2011

Group Colorability and Hamiltonian Properties of Graphs

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Group Colorability and Hamiltonian Properties of Graphs

Hao Li

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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2011

Keywords: Group coloring, Multigraph, Cartesian product, Hamiltonian,
Collapsible, Reduction, P3D, subdivision

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ABSTRACT

Group Colorability and Hamiltonian Properties of Graphs

Hao Li

The research of my dissertation was motivated by the conjecture of Thomassen that every 4-connected line graph is hamiltonian and by the conjecture of Matthews and Sumner that every 4-connected claw-free graph is hamiltonian. Towards the hamiltonian line graph problem, we proved that every 3-edge-connected, essentially 4-edge-connected graph G has a spanning eulerian subgraph, if for every pair of adjacent vertices u and v , $d_G(u) + d_G(v) \geq 9$. A straight forward corollary is that every 4-connected, essentially 6-connected line graph with minimum degree at least 7 is hamiltonian.

We also investigate graphs G such that the line graph $L(G)$ is hamiltonian connected when $L(G)$ is 4-connected. Ryjáček and Vrána recently further conjectured that every 4-connected line graph is hamiltonian-connected. In 2001, Kriesell proved that every 4-connected line graph of a claw free graph is hamiltonian connected. Recently, Lai et al showed that every 4-connected line graph of a quasi claw free graph is hamiltonian connected, and that every 4-connected line graph of an almost claw free graph is hamiltonian connected. In 2009, Broersma and Vumer discovered the P_3 -dominating (P3D) graphs as a superfamily that properly contains all quasi claw free graphs, and in particular, all claw-free graphs. Here we prove that every 4-connected line graph of a P3D graph is hamiltonian connected, which extends several former results in this area.

R. Gould [15] asked what natural graph properties of G and H are sufficient to imply that the product of G and H is hamiltonian. we first investigate the sufficient and necessary conditions for $G \times H$ being hamiltonian or traceable when G is a hamiltonian graph and H is a tree. Then we further investigate sufficient and necessary conditions for $G \times H$ being hamiltonian connected, or edge-pancyclic, or pan-connected.

The problem of group colorings of graphs is also investigated in this dissertation. Group coloring was first introduced by Jeager et al. [21]. They introduced a concept of

group connectivity as a generalization of nowhere-zero flows. They also introduced group coloring as a dual concept to group connectivity. Prior research on group chromatic number was restricted to simple graphs, and considered only Abelian groups in the definition of $\chi_g(G)$. The behavior of group coloring for multigraphs is different to that of simple graphs. Thus we extend the definition of group coloring by considering general groups (both Abelian groups and non-Abelian groups), and investigate the properties of χ_g for multigraphs by proving an analogue to Brooks' Theorem.

Acknowledgements

First and foremost, I am most indebted to my supervisor, Dr. Hong-Jian Lai, for his continued encouragement and support over these last few years. It is a pleasure to work under his supervision. Without him, this paper could not have come about.

I would like to take this opportunity to thank Dr. Mei Lu at Tsinghua University, who brought me to the research field of graph theory.

I would also like to thank my other committee members: Dr. John Goldwasser, Dr. K. Subramani, Dr. Jerzy Wojciechowski, and Dr. Cun-Quan Zhang, for their help during my studies.

And finally, I would like to thank the Department of Mathematics and Eberly College of Arts and Sciences at West Virginia University for providing me with an excellent study environment and support during my study as a graduate student.

DEDICATION

To

my father Jiawang Li , my mother Xiaomei Zhou

and

my wife Xin Gao

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Chapter 1

Preliminaries

1.1 Notation and Terminology

We use [4] for terminology and notations not defined here. All graphs in this thesis are finite and undirected. A graph with at most one vertex is called a **trivial** graph. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G , respectively. Two vertices u, v are adjacent if $uv \in E(G)$. For a vertex $v \in V(G)$, the number of edges incident with v is called the **degree** of v in G , and is denoted by $d_G(v)$ or $d(v)$. We use $N_G(v)$ and $E_G(v)$ to denote the set of vertices adjacent to v and the set of edges incident with v , respectively. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of G , respectively. For each integer $i = 0, 1, 2, \dots$, we let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$. A graph is called a k -**regular** graph if every vertex is of degree k .

A **walk** (of length k) in G is a non-empty alternating sequence $v_0e_0v_1e_1\dots e_{k-1}v_k$ of vertices and edges in G such that $e_i = v_iv_{i+1}$ for all $0 \leq i \leq k-1$. If the edges in a walk are distinct, it defines a **trail** in G . If the vertices in a walk are distinct, it defines a **path** in G . We use P_k to denote a path with k vertices. A graph is **connected** if any two of its vertices are linked by a path. A maximal connected subgraph of a graph is called a **component**, and let $c(G)$ denote the number of components of G . If any two

vertices are adjacent in G , then G is called a **complete graph**. A complete graph with n vertices is denoted by K_n . A nontrivial 2-regular connected graph is called a **circuit**, and a k -circuit, denoted by C_k , is a circuit of k vertices. A **star**, denoted by $K_{1,j}$ ($j \geq 0$), is a graph with $j + 1$ vertices and j edges, in which one vertex is adjacent with other j vertices. The center of a $K_{1,j}$ is the vertex of degree j . We use $H \subseteq G$ to denote the fact that H is a subgraph of G .

For a connected graph G , a **vertex cut** of G is a subset V' of $V(G)$ such that $G - V'$ is disconnected. A k -**vertex cut** is a vertex cut of k elements. A complete graph has no vertex cut; in fact, the only graphs that do not have vertex cuts are those that contain complete graphs as spanning subgraphs. G is called k -**connected** (for some positive integer k) if G has no i -vertex cut for all $1 \leq i \leq k - 1$. The greatest integer k such that G is k -connected is the **connectivity** $\kappa(G)$ of G . In particular, $\kappa(K_n) = n - 1$. Similarly, an **edge cut** of G is a subset E' of $E(G)$ such that $G - E'$ is disconnected. G is called l -**edge-connected** (for some positive integer l) if G has no i -edge cut for all $1 \leq i \leq l - 1$. The greatest integer l such that G is l -edge-connected is the **edge-connectivity** $\kappa'(G)$ of G . An edge cut X of G is **peripheral** if for some $v \in V(G)$, $X = E_G(v)$; and is **essential** if each side of $G - X$ has an edge. A graph G is **essentially k -edge-connected** if $|E(G)| \geq k + 1$ and if for every $E_0 \subseteq E(G)$ with $|E_0| < k$, $G - E_0$ has exactly one component H with $E(H) \neq \emptyset$.

Let A, B be the subsets of $V(G)$ with $A \cap B = \emptyset$. An edge $xy \in E(G)$ is called an A - B edge if $x \in A$ and $y \in B$. The set of all A - B edges in G is denoted by $[A, B]_G$. Instead of $[\{u\}, B]_G$ and $[\{u\}, \{v\}]_G$, we simply write $[u, B]_G$ and $[u, v]_G$. The number of edges in $[A, B]_G$ is denoted by $||A, B||_G$. For two vertex-disjoint subgraphs H_1, H_2 of G , we write $[H_1, H_2]_G$ for $[V(H_1), V(H_2)]_G$. For simplicity, we use $[A, B]$ and $||A, B||$ for $[A, B]_G$ and $||A, B||_G$, respectively.

For $V_1 \subseteq V(G)$, let $G[V_1]$ denote the subgraph induced by V_1 . Let H be a subgraph of G . For $E_1 \subseteq E(H)$, $E_2 \cap E(H) = \emptyset$, $V_1 \subseteq V(H)$, and $V_2 \cap V(H) = \emptyset$, the subgraphs induced by $E(H) - E_1$, $E(H) \cup E_2$, $V(H) - V_1$, $V(H) \cup V_2$ are denoted by $H - E_1$, $H + E_2$, $H - V_1$, $H + V_2$, respectively. For $E_1 = \{e_1\}$, $E_2 = \{e_2\}$, $V_1 = \{v_1\}$, $V_2 = \{v_2\}$, we abbreviate them to $H - e_1$, $H + e_2$, $H - v_1$, and $H + v_2$.

Let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. For convenience, we use G/e for $G/\{e\}$ and $G/\emptyset = G$; and if H is a subgraph of G , we write G/H for $G/E(H)$. Note that even if G is a simple graph, contracting some edges of G may result in a graph with multiple edges. If K is a connected subgraph of G , and if v_K is the vertex in G/K onto which K is contracted, then K is called the **pre-image** of v_K , and is denoted by $PI(v_K)$. A vertex v in a contraction of G is a **nontrivial vertex** if $PI(v)$ has at least one edge.

A subgraph H of G is **spanning** if $V(H) = V(G)$, and a spanning path (or circuit) of G is sometimes called a **hamiltonian path (or circuit)** of G . A graph is **traceable** if it contains a hamiltonian path, and **hamiltonian** if it contains a hamiltonian circuit. For a pair of distinct vertices u and v in G , a (u, v) -path is a path linking u and v . The **distance** between u and v , denoted by $dist_G(u, v)$, is the length of a shortest (u, v) -path in G . G is **hamiltonian connected** if for every pair of distinct vertices u and v , there is a hamiltonian (u, v) -path. G is **edge-pancyclic**, if every edge lies on a circuit of length l for all $l = 3, 4, \dots, |V(G)|$. G is **pan-connected** if for every pair of distinct vertices u and v , there is a (u, v) -path of length l for all $l = dist_G(u, v), \dots, n - 1$.

The **line graph** of a graph G , denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. For a graph G , an induced subgraph H isomorphic to $K_{1,3}$ is called a **claw** of G , and the only vertex of degree 3 of H is the **center** of the claw. A graph G is **claw free** if it does not contain a claw. Beineke ([2]) and Robertson ([33] and [17]) showed that every line graph is also a claw-free graph.

Proposition 1.1.1 *Let G be a nontrivial simple graph. Then $L(G)$ is complete if and only if G is a K_3 or a $K_{1,n}$ for an integer $n \geq 1$.*

Proof. The line graph $L(G)$ is complete if and only if any two edges in G are incident. If $|E(G)| = 1$, $G = K_2 = K_{1,1}$; if $|E(G)| = 2$, $G = P_2 = K_{1,2}$; if $|E(G)| = 3$, $G = K_3$ or $G = K_{1,3}$; if $|E(G)| \geq 4$, $G = K_{1,n}$. $\square$

A subgraph H of a graph G is **dominating** if $E(G - V(H)) = \emptyset$. Let $O(G)$ denote the

set of odd degree vertices of G . A graph G is **eulerian** if $O(G) = \emptyset$ and G is connected. Note that the graph K_1 is also eulerian.

