G'_i = G_i$ if $t = 1$; let $G'_i = G_i \cup \{x_1x_2\}$ if $t = 2$ and let $G'_i = G_i \cup \{x_1x_2, x_2x_3, x_3x_1\}$ if $t = 3$. Let $G' = G'_1 \cup G'_2$ and extend f to $f' \in F(G', A)$ arbitrarily. Note that, since G is not planar, at least one of G'_1 and G'_2 is not planar.

If $t = 1$ or 2 , then G'_1 and G'_2 are K_5 -minor-free. Since $|V(G'_1)| < |V(G)|$ and $|V(G'_2)| < |V(G)|$, by the inductive hypothesis and the result of case 1, c_0 can be extended from H to G'_1 and then from $G'[T]$ to G'_2 . We are done.

Suppose that $t = 3$. Since G is 3-connected, We have the following two Claims.

Claim 1. If $G_i \not\cong K_{1,3}$, then G_i has a subgraph which can be contracted to a K_3 such that the contraction images of the three vertices of this K_3 contain x_1 , x_2 and x_3 , respectively.

Proof. We only consider $i = 1$. The case when $i = 2$ is similar.

Let $L_1, L_2, \dots, L_k$ be the components of $G_1 - T$. Since G is 3-connected, there are at least three edges from each component L_i ($1 \leq i \leq k$) to T which are incident with x_1 , x_2 and x_3 respectively.

If $E[G[T]] \neq \emptyset$, we may assume that $x_2x_3 \in E(G)$. Then $G_1/E(L_1)$ contains a subgraph isomorphic to (a) of Figure 3.5 and so $(G_1/E(L_1))/v_1x_1$ contains a K_3 we want.

Suppose that $E[G[T]] = \emptyset$ and $k \geq 2$. Then $G_1/E(L_1 \cup L_2)$ contains a

subgraph isomorphic to (b) of Figure 3.5 and by contracting v_1x_1 and v_2x_3 , we get a K_3 we want.

Suppose that $E[G[T]] = \emptyset$ and $k = 1$. Since $G_1 \not\cong K_{1,3}$, $|V(G_1 - T)| \geq 2$. Let $v_1 \in V(G_1 - T)$. Since G is 3-connected, we can find a (v_1, x_1) -path, a (v_1, x_2) -path and a (v_1, x_3) -path in G_1 such that they are internally-disjoint. Let $P_1 = (v_1, x_1)$ -path, $P_2 = (v_1, x_2)$ -path and $P_3 = (v_1, x_3)$ -path.

If $V(G_1) - \cup_{i=1}^3 V(P_i) = \emptyset$, since G is 3-connected and $|V(G_1 - T)| \geq 2$, there is an edge u_1u_2 such that u_1 is on one path, u_2 is on another path and $u_1, u_2 \neq v_1$. Now by contracting all edges on paths P_1, P_2 or P_3 which are not incident with v_1 , G_1 can be contracted to a graph which contains a subgraph isomorphic to (a) of Figure 3.5.

Now if $V(G_1) - \cup_{i=1}^3 V(P_i) \neq \emptyset$, let $v_2 \in V(G_1) - \cup_{i=1}^3 V(P_i)$. Since G is 3-connected, we can find a (v_2, x_1) -path, a (v_2, x_2) -path and a (v_2, x_3) -path in G_1 such that they are internally-disjoint. Let $P'_1 = (v_2, x_1)$ -path, $P'_2 = (v_2, x_2)$ -path and $P'_3 = (v_2, x_3)$ -path. By contracting those edges on paths $P_1, P_2, P_3, P'_1, P'_2$ or P'_3 and incident with x_1, x_2 and x_3 , eventually, G_1 can be contracted to a graph which contains a subgraph isomorphic to (b) or (c) of Figure 3.5. Then we contract v_1x_1 and v_2x_3 and get a K_3 we want. We finish the proof of Claim 1.

Note that G is K_5 -minor-free. We have Claim 2.

Claim 2. If $G_i \not\cong K_{1,3}$, then G'_{3-i} is K_5 -minor-free, where $i = 1$ or 2 .

Proof. We shall only prove the case when $i = 1$.

Assume that $G_1 \not\cong K_{1,3}$. If G'_2 is not K_5 -minor-free, then there is a subgraph H' of G'_2 such that a K_5 can be obtained from H' by contracting

Figure 3.5:

some of its edges.

If $E(H') \cap \{x_1x_2, x_2x_3, x_1x_3\} = \emptyset$, then H' must be a subgraph of G , contrary to the assumption that G is K_5 -minor-free.

If $E(H') \cap \{x_1x_2, x_2x_3, x_1x_3\} \neq \emptyset$, let $E_0 = E(H') \cap \{x_1x_2, x_2x_3, x_1x_3\}$. Since $G_1 \not\cong K_{1,3}$, by Claim 1, G_1 has a subgraph which can be contracted to a K_3 such that the contraction images of the three vertices of this K_3 contain x_1, x_2 and x_3 , respectively. Hence, there is a subgraph H'' of G_1 which can be contracted to $H'[E_0]$. Let $H_1 = (H' - E_0) \cup H''$. Then H_1 is also a subgraph of G and H_1 is contractible to a K_5 . This proves Claim 2.

If for $i = 1$ nad 2 , $G_i \not\cong K_{1,3}$, by Claim 2, G'_{3-i} is K_5 -minor-free. Since $|V(G_i) - V(G_{3-i})| > 0$, $|V(G'_i)| = |V(G_i)| < |V(G)|$. By the inductive hypothesis and the result of case 1, c_0 can be extended to an $(A, f'|_{E(G'_1)})$ -coloring c_1 and $c_1|_T$ can then be extended to an $(A, f'|_{E(G'_2)})$ -coloring c_2 . Therefore, we obtain an (A, f') -coloring c of G' extending from c_0 . Since G is a subgraph of G' , c is also an (A, f) -coloring of G .

If $G_2 \cong K_{1,3}$, let $V(G_2) = \{x_1, x_2, x_3, v\}$, $G'' = G \cup \{x_1x_2, x_2x_3\}$ and $G''_1 = G_1 \cup \{x_1x_2, x_2x_3\}$. (See (a) of Figure 3.6.) Then G''_1 is K_5 -minor-

free since G_2 can be contracted to a path $P : x_1x_2x_3$. We extend f to $f'' \in F(G'', A)$ arbitrarily. By the inductive hypothesis, c_0 can be extended from H to G_1'' , then from G_1'' to G'' since $d(v) = 3 < |A|$. So we get an (A, f'') -coloring c of G'' extending from c_0 . Since G is a subgraph of G'' , c is also an (A, f) -coloring of G .

Figure 3.6:

If $G_1 \cong K_{1,3}$, we assume that $V(G_1) = \{v, x_1, x_2, x_3\}$. Let $G'' = G \cup \{x_1x_2, x_2x_3\}$, $G_1'' = G_1 \cup \{x_1x_2, x_2x_3\}$ and $G_2'' = G_2 \cup \{x_1x_2, x_2x_3\}$. (See (b) of Figure 3.6.)

Then G_2'' is K_5 -minor-free since G_1 can be contracted to a path $P : x_1x_2x_3$. Since $H \subseteq G_1$, $H \cong K_2$. We may assume that $H = G[v, x_1]$ ($c_0(x_1) - c(v) \neq f(x_1v)$), and extend f by assigning any element of A to edge x_2x_1 . Now we extend c_0 from H to $G''[v, x_1, x_2]$ by assigning an element, say $c_0(x_2)$, in $A - \{c_0(v) + f(x_2v), c_0(x_1) + f(x_2x_1)\}$ to x_2 and then extend f to $f'' \in F(G'', A)$ by assigning $c_0(v) + f(x_3v) - c_0(x_2)$ to edge x_3x_2 . By the inductive hypothesis, c_0 can be extended to an $(A, f''|_{E(G_2'')})$ -coloring c'' from $G''[x_1, x_2]$ to G_2'' . Since $c''(x_3) - c_0(x_2) \neq c_0(v) + f(x_3v) - c_0(x_2)$, $c''(x_3) - c_0(v) \neq f(x_3v)$

and so we can define an (A, f) -coloring c of G as follows:

$$c(u) = \begin{cases} c_0(u), & \text{if } u = v, x_1 \text{ or } x_2, \\ c''(u), & \text{otherwise} \end{cases} \quad (3.4)$$

□

An easy consequence is given as follows.

COROLLARY 4.1. $\chi_1(G) \leq 5$ for every K_5 -minor-free graph G .

Chapter 4

On Edge-integrity Maximal Graphs

The **edge-integrity** of a graph G is given by $\min_{S \subseteq E(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ denotes the maximum order of a component of $G - S$. Let $I'(G)$ denote the edge-integrity of a graph G . We define a graph G to be I' -maximal if for every edge e in $\overline{G}$, the complement of graph G , $I'(G + e) > I'(G)$. In this chapter, some basic results of I' -maximal graphs are established, the girth of a connected I' -maximal graph is given and lower and upper bounds on the size of I' -maximal connected graphs with given order and edge-integrity are investigated. Also, the I' -maximal trees and unicyclic graphs are completely characterized. Most results of this chapter will be published in [25].

4.1 Introduction

In this chapter we consider finite undirected simple graphs. The edge-integrity of a graph attempts to measure the disruption caused by the removal of edges from the graph. The order of a component or graph is the number of its vertices, and we let $m(H)$ denote the maximum order of a component

of graph H . The edge-integrity is defined as

$$I'(G) = \min_{S \subseteq E} \{|S| + m(G - S)\}.$$

Barefoot, Entringer and Swart [4] proposed this parameter as a measure of how hard it is to disrupt thoroughly a network by edge failures. Properties of edge-integrity were also investigated by Bagga et al., and by others. See [1, 2, 3, 5, 27, 29].

Let G be a graph and $\overline{G}$ be the complement of G . G is I' -**maximal** if and only if $I'(G + e) > I'(G)$, for every edge e of $\overline{G}$. Let $M(k)$ denote the collection of all I' -maximal graphs with edge-integrity k and $M_n(k)$ denote the collection of all I' -maximal graphs with order n and edge-integrity k .

4.2 Some Basic Properties

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E . For $S \subseteq E$, let $I'(S)$ denote $|S| + m(G - S)$. A set $S \subseteq E$ for which $I'(S)$ is minimized is called an I' -set of G . If $H_1, H_2, \dots, H_r$ are the components of $G - S$ with $|V(H_i)| = n_i$ such that $n_1 \leq n_2 \leq \dots \leq n_r$, we say that $G - S$ has type $(n_1, n_2, \dots, n_r)$. For $V_1, V_2 \subseteq V(G)$, let $[V_1, V_2]_G = \{e = x_1x_2 \in E(G) | x_i \in V_i, i = 1, 2\}$. When G can be understood from the context, we write $[V_1, V_2]$ for $[V_1, V_2]_G$.

THEOREM 2.1 Assume $G \in M(k)$. Let $S \subseteq E(G)$ with $I'(G) = I'(S) = k$. Let $H_1, H_2, \dots, H_r$ be the components of $G - S$ with type $(n_1, n_2, \dots, n_r)$. Then $H_i \cong K_{n_i}$ ($i = 1, \dots, r$).

Proof: If there exists an edge $e \in E(\overline{H}_i)$, then in $G + e$, $I'(S) = k$, and so $I'(G + e) \leq k$, a contradiction. $\square$

THEOREM 2.2 Assume $G \in M(k)$ and $S \subseteq E(G)$ is an I' -set. Let $H_1, H_2, \dots, H_r$ be the components of $G - S$ with type $(n_1, n_2, \dots, n_r)$. Assume that $r > 2$. For any i, j , we have

$$n_r + |[V(H_i), V(H_j)]| < n_i + n_j.$$

Proof: We consider two cases.

Case 1: If there exist i, j such that $i, j < r$ and $n_r + |[V(H_i), V(H_j)]| \geq n_i + n_j$, then $|S| - |[V(H_i), V(H_j)]| + n_i + n_j \leq |S| + n_r$.

If $[V(H_i), V(H_j)] \neq \emptyset$, let $S' = S - [V(H_i), V(H_j)]$. Then $I'(S') = |S| - |[V(H_i), V(H_j)]| + \max\{n_i + n_j, n_r\} \leq |S| + n_r = I'(S)$. However, $H_i \cup H_j \cup [V(H_i), V(H_j)]$ can not be a complete subgraph, contrary to Theorem 2.1.

If $[V(H_i), V(H_j)] = \emptyset$, then $|S| + n_i + n_j \leq |S| + n_r$. Hence, for any edge $e \in [V(H_i), V(H_j)]_{\overline{G}}$, we have that $I'(G + e) \leq I'(G)$, contrary to the definition of I' -maximal graph.

Case 2: If $i = r$ or $j = r$, then we only need to prove that, for any $1 \leq i \leq r - 1$, we have

$$|[V(H_r), V(H_i)]| < n_i.$$

Otherwise, suppose that there exists an integer i with $1 \leq i \leq r - 1$ such that $|[V(H_r), V(H_i)]| \geq n_i$. Let $S' = S - |[V(H_r), V(H_i)]|$. We have

$$I'(S') = |S| - |[V(H_r), V(H_i)]| + n_r + n_i \leq |S| + n_r = I'(S).$$

Since $G[V(H_r), V(H_i)]$ is not a complete subgraph of G , we get a contradiction with Theorem 2.1. $\square$

COROLLARY 2.1 Let $G \in M(k)$ and $S \subseteq E(G)$ be an I' -set of G . Assume that $G - S$ has type $(n_1, n_2, \dots, n_r)$ ($r > 2$). Then $n_2 > n_r/2$.

Proof: By Theorem 2.2, we obviously have $n_1 + n_2 > n_r$. If $n_2 \leq n_r/2$, we have $n_1 + n_2 \leq 2n_2 \leq n_r$. Therefore $n_2 > n_r/2$. $\square$

Since the removal of an edge which leaves an isolated vertex reduces the order of some component (not necessarily the largest) by one, we have the following Theorem.

THEOREM 2.3 Let $G \in M(k)$ and $S \subseteq E(G)$ be an I' -set. And let $H_1, H_2, \dots, H_r$ be the components of $G - S$ with type $(n_1, n_2, \dots, n_r)$. If G is connected, then we have $n_1 \geq 2$.

4.3 The girth of I' -maximal graphs

First, we construct two classes of graphs and prove three lemmas which we will use.

Let $H_0, H_1, \dots, H_r$ be $r + 1$ complete graphs with $H_i \cong K_{p_i}$ ($0 \leq i \leq r$) such that $p_0 \geq p_1 \geq p_2 \geq \dots \geq p_r$. For each $i, 0 \leq i \leq r$, we choose one vertex v_i from $V(H_i)$ and construct a new graph $G(p_0; p_1, \dots, p_r)$ as follows:

$$\begin{aligned} V(G(p_0; p_1, \dots, p_r)) &= V(H_0) \cup V(H_1) \cup \dots \cup V(H_r) \\ E(G(p_0; p_1, \dots, p_r)) &= E(H_0) \cup \dots \cup E(H_r) \cup \{v_0v_1, v_0v_2, \dots, v_0v_r\}. \end{aligned}$$

If $p_1 = \dots = p_j = p$, we denote this graph by $G(p_0; (p)_j, p_{j+1}, \dots, p_r)$.

LEMMA 3.1 Suppose that G is a graph with unique I' -set $S \subseteq E(G)$ and $H_1, H_2, \dots, H_r$ are the components of $G - S$ with $H_i \cong K_{n_i}$ ($1 \leq i \leq r$).

If for any i, j such that $1 \leq i, j \leq r$, $n_i + n_j > \max\{n_1, n_2, \dots, n_r\}$, then G is an I' -maximal graph.

Proof: For any $e \in E(\overline{G})$, we consider graph $G + e$. Assume that S' is an I' -set of $G + e$.

Case 1: $e \in S'$. Then $I'(G + e) = |S'| + m(G + e - S') = |S' - e| + 1 + m(G - (S' - e)) \geq I'(G) + 1$.

Case 2: $e \notin S'$. If $S' = S$, then $I'(G + e) = |S| + m(G + e - S) > |S| + m(G - S) = I'(G)$; if $S' \neq S$, then $I'(G + e) = |S'| + m(G + e - S') \geq |S'| + m(G - S') > I'(G)$. $\square$

LEMMA 3.2 Suppose that $p_r \geq 2$ and for any i, j such that $1 \leq i, j \leq r$, $p_i + p_j > p_0$. Then

- (i) $I'(G(p_0; p_1, \dots, p_r)) = p_0 + r$.
- (ii) The I' -set of $G(p_0; p_1, \dots, p_r)$ is unique.
- (iii) $G(p_0; p_1, \dots, p_r)$ is I' -maximal.

Proof: (i) Let $\Delta(G)$ be the maximum degree of a vertex of graph G . We have, for any graph G , $I'(G) \geq \Delta(G) + 1$ (see [3]). Thus we get that $I'(G(p_0; p_1, \dots, p_r)) \geq p_0 + r$. Let $S = \{v_0v_1, v_0v_2, \dots, v_0v_r\}$. Then $I'(S) = p_0 + r$ and so $I'(G(p_0; p_1, \dots, p_r)) = p_0 + r$.

(ii) Suppose that S' is an I' -set of G such that $S' \neq S$. Let c be the number of H'_i s that S' intersects. We might as well assume $S' \cap E(H_{i_t}) \neq \emptyset$, for some $1 \leq t \leq c \leq r + 1$. Note that since $H_{i_t} \cong K_{p_{i_t}}$ and $p_{i_t} \geq 2$, $|S' \cap E(H_{i_t})| \geq p_{i_t} - 1 \geq 1$, ($1 \leq t \leq c$). If $S' \cap E(H_0) = \emptyset$, then the component of $G - S'$ containing H_0 also contains at least one vertex from each H_{i_t} , $1 \leq t \leq c$, and so this component has at least $p_0 + c$ vertices. Hence $I'(S) = I'(S') = |S'| + m(G - S') \geq |S| - c + \sum_{t=1}^c (p_{i_t} - 1) + p_0 + c \geq I'(S) + c >$

$I'(S)$, a contradiction. Thus we assume $H_{i_c} = H_0$ and $X = S' \cap E(H_0) \neq \emptyset$.

Let H'_0 and H''_0 be the two parts of $H_0 - X$ such that $v_0 \in V(H'_0)$. Then since $H_0 \cong K_{p_0}$, $|X| = |V(H'_0)|(p_0 - |V(H'_0)|)$. Since $v_0 \in V(H'_0)$, the component of $G - S'$ containing $V(H'_0)$ also contains at least one vertex from each H_{i_t} , $1 \leq t \leq c - 1$, and so $m(G - S') \geq |V(H'_0)| + c$. Hence $I'(S) = I'(S') = |S'| + m(G - S') \geq |S| - c + \sum_{i=1}^{c-1} (p_{i_t} - 1) + |V(H'_0)|(p_0 - |V(H'_0)|) + |V(H'_0)| + c \geq I'(S) + c - 1 \geq I'(S)$. This implies $c = 1$ and $V(H'_0) = \{v_0\}$.

Let $E_{H_0}(v_0) \subseteq E(H_0)$ be the set of edges in H_0 incident with v_0 . Since $H_0 \cong K_{p_0}$, $|E_{H_0}(v_0)| = p_0 - 1$. Since $c = 1$ and $V(H'_0) = \{v_0\}$, $S' = S \cup E_{H_0}(v_0)$ and $G - S'$ has components $H_1, H_2, \dots, H_r$ and $H_0 - v_0$. Hence $I'(S) = I'(S') \geq |S| + (p_0 - 1) + |V(H_0 - v_0)| = r + 2p_0 - 2 = I'(S) + (p_0 - 2)$, and so $p_0 = 2$. Since $p_r \geq 2$, we conclude that $p_0 = p_1 = \dots = p_r = 2$. Therefore $I'(S) = I'(S') = |S'| + m(G - S') = (r + 1) + 2 > r + 2 = I'(S)$, a contradiction. This proves (ii).

(iii) This result follows from (ii) and Lemma 3.1. $\square$

Let $T(d_1, d_2)$ be a tree with vertex set $\{u_1, u_2, v_i, w_i, v'_j, w'_j | 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$ and edge set $\{u_1u_2, u_1w_i, w_iw'_i, u_2v_j, v_jv'_j | 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$; let Δ_1 be a graph obtained from K_3 with vertex set $V(K_3) = \{v_1, v_2, v_3\}$ by adding three new vertices: u_1, u_2, u_3 , and three new edges: u_1v_1, u_2v_2, u_3v_3 ; and let $\Delta(d_1, d_2, d_3)$ be a graph obtained from K_3 with vertex set $V(K_3) = \{v_1, v_2, v_3\}$ by adding a new vertex set $\{x_i, x'_i, y_j, y'_j, z_k, z'_k | 1 \leq i \leq d_1, 1 \leq j \leq d_2 \text{ and } 1 \leq k \leq d_3\}$ and a new edge set $\{v_1x_i, x_ix'_i, v_2y_j, y_jy'_j, v_3z_k, z_kz'_k | 1 \leq i \leq d_1, 1 \leq j \leq d_2 \text{ and } 1 \leq k \leq d_3\}$.

LEMMA 3.3 If one of d_1, d_2 and d_3 is equal to zero and at least one of them is not equal to zero, then $\Delta(d_1, d_2, d_3)$ is I' -maximal.

Proof: Without loss of generality, we may assume that $d_3 = 0$. Let

Figure 4.1: The graphs $T(2, 3)$ and $\Delta(3, 2, 1)$

$G = \Delta(d_1, d_2, 0)$. Assume that $S \subseteq E(G)$ is an I' -set of G . Let $S_1 = \{v_1x_i, v_2y_j | 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$. We consider three cases.

Case 1: Only one edge e of v_1v_2, v_2v_3 and v_3v_1 belongs to S . We easily check that $S_1 \subseteq S$ and so $I'(S_1) < I'(S)$, a contradiction.

Case 2: Only two edges e_1 and e_2 of v_1v_2, v_2v_3 and v_3v_1 belong to S . Obviously, $\{e_1, e_2\} \neq \{v_2v_3, v_3v_1\}$. We might as well assume that $\{e_1, e_2\} = \{v_1v_2, v_3v_1\}$ and $S = \{v_1x_i, v_2y_j | 1 \leq i \leq r_1, 1 \leq j \leq r_2\} \cup \{e_1, e_2\}$, where $r_1 \leq d_1, r_2 < d_2$. Then $I'(S) = 2 + r_1 + r_2 + \max\{2(d_1 - r_1) + 2, 2(d_2 - r_2) + 1\} = \max\{4 + 2d_1 - r_1 + r_2, 3 + 2d_2 + r_1 - r_2\} > I'(S_1)$, a contradiction.

Case 3: $\{v_1v_2, v_2v_3, v_3v_1\} \subseteq S$. We might as well assume that $S = \{v_1x_i, v_2y_j | 1 \leq i \leq r_1, 1 \leq j \leq r_2\} \cup \{v_1v_2, v_2v_3, v_3v_1\}$, where $r_1 < d_1, r_2 < d_2$. Then $I'(S) > I'(S_1)$, a contradiction.

Hence, S does not contain v_1v_2, v_2v_3 or v_3v_1 . Obviously, S can be only S_1 . The result follows from Lemma 3.1. $\square$

THEOREM 3.1 Let G be a connected I' -maximal graph. Suppose that there exists an I' -set $S \subseteq E(G)$ such that $G - S$ has components: $H_1, H_2, \dots, H_c$ with $c = \lfloor \frac{n}{2} \rfloor$.

- (i) If n is even, then $G \cong G(2; (2)_{c-1})$ or Δ_1 .
- (ii) If n is odd, then $G \cong \Delta(d_1, d_2, 0)$, where $d_1 + d_2 = c - 1$.

Proof: (i) If n is even, then $c = n/2$ and so each component $H_1, H_2, \dots, H_c$ of $G - S$ is a K_2 .

Since G is connected, we might as well assume $||V(H_1), V(H_2)|| \neq 0$.

Claim 1: For any $i, j \geq 3$ ($i \neq j$), $[V(H_i), V(H_j)] = \emptyset$.

If not, let $e \in [V(H_1), V(H_2)]$ and $e_1 \in [V(H_i), V(H_j)]$ for some $i, j \geq 3$ ($i \neq j$). Let $S' = S - \{e, e_1\}$. Then $I'(S') = |S| - 2 + (2+2) = |S| + 2 = I'(S)$, contrary to Theorem 2.1.

Claim 2: If $||V(H_1), V(H_i)|| \neq 0$ ($3 \leq i \leq c$), then $||V(H_2), V(H_j)|| = 0$, for any j ($3 \leq j \leq c$ and $j \neq i$); or if $||V(H_2), V(H_i)|| \neq 0$ ($3 \leq i \leq c$), then $||V(H_1), V(H_j)|| = 0$, for any j ($3 \leq j \leq c$ and $j \neq i$).

Otherwise, we assume $e_1 \in [V(H_1), V(H_i)], e_2 \in [V(H_2), V(H_j)]$, where $3 \leq i, j \leq c$, and $i \neq j$. Let $S' = S - \{e_1, e_2\}$. Then $I'(S') = |S| - 2 + 4 = |S| + 2 = I'(S)$, contrary to Theorem 2.1.

Claim 3: For any $1 \leq i, j \leq c$, and $i \neq j$, $||V(H_i), V(H_j)|| \leq 1$.