A well known relationship between dominating Eulerian subgraphs in G and hamiltonian circuits in $L(G)$ is given by Harary and Nash-Williams.

Theorem 1.1.2 (*Harary and Nash-Williams, [18]*) *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

The following follows by a similar argument in the proof of Theorem 1.1.2.

Proposition 1.1.3 *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'') -trail.*

Let G be a connected, essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The core of the graph G , denoted by G_0 , is obtained by applying the following two operations repeatedly:

Operation 1. Delete a vertex of degree 1.

Operation 2. For each vertex y of degree 2 with $\Theta(y) = \{xy, yz\}$, contract exactly one edge in $\Theta(y)$. This amounts to deleting the vertex y for the path xyz in G with $d_G(y) = 2$ and adding the new edge xz .

Shao [38] proved Theorem 1.1.4 (a)-(c). In a similar way as Theorem 1.1.4 (c), one can prove Theorem 1.1.4 (d).

Theorem 1.1.4 (*Shao, [38]*) *Let G be a connected and essentially 3-edge-connected graph and let G_0 be the core of graph G , then each of the following holds:*

(a) G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$;

(b) G_0 is well defined;

(c) If G_0 has a spanning eulerian subgraph, then G has a dominating eulerian subgraph;

(d) If G_0 has a dominating eulerian subgraph containing all nontrivial vertices and both endvertices of each nontrivial edges, then G has a dominating eulerian subgraph.

1.2 Catlin's Reduction Method

In [7] Catlin defined **collapsible** graphs. Given a subset R of $V(G)$, a subgraph Γ of G is called an R -subgraph if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is collapsible if for any even subset R of $V(G)$, G has an R -subgraph. In particular, K_1 is collapsible.

For collapsible graphs, there is another equivalent definition: a graph G is **collapsible** if for any even subset X of $V(G)$, G has a spanning connected subgraph R_X of G such that $O(R_X) = X$.

Catlin [7] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \dots, H_k$ such that $\bigcup_{i=1}^k V(H_i) = V(G)$. The c -**reduction** of G is the graph obtained from G by successively contracting $H_1, H_2, \dots, H_k$. This contraction is called the c -contraction. If H_i is c -contracted to v_i , then H_i is called the c -**preimage** of v_i , and denoted by $\mathcal{CP}(v_i)$. Note that if G has an $O(G)$ -subgraph Γ , then $G - E(\Gamma)$ is a spanning eulerian subgraph of G . Therefore, every collapsible graph is supereulerian.

Theorem 1.2.1 *Let G be a graph and let H be a collapsible subgraph of G . Let v_H denote the vertex onto which H is contracted in G/H . Each of the following holds.*

- (i) (Catlin, Theorem 3 of [7]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction of G is K_1 .
- (ii) (Catlin, Theorem 8 of [7]) 2-circuits and 3-circuits are collapsible.
- (iii) ([27]) If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v) -trail.

- (iv) ([27]) For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v) -trail, then G has a spanning (u, v) -trail.
- (v) (Catlin, Theorem 5 of [7]) Any subgraph of a reduced graph is reduced.
- (vi) ([7]) If G is collapsible, and if $e \in E(G)$, then G/e is also collapsible.
- (vii) (Lemma 1 of [8]) $K_{3,3} - e$ is collapsible.
- (viii) (Catlin, Theorem 3 of [7]) Let H be a collapsible subgraph of G . Then G is supereulerian if and only if G/H is supereulerian.

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G . We assume that $\tau(K_1) = \infty$. Catlin showed the relationship between $\tau(G)$ and the edge-connectivity $\kappa'(G)$.

Theorem 1.2.2 *Let G be a graph, H be a subgraph of G , and $k > 0$ be an integer.*

- (i) (Catlin, Theorem 5.1 of [8]) $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.
- (ii) If $\tau(H) \geq k$ and if $\tau(G/H) \geq k$, then $\tau(G) \geq k$.

Theorem 1.2.3 (Catlin and Lai, Theorem 4 of [12]) *Let G be a graph with $\tau(G) \geq 2$ and let $e', e'' \in E(G)$. Then G has a spanning (e', e'') -trail if and only if $\{e', e''\}$ is not an essential edge cut of G .*

We define $F(G)$ be the minimum number of additional edges that must be added to G such that the resulting graph has two edge-disjoint spanning trees.

Theorem 1.2.4 *Let G be a graph.*

- (i) (Catlin, Han and Lai, Lemma 2.3 of [10]) *If for any $H \subset G$ with $|V(H)| < |V(G)|$, H is reduced, and if $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.*
- (ii) (Catlin, Theorem 7 of [7]) *If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$.*
- (iii) (Catlin, Han and Lai, Theorem 1.3 of [10]) *Let G be a connected graph and t an integer. If $F(G) \leq 2$, then G is collapsible if and only if the c -reduction of G is not a member in $\{K_2\} \cup \{K_{2,t} : t \geq 1\}$.*

(iv) (Catlin, Theorem 2 of [7]) If $F(G) = 0$, then G is collapsible and hence G is supereulerian.

Theorem 1.2.5 (Lai, [24]) Let G be a 2-connected graph with $\delta(G) \geq 3$. If every edge of G is in a circuit of length at most 4, then G is collapsible.

It is known that all complete graphs of order at least 3 are collapsible and any circuit of length at least 4 is not collapsible. If G contains a 4-circuit $C = uvzvu$ with a partition $\pi = \langle \{u, z\}, \{v, w\} \rangle$, then we can follow Catlin [6] and define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying u and z to form a new vertex x , identifying v and w to form a new vertex y , and adding an edge $e_\pi = xy$.

Theorem 1.2.6 (Catlin, [6]) Let G be a graph containing a 4-circuit C and let $G/\pi(C)$ be defined as above. Each of the following holds:

- (i) If $G/\pi(C)$ is collapsible, then G is collapsible;
- (ii) If $G/\pi(C)$ has a spanning eulerian subgraph, then G has a spanning eulerian subgraph, i.e., if $G/\pi(C)$ is supereulerian, then G is supereulerian.

1.3 Main Results

In the consequent chapters, we will present the following main results.

(1) R. Gould [15] asked what natural graph properties of G and H are sufficient to imply that the product of G and H is hamiltonian. we first investigate the sufficient and necessary conditions for $G \times H$ being hamiltonian or traceable when G is a hamiltonian graph and H is a tree. Then we further investigate sufficient and necessary conditions for $G \times H$ being hamiltonian connected, or edge-pancyclic, or pan-connected for more generic graphs G and H .

(2) The supereulerian graph problem, raised by Boesch, Suffel, and Tindell [3], is to determine when a graph has a spanning eulerian subgraph. Pulleyblank showed that such

a decision problem, even when restricted to planar graphs, is NP-complete. Jaeger [20] and Catlin [7] independently showed that every 4-edge-connected graph has a spanning eulerian subgraph. In 1992, Zhan [44] showed that every 3-edge-connected, essentially 7-edge-connected graph has a spanning eulerian subgraph. It has been conjectured in 1995 that every 3-edge-connected, essentially 5-edge-connected graph has a spanning eulerian subgraph. Here we show that if G is a 3-edge-connected, essentially 4-edge-connected graph and if for every pair of adjacent vertices u and v , $d_G(u) + d_G(v) \geq 9$, then G has a spanning eulerian subgraph.

(3) Mathews and Sumner [31], and Tommasen [40] conjectured that every 4-connected line graph is hamiltonian. We investigate graphs G such that the line graph $L(G)$ is hamiltonian connected when $L(G)$ is 4-connected. Ryjáček and Vrána [37] recently further conjectured that every 4-connected line graph is hamiltonian-connected. In 2001, Kriesell [23] proved that every 4-connected line graph of a claw free graph is hamiltonian connected. Recently, Lai et al [27] showed that every 4-connected line graph of a quasi claw free graph is hamiltonian connected, and that every 4-connected line graph of an almost claw free graph is hamiltonian connected. In 2009, Broersma and Vumer [5] discovered the P_3 -dominating (P3D) graphs as a superfamily that properly contains all quasi claw free graphs, and in particular, all claw-free graphs. Here we prove that every 4-connected line graph of a P3D graph is hamiltonian connected, which extends several former results in this area.

(4) Group coloring was first introduced by Jeager et al. [21], who introduced group coloring of graphs. Prior research on group chromatic number was restricted to simple graphs, and considered only Abelian groups in the definition of $\chi_g(G)$. The behavior of group coloring for multigraphs is different to that of simple graphs. Thus we extend the definition of group coloring by considering general groups (both Abelian groups and non-Abelian groups), and investigate the properties of χ_g for multigraphs by proving an analogue to Brooks' Theorem.

Chapter 2

Hamiltonian properties in Cartesian product

2.1 Introduction

All graphs considered in this chapter are finite loopless simple graphs.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The **Cartesian product** of G_1 and G_2 , denoted by $G_1 \times G_2$, is the graph with vertex set $V_1 \times V_2$ such that two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2 \in V_1$ with $y_1 y_2 \in E_2$, or $y_1 = y_2 \in V_2$ with $x_1 x_2 \in E_1$. It follows from the definition that for any $(x, y) \in V(G_1 \times G_2)$,

$$d_{G_1 \times G_2}(x, y) = d_{G_1}(x) + d_{G_2}(y).$$

For any $y \in V_2$, define $(G_1)_y$ to be the graph with vertex set $(V_1)_y = \{(x, y) | x \in V_1\}$ and edge set $(E_1)_y = \{(x_1, y)(x_2, y) | x_1 x_2 \in E_1\}$. Similarly, For any $x \in V_1$, define $(G_2)_x$ to be the graph with vertex set $(V_2)_x = \{(x, y) | y \in V_2\}$ and edge set $(E_2)_x = \{(x, y_1)(x, y_2) | y_1 y_2 \in E_2\}$.