Otherwise, we can choose two edges $e_1, e_2 \in [V(H_i), V(H_j)]$. Let $S' = S - \{e_1, e_2\}$. Similarly, we get a contradiction.

Now we consider three cases.

Case 1: $c \geq 4$.

Without loss of generality, we assume $||V(H_2), V(H_i)|| = 0$, for any i ($3 \leq i \leq c$), and $V(H_1) = \{u_1, u_2\}$. Let $d(u_1) = d_1 + 1, d(u_2) = d_2 + 1$, where $d_1 + d_2 = c - 1$. Then this I' -maximal graph is isomorphic to tree $T(d_1, d_2)$. We claim that $d_1 = 0$ or $d_2 = 0$. Otherwise, assume that $[u_1, V(H_i)] \neq \emptyset$ and $[u_2, V(H_j)] \neq \emptyset$, where $i \neq j$. Let $e_1 \in [u_1, V(H_i)]$ and $e_2 \in [u_2, V(H_j)]$. Let $S' = S \cup \{u_1 u_2\} - \{e_1, e_2\}$. Then $I'(S') = I'(S)$, and so S' is also an I' -set of $T(d_1, d_2)$, which yields a contradiction to Theorem 2.1. Thus G is isomorphic to $G(2; (2)_{c-1})$.

Case 2: $c = 3$.

If $||[V(H_1), V(H_3)]|| = 0$ or $||[V(H_2), V(H_3)]|| = 0$, then this case returns to case 1 and G is isomorphic to $G(2; (2)_2)$. If $||[V(H_1), V(H_3)]|| \neq 0$ and $||[V(H_2), V(H_3)]|| \neq 0$, then by claim 3, $||[V(H_1), V(H_3)]|| = 1$ and $||[V(H_2), V(H_3)]|| = 1$. Hence, we need only consider four graphs: G_1, G_2, C_6 and Δ_1 , where G_1 is a graph with vertex set $\{v_1, v_2, v_3, v_4, u_1, u_2\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, u_1v_1, u_2v_2\}$, G_2 is a graph obtained from a 5-cycle C_5 with vertex set $\{v_i | 1 \leq i \leq 5\}$ by adding a new vertex v_0 and a new edge v_0v_1 and C_6 is a 6-cycle. We easily check that G_1, G_2 and C_6 are not I' -maximal and Δ_1 is I' -maximal.

Case 3: $c = 2$.

If $r = 2$, this I' -maximal graph is a path with order 4, namely graph $G(2; 2)$.

Hence, the result follows from Lemma 3.2.

(ii) If n is odd, then $c = (n - 1)/2$. Hence, one of $H_1, H_2, \dots, H_c$ is isomorphic to K_3 and the others are isomorphic to K_2 .

Without loss of generality, we suppose $H_c \cong K_3$ and $H_i \cong K_2$, where $1 \leq i \leq c - 1$. We claim that $[V(H_i), V(H_j)] = \emptyset, 1 \leq i, j \leq c - 1$ and $i \neq j$. Otherwise, let $e \in [V(H_i), V(H_j)]$ and $S' = S - e$. We have $I'(S') = I'(S)$, contrary to Theorem 2.1. Now we prove that, for any i ($1 \leq i \leq c - 1$), $||[V(H_i), V(H_c)]|| = 1$. If not, let $e_1, e_2 \in [V(H_i), V(H_c)]$ and $S' = S - \{e_1, e_2\}$. Then $I'(S') = I'(S)$, contrary to Theorem 2.1.

Hence, G is isomorphic to one of three kinds of graphs: $\Delta(d_1, 0, 0)$, $\Delta(d_1, d_2, 0)$ and $\Delta(d_1, d_2, d_3)$, where $d_1, d_2, d_3 \neq 0$, and so we need only to consider graphs: $\Delta(d_1, 0, 0)$, $\Delta(d_1, d_2, 0)$ and $\Delta(d_1, d_2, d_3)$.

Since $d_1, d_2 \neq 0$, by Lemmas 3.2 and 3.3, $\Delta(d_1, 0, 0)$ and $\Delta(d_1, d_2, 0)$ are I' -maximal. For graph $\Delta(d_1, d_2, d_3)$ (d_1, d_2 and $d_3 \neq 0$), let $S' = S \cup \{v_1v_2, v_2v_3, v_3v_1\} - \{v_1x_1, v_2y_1, v_3z_1\}$. Then $I'(S') = I'(S)$, contrary to Theorem 2.1. $\square$

COROLLARY 3.1 Let T be a tree. Then T is I' -maximal if and only if $T \cong G(2; (2)_d)$, ($d \geq 1$).

Proof: Assume tree $T \in M(k)$. Let $S \subseteq E(T)$ be an I' -set of T . Let $H_1, H_2, \dots, H_c$ be the components of $T - S$. By Theorems 2.1 and 2.4, we have $H_i \cong K_2$ ($i = 1, 2, \dots, c$). Hence, we know that any tree with an odd number of vertices is not I' -maximal. By Theorem 3.1, this I' -maximal tree must be isomorphic to tree $G(2; (2)_d)$, where $d = c - 1$. By Lemma 3.2, for each integer $d \geq 1$, $G(2; (2)_d)$ is an I' -maximal tree, and so the proof is complete. $\square$

COROLLARY 3.2 Any connected I' -maximal graph, except $G(2; (2)_d)$ ($d \geq 1$), has girth 3.

Proof: Let $G = (V, E)$ be an I' -maximal graph with girth larger than 3 and $S \subseteq E(G)$ be an I' -set of G . Let $H_1, H_2, \dots, H_c$ be the components of $G - S$.

By the assumption that G has no 3-cycle and by Theorems 2.1 and 2.4, we know that $H_i \cong K_2$ ($i = 1, 2, \dots, c$), where $c = |V(G)|/2$. By Theorem 3.1, G is isomorphic to the tree $G(2; (2)_d)$, where $d = c - 1$. $\square$

COROLLARY 3.3 Let G be a unicyclic connected graph. G is I' -maximal if and only if $G \cong \Delta_1$ or $\Delta(d_1, d_2, 0)$, where d_1 and d_2 are not both zero.

Proof: Let $S \subseteq E(G)$ be an I' -set of G and $H_1, H_2, \dots, H_c$ be the components of $G - S$. Since G is a unicyclic connected graph, $c = \lfloor \frac{n}{2} \rfloor$. This corollary follows from Theorem 3.1. $\square$

4.4 Minimum size of an I' -maximal graph

Let $m(n, k) = \min\{|E(G)| : G \in M_n(k) \text{ and is connected}\}$.

LEMMA 4.1 Assume $G \in M_n(k)$. Let $S \subseteq E(G)$ be an I' -set of G . Let $H_1, H_2, \dots, H_c$ be the components of $G - S$ with type $(n_1, n_2, \dots, n_c)$. Then

$$m(n, k) \geq \left\lceil \frac{(n-1)^2}{2c} - \frac{(n+1)}{2} + k \right\rceil.$$

Proof: By Theorem 2.1, we have

$$|E(G)| = \sum_{i=1}^c \binom{n_i}{2} + |S| = \sum_{i=1}^c \binom{n_i}{2} + k - n_c.$$

Combining the constraint $n_1 + n_2 + \dots + n_c = n$, the minimum can be determined by using calculus. When $n_1 = n_2 = \dots = n_{c-1} = (n-1)/c, n_c = (n-1+c)/c$, it is attained, namely, $|E(G)| \geq \frac{(n-1)^2}{2c} - \frac{(n+1)}{2} + k$. Since $m(n, k)$ is an integer, $m(n, k) \geq \left\lceil \frac{(n-1)^2}{2c} - \frac{(n+1)}{2} + k \right\rceil$. $\square$

Note that the maximum value of c in Lemma 4.1 is $\lfloor \frac{n}{2} \rfloor$. When $c = \lfloor \frac{n}{2} \rfloor$, the I' -maximal graphs are what Theorem 3.1 described. Hence, by Theorem 3.1, we have

COROLLARY 4.1

$$m(n, k) = \begin{cases} 2r - 1, & \text{if } (n, k) = (2r, r + 1), r \geq 2 \\ 2r + 1, & \text{if } (n, k) = (2r + 1, r + 2), r \geq 2 \\ 6, & \text{if } (n, k) = (6, 5). \end{cases} \quad (4.1)$$

LEMMA 4.2 Let G be an I' -maximal graph with order n and $S \subseteq E(G)$ be an I' -set of G . Suppose that $G - S$ has type $(n_1, n_2, \dots, n_c)$ and $c = \lfloor \frac{n}{2} \rfloor - 1$.

- (i) If n is even and $n \geq 6$, then $G - S$ has type $(2, 2, \dots, 2, 3, 3)$ or $(2, 4)$.
- (ii) If n is odd and $n \geq 7$, then $G - S$ has type $(2, 2, \dots, 2, 3, 3, 3)$ ($n \geq 9$), or $(2, 3, 4)$, or $(2, 5)$ or $(3, 4)$.

Proof:(i) Suppose n is even and $n \geq 6$. Since G is I' -maximal, by Theorem 2.3, $n_1 \geq 2$.

Claim: If $c = n/2 - 1$, then $2 < n_c < 5$.

If $n_c = 2$, then $c = n/2$; if $n_c \geq 5$, then $c - 1 \leq (n - n_c)/2 \leq (n - 5)/2$ and so $c \leq (n - 3)/2 < n/2 - 1$, contrary to the assumption that $c = n/2 - 1$. This proves the Claim by contradiction.

Hence $n_c = 4$ or 3 . If $n_c = 4$, since $n - 4 = 2(c - 1)$, then $n_{c-1} = 2$. By Corollary 2.1, we know that its type must be $(2, 4)$. If $n_c = 3$, since n is even, then $n_{c-1} = 3$. Since $n - 6 = 2(c - 2)$, $n_{c-2} = 2$. Hence its type is $(2, 2, \dots, 2, 3, 3)$.

(ii) Suppose n is odd and $c = \frac{n-1}{2} - 1$, where $n \geq 7$. By similar argument, we know that $n_c = 5, 4$ or 3 . By Corollary 2.1, if $n_c = 5$, then its type is $(2, 5)$; if $n_c = 4$, then its type is $(2, 3, 4)$ or $(3, 4)$; if $n_c = 3$ and $(n \geq 9)$, then its type is $(2, 2, \dots, 2, 3, 3, 3)$. $\square$

THEOREM 4.1 Suppose that $(n, k) \neq (2c, c + 1), (2c + 1, c + 2)$ and $(6, 5)$, where $c \geq 2$.

(i)

$$m(n, k) \geq \begin{cases} \left\lceil \frac{(n-1)^2}{2\lfloor \frac{n}{2} \rfloor - 2} - \frac{(n-1)}{2} + \lfloor \frac{n}{2} \rfloor \right\rceil, & \text{if } \lfloor \frac{n}{2} \rfloor \leq k-1, \\ \left\lceil \frac{(n-1)^2}{2(k-2)} - \frac{(n+1)}{2} + k \right\rceil, & \text{if } \lfloor \frac{n}{2} \rfloor > k-1. \end{cases} \quad (4.2)$$

(ii) If $\lfloor \frac{n}{2} \rfloor \leq k-1$, then equality in (2) holds if and only if $(n, k) = (2c+3, c+2)$ ($c \geq 3$).

(iii) If $\lfloor \frac{n}{2} \rfloor > k-1$, then equality in (2) holds if and only if $(n, k) = (3c, c+2)$ ($c \geq 4$), $(3c-1, c+2)$ ($c \geq 5$), $(16, 8)$, $(18, 9)$, $(19, 9)$, $(20, 10)$ or $(22, 10)$.

Proof: (i) Assume $G \in M_n(k)$. Let $S \subseteq E(G)$ be an I' -set of G . Let $H_1, H_2, \dots, H_c$ be the components of $G - S$ with type $(n_1, n_2, \dots, n_c)$. By Theorem 3.1 and Corollary 4.1, we know $c \leq \lfloor \frac{n}{2} \rfloor - 1$ and so $n_c \geq 3$. Since G is connected, $k \geq (c-1) + 3$, so $c \leq k-2$. Hence, we have

$$c \leq \min\{\lfloor \frac{n}{2} \rfloor - 1, k-2\}. \quad (4.3)$$

If $\lfloor \frac{n}{2} \rfloor \leq k-1$, then $c \leq \lfloor \frac{n}{2} \rfloor - 1$ and $k \geq \lfloor \frac{n}{2} \rfloor + 1$; if $\lfloor \frac{n}{2} \rfloor > k-1$, then $c \leq k-2$. Hence the inequality (2) follows from Lemma 4.1.

Next, we shall determine all values of n and k for which $m(n, k)$ reaches its minimum.

(ii) Suppose that $\lfloor \frac{n}{2} \rfloor \leq k-1$.

Claim 1: If equality holds in (2), then $(n, k) = (2c+3, c+2)$ ($c \geq 3$).