Note that $(G_1)_y$ is isomorphic to graph G_1 , for any $y \in V_2$; and that $(G_2)_x$ is isomorphic

to graph G_2 for any $x \in V_1$. It is clear that

$$\begin{aligned} (V_1)_y \cap (V_1)_{y'} &= \emptyset \quad (E_1)_y \cap (E_1)_{y'} = \emptyset \quad \text{for } y \neq y'; \\ (V_2)_x \cap (V_2)_{x'} &= \emptyset \quad (E_2)_x \cap (E_2)_{x'} = \emptyset \quad \text{for } x \neq x'; \\ (V_1)_y \cap (V_2)_x &= \{(x, y)\} \quad \text{for } x \in V_1; \quad y \in V_2; \\ E(G_1 \times G_2) &= (\cup_{y \in V_2} (E_1)_y) \cup (\cup_{x \in V_1} (E_2)_x); \\ V(G_1 \times G_2) &= \cup_{y \in V_2} (V_1)_y = \cup_{x \in V_1} (V_2)_x. \end{aligned}$$

Define a graph to be **weakly edge-pancyclic** if every edge e lies on a circuit of length l for all $l = k, k+1, \dots, |V(G)|$, where k is the length of a minimum circuit containing e .

Our research is motivated by the following open problem posed in [15]:

Problem 2.1.1 (*Problem 6 of [15]*) *What natural graph properties of G and H are sufficient to imply that the product of G and H is hamiltonian?*

In this chapter, we first investigate the sufficient and necessary conditions for $G \times H$ being hamiltonian or traceable when G is a hamiltonian graph and H is a tree. Define $\mathfrak{F}$ to be the set of such graph $H^{(n)}$ which is obtained by identifying every degree 1 vertex of H with the center of a $K_{1,n}$, where H is a subdivision of $K_{1,3}$. Specially, $K_{1,3}^{(n)} \in \mathfrak{F}$. Our main theorems are as follows:

Theorem 2.1.2 *Let G_1 be a hamiltonian graph with order n and let T be a tree with maximum degree Δ . Then the graph $G = G_1 \times T$ is hamiltonian if and only if $\Delta \leq n$.*

Theorem 2.1.3 *Let G_1 be a hamiltonian graph with order n and let T be a tree with maximum degree Δ . Then the graph $G = G_1 \times T$ is traceable if and only if either $\Delta \leq n$, or $\Delta = n+1$ and T contains no member of $\mathfrak{F}$ as a subgraph.*

If we strengthen the condition of G in $G \times H$, then the Cartesian product will have better hamiltonian properties.

Theorem 2.1.4 *Let G be a Hamiltonian connected graph of order n and let T be a tree. Then $G \times T$ is Hamiltonian connected if and only if $\Delta(T) \leq n - 1$.*

Theorem 2.1.5 *Let G be an edge-pancyclic graph of order n and let T be a tree. Then $G \times T$ is weakly edge-pancyclic if and only if $\Delta(T) \leq n - 1$.*

Theorem 2.1.6 *Let G be a pan-connected graph of order n and let T be a tree. Then $G \times T$ is pan-connected if and only if $\Delta(T) \leq n - 1$.*

In Section 2.2, we will prove some lemmas. Then from Section 2.3 to Section 2.7, we will prove Theorem 2.1.2, Theorem 2.1.3, Theorem 2.1.4, Theorem 2.1.5 and Theorem 2.1.6, respectively.

2.2 Preliminaries

In this section, we will show some lemmas that will be used in the proofs of our main results. The following two theorems are well known as "toughness conditions".

Theorem 2.2.1 ([4]) *Let S be a set of vertices of a hamiltonian graph G . Then $c(G - S) \leq |S|$.*

Theorem 2.2.2 ([4]) *Let S be a set of vertices of a graph G . If G is traceable, then $c(G - S) \leq |S| + 1$.*

Lemma 2.2.3 *Let $K_{1,m}$ be a star ($m \geq 0$) and let C_n be a circuit ($n \geq 3$). If $m \leq n$, then the graph $C_n \times K_{1,m}$ is Hamiltonian.*

Proof. We denote $G = C_n \times K_{1,m}$. If $m = 0$, then the result holds trivially since G is isomorphic to C_n . So we will assume that $m \geq 1$ in the following proof. Assume that

$V(C_n) = \{x_1, x_2, \dots, x_n\}$ and $V(K_{1,m}) = \{y_0, y_1, \dots, y_m\}$, where $d(y_0) = m$ and $d(y_i) = 1$ for $1 \leq i \leq m$. Note that $(G_1)_{y_i} \cong C_n$ and that $(G_1)_{y_i} = (x_1, y_i)(x_2, y_i) \cdots (x_n, y_i)(x_1, y_i)$ for $i = 0, 1, \dots, m$. Then

$$\begin{aligned} C &= ((E_1)_{y_1} - \{(x_1, y_1)(x_2, y_1)\}) \cup \cdots \cup ((E_1)_{y_i} - \{(x_i, y_i)(x_{i+1}, y_i)\}) \cup \cdots \\ &\cup ((E_1)_{y_m} - \{(x_m, y_m)(x_{m+1}, y_m)\}) \\ &\cup ((E_1)_{y_0} - \{(x_1, y_0)(x_2, y_0), (x_2, y_0)(x_3, y_0), \dots, (x_m, y_0)(x_{m+1}, y_0)\}) \\ &\cup \{(x_1, y_0)(x_1, y_1), (x_2, y_0)(x_2, y_1), \dots, (x_i, y_0)(x_i, y_i), (x_{i+1}, y_0)(x_{i+1}, y_i), \dots, \\ &\quad (x_m, y_0)(x_m, y_m), (x_{m+1}, y_0)(x_{m+1}, y_m)\} \end{aligned}$$

is a Hamiltonian circuit of G , where all indexes are taken modulo n . $\square$

Note 1 By the proof of Lemma 1, $|(E_1)_{y_0} \cap E(C)| = \emptyset$ if $n = m$. If $n > m$, then $(x_i, y_0)(x_{i+1}, y_0) \in E(C)$ for $m+1 \leq i \leq n$, that is, $|(E_1)_{y_0} \cap E(C)| = n - m$. On the other hand, we can construct another Hamiltonian circuit C' such that $(x_i, y_0)(x_{i+1}, y_0) \in E(C')$ for $m+j \leq i \leq n+j$ by $(G_1)_{y_i} \cong C_n$ for $i = 0, 1, \dots, m$, where $1 \leq j \leq n$ and all indexes are taken modulo n .

Lemma 2.2.4 *Let C_n be a circuit of n vertices ($n \geq 3$) and let $K_{1,n+1}$ be a star with center y_0 . Then in the graph $C_n \times K_{1,n+1}$, for any pair of adjacent vertices $u_1, u_2 \in V(C_n)$ and any pair of distinct degree one vertices $v_1, v_2 \in V(K_{1,n+1})$, there is a Hamiltonian path P of $C_n \times K_{1,n+1}$ satisfying:*

- (i) (u_1, v_1) and (u_2, v_2) are endpoints of P ;
- (ii) For $i = 1$ or 2 , $P \setminus (G_1)_{v_i}$ is a Hamiltonian path of $C_n \times (K_{1,n+1} - v_i)$ with endpoints (u_{3-i}, v_{3-i}) and (u_{3-i}, y_0) .

Proof. Let $\{u_1, u_2, v_1, v_2\}$ be given vertices such that $u_1 u_2 \in E(C_n)$ and $v_1, v_2 \in D_1(K_{1,n+1})$. Let $V(C_n) = \{x_1, x_2, \dots, x_n\}$ and $V(K_{1,n+1}) = \{y_0, y_1, \dots, y_{n+1}\}$, where $d(y_0) = n+1$ and $d(y_i) = 1$ for $1 \leq i \leq n+1$. By the symmetry of C_n and $K_{1,n+1}$, we

can assume that $u_1 = x_1$, $u_2 = x_2$, $v_1 = y_1$ and $v_2 = y_{n+1}$. Then

$$\begin{aligned} P &= ((E_1)_{y_1} - \{(x_1, y_1)(x_2, y_1)\}) \cup \cdots \cup ((E_1)_{y_i} - \{(x_i, y_i)(x_{i+1}, y_i)\}) \cup \cdots \\ &\cup ((E_1)_{y_{n+1}} - \{(x_1, y_{n+1})(x_2, y_{n+1})\}) \\ &\cup \{(x_2, y_0)(x_2, y_1), (x_2, y_0)(x_2, y_2), \dots, (x_i, y_0)(x_i, y_{i-1}), (x_i, y_0)(x_i, y_i), \dots, \\ &\quad (x_n, y_0)(x_n, y_{n-1}), (x_n, y_0)(x_n, y_n), (x_1, y_0)(x_1, y_n), (x_1, y_0)(x_1, y_{n+1}), \} \end{aligned}$$

is a required Hamiltonian path of $C_n \times K_{1,n+1}$. $\square$

Let $K_{1,3}^{(n)}$ be the graph obtained by identifying every degree 1 vertex of a $K_{1,3}$ with the center of a $K_{1,n}$. Note that $\Delta(K_{1,3}^{(n)}) = n + 1$.

Lemma 2.2.5 *Let C_n be a circuit ($n \geq 3$). Then $C_n \times K_{1,3}^{(n)}$ is not traceable.*

Proof. We denote $G = C_n \times K_{1,3}^{(n)}$. By way of contradiction, we may assume that P is a Hamiltonian path of G . Referring to the structure of $K_{1,3}^{(n)}$, we may assume that T_1 , T_2 and T_3 are the three copies of $K_{1,n}$ whose centers were identified with the degree one vertices of $K_{1,3}$. Let y_i be the center of T_i ($i = 1, 2, 3$). Note that the T_i can be also viewed as a sub-tree of $K_{1,3}^{(n)}$, and then $C_n \times T_i$ can be viewed as a subgraph of G ($i = 1, 2, 3$).

Suppose $C_n \times T_1$ contains no endpoints of P . Assume that $\{z_1, z_2, \dots, z_n\}$ are the degree one vertices in T_1 . Then $(G_1)_{z_1}, (G_1)_{z_2}, \dots, (G_1)_{z_n}$ and $C_n \times (K_{1,3}^{(n)} - T_1)$ are the $n + 1$ components of $G - (G_1)_{y_1}$. Since P is a spanning subgraph of G , $c(P - (G_1)_{y_1}) \geq c(G - (G_1)_{y_1}) = n + 1$. By Theorem 2.2.2, $c(P - (G_1)_{y_1}) \leq |(V_1)_{y_1}| + 1 = n + 1$. Thus, $c(P - (G_1)_{y_1}) = n + 1$. Note that every $(G_1)_{z_i}$ ($1 \leq i \leq n$) contains at least one component of $P - (G_1)_{y_1}$, and $C_n \times (K_{1,3}^{(n)} - T_1)$ contains at least two components of $P - (G_1)_{y_1}$ as it contains both endpoints of P . Therefore, $c(P - (G_1)_{y_1}) \geq n + 2$, which contradicts to $c(P - (G_1)_{y_1}) = n + 1$. So $C_n \times T_1$ contains at least one endpoint of P . Similarly, each of $C_n \times T_2$ and $C_n \times T_3$ contains at least one endpoint of P . Since $C_n \times T_1$, $C_n \times T_2$ and $C_n \times T_3$ are disjoint subgraphs of G , P has at least three endpoints, a contradiction. $\square$

Corollary 2.2.6 *Let C_n be a circuit, where $n \geq 3$. For any tree T , if T contains a member of $\mathfrak{F}$ as a subgraph, then $G = C_n \times T$ is not traceable.*

The proof of this corollary is similar to the proof of Lemma 2.2.5. So it is omitted.

Let $P_m = p_1 p_2 \cdots p_m$ be a path with m vertices. Denote $P_m^{(n)}$ to be the graph obtained by identifying p_i with the center of a star T_i where $T_i = K_{1,n-1}$ for $i = 1$ or m and $T_i = K_{1,n-2}$ for $2 \leq i \leq m-1$ if $m \geq 2$ and $P_m^{(n)} = K_{1,n}$ if $m = 1$. Thus every vertex of $P_m^{(n)}$ is either of degree one or of degree n . Note that P_m and T_i ($1 \leq i \leq m$) can be viewed as subgraphs of $P_m^{(n)}$.

Let C_n be a circuit ($n \geq 3$). Let u_1 and u_2 be adjacent vertices of C_n , and let v_1 and v_2 be degree one vertices of $P_m^{(n+1)}$ where $v_1 \in T_1$ and $v_2 \in T_m$. In $C_n \times P_m^{(n+1)}$, a path P is called a Hamiltonian path with respect to $\{u_1, u_2, v_1, v_2\}$ if P satisfies:

- (i) P is a Hamiltonian path where (u_2, v_2) is an endpoint and the other endpoint is in $(C_n)_{v_1}$;
- (ii) $P - (C_n)_{v_1}$ is a Hamiltonian path of $C_n \times (P_m^{(n+1)} - v_1)$ with an endpoint in $(C_n)_{p_1}$;
- (ii) $P - (C_n)_{v_2}$ is a Hamiltonian path of $C_n \times (P_m^{(n+1)} - v_2)$ with an endpoint (u_1, p_m) .

Lemma 2.2.7 *Let C_n be a circuit ($n \geq 3$) and $G = C_n \times P_m^{(n+1)}$. Then G is traceable. Moreover, for any two adjacent vertices $u_1, u_2 \in V(C_n)$ and any two degree one vertices v_1, v_2 of $P_m^{(n+1)}$ where $v_1 \in T_1$ and $v_2 \in T_m$, there is a Hamiltonian path P of G with respect to $\{u_1, u_2, v_1, v_2\}$.*

Proof. We argue by induction on m . If $m = 1$, then the result follows by Lemma 2.2.4. So assume $m \geq 2$ and the result holds for smaller values of m . Let $\{u_1, u_2, v_1, v_2\}$ be given vertices satisfying the hypotheses.

As $T_m \subseteq P_m^{(n+1)}$, denote $H_1 = P_m^{(n+1)}/T_m$ and $H_2 = P_m^{(n+1)}/(P_m^{(n+1)} - T_m)$. Then $H_1 \cong P_{m-1}^{(n+1)}$ and $H_2 \cong P_1^{(n+1)} = K_{1,n+1}$. Assume that $v_0 \in V(H_1)$ is the vertex to which T_m contracts and $v'_0 \in V(H_2)$ is the vertex to which $P_m^{(n+1)} - T_m$ contracts.

By induction hypotheses, $C_n \times H_1$ has a Hamiltonian path P_1 with respect to $\{u_2, u_1, v_1, v_0\}$. Note that (u_2, p_{m-1}) is an endpoint of $P_1 - (C_n)_{v_0}$. By Lemma 2.2.4, $C_n \times H_2$ has a Hamiltonian path P_2 with endpoint (u_1, v'_0) and (u_2, v_2) such that (u_2, p_m) is an endpoint

of $P_2 - (C_n)_{v'_0}$ and (u_1, p_m) is an endpoint of $P_2 - (C_n)_{v_2}$. Therefore

$$P = (P_1 - (C_n)_{v_0}) \cup (P_2 - (C_n)_{v'_0}) \cup \{(u_2, p_{m-1})(u_2, p_m)\}$$

is a Hamiltonian path of G with respect to $\{u_1, u_2, v_1, v_2\}$. $\square$

Corollary 2.2.8 *Let C_n be a circuit ($n \geq 3$) and let T be a tree obtained by subdividing some edges of P_m in $P_m^{(n+1)}$ ($m \geq 2$). Then $C_n \times T$ is traceable. Moreover, for any two adjacent vertices $u_1, u_2 \in V(C_n)$ and any two degree one vertices v_1, v_2 of $P_m^{(n+1)}$ where $v_1 \in T_1$ and $v_2 \in T_m$, $C_n \times T$ admits a Hamiltonian path P with respect to $\{u_1, u_2, v_1, v_2\}$.*

Proof. Let $\{u_1, u_2, v_1, v_2\}$ be given vertices satisfying the hypotheses. It's sufficient to show that the result holds when T is obtained by subdividing one edge of P_m . Assume that T is obtained by subdividing $p_k p_{k+1} \in E(P_m)$ ($k \geq 1$) and v_0 is the subdividing vertex.

Let C_1 and C_2 be the two components of $T - v_0$ where $p_k \in V(C_1)$. Denote H_i to be the induced subgraph of $V(C_i) \cup v_0$ in T ($i = 1, 2$). Then $V(H_1) \cap V(H_2) = v_0$, and $H_1 \cong P_k^{n+1}$, $H_2 \cong P_{m-k-1}^{n+1}$. Therefore, by Lemma 2.2.5, $C_n \times H_2$ has a Hamiltonian path P_2 with respect to $\{u_1, u_2, v_0, v_2\}$. Assume that (x_j, p_{k+1}) is the endpoint of $P_2 - (C_n)_{v_0}$ in $(C_n)_{p_{k+1}}$ ($x_j \in V(C_n)$). Let x_{j+1} be a neighbor of x_j in C_n . By Lemma 2.2.5, $C_n \times H_1$ has a Hamiltonian path P_1 with respect to $\{x_{j+1}, x_j, v_1, v_0\}$. Hence,

$$\begin{aligned} P = & (P_1 - (C_n)_{v_0}) \cup (P_2 - (C_n)_{v_0}) \cup (E((C_n)_{v_0}) - (x_j, v_0)(x_{j+1}, v_0)) \\ & \cup \{(x_{j+1}, p_k)(x_{j+1}, v_0), (x_j, v_0)(x_j, p_{k+1})\} \end{aligned}$$

is a Hamiltonian path of $C_n \times T$ with respect to $\{u_1, u_2, v_1, v_2\}$. $\square$

Lemma 2.2.9 *Let C_n be a circuit (where $n \geq 3$). Let T be a tree where $y_1 y_2 \in E(T)$ and $y_1 \in D_1(T)$. If $d_T(y_2) \leq n$ and $C_n \times (T - y_1)$ is traceable, then $C_n \times T$ is traceable.*

Proof. Let P be a Hamiltonian path of $C_n \times (T - y_1)$. Since $d_{T-y_1}(y_2) \leq n - 1$, at least one edge of $(C_n)_{y_2}$ lies in P . Without loss of generality, we can assume that

$(x_1, y_2)(x_2, y_2) \in E((C_n)_{y_2}) \cap E(P)$ where $x_1x_2 \in E(C_n)$. Hence

$$\begin{aligned} P' &= (P - \{(x_1, y_2)(x_2, y_2)\}) \cup (E((C_n)_{y_1}) - \{(x_1, y_1)(x_2, y_1)\}) \\ &\cup \{(x_1, y_1)(x_1, y_2), (x_2, y_1)(x_2, y_2)\} \end{aligned}$$

is a Hamiltonian path of $C_n \times T$. $\square$

Lemma 2.2.10 *Let G be a Hamiltonian connected graph of order n . Then $G \times K_{1,n}$ is not Hamiltonian connected.*

Proof. Let $G_0 = G \times K_{1,n}$. Suppose G_0 is Hamiltonian connected. Let y be the center of $K_{1,n}$. Choose $(x_1, y), (x_2, y) \in V(G_0)$. Then there is an $((x_1, y), (x_2, y))$ -Hamiltonian path P of G_0 . Since $c(G_0 - V((G)_y)) = n$ and $P - V((G)_y)$ is a spanning subgraph of $G_0 - V((G)_y)$, $c(P - V((G)_y)) \geq n$. Note that $P - \{(x_1, y), (x_2, y)\}$ is still a path. Thus by Theorem 2.2.2,

$$\begin{aligned} c(P - V((G)_y)) &= c((P - \{(x_1, y), (x_2, y)\}) - (V((G)_y) \setminus \{(x_1, y), (x_2, y)\})) \\ &\leq |V((G)_y) \setminus \{(x_1, y), (x_2, y)\}| + 1 \\ &= n - 1, \end{aligned}$$

which contradicts to $c(P - V((G)_y)) \geq n$. $\square$

Lemma 2.2.11 *Let G be a Hamiltonian connected graph of order n . Let H be a path of length one. Then $G \times H$ is Hamiltonian connected.*

Proof. We may assume that $H = z_1z_2$. Let (x_1, y_1) and (x_2, y_2) be a pair of distinct vertices in $G \times H$. Without loss of generality, we may assume that $y_1 = z_1$.

If $y_1 \neq y_2$, then $y_2 = z_2$. Choose $x_3 \in V(G) \setminus \{x_1, x_2\}$. Since $(G)_{z_i} \cong G$ is Hamiltonian connected, there is a Hamiltonian $((x_i, z_i), (x_3, z_i))$ -path P_i of $(G)_{z_i}$ ($i = 1, 2$). Then $P_1 + P_2 + (x_3, z_1)(x_3, z_2)$ is a Hamiltonian $((x_1, y_1), (x_2, y_2))$ -path of $G \times H$.