Let $G \in M_n(k)$ and $S \subseteq E(G)$ be an I' -set of G such that $G - S$ has type $(n_1, n_2, \dots, n_c)$. We shall show that if $|E(G)|$ reaches the lower bound in (2), then $c = \lfloor \frac{n}{2} \rfloor - 1$, $|S| = \lfloor \frac{n}{2} \rfloor - 2$ and $n_c = 3$.

If $\lfloor \frac{n}{2} \rfloor \leq k - 1$, then $c \leq \lfloor \frac{n}{2} \rfloor - 1$ and $k \geq \lfloor \frac{n}{2} \rfloor + 1$. Note that a decrease of c by 1 or an increase of k by 1 must cause an increase of the lower bound in Lemma 4.1 by at least 1. Hence, by Lemma 4.1 and (2), if $G \in M_n(k)$ and $|E(G)|$ reaches the lower bound, then $c = \lfloor \frac{n}{2} \rfloor - 1$ and $k = \lfloor \frac{n}{2} \rfloor + 1$. Since G is connected, $|S| \geq c - 1 = \lfloor \frac{n}{2} \rfloor - 2$. On the other hand, $|S| = k - n_c \leq \lfloor \frac{n}{2} \rfloor + 1 - 3 = \lfloor \frac{n}{2} \rfloor - 2$. So $|S| = \lfloor \frac{n}{2} \rfloor - 2$ and $n_c = 3$.

Thus $k = |S| + n_c = c + 2$. By Lemma 4.2, since $n_c = 3$, $n = 2c + 2$ or $2c + 3$. Therefore $(n, k) = (2c + 2, c + 2)$ ($c \geq 2$), or $(2c + 3, c + 2)$ ($c \geq 3$). Note that $(n, k) \neq (2r, r + 1)$, where $r \geq 2$. Hence $(n, k) = (2c + 3, c + 2)$ ($c \geq 3$).

Claim 2: If $(n, k) = (2c + 3, c + 2)$ ($c \geq 3$), equality in (2) holds.

By Lemma 3.2, $G(3; 3, 3, 2, \dots, 2)$ are I' -maximal connected graphs with $(n, k) = (2c + 3, c + 2)$ ($c \geq 3$). The size of $G(3; 3, 3, 2, \dots, 2)$ is $2c + 5$ ($c \geq 3$). On the other hand, when $(n, k) = (2c + 3, c + 2)$ ($c \geq 3$), $\left\lceil \frac{(n-1)^2}{2\lfloor \frac{n}{2} \rfloor - 2} - \frac{(n-1)}{2} + \lfloor \frac{n}{2} \rfloor \right\rceil = 2c + 5$. Hence, when $(n, k) = (2c + 3, c + 2)$ ($c \geq 3$), $m(n, k)$ reaches the lower bound.

(iii) Suppose that $\lfloor \frac{n}{2} \rfloor > k - 1$.

Claim 3: If equality holds in (2), then $(n, k) = (3c, c + 2)$ ($c \geq 4$), $(3c - 1, c + 2)$ ($c \geq 5$), $(16, 8)$, $(18, 9)$, $(19, 9)$, $(20, 10)$ or $(22, 10)$.

Let $G \in M_n(k)$ and $S \subseteq E(G)$ be an I' -set of G such that $G - S$ has type $(n_1, n_2, \dots, n_c)$. We shall first prove that if equality holds in (2), then $n_c = 3$ and $|S| = c - 1$.

If $\lfloor \frac{n}{2} \rfloor > k - 1$, then $c \leq k - 2$. Note that a decrease of c by 1 must cause an increase of the lower bound in Lemma 4.1 by at least 1. Hence, if equality holds in (2), then $c = k - 2$, and so $n_c = k - |S| = c + 2 - |S| \leq (c + 2) - (c - 1)$, that is $n_c \leq 3$. Since $n_c \geq 3$, $n_c = 3$ and so $|S| = c - 1$.

Suppose that $n_1 = n_2 = \cdots = n_{c_1} = 2, n_{c_1+1} = n_{c_1+2} = \cdots = n_c = 3$. Let $c - c_1 = c_2$. Then $n = 3c_2 + 2c_1$, $c = c_1 + c_2$, $k = c_1 + c_2 + 2$ and $|S| = c_1 + c_2 - 1$, so $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{3c_2+2c_1}{2} \rfloor = c_1 + c_2 + \lfloor \frac{c_2}{2} \rfloor$. Since $\lfloor \frac{n}{2} \rfloor > k - 1$, we have $c_1 + c_2 + \lfloor \frac{c_2}{2} \rfloor > c_1 + c_2 + 1$, which implies $c_2 \geq 4$.

Since equality holds in (2), we have

$$\left\lceil \frac{(n-1)^2}{2(c_1+c_2)} - \frac{(n+1)}{2} + k \right\rceil = 3c_2 + c_1 + c - 1. \quad (4.4)$$

Substituting $n = 3c_2 + 2c_1$ in (4), we get

$$\left\lceil \frac{(3c_2+2c_1-1)^2}{2(c_1+c_2)} - \frac{(3c_2+2c_1+1)}{2} + c_1 + c_2 + 2 \right\rceil = 4c_2 + 2c_1 - 1, \quad (4.5)$$

or

$$\left\lceil \frac{1+c_1-c_1c_2-c_2}{2(c_1+c_2)} + 4c_2 + 2c_1 - 1 \right\rceil = 4c_2 + 2c_1 - 1.$$

Hence $-1 < \frac{1+c_1-c_1c_2-c_2}{2(c_1+c_2)} \leq 0$, and so $c_2 \geq 1$ and $c_2 < (1+3c_1)/(c_1-1)$, where $c_1 > 1$.

Therefore, if $c_1 > 1$, then $4 \leq c_2 < (1+3c_1)/(c_1-1)$, and so $c_1 < 5$.

We conclude that, if $c_1 = 0$ or 1 , then $c_2 \geq 4$; if $c_1 = 2$, then $4 \leq c_2 \leq 6$; if $c_1 = 3$ or 4 , then $c_2 = 4$. Hence, (n, k) must be $(3c, c+2)$ ($c \geq 4$), $(3c-1, c+2)$ ($c \geq 5$), $(16, 8)$, $(18, 9)$, $(19, 9)$, $(20, 10)$ or $(22, 10)$.

Claim 4: If $(n, k) = (3c, c+2)$ ($c \geq 4$), $(3c-1, c+2)$ ($c \geq 5$), $(16, 8)$, $(18, 9)$, $(19, 9)$, $(20, 10)$ or $(22, 10)$, equality in (2) holds.

For each (n, k) , if we can find a I' -maximal graph $G \in M_n(k)$ whose edge number reaches the lower bound, then we are done.

By Lemma 3.2, we know that the following graphs are I' -maximal.

$G(3; (3)_i)$ ($i \geq 3$) with $c_1 = 0$,

$G(3; (3)_i, 2)$ ($i \geq 3$) with $c_1 = 1$,

$G(3; (3)_i, 2, 2)$ ($3 \leq i \leq 5$) with $c_1 = 2$,

$G(3; (3)_3, 2, 2, 2)$ with $c_1 = 3$,

$G(3; (3)_3, 2, 2, 2, 2)$ with $c_1 = 4$.

For these graphs, $(n, k) = (3c, c + 2)$, where $c \geq 4$, $(3c - 1, c + 2)$, where $c \geq 5$, $(16, 8)$, $(18, 9)$, $(19, 9)$, $(20, 10)$ or $(22, 10)$, respectively. We can verify that their edge numbers reach the lower bound. $\square$

4.5 Maximum size of an I' -maximal graph

Let $M(n, k) = \max\{|E(G)| : G \in M_n(k) \text{ and is connected}\}$. In order to find $M(n, k)$, first we introduce four Lemmas.

LEMMA 5.1 For any positive integer n , we have

$$\left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil + \lfloor \sqrt{n} \rfloor = \lceil 2\sqrt{n} \rceil. \quad (4.6)$$

Proof: If n is the square of a positive integer, then (3.6) holds obviously. So we may assume that $(a - 1)^2 < n < a^2$, where $a > 1$ is an integer. Then $\lfloor \sqrt{n} \rfloor = a - 1$. We consider two cases.

Case 1: $(a - 1)^2 < n \leq a(a - 1)$. Then $(a - 1) < \frac{n}{a - 1} \leq a$ and so $\left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil = a$. Hence $\left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil + \lfloor \sqrt{n} \rfloor = a + a - 1 = 2a - 1$.

On the other hand, since $(a - 1)^2 < n \leq a(a - 1)$, $a - 1 < \sqrt{n} \leq \sqrt{a(a - 1)} < a - 1/2$, and so $\lceil 2\sqrt{n} \rceil = 2a - 1$.

Case 2: $a(a-1)+1 \leq n \leq a^2-1$. This can be treated similarly and the proof is omitted. $\square$

LEMMA 5.2 Assume that $G \in M_n(k)$ and there exists an I' -set $S \subseteq E(G)$ such that $|S| = r$. Let $n = m(k-r) + b$, where m is a positive integer, $0 \leq b < k-r$.

(i) If $b \neq 1$, then

$$|E(G)| \leq m \binom{k-r}{2} + \binom{b}{2} + r. \quad (4.7)$$

(ii) If $b = 1$, then

$$|E(G)| \leq (m-1) \binom{k-r}{2} + \binom{k-r-1}{2} + 1 + r. \quad (4.8)$$

Proof: (i) We consider the following three cases.

Case 1: Suppose that $k-r = 2$. By Theorem 3.1(i), $G \cong G(2; (2)_{c-1})$ or Δ_1 , where $c = \frac{n}{2}$. We can check that the equality in (3.7) holds.

Case 2: Suppose $G-S$ has type $(n_1, \dots, n_j, \dots, n_d, \dots, n_c)$, where $n_1 = \dots = n_{j-1} = 2$, $n_j > 2$, $n_d < k-r$ and $n_{d+1} = \dots = n_c = k-r$. Since $n_j \leq n_d$,

$$\sum_{i=1}^c \binom{n_i}{2} < \binom{n_j-1}{2} + \binom{n_d+1}{2} + \sum_{i \neq j,d} \binom{n_i}{2}. \quad (4.9)$$

Case 3: Suppose $G-S$ has type $(n_1, \dots, n_j, n_{j+1}, \dots, n_c)$, where $n_1 = \dots = n_{j-1} = 2$, $2 \leq n_j < k-r$ and $n_{j+1} = \dots = n_c = k-r$. If $j \geq 3$, we

have

$$\sum_{i=1}^c \binom{n_i}{2} < \binom{n_{j-1}+1}{2} + \binom{n_j+1}{2} + \sum_{i \neq 1, j-1, j} \binom{n_i}{2}. \quad (4.10)$$

If $j = 2$, since $b \neq 1$, then $n_j \neq k - r - 1$, and so $n_1 + n_2 \leq k - r$ and

$$\sum_{i=1}^c \binom{n_i}{2} < \binom{n_1+n_2}{2} + \sum_{i=3}^c \binom{n_i}{2}. \quad (4.11)$$

Note that if $G - S$ has type $(n_1, n_2, \dots, n_c)$, then $|E(G)| = \sum_{i=1}^c \binom{n_i}{2} + |S|$. Hence, by (3.9), (3.10) and (3.11), when $G - S$ has type $(b, k - r, \dots, k - r)$, $|E(G)|$ can reach its maximum and this completes the proof of (i).

(ii) If $b = 1$, by similar argument, when $G - S$ has type $(2, k - r - 1, k - r, \dots, k - r)$, $|E(G)|$ can reach its maximum. The result follows from Theorem 2.1. $\square$

LEMMA 5.3 For any given n and k , suppose that $n = m(k - r) + b$, where m is a positive integer, $0 \leq b < k - r$. let

$$f(r) = \begin{cases} m \binom{k-r}{2} + \binom{b}{2} + r, & \text{if } b \neq 1, \\ (m-1) \binom{k-r}{2} + \binom{k-r-1}{2} + 1 + r, & \text{otherwise.} \end{cases} \quad (4.12)$$

Then when $2 \leq r \leq k - r$, $f(r) \leq f(r - 1)$.

Proof: Let $g(r) = f(r) - r$. Note that, for any two positive integers n_1 and n_2 with $n_1 \leq n_2$, $\binom{n_1}{2} + \binom{n_2}{2} \leq \binom{n_1-1}{2} + \binom{n_2+1}{2} - 1$, and

the equality holds if and only if $n_1 = n_2$. Suppose $n = m_1(k - r + 1) + b_1$, $0 \leq b_1 < k - r + 1$. Now we consider four cases.

Case 1: $b \neq 1$ and $b_1 \neq 1$. Then

$$\begin{aligned}
 g(r) &= m \binom{k-r}{2} + \binom{b}{2} \\
 &\leq \binom{k-r+1}{2} + (m-1) \binom{k-r}{2} + \binom{b-1}{2} - 1 \\
 &\leq \binom{k-r+1}{2} + \binom{k-r+1}{2} + (m-2) \binom{k-r}{2} + \binom{b-2}{2} - 2 \\
 &\leq \dots \\
 &\leq m_1 \binom{k-r+1}{2} + \binom{b_1}{2} - m_1 \\
 &= g(r-1) - m_1.
 \end{aligned}$$

Similarly, we can show the results of other cases.

$$\begin{aligned}
 \text{Case 2: } b \neq 1 \text{ and } b_1 = 1. \text{ Then } g(r) &= m \binom{k-r}{2} + \binom{b}{2} \leq \\
 (m_1 - 1) \binom{k-r+1}{2} + \binom{k-r}{2} + 1 - (m_1 - 1) &= g(r-1) - (m_1 - 1).
 \end{aligned}$$

$$\begin{aligned}
 \text{Case 3: } b = 1 \text{ and } b_1 \neq 1. g(r) &= (m-1) \binom{k-r}{2} + \binom{k-r-1}{2} + 1 \leq \\
 m_1 \binom{k-r+1}{2} + \binom{b_1}{2} - m_1 &= g(r-1) - m_1.
 \end{aligned}$$

$$\begin{aligned}
 \text{Case 4: } b = 1 \text{ and } b_1 = 1. g(r) &= (m-1) \binom{k-r}{2} + \binom{k-r-1}{2} + 1 \leq \\
 (m_1 - 1) \binom{k-r+1}{2} + \binom{k-r}{2} + 1 - (m_1 - 1) &= g(r-1) - (m_1 - 1).
 \end{aligned}$$

Note that $n > (k - r) + 2$. If not, $n + r - 2 \leq k \leq n - 1$ and so $r \leq 1$, contrary to the assumption of the Lemma. Hence, $n > (k - r + 1) + 1$, and so $m_1 \geq 1$ and if $m_1 = 1$, then $b_1 \neq 1$. For Cases 1 and 3, $f(r) = g(r) + r \leq g(r-1) + r - m_1 = f(r-1) + 1 - m_1$ and so $f(r-1) - f(r) \geq m_1 - 1 \geq 0$. For Cases 2 and 4, since $b_1 = 1$, $m_1 \geq 2$. Therefore, $f(r) = g(r) + r \leq g(r-1) + r - (m_1 - 1) = f(r-1) + 2 - m_1$ and so $f(r-1) - f(r) \geq m_1 - 2 \geq 0$. $\square$

LEMMA 5.4 Let n, k be two given positive integers such that $\lceil 2\sqrt{n} \rceil - 1 \leq k \leq n - 1$.

- (i) There is a unique integer r such that $1 \leq r \leq \lfloor \sqrt{n} \rfloor - 1$ and $\lceil \frac{n}{r+1} \rceil + r \leq k \leq \lceil \frac{n}{r} \rceil + r - 2$.
- (ii) If r satisfies (i), then $n = r(k - r) + b$, $2 \leq b \leq k - r$.

Proof: (i) Let $T_r = \{k | \lceil \frac{n}{r+1} \rceil + r \leq k \leq \lceil \frac{n}{r} \rceil + r - 2\}$. Then, for any i, j such that $i \neq j$ and $1 \leq i, j \leq \lfloor \sqrt{n} \rfloor - 1$, $T_i \cap T_j = \emptyset$ and $T_1 \cup T_2 \cup \dots \cup T_{\lfloor \sqrt{n} \rfloor - 1} = \{\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil + \lfloor \sqrt{n} \rfloor - 1, \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil + \lfloor \sqrt{n} \rfloor, \dots, n - 1\} = \{\lceil 2\sqrt{n} \rceil - 1, \lceil 2\sqrt{n} \rceil, \dots, n - 1\}$.

- (ii) Since $\lceil \frac{n}{r+1} \rceil + r \leq k \leq \lceil \frac{n}{r} \rceil + r - 2$ ($1 \leq r \leq \lfloor \sqrt{n} \rfloor - 1$), $r(k - r) + r < n \leq (k - r)(r + 1)$ and so $r(k - r) + 2 \leq n \leq (k - r)(r + 1)$. $\square$

THEOREM 5.1 Let n and k are two positive integers such that there exists a connected graph $G \in M_n(k)$. Suppose $\lceil \frac{n}{r+1} \rceil + r \leq k \leq \lceil \frac{n}{r} \rceil + r - 2$, where $1 \leq r \leq \lfloor \sqrt{n} \rfloor - 1$. Then

$$M(n, k) = r \binom{k - r}{2} + \binom{n - r(k - r)}{2} + r.$$

Proof: Let $G \in M_n(k)$ and S be an I' -set of G . Note that, since G is connected, $k \geq \lceil 2\sqrt{n} \rceil - 1$ (see [3]). Hence, by Lemma 5.3, $k \geq \lceil 2\sqrt{n} \rceil - 1 = \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil + \lfloor \sqrt{n} \rfloor - 1$. By Lemmas 5.2 and 5.3, the smaller $|S|$ is, the larger the

upper bound of $|E(G)|$ becomes. When $|S| = r$, $G - S$ has at most $r + 1$ components and so $k \geq \lceil \frac{n}{r+1} \rceil + r$, and when $|S| = r - 1$, $k \geq \lceil \frac{n}{r} \rceil + r - 1$. Hence, for given n and k , if $\lceil \frac{n}{r+1} \rceil + r \leq k \leq \lceil \frac{n}{r} \rceil + r - 2$, then $|S| \geq r$ and so $|E(G)| \leq f(|S|) \leq f(r)$. By Lemma 5.4(ii), $n = r(k - r) + b$, where $2 \leq b \leq k - r$. Hence, by Lemma 5.2,

$$|E(G)| \leq r \binom{k - r}{2} + \binom{b}{2} + r = r \binom{k - r}{2} + \binom{n - r(k - r)}{2} + r.$$

We shall show that this upper bound can be attained.

For given n and k such that $\lceil \frac{n}{r+1} \rceil + r \leq k \leq \lceil \frac{n}{r} \rceil + r - 2$ ($1 \leq r \leq \lfloor \sqrt{n} \rfloor - 1$), we construct graph $G(n, k)$ as follows:

$$G(n, k) = G(k - r; (k - r)_{r-1}, n - r(k - r)).$$

By Lemma 5.4(ii), $n - r(k - r) \neq 1$. So by Lemma 3.2, each $G(n, k)$ defined above is an I' -maximal graph. It is straightforward to check that $|E(G(n, k))| = r \binom{k - r}{2} + \binom{n - r(k - r)}{2} + r$, and so this concludes the proof of Theorem 5.1. $\square$

Chapter 5

Spanning trails that join given edges in 3-edge-connected graphs

For given edges e_1, e_2 in $E(G)$, a spanning trail of G with e_1 as the first edge and e_2 as the last edge is called a spanning (e_1, e_2) -trail. In this chapter, we consider best possible degree conditions to assure the existence of these trails for every pair of edges in a 3-edge-connected graph G . The main results of this chapter will be published in [26].

5.1 Introduction

The graphs in this note are finite and loopless. For terms not defined here, see Bondy and Murty [7]. Let G be a graph and let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G , then we use G/H for $G/E(H)$. The **line graph** of G , denoted by $L(G)$, has vertex-set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

Given edges $e_1, e_2 \in E(G)$, a trail T of G with e_1 as the first edge and e_2 as the last edge is called an (e_1, e_2) -**trail**. An (e_1, e_2) -trail T is a **spanning** (e_1, e_2) -trail if every vertex of G is either an internal vertex of T or both the origin and the terminus of T . Lesniak and Williamson [28] showed that if G has a spanning (e_1, e_2) -trail, then $L(G)$ has a hamilton (e_1, e_2) -path. Note that if $G = C_4$, the 4-cycle, and if e_1 and e_2 are two nonadjacent edges in C_4 , then C_4 has an (e_1, e_2) -trail that is spanning in C_4 but $L(C_4)$ does not have a hamilton (e_1, e_2) -path. This explains why a spanning (e_1, e_2) -trail is defined in the way above.

It has been shown ([24], Theorem 3) that the problem to determine if a graph G has a spanning (e_1, e_2) -trail, for given $e_1, e_2 \in E(G)$, is NP-complete. We list some other prior results below.

From the proof of Lemma 6 in [40], we easily know that the following theorem holds.

THEOREM A (Zhan [40]) If G is 4-edge-connected, then for any pair of distinct edges $e_1, e_2 \in E(G)$, G has a spanning (e_1, e_2) -trail.

A bond $X \subset E(G)$ is called **essential edge-cut** if each component of $G - X$ has an edge. It is clear that if $\{e_1, e_2\}$ is an essential edge-cut of G , then G cannot have a spanning (e_1, e_2) -trail.

Since that G is 4-edge-connected implies that G has 2 edge-disjoint spanning trees, (see [23]), Theorem B below improves Theorem A.

THEOREM B (Catlin and Lai [13], Theorem 4) If G has two edge-disjoint spanning trees, then for every pair of distinct edges $e_1, e_2 \in E(G)$, exactly one of the following holds:

- (i) G has a spanning (e_1, e_2) -trail.
- (ii) $\{e_1, e_2\}$ is an essential edge-cut of G .

THEOREM C (Lai and Zhang [24]) Let G be a 2-edge-connected simple graph with $n \geq 27$ vertices. If for every pair of nonadjacent vertices $u, v \in V(G)$,

$$\deg(u) + \deg(v) > \frac{2n}{3} - 2,$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, exactly one of the following holds:

- (i) G has a spanning (e_1, e_2) -trail.
- (ii) $\{e_1, e_2\}$ is an essential edge-cut of G .

THEOREM D (Lai and Zhang [24]) Let G be a 2-edge-connected simple graph with $n \geq 33$ vertices and without 3-cycles. If for every pair of nonadjacent vertices $u, v \in V(G)$,

$$\deg(u) + \deg(v) > \frac{n}{3},$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, exactly one of the following holds:

- (i) G has a spanning (e_1, e_2) -trail.
- (ii) $\{e_1, e_2\}$ is an essential edge-cut of G .

5.2 Main Results

In this note we consider 3-edge-connected graphs. In order to exclude essential edge-cut of size 2, a subgraph of the Petersen graph emerges. In the rest of this paper, we let P denote the Petersen graph and let $P - e$ denote the subgraph of P by deleting an edge.

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. This procedure is called **subdividing** e . For distinct edges $e_1, e_2 \in E(G)$, let $G(e_1, e_2)$ denote the graph obtained from G by subdividing e_1 and e_2 . Thus

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

The following lemmas 1 and 2 follow easily from the definitions of collapsibility (See next section) and spanning (e_1, e_2) -trails. Proofs can be found in [24].

LEMMA 1 (Lai and Zhang [24], Lemma 3) Let G be a graph and let G' denote the reduction of G . For vertices $u, v \in V(G)$, define u', v' to be the vertices in G' whose preimages contain u and v , respectively. (Note that even $u \neq v$, it is still possible that $u' = v'$). Then G has a spanning (u, v) -trail if and only if G' has a spanning (u', v') -trail.

LEMMA 2 (Lai and Zhang [24], Lemma 1) Let G be a graph with $e_1, e_2 \in E(G)$. G has a spanning (e_1, e_2) -trail if and only if either $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$ -trail, or both e_1 and e_2 are incident with a common vertex v such that $G(e_1, e_2) - v$ has a spanning $(v(e_1), v(e_2))$ -trail.

THEOREM 1 Let G be a simple 3-edge-connected graph with $n \geq 77$ vertices. If for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$,

$$\deg(u) + \deg(v) \geq \frac{n}{4} - 2, \quad (5.1)$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, one of the following holds:

- (i) G has a spanning (e_1, e_2) -trail.
- (ii) Equality holds in (1) for some vertices $u, v \in V(G)$ with $uv \notin E(G)$ and $G(e_1, e_2)$ can be contracted to $P - e$ with $v(e_1), v(e_2)$ being the two vertices of degree 2 in $P - e$.

THEOREM 2 Let G be a simple 3-edge-connected graph with $n \geq 117$ vertices and without 3-cycles. If for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$,

$$\deg(u) + \deg(v) \geq \frac{n}{8}, \quad (5.2)$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, one of the following holds:

- (i) G has a spanning (e_1, e_2) -trail.
- (ii) Equality holds in (2) for some vertices $u, v \in V(G)$ with $uv \notin E(G)$ and $G(e_1, e_2)$ can be contracted to $P - e$ with $\{v(e_1), v(e_2)\}$ being the only two vertices of degree 2 in $P - e$.

5.3 Collapsible Graphs and Reductions

In this section we summarize some mechanism for the proofs. Let G be a graph and let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the ends of each edge in X and deleting the resulting loops. If H is a subgraph of G , then we use G/H for $G/E(H)$.