So we could assume that $y_1 = y_2 = z_1$. Then $x_1 \neq x_2$. Since $(G)_{z_1} \cong G$ is Hamiltonian connected, there is a $((x_1, z_1), (x_2, z_1))$ -Hamiltonian path P_1 of $(G)_{z_1}$. Choose

$(x_3, z_1)(x_4, z_1) \in E(P_1)$. Since $(G)_{z_2} \cong G$ is Hamiltonian connected, there exists a $((x_3, z_2), (x_4, z_2))$ -Hamiltonian path P_2 of $(G)_{z_2}$. Thus

$$P = (P_1 - (x_3, z_1)(x_4, z_1)) + P_2 + \{(x_3, z_1)(x_3, z_2), (x_4, z_1)(x_3, z_2)\}$$

is a Hamiltonian $((x_1, y_1), (x_2, y_2))$ -path of $G \times H$. $\square$

Lemma 2.2.12 *Let G be an edge-pancyclic graph of order n . Then $G \times K_{1,n}$ is not weakly edge-pancyclic.*

Proof. Let $G_0 = G \times K_{1,n}$. Suppose G_0 is weakly edge-pancyclic. Let y be the center of $K_{1,n}$. Choose $x_1x_2 \in E(G)$. Then by the definition of weakly edge-pancyclic graph, there is a Hamiltonian circuit C of G_0 containing $(x_1, y)(x_2, y)$. Since $c(G_0 - V((G)_y)) = n$, $c(C - V((G)_y)) \geq n$. Note that $C - \{(x_1, y), (x_2, y)\}$ is a path. Thus by Theorem 2.2.2,

$$\begin{aligned} c(C - V((G)_y)) &= c((C - \{(x_1, y), (x_2, y)\}) - (V((G)_y) \setminus \{(x_1, y), (x_2, y)\})) \\ &\leq |V((G)_y) \setminus \{(x_1, y), (x_2, y)\}| + 1 \\ &= n - 1, \end{aligned}$$

which contradicts to $c(C - V(G_y)) \geq n$. $\square$

Lemma 2.2.13 *Let G be an edge-pancyclic graph of order n and let $H = y_1y_2$ be a path of length 1. Then for $\forall x \in V(G)$, $(x, y_1)(x, y_2)$ is on a circuit of length l in $G \times H$ for all $l = 4, 5, \dots, 2n$.*

Proof. Fix $x \in V(G)$. We will show that for all $l = 4, 5, \dots, 2n$, there is a circuit C_l of length l containing $(x, y_1)(x, y_2)$.

Since G is edge-pancyclic, there is a triangle xz_1z_2 in G , where $z_1, z_2 \in N_G(x)$. Then $C_4 = (x, y_1)(x, y_2)(z_1, y_2)(z_1, y_1)$ and $C_5 = (x, y_1)(x, y_2)(z_1, y_2)(z_1, y_1)(z_2, y_1)$. So we may assume that $l \geq 6$. Let $l_1 = \lfloor l/2 \rfloor$ and $l_2 = l - l_1$. Note that both l_1 and l_2 are at least 3. Since $(G)_{y_i} \cong G$ is edge-pancyclic, there is a circuit $C_{y_i}^{l_i}$ of length l_i containing $(x, y_i)(z_1, y_i)$ in $(G)_{y_i}$ ($i = 1, 2$). Thus $C_l = (C_{y_1}^{l_1} - (x, y_1)(z_1, y_1)) + (C_{y_2}^{l_2} - (x, y_2)(z_1, y_2)) + \{(x, y_1)(x, y_2), (z_1, y_1)(z_1, y_2)\}$. $\square$

Lemma 2.2.14 *Let n and k be two positive integers such that $n \geq 3$ and $k \leq n - 1$. Let $\{n_1, n_2, \dots, n_k\}$ be a sequence of positive integers. Then for each integer $l \in [n + 2, n + \sum_{i=1}^k (n_i * n)]$, there is a sequence of integers $\{l_1, l_2, \dots, l_k\}$ such that $l = n + \sum_{i=1}^k l_i$, and l_i is either 0 or in the interval $[2, n_i * n]$ for all $i = 1, 2, \dots, k$.*

Proof. If $n + 2 \leq l \leq n + n_1 * n$, then $\{l_1, l_2, \dots, l_k\} = \{l - n, 0, \dots, 0\}$. So we assume that $l \geq n + n_1 * n + 1$. If $l = n + \sum_{i=1}^j (n_i * n) + 1$ ($1 \leq j \leq k - 1$), then $\{l_1, l_2, \dots, l_k\} = \{n_1 * n - 1, n_2 * n, \dots, n_j * n, 2, 0, \dots, 0\}$. If $l = n + \sum_{i=1}^j (n_i * n) + l'$ ($1 \leq j \leq k - 1$ and $2 \leq l' \leq n_{j+1} * n$), then $\{l_1, l_2, \dots, l_k\} = \{n_1 * n, n_2 * n, \dots, n_j * n, l', 0, \dots, 0\}$. $\square$

2.3 Hamiltonian circuit in $G \times T$

In this section, we will prove Theorem 2.1.2. We first prove the following theorem, and Theorem 2.1.2 will be its corollary.

Theorem 2.3.1 *Let T be a tree with maximum degree Δ . Then the graph $G = C_n \times T$ is Hamiltonian if and only if $\Delta \leq n$.*

Proof of Theorem 2.3.1 Let $G = C_n \times T$ be a graph with a Hamiltonian circuit. If there exists $y \in V(T)$ such that $d_T(y) \geq n + 1$, then $c(G - (C_n)_y) = c(T - y) = d_T(y) \geq n + 1$. However, by Theorem 2.2.1, $c(G - (C_n)_y) \leq |V((C_n)_y)| = n$, a contradiction. Therefore $d_T(y) \leq n$ for every $y \in V(T)$ and $\Delta \leq n$.

So it is sufficient to show that if $\Delta \leq n$, then $G = C_n \times T$ has a Hamiltonian circuit. If T is a star, then it follows Lemma 2.2.3. Therefore we may assume that T is not a star. By way of contradiction, choose a tree T with minimal number of vertices such that $\Delta \leq n$ and $G = C_n \times T$ has no Hamiltonian circuit.

Note that T can be viewed as a graph obtained from finite stars $T_1, T_2, \dots, T_k$ by connecting their centers with edges and there exists such a star T_i that is connected to the other stars with only one edge. Without loss of generality, we may assume that T_1

is only connected to T_2 . Let y_i be the center of T_i ($i = 1, 2$). Since $T - T_1$ is also a tree and $|V(T - T_1)| < |V(T)|$, $C_n \times (T - T_1)$ has a Hamiltonian circuit C . Since $d_{T-T_1}(y_2) \leq n-1$, at least one edge of $(C_n)_{y_2}$ lies in C . Without loss of generality, we can assume that $(x_1, y_2)(x_2, y_2) \in E((C_n)_{y_2}) \cap E(C)$ where $x_1x_2 \in E(C_n)$. Since $\Delta(T_1) \leq n-1$, by Lemma 2.2.3, there is a Hamiltonian circuit C' of $C_n \times T_1$. By Note 1, we can assume that $(x_1, y_1)(x_2, y_1) \in E(C')$. Hence

$$\begin{aligned} C'' &= (C - \{(x_1, y_2)(x_2, y_2)\}) \cup (C' - \{(x_1, y_1)(x_2, y_1)\}) \\ &\cup \{(x_1, y_1)(x_1, y_2), (x_2, y_1)(x_2, y_2)\} \end{aligned}$$

is a Hamiltonian circuit of G , a contradiction to the choice of T . This completes the proof of Theorem 2.3.1.

Theorem 2.1.2 will be a corollary of Theorem 2.3.1.

2.4 Hamiltonian path in $G \times T$

Recall that $\mathfrak{F}$ is the set of such graph $H^{(n)}$ which is obtained by identifying every degree 1 vertex of H with the center of a $K_{1,n}$, where H is a subdivision of $K_{1,3}$. Specially, $K_{1,3}^{(n)} \in \mathfrak{F}$. In this section, we will prove Theorem 2.1.3. We first prove the following theorem, and Theorem 2.1.2 will be its corollary.

Theorem 2.4.1 *Let T be a tree with maximum degree Δ . Then the graph $G = C_n \times T$ is traceable if and only if $\Delta \leq n$ or $\Delta = n+1$ and T contains no member of $\mathfrak{F}$ as a subgraph.*

Proof of Theorem 2.4.1 By Theorem 2.3.1, we only need to consider T with $\Delta \geq n+1$.

Let $G = C_n \times T$ be a graph with a Hamiltonian path P and $\Delta \geq n+1$. If $\Delta \geq n+2$, choose $y \in V(T)$ such that $d_T(y) \geq n+2$. Then $c(G - (C_n)_y) = c(T - y) = d_T(y) \geq n+2$. However, by Theorem 2.1.2, $c(G - (C_n)_y) \leq |V_{1y}| + 1 = n+1$, a contradiction. Therefore $\Delta = n+1$. By Corollary 2.2.6, T contains no member of $\mathfrak{F}$ as a subgraph.

So it is sufficient to show that if $\Delta = n + 1$ and T contains no member of $\mathfrak{F}$ as a subgraph, then $G = C_n \times T$ has a Hamiltonian path. If T is a star, then it follows Lemma 2.2.4. So we may assume that T is not a star. By way of contradiction, let T be a tree with minimum vertices such that T contains no member of $\mathfrak{F}$ as a subgraph and $G = C_n \times T$ is not traceable.

Claim 1 $|D_{n+1}(T)| \geq 2$.

Suppose $|D_{n+1}(T)| = |\{y\}| = 1$. Let y_1 be a vertex of T such that the distance between y and y_1 in T is maximum. Then $y_1 \in D_1(T)$. Assume that y_2 is the only neighbor of y_1 in T . Since T is not a star, $y_2 \neq y$, and then $d_T(y_2) \leq n$. Since $T - y_1$ is also a tree and $|V(T - y_1)| < |V(T)|$, $C_n \times (T - y_1)$ has a Hamiltonian path P . By Lemma 2.2.9, $C_n \times T$ is traceable, a contradiction. This completes the proof of Claim 1.

Choose $z_1, z_2 \in D_{n+1}(T)$ such that the distance between z_1 and z_2 in T is maximal. Let P_0 be the path between z_1 and z_2 in T .

Claim 2 $D_{n+1}(T) \subseteq V(P_0)$.

If $D_{n+1}(T) - V(P_0) \neq \emptyset$, then T contains no member of $\mathfrak{F}$ as a subgraph, a contradiction. This completes the proof of Claim 2.

Claim 3 For every $y \in D_1(T)$, the distance from y to P_0 is one.