Let $O(G)$ denote the set of vertices of odd degree in G . Let $R \subseteq V(G)$ be a subset with $|R|$ even. If G is connected and if $O(G) = \emptyset$, then G is **eulerian**. A graph is **supereulerian** if it has a spanning eulerian subgraph. An **R -subgraph** Γ of G satisfies that $G - E(\Gamma)$ is connected and that $O(\Gamma) = R$. A graph G is **collapsible** if for every $R \subseteq V(G)$ with $|R|$ even, G has an R -subgraph. Thus by definition, K_1 is collapsible, and cycles of length less than 4 are collapsible. In [10], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \dots, H_c$ such that $\cup_{i=1}^c V(H_i) = V(G)$. The **reduction** of G , denoted by G' , is the graph obtained from G by contracting each of the maximal collapsible subgraph $H_i, (1 \leq i \leq c)$, into a single vertex v_i . A graph G is **reduced** if it is the reduction of some graph. Each subgraph $H_i, (1 \leq i \leq c)$,

is called the **preimage** of the vertex v_i of G' . A vertex v in the reduction of G is **trivial** if the preimage in G under the contraction is a K_1 in G .

Let $a(G)$ denote the minimum number of edge-disjoint forests whose union equals G . Nash-Williams [30] showed that

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil.$$

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. It was shown (Catlin [9], Theorem 7) that if G is reduced, then $a(G) \leq 2$; and if $a(G) \leq 2$, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \quad (5.3)$$

THEOREM E (Catlin [10]) Let G be a graph.

- (i) (Theorem 5 of [10]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
- (ii) (Theorem 8 of [10]) If G is reduced, then G is simple and K_3 -free and for any $H \subseteq G$, either $H \in \{K_1, K_2\}$ or $|E(H)| \leq 2|V(H)| - 4$.
- (iii) (Theorems 2 and 7 of [10]) If $F(G) = 0$, then G is collapsible. If $F(G) = 1$, then G is collapsible if and only if G is 2-edge-connected.
- (iv) (Theorem 3 of [10]) Let H be a collapsible subgraph of G . Then G is collapsible if and only if G/H is collapsible, and G is supereulerian if and only if G/H is supereulerian.

THEOREM F (Catlin, Han and Lai [12], Theorem 1.3) If G is a connected graph with $F(G) \leq 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$, ($t \geq 1$).

THEOREM G (Chen [16], Theorem 1) If G is 3-edge-connected graph with at most 11 vertices, then either G is collapsible or G is the Petersen graph.

5.4 Associate Result

We need the following technique developed by Catlin [9].

THEOREM H Let G be a graph, let $wxyzw$ be a 4-cycle in G , and define the partition $\pi = \{w, y\} \cup \{x, z\}$. Denote G/π to be the graph obtained from $G - \{wx, xy, yz, zw\}$ by identifying w and y to form a single vertex u , by identifying x and z to form a single vertex v , and by adding an extra edge uv . Each of the following holds:

- (a) (Catlin [9], Corollary 1) If G/π is collapsible, then G is collapsible.
- (b) (Catlin [9], Corollary 2) If G/π is supereulerian, then G is supereulerian.
- (c) If G is reduced, then $F(G/\pi) = F(G) - 1$.

Proof of (c) of Theorem H Suppose $a(G/\pi) \leq 2$. Since G is reduced, we also have $a(G) \leq 2$. Apply (3) to G/π and G to get

$$F(G/\pi) = 2|V(G/\pi)| - |E(G/\pi)| - 2$$

and

$$F(G) = 2|V(G)| - |E(G)| - 2.$$

By the definition of G/π , we have $|E(G)| - 3 = |E(G/\pi)|$ and $|V(G)| - 2 = |V(G/\pi)|$. Combine these equalities to get $F(G/\pi) = F(G) - 1$.

Suppose $a(G/\pi) > 2$. Since $a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$, G/π has a non-trivial subgraph H_π satisfying $|E(H_\pi)| \geq 2|V(H_\pi)| - 1$, and hence $|E(H_\pi)| \geq 3$. Since G is reduced, (i) and (ii) of Theorem E imply that subgraph H of size at least 3 is also reduced and thus satisfies $|E(H)| \leq 2|V(H)| - 4$.

Case 1 $|V(H_\pi) \cap \{u, v\}| = k$ for some $k \in \{0, 1\}$. Define $H = G[E(H_\pi)]$. Then $|E(H)| = |E(H_\pi)| \geq 3$, and $|V(H_\pi)| = |V(H)| - k$ and so

$$|E(H)| = |E(H_\pi)| \geq 2|V(H_\pi)| - 1 = 2(|V(H)| - k) - 1 \geq 2|V(H)| - 3,$$

a contradiction.

Case 2 $\{u, v\} \subseteq V(H_\pi)$. Define $H = G[E(H_\pi - uv) \cup \{wx, xy, yz, zw\}]$. Then a similar contradiction arises: $|E(H)| = |E(H_\pi)| + 3 \geq 2|V(H_\pi)| + 2 = 2|V(H)| - 2$.

Hence $a(G/\pi) > 2$ can not hold. This proves (c) of Theorem H. $\square$

COROLLARY I Let G be a graph with a 4-cycle $wxyzw$ and let $u, v, G/\pi$ be defined as in Theorem H. Let v_1, v_2 be two vertices (not necessary distinct) in $V(G)$ and let v'_1, v'_2 be the corresponding vertices in G/π such that either $v_i = v'_i$, or $v_i \in \{w, y\}$ and $v'_i = u$, or $v_i \in \{x, z\}$ and $v'_i = v$. If G/π has a spanning (v'_1, v'_2) -trail, then G has a spanning (v_1, v_2) -trail.

Proof: If $v_1 = v_2$, then Corollary I follows from (b) of Theorem H. Hence we assume that $v_1 \neq v_2$. Let v be a vertex not in $V(G)$ and let $G' = G + v_1vv_2$. thus G has a spanning (v_1, v_2) -trail if and only if G' is supereulerian. Note also that G/π has a spanning (v'_1, v'_2) -trail implies that G'/π is supereulerian, and so by (b) of Theorem H, G' is supereulerian. Thus G has a spanning (v_1, v_2) -trail. $\square$

For integer $i \geq 1$, let $D_i(G) = \{v \in V(G) : \deg(v) = i\}$.

By Theorem 2.4 in [17], an immediate corollary is given as follows:

COROLLARY J If G is a 2-edge-connected reduced graph with $F(G) \leq 3$, $|D_2(G)| \leq 3$ and $|V(G)| \leq 11$, then G is the Petersen graph or one of the following graphs:

THEOREM K ([15], [[17], Theorem 3.2]) Let G be a connected simple graph with $|V(G)| \leq 13$ and $\delta(G) \geq 3$. Then either G is a supereulerian

Figure 5.1:

graph with 12 vertices and with an odd cycle, or the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{1,3}, P\}$.

THEOREM L (Chen [16], Lemma 1) Let G be a simple 2-edge-connected graph of order at most 7, with $\delta(G) \geq 2$ and $|D_2(G)| \leq 2$. Then G is collapsible.

THEOREM M (Catlin and Chen [11], Lemma 3) Let G be a simple with $\kappa'(G) \geq 2$, $|V(G)| \leq 8$ and $|D_2(G)| \leq 1$. Then G is collapsible.

LEMMA 3 Let G be a 2-edge-connected graph with at most 9 vertices with $2 \leq \delta(G) \leq \Delta(G) \leq 3$. If $|D_2(G)| \leq 1$, then G is collapsible.

Proof: By Theorem M, we shall consider only $|V(G)| = 9$. If $|D_2(G)| = 0$, then by Theorem K and by $\kappa'(G) \geq 2$, the reduction of G must be K_1 and so G is collapsible. Hence, we assume that $|V(G)| = 9$ and $|D_2(G)| = 1$.

Suppose that G is reduced. Since $|D_2(G)| = 1$ and $|D_3(G)| = 8$, $|E(G)| = 13$. By (3), $F(G) = 2|V(G)| - |E(G)| - 2 = 3$. By Corollary J, G must be one of those graphs in Figure 1, contrary to the assumption that $|V(G)| = 9$. Hence, G has at least one nontrivial subgraph which is collapsible. Let H be a maximal nontrivial collapsible subgraph of G . Since $|V(H)| \geq 3$, $|V(G/H)| \leq 7$. Since G/H is a simple graph with $\kappa'(G/H) \geq 2$ and $|D_2(G/H)| \leq 2$, by Theorem L, G/H is collapsible and so G is collapsible.

ble. $\square$

THEOREM 3 Suppose that G is a 2-edge-connected reduced graph with at most 10 vertices with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ and with $D_2(G) = \{v_1, v_2\}$. If G does not have a spanning (v_1, v_2) -trail, then v_1v_2 is not an edge in G and $G + v_1v_2$ is the Petersen graph P .

Proof: We shall show first that $v_1v_2 \notin E(G)$. By contradiction, we assume that $v_1v_2 \in E(G)$. Then by Lemma 3, $G/\{v_1v_2\}$ is collapsible and so it is supereulerian too. Since $v_1v_2 \in E(G)$, any spanning eulerian trail in $G/\{v_1v_2\}$ induces a spanning (v_1, v_2) -trail in G , contrary to the assumption that G does not have a spanning (v_1, v_2) -trail.

Consider the graph $G + v_1v_2$. If $G + v_1v_2$ is not supereulerian, then it is not collapsible either. Since G is 2-edge-connected, by Theorem K, $G + v_1v_2$ must be isomorphic to the Petersen graph P . Hence, we only need to prove that $G + v_1v_2$ is not supereulerian.

We may assume that $G + v_1v_2$ is supereulerian. Since G does not have a spanning (v_1, v_2) -trail, G must be supereulerian. Since $\Delta(G) \leq 3$, G is hamiltonian.

Claim 1 If G does not have a spanning (v_1, v_2) -trail, then every edge-cut X with $|X| = 2$ must consist of edges incident with a vertex in $D_2(G)$.

Proof: By contradiction, we assume that G has an edge-cut X with $|X| = 2$ such that each component of $G - X$ has at least 2 vertices. Let G_1 and G_2 be the two components of $G - X$ with $|V(G_1)| \leq |V(G_2)|$. By the following conditions: G is hamiltonian, $|D_2(G)| = 2$, $v_1v_2 \notin E(G)$ and G has no 3-cycles (by (ii) of Theorem E), we can get $|V(G_1)| \geq 5$. Since $|V(G)| \leq 10$, $|V(G_1)| = |V(G_2)| = 5$. Hence, G must be the following graph:

Figure 5.2:

Obviously, G has a spanning (v_1, v_2) -trail, a contradiction. This proves Claim 1.

Claim 2 G has a 4-cycle.

Proof: By contradiction and by (ii) of Theorem E, we may assume that

$$G \text{ has no cycles of length less than 5.} \quad (5.4)$$

Let $m = |V(G)| \leq 10$ and let $C = u_1 u_2 \dots u_m u_1$ be a hamilton cycle of G . Without loss of generality, we assume that $u_1 = v_1$. Since $v_1 v_2 \notin E(G)$,

$$u_2, u_m \in D_3(G). \quad (5.5)$$

Let $u_i, i \notin \{1, m-1\}$, be a neighbor of u_m . By (4), $m-4 \geq i \geq 4$, and so $m \geq 8$. Note that since $|D_2(G)| = 2$, $m \neq 9$

If $m = 8$, then by (4), $u_5 = v_2$ and $u_4 u_8 \in E(G)$. Since $|D_2(G)| = 2$, $u_3 \in D_3(G)$, and so by (4), $u_3 u_7 \in E(G)$. Thus a 4-cycle $u_3 u_4 u_8 u_7 u_3$ exists, contrary to (4).

Hence $m = 10$. Since $u_1 \in D_2(G)$ and since $|D_2(G)| = 2$, not both u_5 and u_7 are in $D_2(G)$, and so we may assume that $u_5 \in D_3(G)$. By (4), either $u_5u_9 \in E(G)$ or $u_5u_{10} \in E(G)$.

Assume first that $u_5u_9 \in E(G)$. Note that by (4), the neighbors of u_{10} must be in $\{u_1, u_4, u_5, u_6, u_9\}$ and so G must have a cycle of length less than 5, contrary to (4).

Thus $u_5u_{10} \in E(G)$. If $u_9 \in D_2(G)$, then $u_4 \in D_3(G)$ and so by (4), $u_4u_8 \in E(G)$. Since $u_9 \in D_2(G)$ now, $u_3 \in D_3(G)$ and so u_3 must be adjacent to u_7 , forming a 4-cycle $u_3u_7u_8u_4u_3$, contrary to (4). So we assume that $u_9 \in D_3(G)$. Since $u_5u_{10} \in E(G)$ and by (4), $u_9u_3 \in E(G)$. Since $|D_2(G)| = 2$ and $u_1 \in D_2(G)$, either u_4 or u_8 is in $D_3(G)$. Since by (4), the neighbors of u_4 are in $\{u_3, u_5, u_8\}$, and the neighbors of u_8 are in $\{u_7, u_9, u_4\}$, G must have a cycle of length less than 5, contrary to (4). This proves Claim 2.