By way of contradiction, assume that there exists a vertex $y_1 \in D_1(T)$ such that the distance from v to P is greater than 1. Assume that y_2 is the only neighbor of y_1 in T . Note that $y_2 \notin V(P)$, and then $d_T(y_2) \leq n$. Since $T - y_1$ is also a tree and $|V(T - y_1)| < |V(T)|$, $C_n \times (T - y_1)$ has a Hamiltonian path P . By Lemma 2.2.9, $C_n \times T$

is traceable, a contradiction. This completes the proof of Claim 3.

Claim 4 $V(T) - V(P) = D_1(T)$.

Claim 4 follows directly from Claim 3.

Claim 5 $V(P_0) = D_2(T) \cup D_{n+1}(T)$.

Suppose there exists $y_2 \in V(P_0) \setminus (D_2(T) \cup D_{n+1}(T))$. Then there exists $y_1 \in N_T(y_2)$ such that $y_1 \notin V(P_0)$. By Claim 4, $y_1 \in D_1(T)$. Since $T - y_1$ is also a tree and $|V(T - y_1)| < |V(T)|$, $C_n \times (T - y_1)$ has a Hamiltonian path P . By Lemma 2.2.9, $C_n \times T$ is traceable, a contradiction. This completes the proof of Claim 5.

By Claim 4 and 5, T is either P_m^{n+1} or a subdivision of P_m^{n+1} by subdividing some edges of P_m ($m \geq 2$). By Lemma 2.2.7 and Corollary 2.2.8, $G = C_n \times T$ is traceable, a contradiction. This completes the proof of Theorem 2.4.1.

Theorem 2.1.2 will be a corollary of Theorem 2.3.1.

2.5 Hamiltonian connected Cartesian product

In this section, we will prove Theorem 2.1.4.

Proof of Theorem 2.1.4 Let G be a Hamiltonian connected graph of order n and let T be a tree. The necessary condition follows by Lemma 2.2.10. So it is sufficient to show that $G \times T$ is Hamiltonian connected if $\Delta(T) \leq n - 1$.

We argue by induction on $|V(T)|$. The result holds trivially when $|V(T)| = 1$, and is

also true when $|V(T)| = 2$ by Lemma 2.2.11. Let $|V(T)| = m(m \geq 3)$ and suppose the result holds for trees of at most $m - 1$ vertices. Let $u = (x_u, y_u)$ and $v = (x_v, y_v)$ be a pair of distinct vertices in $G \times T$. Let $N_T(y_u) = \{y_1, y_2, \dots, y_k\}$ ($1 \leq k \leq n - 1$). Denote $T_1, T_2, \dots, T_k$ to be components of $T - y_u$ where $y_i \in V(T_i)$ ($1 \leq i \leq k$).

If $y_u = y_v$, then $x_u \neq x_v$. Since $(G)_{y_u} \cong G$ is Hamiltonian connected, there is a Hamiltonian (u, v) -path P_{y_u} of $(G)_{y_u}$. Denote $P_{y_u} = (x_1, y_u)(x_2, y_u) \cdots (x_n, y_u)$ where $x_1 = x_u$ and $x_n = x_v$. In $G \times T_i$ ($1 \leq i \leq k$), by induction, there is a Hamiltonian $((x_i, y_i), (x_{i+1}, y_i))$ -path P_i . Thus

$$P = (P_{y_u} - \{(x_i, y_u)(x_{i+1}, y_u) : 1 \leq i \leq k\}) + \sum_{i=1}^k P_i + \{(x_i, y_u)(x_i, y_i), (x_{i+1}, y_u)(x_{i+1}, y_i) : 1 \leq i \leq k\}$$

is a Hamiltonian (u, v) -path of $G \times T$.

So we may assume that $y_u \neq y_v$. Without loss of generality, we may assume that $y_v \in V(T_k)$. Choose $x_0 \in V(G) \setminus \{x_u, x_v\}$. Note that (x_0, y_k) and v are distinct vertices. By induction, there is a Hamiltonian $(u, (x_0, y_u))$ -path P' of $G \times (T - T_k)$ and there is a Hamiltonian $((x_0, y_k), v)$ -path P'' of $G \times T_k$. Thus $P' + (x_0, y_u)(x_0, y_k) + P''$ is a Hamiltonian (u, v) -path of $G \times T$. This completes the proof of Theorem 2.1.4.

2.6 Weakly edge-pancyclic Cartesian product

In this section, we will prove Theorem 2.1.5.

Proof of Theorem 2.1.5 Let G be an edge-pancyclic graph of order n and let T be a tree. The necessary condition follows by Lemma 2.2.12. So it is sufficient to show that $G \times T$ is weakly edge-pancyclic if $\Delta(T) \leq n - 1$. By the definition of weakly edge-pancyclic, we actually need to prove the following theorem.

Theorem 2.6.1 *Let G be an edge-pancyclic graph of order n . Let T be a tree with $\Delta(T) \leq n - 1$. Then in $G \times T$, each of the following holds:*

- (a) for $\forall y \in V(T)$, $\forall e \in E((G)_y)$ and $\forall l$ ($3 \leq l \leq |V(G \times T)|$), there is a circuit of length l containing e ;
- (b) for $\forall x \in V(G)$, $\forall e \in E((T)_x)$ and $\forall l$ ($4 \leq l \leq |V(G \times T)|$), there is a circuit of length l containing e .

Proof. (a) We use induction on $|V(T)|$. The result holds trivially when $|V(T)| = 1$. Let $|V(T)| = m$ ($m \geq 2$) and suppose the result holds for trees with at most $m - 1$ vertices.

Let $y \in V(T)$ and $e \in E(G_y)$ be given. Denote (x_1, y) and (x_n, y) to be the endpoints of e . Suppose $N_T(y) = \{y_1, y_2, \dots, y_k\}$ (where $1 \leq k \leq n - 1$). Denote $T_1, T_2, \dots, T_k$ to be components of $T - y$ where $y_i \in V(T_i)$ and $|V(T_i)| = n_i$ ($1 \leq i \leq k$). Note that $\sum_{i=1}^k n_i = m - 1$. We will show that for $\forall l$ ($3 \leq l \leq n * m$), there is a circuit C_l of length l containing e in $G \times T$.

If $3 \leq l \leq n$, since $(G)_y \cong G$ is edge-pancyclic, there is a circuit C_y^l of length l containing e in $(G)_y$, which can also be viewed as a circuit C_l in $G \times T$. Specially, denote

$$C_y^m = (x_1, y)(x_2, y) \cdots (x_{n-1}, y)(x_n, y),$$

$$\text{and } C_y^{m-1} = (x_1, y)(x'_2, y) \cdots (x'_{n-2}, y)(x_n, y).$$

If $l = n + 1$, then

$$C_l = (C_y^{m-1} - (x_1, y)(x'_2, y)) + \{(x_1, y)(x_1, y_1), (x_1, y_1)(x'_2, y_1), (x'_2, y)(x'_2, y_1)\}.$$

So we may assume that $l \geq n + 2$. By Lemma 2.2.14, there is a sequence of k integers $\{l_1, l_2, \dots, l_k\}$ such that $l = n + \sum_{i=1}^k l_i$, and l_i is either 0 or in the interval $[2, n_i * n]$ (for all $1 \leq i \leq k$). Let $I_1 = \{i : l_i \geq 3\}$ and $I_2 = \{j : l_j = 2\}$ be two disjoint subscribe sets. For $\forall i \in I_1$, by induction, there is a circuit $C_{T_i}^{l_i}$ of length l_i containing $(x_i, y_i)(x_{i+1}, y_i)$ in $G \times T_i$. Thus

$$\begin{aligned} C_l &= (C_y^m - \{(x_i, y)(x_{i+1}, y) : i \in I_1 \cup I_2\}) + \\ &\quad \sum_{i \in I_1} (C_{T_i}^{l_i} - (x_i, y_i)(x_{i+1}, y_i)) + \sum_{j \in I_2} \{(x_j, y_j)(x_{j+1}, y_j)\} + \\ &\quad \{(x_i, y)(x_i, y_i), (x_{i+1}, y)(x_{i+1}, y_i) : i \in I_1 \cup I_2\}. \end{aligned}$$

(b) We use induction on $|V(T)|$. The result holds trivially when $|V(T)| = 1$, and it's also true when $|V(T)| = 2$ by Lemma 2.2.13. Let $|V(T)| = m (m \geq 3)$ and suppose the result holds for trees with at most $m - 1$ vertices. Let $x \in V(G)$ and $e \in E(T_x)$. Denote (x, y_1) and (x, y_2) to be the endpoints of e . We will show that for $\forall l$ ($4 \leq l \leq n * m$), there is a circuit C_l of length l containing e in $G \times T$.

Since G is pancyclic, there is a triangle xz_1z_2 in G , where $z_1, z_2 \in N_G(x)$. Then

$$C_4 = (x, y_1)(x, y_2)(z_1, y_2)(z_1, y_1)$$

$$\text{and } C_5 = (x, y_1)(z_2, y_1)(z_1, y_1)(z_1, y_2)(x, y_2).$$

So we may assume that $l \geq 6$. Let T_1 and T_2 be two components of $T - y_1y_2$ such that $y_i \in T_i$ ($i = 1, 2$). Let $l_1 = \min\{\lfloor l/2 \rfloor, n|V(T_1)|\}$ and $l_2 = l - l_1$. Note that both l_1 and l_2 are at least 3. By result of (a), there is a circuit C_i of length l_i containing $(x, y_i)(z_1, y_i)$ in $G \times T_i$ ($i = 1, 2$). Thus

$$\begin{aligned} C_l &= (C_1 - (x, y_1)(z_1, y_1)) + (C_2 - (x, y_2)(z_1, y_2)) + \\ &\quad \{(x, y_1)(x, y_2), (z_1, y_1)(z_1, y_2)\}. \quad \square \end{aligned}$$

2.7 Pan-connected Cartesian product

In this section, we will prove Theorem 2.1.6.

Proof of Theorem 2.1.6 By the definition of pan-connected and Hamiltonian connected, a pan-connected graph is also Hamiltonian connected. Therefore, the necessary condition follows by Theorem 2.1.4. So it is sufficient to show that $G \times T$ is pan-connected if $\Delta(T) \leq n - 1$.