By Claim 2, G has a 4-cycle. Let $C = xyzwx$ be a 4-cycle of G such that

$$|V(C)| \text{ contains as many vertices in } D_2(G) \text{ as possible.} \quad (5.6)$$

Define G/π , u, v as in Theorem H.

Claim 3: $\kappa'(G/\pi) \geq 2$.

Proof: Since $\kappa'(G) \geq 2$, $\kappa'(G/\pi) \leq 1$ if and only if uv is a cut-edge in G/π , if and only if $E(C)$ is an edge-cut. By contradiction, we assume that $E(C)$ is an edge-cut and G_1 and G_2 are the two sides of $G - E(C)$ with $w, y \in V(G_1)$ and with $x, z \in V(G_2)$. Assume also that $|V(G_1)| \leq |V(G_2)|$. Thus $|V(G_1)| \leq 5$.

If $V(G_1) = \{w, y\}$, then G cannot be hamiltonian, a contradiction. If G_1 has a vertex $v' \in D_3(G)$. Since $|V(G_1)| \leq 5$ and since G has no K_3 ((ii)

of Theorem E), v' must be adjacent to some vertex in G_2 , contrary to the assumption of $\kappa'(G/\pi) \leq 1$. Thus $V(G_1) - \{w, y\} \subseteq D_2(G)$. But then since $v_1 v_2 \notin E(G)$, $|V(G_1) - \{w, y\}| = 1$. We may assume that $V(G_1) = \{w, y, v_1\}$. If $v_2 \in \{w, y\}$, then G must have a 4-cycle containing both vertices of degree 2 in G , contrary to the choice of C ; if $v_2 \notin \{w, y\}$, then G must have a 4-cycle containing at least one vertex of $\{v_1, v_2\}$, contrary to the choice of C . This proves Claim 3.

Since $|D_2(G)| = 2$, $3|V(G)| = 2(|E(G)| + 1)$. Hence it follows by (3) that $F(G) = |E(G)| - |V(G)|$. Since $|V(G)| \leq 10$, $V(G/\pi) \leq 8$ and $F(G) \leq 4$. Let G' denote the reduction of G/π . By (c) of Theorem H, $F(G') \leq F(G/\pi) \leq 3$. Note that G/π has two vertices with degree 2. If G' has at most three vertices with degree 2, by Corollary J, we know that G' must be one of those graphs in Figure 1, namely, G' has three vertices with degree 2. By inspection, G' must have a spanning (v'_1, v'_2) -trail for any $v'_1, v'_2 \in D_2(G')$, and by Lemma 1 and Corollary I, G has a spanning (v_1, v_2) -trail, contrary to the assumption that G does not have any spanning (v_1, v_2) -trails. If G' has four vertices with degree 2, then $G' = C_4$ and G/π has two collapsible subgraphs K_3 . And so G/π has four vertices with degree 2, a contradiction. Hence, $G + v_1 v_2$ can not be supereulerian. This completes the proof of Theorem 3. $\square$

5.5 The Proofs of Main Results

LEMMA 4 If G is reduced, then

$$2F(G) + 4 = \sum_{i=1}^{|V(G)|-1} (4-i)|D_i(G)|. \quad (5.7)$$

Proof: This follows by (3) and by counting the incidences of G . $\square$

Proof of Theorem 1: Let G'' be the reduction of $G(e_1, e_2)$. Suppose first that $F(G'') \leq 2$. Since $\kappa'(G) \geq 3$, it follows that $\kappa'(G'') \geq 2$ and so by

(iv) of Theorem E, G'' is collapsible if $F(G'') \leq 1$. It follows by Lemma 1 that $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$ -trail. Hence by Lemma 2, G has a spanning (e_1, e_2) -trail. If $F(G'') = 2$, then by Theorem F, $G'' = K_{2,t}$ for some $t \geq 2$. Since $\kappa'(G) \geq 3$, the two edges incident with each of $v(e_1), v(e_2)$ are the only edge-cuts of size 2 in $G(e_1, e_2)$ and so $t = 2$. But then it would follow that $\{e_1, e_2\}$ is an edge-cut of G , contrary to $\kappa'(G) \geq 3$. Hence we may assume that $F(G'') \geq 3$.

By Lemma 4 and since $D_1(G'') = \emptyset$, $D_2(G'') \subseteq \{v(e_1), v(e_2)\}$, we have

$$|D_3(G'')| \geq 6, \quad (5.8)$$

with equality if and only if $D_2(G'') = \{v(e_1), v(e_2)\}$, $\Delta(G'') \leq 4$ and $F(G'') = 3$. By definition, G'' is reduced. By (ii) of Theorem E, G'' is simple and has no 3-cycles. It follows that if $D_3(G'')$ has three trivial vertices, then two of them must be nonadjacent, and so by (1), $n/4 - 2 \leq 6$, which implies that $n \leq 32$, contrary to the assumption that $n \geq 77$. Thus we have

$$D_3(G'') \text{ has at least 4 nontrivial vertices.} \quad (5.9)$$

Claim 1: For any nontrivial vertex $v \in D_i(G'')$, $3 \leq i \leq 6$, let H_v denote the preimage of the v . If $\cup_{i=3}^6 D_i(G'')$ has one vertex ($\neq v$) whose preimage in G includes vertex v_0 with $3 \leq \deg(v_0) \leq 6$, then $|V(H_v)| \geq \frac{n}{4} - (1 + \deg(v_0))$. In particular, if $D_3(G'')$ has one trivial vertex, then $|V(H_v)| \geq \frac{n}{4} - 4$.

Proof: For any nontrivial vertex $v \in D_i(G'')$, $3 \leq i \leq 6$, one can choose $w \in V(H_v)$ so that w is incident with at most three edges in $E(G'')$ and is not adjacent to v_0 in G . Thus by (1),

$$|V(H_v)| \geq \deg(w) - 3 + 1 = \deg(w) + \deg(v_0) - \deg(v_0) - 2 \geq \frac{n}{4} - 10.$$

Since $n \geq 77$, we may choose $w' \in V(H_v)$ such that w' is not incident with any edge in $E(G'')$ and is not adjacent to v_0 in G . Hence

$$|V(H_v)| \geq \deg(w') + 1 = \deg(w') + \deg(v_0) - \deg(v_0) + 1 \geq \frac{n}{4} - \deg(v_0) - 1.$$

Claim 2: If $D_3(G'')$ has one trivial vertex v_0 , then $\cup_{i=3}^6 D_i(G'')$ has at most four nontrivial vertices.

Otherwise, by Claim 1, $(n-1) \geq 5(\frac{n}{4}-4) = n + \frac{n}{4} - 20$. and so $n \leq 76$.

Claim 3: Suppose that all vertices in $D_3(G'')$ are nontrivial and $|D_3(G'')|$ equals to m . For any $v \in \cup_{i=3}^6 D_i(G'')$, let H_v denote the preimage of the v .

(i) If $m \geq 6$, $v \in D_3(G'') \cup D_4(G'')$, then H_v has a vertex that incident with no edges in G'' .

(ii) If $m \geq 7$, $v \in D_5(G'')$, then H_v has a vertex that incident with no edges in G'' .

(iii) If $m \geq 8$, $v \in D_6(G'')$, then H_v has a vertex that incident with no edges in G'' .

Proof: (i) Suppose that $v \in D_3(G'')$.

Let v_i , $(1 \leq i \leq m)$ denote the vertices in $D_3(G'')$, and let H_i , $(1 \leq i \leq m)$ denote the preimages of the v_i 's, respectively.

Without loss of generality, we may assume that $v = v_1$. If each vertex in H_1 is incident with at least one edge in G'' , then since $v_1 \in D_3(G'')$, it follows that $H_1 = K_3$ and so by Claim 1, we have

$$n-3 \geq \sum_{i=2}^m |V(H_i)| \geq 5(\frac{n}{4}-4) = \frac{5n}{4} - 20.$$

It follows that $n \leq 68$.

Suppose that $v \in D_4(G'')$.

If each vertex in H_v is incident with at least one edge in G'' , then since $v \in D_4(G'')$, it follows that we can find $v_0 \in V(H_v)$ such that $\deg(v_0) \leq 4$.

By Claim 1,

$$n - 1 \geq 6\left(\frac{n}{4} - 5\right) = \frac{6n}{4} - 30.$$

It follows that $n \leq 58$.

(ii) Suppose that $m \geq 7$, $v \in D_5(G'')$.

If each vertex in H_v is incident with at least one edge in G'' , then since $v \in D_5(G'')$, it follows that we can find $v_0 \in V(H_v)$ such that $\deg(v_0) \leq 5$.

By Claim 1,

$$n - 1 \geq 7\left(\frac{n}{4} - 6\right) = \frac{7n}{4} - 42.$$

It follows that $n \leq 55$.

(iii) Suppose that $m \geq 8$, $v \in D_6(G'')$.

If each vertex in H_v is incident with at least one edge in G'' , then since $v \in D_6(G'')$, it follows that we can find $v_0 \in V(H_v)$ such that $\deg(v_0) \leq 6$.

By Claim 1,

$$n - 1 \geq 8\left(\frac{n}{4} - 7\right) = 2n - 56.$$

It follows that $n \leq 55$.

By Claim 2, we only need to consider the following two cases.

Case 1: $D_3(G'')$ has two trivial vertices, say v'_0 and v''_0 , and four nontrivial vertices.

By Lemma 4, when $i \geq 5$, $|D_i(G'')| = 0$. By Claim 2, all vertices in $D_4(G'')$ are trivial. If $|D_4(G'')| \neq 0$, let $u \in D_4(G'')$. Since G'' is simple and has no 3-cycle, two vertices in $\{u, v'_0, v''_0\}$ must be nonadjacent, and so by (1), $n/4 - 2 \leq 7$, namely, $n \leq 36$. Hence, $|D_4(G'')| = 0$. Since $|D_3(G'')| = 6$, by lemma 4, $|D_2(G'')| = 2$, namely, $D_2(G'') = \{v(e_1), v(e_2)\}$.

Hence, $|V(G'')| = |D_2(G'')| + |D_3(G'')| = 8$. By Theorem 3, G'' has a spanning $(v(e_1), v(e_2))$ -trail. By Lemmas 1 and 2, G has a spanning (e_1, e_2) -trail.

Case 2: All vertices in $D_3(G'')$ are nontrivial.

Let v_i , $(1 \leq i \leq m)$ denote the vertices in $D_3(G'')$ and H_i , $(1 \leq i \leq m)$ denote the preimages of the v_i 's, respectively. By Claim 3, for each i , $(1 \leq i \leq m)$, H_i has a vertex w_i that is incident with no edges in G'' . Then the w_i 's form an independent set in G .

If $m \geq 9$, then by (1),

$$2n \geq 2 \sum_{i=1}^m |V(H_i)| \geq 2 \sum_{i=1}^m (\deg(w_i) + 1) \geq \frac{9n}{4} \quad (5.10)$$

a contradiction. Note that $m \geq 6$. We consider the following six subcases.

Subcase 1: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 6$ and when $i \geq 5$, $|D_i(G'')| = 0$.

We first show that $|D_4(G'')| \leq 2$. Otherwise, for each $v \in D_3(G'') \cup D_4(G'')$, by Claim 3(i), H_v , the preimage of vertex v , has a vertex w that is incident with no edges in G'' . Then the w 's form an independent set in G . Hence,

$$2n \geq 2 \sum_{v \in D_3(G'') \cup D_4(G'')} |V(H_v)| \geq 2 \sum_{v \in D_3(G'') \cup D_4(G'')} (\deg(w) + 1) \geq \frac{9n}{4},$$

a contradiction. It follows that $|V(G'')| \leq 10$ and $D_2(G'') = \{v(e_1), v(e_2)\}$. By Corollary J, graph G'' does not exist.

Subcase 2: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 7$, $|D_5(G'')| = 1$ and when $i \geq 6$, $|D_i(G'')| = 0$.

Since $|D_3(G'')| = 7$ and $|D_5(G'')| = 1$, by Claim 3 (i), we similarly show that $|D_4(G'')| = 0$. Since $D_2(G'') = \{v(e_1), v(e_2)\}$, by Corollary J, graph G'' does not exist.

Subcase 3: $F(G'') = 4$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

Since $|D_3(G'')| = 8$, by Claim 3 (i), we have that $|D_4(G'')| = 0$. Note that $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = 10$. Since $m = 8$, (10) becomes

$$2n \geq 2 \sum_{i=1}^8 |V(H_i)| \geq 2 \sum_{i=1}^8 (\deg(w_i) + 1) \geq \frac{8n}{4} = 2n, \quad (5.11)$$

and so equality holds in (1) for some pairs of nonadjacent vertices in $V(G)$. By Lemma 2, if G does not have a spanning (e_1, e_2) -trail, then $G(e_1, e_2)$ does not have a spanning $(v(e_1), v(e_2))$ -trail either, and so, by Lemma 1 Theorem 3, we know that (ii) of Theorem 1 must hold.