We use induction on $|V(T)|$. The result holds trivially when $|V(T)| = 1$. Let $|V(T)| = m (m \geq 2)$ and suppose the result holds for trees with at most $m - 1$ vertices. Let $u = (x_u, y_u)$ and $v = (x_v, y_v)$ be two vertices of $G \times T$. Note that $\text{dist}_{G \times T}(u, v) = \text{dist}_G(x_u, x_v) + \text{dist}_T(y_u, y_v)$. Suppose $N_T(y_u) = \{y_1, y_2, \dots, y_k\}$ (where $1 \leq k \leq n - 1$).

Denote $T_1, T_2, \dots, T_k$ to be components of $T - y_u$ where $y_i \in V(T_i)$ and $|V(T_i)| = n_i$ ($1 \leq i \leq k$). Note that $\sum_{i=1}^k n_i = m - 1$. We will show that for $\forall l$ ($\text{dist}_{G \times T}(u, v) \leq l \leq n * m - 1$), there is a (u, v) -path P_l of length l in $G \times T$.

Case 1 $y_u = y_v$

In this case, $\text{dist}_{G \times T}(u, v) = \text{dist}_G(x_u, x_v)$. If $\text{dist}_{G \times T}(u, v) \leq l \leq n - 1$, since $(G)_{y_u} \cong G$ is pan-connected, there is a (u, v) -path $P_{y_u}^l$ of length l in $(G)_{y_u}$, which can also be viewed as a (u, v) -path P_l in $G \times T$. Specially, denote

$$P_{y_u}^{n-1} = (x_1, y_u)(x_2, y_u) \cdots (x_{n-1}, y_u)(x_n, y_u),$$

$$\text{and } P_{y_u}^{n-2} = (x_1, y_u)(x'_2, y_u) \cdots (x'_{n-2}, y_u)(x_n, y_u),$$

where $x_1 = x_u$ and $x_n = x_v$.

If $l = n$, then

$$P_l = (P_{y_u}^{n-2} - (x_1, y_u)(x'_2, y_u)) + \{(x_1, y_u)(x_1, y_1), (x_1, y_1)(x'_2, y_1), (x'_2, y_u)(x'_2, y_1)\}.$$

If $l = n + 1$, then

$$P_l = (P_{y_u}^{n-1} - (x_1, y_u)(x_2, y_u)) + \{(x_1, y_u)(x_1, y_1), (x_1, y_1)(x_2, y_1), (x_2, y_u)(x_2, y_1)\}.$$

So we may assume that $n + 2 \leq l \leq n * m - 1$. By Lemma 2.2.14, there is a sequence of k integers $\{l_1, l_2, \dots, l_k\}$ such that $l = n + \sum_{i=1}^k l_i$, and l_i is either 0 or in the interval $[2, n_i * n]$ for $\forall i \in [1, k]$. Let $I = \{i : l_i \neq 0\}$ be a set of subscribes. Choose $i_0 \in I$. By induction, there is a $((x_i, y_i), (x_{i+1}, y_i))$ -path $P_i^{l_i-1}$ of length $l_i - 1$ in $G \times T_i$ for every $i \in I - i_0$, and there is a $((x_{i_0}, y_{i_0}), (x_{i_0+1}, y_{i_0}))$ -path $P_{i_0}^{l_{i_0}}$ of length l_{i_0} in $G \times T_{i_0}$. Thus

$$\begin{aligned} P_l &= (P^{n-1} - \{(x_i, y)(x_{i+1}, y) : i \in I\}) + \\ &\quad \sum_{i \in I - i_0} P_i^{l_i-1} + P_{i_0}^{l_{i_0}} + \{(x_i, y_u)(x_i, y_i), (x_{i+1}, y_u)(x_{i+1}, y_i) : i \in I\}. \end{aligned}$$

Case 2 $y_u \neq y_v$

Without loss of generality, we may assume that $y_v \in V(T_k)$. If $\text{dist}_{G \times T}(u, v) \leq l \leq n + n_k * n - 1$, by induction, there is a (u, v) -path P_k^l of length l in $G \times (y_u \cup T_k)$, which can also be viewed as a (u, v) -path P_l in $G \times T$.

So we may assume that $l \geq n + n_k * n$. Choose $z \in N_G(x_u) \setminus \{x_v\}$. Note that (z, y_k) and v are distinct vertices. By induction, there is a $(u, (z, y_k))$ -path P^{l-n_k*n} of length $l - n_k * n$ in $G \times (T - T_k)$, and there is a $((z, y_k), v)$ -path P^{n_k*n-1} of length $n_k * n - 1$ in $G \times T_k$. Thus $P_l = P^{l-n_k*n} + \{(z, y_u)(z, y_k)\} + P^{n_k*n-1}$. This completes the proof of Theorem 2.1.6.

Chapter 3

On 3-edge-connected Supereulerian Graphs

3.1 Introduction

Graphs in this section may have multiple edges or loops.

A graph G is eulerian if G is connected with $O(G) = \emptyset$, and G is supereulerian if G has a spanning eulerian subgraph. In [3], Boesch et al raised a problem to determine when a graph is supereulerian, and they remarked that such a problem would be a difficult one. In [32], Pulleyblank confirmed the remark by showing that the problem to determine if a graph is supereulerian, even within planar graphs, is NP-complete. For more literature on supereulerian graphs, see Catlin's excellent survey [8] and its supplement [13]. Jaeger and Catlin independently proved the following theorem.

Theorem 3.1.1 (*Catlin [7] and Jaeger [20]*) *Every 4-edge-connected graph is supereulerian.*

It is well known that there exist many 3-edge-connected graphs such as the Petersen graph, and any 3-connected cubic graph that does not have a proper 3-edge-coloring

is not supereulerian. Therefore, it is natural to ask when a 3-edge-connected graph is supereulerian. In [44], Zhan proved the following.

Theorem 3.1.2 (Zhan [44]) *Every 3-edge-connected, essentially 7-edge-connected graph is supereulerian.*

As seen above, there exist many 3-edge-connected, essentially 4-edge-connected non supereulerian graphs. The following conjecture has been posted in [13].

Conjecture 3.1.3 (Chen and Lai [13]) *Every 3-edge-connected, essentially 5-edge-connected graph is supereulerian.*

In this chapter, we will investigate when a 3-edge-connected essentially 4-edge-connected graph is supereulerian. Let $\sigma(G) = \min\{d_G(u) + d_G(v) | uv \in E(G)\}$. The main result of this paper is the following.

Theorem 3.1.4 *Let G be a 3-edge-connected, essentially 4-edge-connected graph. If $\sigma(G) \geq 9$, then G is supereulerian.*

Since the Petersen graph is 3-edge-connected, essentially 4-edge-connected with $\sigma(G) = 6$, and since the Petersen graph is not supereulerian, we propose the following conjecture.

Conjecture 3.1.5 *Let G be a 3-edge-connected, essentially 4-edge-connected graph. If $\sigma(G) \geq 7$, then G is supereulerian.*

The line graph of a graph G , denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. In 1986, Thomassen [40] conjectured that every 4-connected line graph is hamiltonian. Zhan [44] verified the Thomassen's conjecture for all 7-connected line graphs. Theorem 3.1.4 has the following corollaries for line graphs.

Corollary 3.1.6 *Every 4-connected, essentially 6-connected line graph with minimum degree at least 7 is hamiltonian.*

If a graph is 6-connected, then it must also be 4-connected and essentially 6-connected. So we have the following.

Corollary 3.1.7 *Every 6-connected line graph with minimum degree at least 7 is hamiltonian.*

Ryjáček [35] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian, where $cl(G)$ is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity and minimum degree of a graph, the following corollary is obtained.

Corollary 3.1.8 *Every 6-connected claw-free graph with minimum degree at least 7 is hamiltonian.*

3.2 Preliminary

Let $X \subseteq E(G)$ be an edge subset. The contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. When $X = \{e\}$, we use G/e for $G/\{e\}$. If K is a subgraph of G , then we write G/K for $G/E(K)$.

Theorem 3.2.1 (Zhan [44]) *Let G be a 3-edge-connected, essentially 7-edge-connected graph. Then each of the following holds.*

- (i) *For any edge $e \in E(G)$, $F(G - e) = 0$.*
- (ii) *For any edges $e, e' \subseteq E(G)$ such that $\kappa'(G - \{e, e'\}) \geq 2$, $F(G - \{e, e'\}) = 0$.*

Corollary 3.2.2 *Let G be a 3-edge-connected, essentially 7-edge-connected graph. Then each of the following holds.*

- (i). *For any vertex $v \in D_3(G) \cup D_4(G)$, $F(G - v) = 0$, and so G is collapsible.*
- (ii). *For any vertex $v \in D_5(G)$, $F(G - v) \leq 1$. Furthermore, either $G - v$ is collapsible, or $G - v = K_2$ and G is L_1 in Figure 1.*
- (iii). *For any vertex $v \in D_6(G)$, $F(G - v) \leq 2$. Furthermore, either $G - v$ is collapsible, or $G - v = K_2$ and G is either L_2 or L_3 in Figure 1, or $G - v = K_{2,6}$ and $G = K_{3,6}$.*

Proof. (i) Let $E_G(v) = \begin{cases} \{e, e_1, e_2\}, & \text{if } d_G(v) = 3 \\ \{e', e'', e_1, e_2\}, & \text{if } d_G(v) = 4 \end{cases}$. Since G is essentially 7-edge-connected, $\kappa'(G - \{e', e''\}) \geq 2$. Let $G_1 = \begin{cases} G - e, & \text{if } d_G(v) = 3 \\ G - \{e', e''\}, & \text{if } d_G(v) = 4 \end{cases}$. By Theorem 3.2.1, $F(G_1) = 0$. We assume that T_1, T_2 are two edge-disjoint spanning trees of G_1 . Without loss of generality, we also assume that $e_1 \in T_1, e_2 \in T_2$. Then $T_1 - v, T_2 - v$ are two edge-disjoint spanning trees of $G - v$.

(ii) Let $E_G(v) = \{e', e'', e_1, e_2, e_3\}$. Clearly, $\kappa'(G - \{e', e''\}) \geq 2$. By Theorem 3.2.1(ii), $F(G - \{e', e''\}) = 0$. We assume that T_1, T_2 are two edge-disjoint spanning trees of $G - \{e', e''\}$ such that $e_1 \in T_1$ and $e_2 \in T_2$. If $e_3 \notin E(T_1) \cup E(T_2)$, $F(G - v) = 0$. Otherwise, $F(G - v) \leq 1$. By Theorem 2.1(iii), either $G - v$ is collapsible or the c-reduction of $G - v$ is a K_2 . If the c-reduction of $G - v$ is a K_2 , let $V(K_2) = \{w_1, w_2\}$. Since G is essentially 7-edge-connected and 3-edge-connected, $|V(\mathcal{CP}(w_1))| = |V(\mathcal{CP}(w_2))| = 1$. Thus G is L_1 in Figure 1.