Subcase 4: $F(G'') = 3$, $|D_2(G'')| = 1$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

By Claim 3 (i), we can show that $|D_4(G'')| = 0$. Since $|D_2(G'')| = 1$, by Corollary J, graph G'' does not exist.

Subcase 5: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$, $|D_5(G'')| = 2$ and when $i \geq 6$, $|D_i(G'')| = 0$.

For each $v \in D_3(G'') \cup D_5(G'')$, by Claim 3 (i) and (ii), H_v , the preimage of vertex v , has a vertex w that is incident with no edges in G'' . Then the w 's form an independent set in G . Hence,

$$2n \geq 2 \sum_{v \in D_3(G'') \cup D_5(G'')} |V(H_v)| \geq 2 \sum_{v \in D_3(G'') \cup D_5(G'')} (\deg(w) + 1) \geq \frac{10n}{4},$$

a contradiction.

Subcase 6: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$, $|D_5(G'')| = 0$, $|D_6(G'')| = 1$ and when $i \geq 7$, $|D_i(G'')| = 0$.

Similarly, we may show that graph G'' does not exist. This completes the proof of Theorem 1. $\square$

Proof of Theorem 2: Let G'' denote the reduction of $G(e_1, e_2)$. Similar to the arguments in the proof of Theorem 1, we may assume that $F(G'') \geq 3$, and so by (4),

$$|D_3(G'')| \geq 6, \quad (5.12)$$

where equality holds if and only if $D_2(G'') = \{v(e_1), v(e_2)\}$, $\Delta(G'') \leq 4$ and $F(G'') = 3$. Since G'' is reduced, by (ii) of Theorem E, G'' is simple and has no 3-cycles. Thus if $D_3(G'')$ has 3 trivial vertices, then two of them must be nonadjacent and so by (2), $n/8 \leq 6$, contrary to $n \geq 117$. This implies that

$$D_3(G'') \text{ has at least 4 nontrivial vertices.} \quad (5.13)$$

Claim 1 If v_H is a nontrivial vertices in $\cup_{i=3}^6 D_i(G'')$ whose preimage in G is H , then

$$|V(H)| \geq \frac{n}{8}.$$

Proof: For any vertex $v \in V(G)$, let $N(v)$ denote its neighborhood in G . Since G is K_3 -free, there are a pair of nonadjacent vertices $u, v \in V(H)$. Without loss of generality, we assume that $\deg(v) \leq \deg(u)$. By (2), we have that $\deg(u) \geq n/16$. Since $n \geq 117$, $\deg(u) \geq 8$. Hence, we can find two vertices $u', v' \in N(u) \cap V(H)$ such that $\deg(v') \leq \deg(u')$ (So $\deg(u') \geq n/16$). Since G is K_3 -free, $u'v' \notin E(G)$ and $|N(u) \cap N(u')| = 0$. Thus by (2), $|V(H)| \geq \deg(u) + \deg(u') - 6 \geq n/16 + n/16 - 6 = n/8 - 6$. As $n \geq 117$, we may choose u, v nonadjacent in G and not incident with any edges in G'' . Since $\deg(u) \geq 8$, we may choose u', v' which are not incident with any edges

in G'' . Hence, $|V(H)| \geq \deg(u) + \deg(u') \geq n/8$.

Claim 2 (i) $\cup_{i=3}^6 D_i(G'')$ has at most two trivial vertices.

(ii) Let v_H be a nontrivial vertex in $\cup_{i=3}^6 D_i(G'')$ whose preimage in G is H . If $\cup_{i=3}^6 D_i(G'')$ has one trivial vertex with degree j , ($3 \leq j \leq 6$), then $|V(H)| \geq \frac{n}{4} - 2j$.

Proof: (i) If $\cup_{i=3}^6 D_i(G'')$ has three trivial vertices, then two of them must be nonadjacent and by (2), $n/8 \leq 12$, namely, $n \leq 96$, contrary to the assumption that $n \geq 117$.

(ii) From the proof of Claim 1, we know that H contains two vertices $u, u' \in V(G)$ which are not adjacent to any edge in G'' . Since G is K_3 -free and by (2),

$$|V(H)| \geq \deg(u) + \deg(u') \geq 2\left(\frac{n}{8} - j\right) = \frac{n}{4} - 2j.$$

Claim 3 If $D_3(G'')$ has one trivial vertex, then $\cup_{i=3}^6 D_i(G'')$ has at most four nontrivial vertices.

Otherwise, by Claim 2, $n - 1 \geq 5(n/4 - 6)$, namely, $n \leq 116$.

Hence, similar to the proof of Theorem 2, we need to consider the following two cases.

Case 1: $D_3(G'')$ has two trivial vertices, say v'_0 and v''_0 , and four nontrivial vertices.

By Lemma 4, when $i \geq 5$, $|D_i(G'')| = 0$. By Claim 3, all vertices in $D_4(G'')$ are trivial. By Claim 2 (i), $|D_4(G'')| = 0$. Since $|D_3(G'')| = 6$, by lemma 4, $|D_2(G'')| = 2$, namely, $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = |D_2(G'')| + |D_3(G'')| = 8$. By Theorem 3, G'' has a spanning $(v(e_1), v(e_2))$ -

trail. By Lemmas 1 and 2, G has a spanning (e_1, e_2) -trail.

Case 2: All vertices in $D_3(G'')$ are nontrivial.

Suppose that $|D_3(G'')| = m$. Let $v_i (1 \leq i \leq m)$ denote the vertices in $D_3(G'')$, and let $H_i (1 \leq i \leq m)$ denote the preimages of the v_i 's, respectively. By Claim 1, if $m \geq 9$, then $n \geq \sum_{i=1}^m |V(H_i)| \geq \frac{9n}{8}$, a contradiction. Note that $m \geq 6$. We consider the following subcases.

Subcase 1: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 6$ and when $i \geq 5$, $|D_i(G'')| = 0$.

If there exists a trivial vertex in $D_4(G'')$, by Claim 2 (ii), $n-1 \geq 6(n/4-8)$, namely, $n \leq 94$, a contradiction. By Claim 1, $D_4(G'')$ has at most two non-trivial vertices. It follows that $|V(G'')| \leq 10$ and $D_2(G'') = \{v(e_1), v(e_2)\}$. By Corollary J, graph G'' does not exist.

Subcase 2: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 7$, $|D_5(G'')| = 1$ and when $i \geq 6$, $|D_i(G'')| = 0$.

If there exists a trivial vertex in $D_4(G'') \cup D_5(G'')$, by Claim 2 (ii), $n-1 \geq 7(n/4-10)$, namely, $n \leq 92$, a contradiction. Hence, by Claim 1, $|D_4(G'')| = 0$ and $D_5(G'')$ has a nontrivial vertex. Since $D_2(G'') = \{v(e_1), v(e_2)\}$, by Corollary J, graph G'' does not exist.

Subcase 3: $F(G'') = 4$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

By Claim 1 and Claim 2 (ii), we easily show that $|D_4(G'')| = 0$. Note that $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = 10$. Let $v_i (1 \leq i \leq 8)$ denote the vertices in $D_3(G'')$ and $H_i (1 \leq i \leq 8)$ denote the preimages of

the v_i 's, respectively. By Claim 1 and by $n \geq 117$, for each i , ($1 \leq i \leq m$), H_i contains an edge $u_i u'_i \in E(G)$ that is not adjacent to any edge in G'' . Since G is K_3 -free and by (2), we have

$$n \geq \sum_{i=1}^8 |V(H_i)| \geq \sum_{i=1}^8 (\deg(u_i) + \deg(u'_i)) \geq \frac{8n}{8} = n, \quad (5.14)$$

and so for each i , ($1 \leq i \leq 8$),

$$|V(H_i)| = \frac{n}{8}. \quad (5.15)$$

Hence, equality holds in (2) for some pairs of nonadjacent vertices in $V(G)$. By Lemmas 1, 2 and Theorem 3, if G does not have a spanning (e_1, e_2) -trail, then (ii) of Theorem 2 must hold.

Subcase 4: $F(G'') = 3$, $|D_2(G'')| = 1$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

Similarly, we may show that $|D_4(G'')| = 0$. Since $|D_2(G'')| = 1$, by Corollary J, graph G'' does not exist. This completes the proof of Theorem 2. $\square$

(ii) of Theorem 1 and (ii) Theorem 2 can hold. In other words, for every pair of edges e_1, e_2 in a simple 3-edge-connected graph G with $V(G) \geq 77$, the best possible degree conditions to assure the existence of (e_1, e_2) -trail are: for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$, $\deg(u) + \deg(v) > n/4 - 2$; if G is a 3-edge-connected triangle-free simple graph with $V(G) \geq 117$, then the best possible degree conditions are: for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$, $\deg(u) + \deg(v) > n/8$.

Let u_0, v_0 be two vertices of degree 2 and $u_1 u_0 u_2, v_1 v_0 v_2$ be two paths with length 2 in $P - e$. Graph G is obtained by replacing two paths $u_1 u_0 u_2, v_1 v_0 v_2$ in $P - e$ by two edges e_1, e_2 , respectively. Let $s \geq 8$ be an integer and $n = 16s$. Now we construct graph G_1, G_2 by replacing each vertex in G by a complete subgraph K_{2s} , a complete bipartite subgraph $K_{s,s}$, respectively.

Then for every pair of nonadjacent $u, v \in V(G_1)$, $\deg(u) + \deg(v) \geq n/4 - 2$; for every pair of nonadjacent $u, v \in V(G_2)$, $\deg(u) + \deg(v) \geq n/8$. Note that complete subgraph and complete bipartite subgraph are collapsible. We easily check that for edges $e_1, e_2 \in E(G_1)$, (ii) of Theorem 1 holds; for edges $e_1, e_2 \in E(G_2)$, (ii) of Theorem 2 holds.

Appendix A

Glossary of Terms

arboricity: minimum number of forests covering the edges of G

arc: directed edge

arborescence: a directed forest in which every vertex has out-degree at most one

binomial coefficient $\binom{n}{k}$: the number of ways to choose a subset of size k from an n -element set, equal to $n!/[k!(n-k)!]$

bipartite graph: a graph whose vertices can be covered by two independent sets

bond: a minimal edge cut

Brooks' Theorem: $\chi(G) \leq \Delta(G)$ for connected graphs, except for complete graphs and odd cycles

choice number: choosability

choosability: minimum k such that G is k -choosable

k -choosable: for all lists of size k assigned to vertices of G , there is a proper coloring that selects a color for each vertex from its list

chromatic number $\chi(G)$: minimum number of colors graph G takes to color its vertices so that adjacent vertices have different colors.

critical graph: used with respect to many graph properties, indicating that the deletion of any vertex (or edge, depending on context) destroys the property

cycle matroid $M(G)$: the matroid whose circuits are the cycle of G

Eulerian trail: a trail containing every edge

Five Color Theorem: Every planar graph is 5-colorable

forest: a disjoint union of trees, an acyclic graph

Four Color Theorem: Every planar graph is 4-colorable

fundamental cycle: for a spanning tree, a cycle formed by adding an edge to it

greedy algorithm: a fast non-backtracking algorithm to find a good feasible solution by iteratively making a heuristically good choice

greedy coloring: with respect to some vertex ordering, color each vertex with the least indexed color not already appearing among the neighbors of the vertex being colored

Hadwiger conjecture: a k -chromatic graph has a subgraph contractible to K_k

Hajós conjecture: a k -chromatic graph contains a subdivision of K_k (false beyond $k = 5$)

list chromatic number: choosability

matroid: a hereditary system satisfying any of many equivalent and useful properties

maximal (instance) satisfying P : no proper superset or supergraph also has property P

maximum (instance) satisfying P : no larger set or graph also has property P

minimal (instance) satisfying P : no proper subset or subgraph also has property P

minimum (instance) satisfying P : no smaller set or graph also has property P

NP: the class of problems solvable by nondeterministic polynomial algorithms

NP-complete: NP-hard and in NP

NP-hard: polynomial time reducible from any problem in NP

orientation: an assignment of order to each of the edge pairs in an undirected graph, making it a directed graph

planar: embeddable in the plane

plane graph: a particular planar embedding of a planar graph

torus: the (orientable) surface with one handle

trail: a walk in which no edge appears more than once

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APPROVAL OF EXAMINING COMMITTEE

 Hong-Jian Lai Ph.D., Chair

.....
 William F. Klostermeyer Ph.D.

.....
 Jerzy Wojciechowski Ph.D.

.....
 Harumi Hattori Ph.D.

.....
 Cun-Quan Zhang Ph.D.

December 4, 1998

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 Date