(iii) Let $E_G(v) = \{e', e'', e_1, e_2, e_3, e_4\}$. Then $\kappa'(G - \{e', e''\}) \geq 2$. By Theorem 3.2.1(ii), $F(G - \{e', e''\}) = 0$. We assume that T_1, T_2 are two edge-disjoint spanning trees of $G - \{e', e''\}$ such that $e_1 \in T_1$ and $e_2 \in T_2$. If $e_3, e_4 \notin E(T_1) \cup E(T_2)$, $F(G - v) = 0$.

Otherwise, $F(G - v) = 1$ or 2 . By Theorem 1.2.4 (iii), either $G - v$ is collapsible or the c -reduction of $G - v$ is a K_2 or a $K_{2,t}$ ($t \geq 1$). Let G' be the c -reduction of $G - v$.

If $G' = K_2$, let $V(G') = \{w_1, w_2\}$. Since G is essentially 7-edge-connected and 3-edge-connected, $|V(\mathcal{CP}(w_1))| = |V(\mathcal{CP}(w_2))| = 1$. Thus G is either L_2 or L_3 in Figure 1. Next we assume that $G' = K_{2,t}$ ($t \geq 1$). Let $V(G') = \{u_1, u_2, w_1, w_2, \dots, w_t\}$ with $d_{G'}(u_i) = t$ ($i = 1, 2$) and $d_{G'}(w_j) = 2$ ($j = 1, \dots, t$). If $[v, \mathcal{CP}(w_j)] = 0$, then $|V(\mathcal{CP}(w_j))| \geq 2$ since $\kappa'(G) \geq 3$. Thus $[\mathcal{CP}(w_j), G - \mathcal{CP}(w_j)]$ is an essential 2-edge cut in G , a contradiction. Thus $||v, \mathcal{CP}(w_j)|| \geq 1$. As $d_G(v) = 6$, $t \leq 6$.

Assume that $||v, \mathcal{CP}(w_j)|| \geq 2$. If $t \geq 2$, $([\mathcal{CP}(w_j), G - \mathcal{CP}(w_j)] \cup [v, G - v]) - [v, \mathcal{CP}(w_j)]$ is an essential edge cut of size at most 6 in G ; if $t = 1$, then $||v, \mathcal{CP}(u_1)|| = ||v, \mathcal{CP}(u_2)|| \geq 2$ since G is 3-edge-connected. Thus $||v, \mathcal{CP}(u_1)|| = ||v, \mathcal{CP}(u_2)|| = ||v, \mathcal{CP}(w_1)|| = 2$, and so $[\mathcal{CP}(w_1) \cup \mathcal{CP}(u_1), G - (\mathcal{CP}(w_1) \cup \mathcal{CP}(u_1))]$ is an essential 5-edge cut. It contradicts the hypothesis that G is essentially 7-edge-connected. So $||v, \mathcal{CP}(w_j)|| = 1$. Since G is again essentially 7-edge-connected, $|V(\mathcal{CP}(w_i))| = 1$.

If $t \leq 5$, then $||v, \mathcal{CP}(u_1) \cup \mathcal{CP}(u_2)|| \geq 1$. Without loss of generality, we assume that $||v, \mathcal{CP}(u_1)|| \geq ||v, \mathcal{CP}(u_2)||$. Then $||\{w_1\} \cup \mathcal{CP}(u_2), G - (\{w_1\} \cup \mathcal{CP}(u_2))|| \leq 6$, contrary to the hypothesis that G is essentially 7-edge-connected. So $t = 6$. Thus $vw_j \in E(G)$ ($j = 1, \dots, 6$). Since G is essential 7-edge-connected again, $|V(\mathcal{CP}(u_i))| = 1$ ($i = 1, 2$). Therefore, $G = K_{3,6}$. $\square$

3.3 Proofs of Theorems

In this section, we will prove Theorem 3.1.4 and Corollary 3.1.6.

Proof of Theorem 3.1.4. By contradiction, we assume that Theorem 3.1.4 is false. Then there is a graph G satisfying the hypothesis of Theorem 3.1.4 but G is not supereulerian. Choose G such that

- (1) G is a counterexample of Theorem 3.1.4;

(2) subject to (1), $|V(G)|$ is minimized.

If G is essentially 7-edge-connected, then, by Theorem 3.2.1, $F(G) = 0$. By Theorem 1.2.4(iv), G is supereulerian, contrary to (1). Thus G has an essential edge cut X with $4 \leq |X| \leq 6$. Let G_1 and G_2 denote two nontrivial connected components of $G - X$ with $|V(G_1)| \leq |V(G_2)|$. We choose G and X so that

(3) subject to (1) and (2), $|V(G_1)|$ is as small as possible.

Let $G'_1 = G/G_2$ and $G'_2 = G/G_1$, and let v_{G_1} and v_{G_2} denote the vertices onto which G_1 and G_2 are contracted, respectively (see Figure 2). Then $X = [v_{G_2}, G'_1 - v_{G_2}]_{G'_1}$ and $X = [v_{G_1}, G'_2 - v_{G_1}]_{G'_2}$. As contraction will not decrease edge-connectivity and essential edge-connectivity, both G'_1 and G'_2 are also 3-edge-connected and essentially 4-edge-connected. By (2), G'_2 is supereulerian.

If G'_1 has an essential edge cut Y with $|Y| \in \{4, 5, 6\}$, let G_{11} and G_{12} be the components of $G - Y$. Also, we assume that $v_{G_2} \in V(G_{12})$. Thus Y is also an essential edge cut of G such that G_{11} and $G[V(G_{12}) \cup V(G_2)]$ are nontrivial components of $G - Y$, contrary to (3). So G'_1 is 3-edge-connected and essentially 7-edge-connected.

Figure 2.

Note that $d_{G'_1}(v_{G_2}) = |X| \leq 6$ and $G'_1 - v_{G_2} = G_1$. By Corollary 3.2.2, either G_1 is collapsible, or $G_1 = K_2$, or $G_1 = K_{2,6}$. Note that G'_2 is supereulerian, and $G'_2 = G/G_1$. If G_1 is collapsible, then G is supereulerian by Theorem 1.2.1(viii), contrary to (1). So we have either $G_1 = K_2$ or $G_1 = K_{2,6}$.

Claim 1. $\kappa'(G_1) \geq 2$. Therefore, $G_1 = K_{2,6}$ and $G'_1 = K_{3,6}$.

By contradiction, we assume that $\kappa'(G_1) \leq 1$. If G_1 is not connected, let G_{11} and

G_{12} be the components of G_1 . Since G is essentially 4-edge-connected with $\kappa'(G) \geq 3$ and $|X| \leq 6$, we have $[G_2, G_{1i}] = 3$, $|E(G_{1i})| = 0$, and $|V(G_{1i})| = 1$ for $i = 1, 2$. This contradicts the hypothesis that X is an essential edge cut. Thus G_1 must be connected, and so G_1 has a cut edge e . Let G'_{11} and G'_{12} be the two components of $G_1 - e$. Then $[G'_{11}, G - G'_{11}] \cup [G'_{12}, G - G'_{12}] = X \cup \{e\}$ and $[G'_{11}, G - G'_{11}] \cap [G'_{12}, G - G'_{12}] = \{e\}$. It follows that $\|G'_{11}, G - G'_{11}\| + \|G'_{12}, G - G'_{12}\| = |X \cup \{e\}| + 1 \leq 8$. Since $\kappa'(G) \geq 3$, $\max\{\|G'_{11}, G - G'_{11}\|, \|G'_{12}, G - G'_{12}\|\} \leq 5$. By (3), neither G'_{11} nor G'_{12} can have an edge, and so we may assume that $V(G'_{11}) = \{u\}$ and $V(G'_{12}) = \{v\}$. Thus $d_G(u) + d_G(v) \leq 8$, contrary to the hypothesis that $\sigma(G) \geq 9$. This proves Claim 1.

Figure 3.

By Claim 1, $G_1 = K_{2,6}$ and $G'_1 = K_{3,6}$. Thus $|X| = 6$. Let $W = \{w_1, w_2, \dots, w_6\}$ and $\{v_1, v_2\}$ be the sets of vertices of degree 2 and degree 6 in G_1 , respectively (see Figure 3). Note that G'_2 is supereulerian. Let T be a spanning eulerian subgraph of G'_2 , and $H = G[E(T)]$.

Assume $|V(H) \cap W| = 6$. If $T - v_{G_1}$ is not connected in G'_2 , then H is not connected in G . Let $H_1, \dots, H_s$ be the components of H , where $s \in \{2, 3\}$. Thus there is an H_i , say H_1 , such that $|V(H_1) \cap W| = 2$. Without loss of generality, we assume that $V(H_1) \cap W = \{w_1, w_2\}$. Then $H' = G[E(H) \cup \{v_1w_2, v_1w_4, v_1w_5, v_1w_6, v_2w_1, v_2w_3\}]$ is a spanning eulerian subgraph of G , contrary to (1). If $T - v_{G_1}$ is connected in G'_2 , then H is also connected in G . Thus $H' = G[E(H) \cup \{v_1w_2, v_1w_4, v_1w_5, v_1w_6, v_2w_1, v_2w_3\}]$ is also a spanning eulerian subgraph of G , contrary to (1).

Assume $|V(H) \cap W| = 4$. If $T - v_{G_1}$ is not connected in G'_2 , then H is not connected in

G . Let H_1, H_2 be the components of H . Assume that $V(H_1) \cap W = \{w_1, w_2\}$ and $V(H_2) \cap W = \{w_3, w_4\}$. Then $H' = G[E(H) \cup \{v_1w_1, v_1w_4, v_1w_5, v_1w_6, v_2w_2, v_2w_3, v_2w_5, v_2w_6\}]$ is a spanning eulerian subgraph of G , contrary to (1). If $T - v_{G_1}$ is connected in G'_2 , then $H' = G[E(H) \cup \{v_1w_1, v_1w_4, v_1w_5, v_1w_6, v_2w_2, v_2w_3, v_2w_5, v_2w_6\}]$ is also a spanning eulerian subgraph of G , contrary to (1).

If $|V(H) \cap W| = 2$, without loss of generality, we assume that $V(H) \cap W = \{w_1, w_2\}$. Then $H' = G[E(H) \cup \{v_1w_1, v_1w_2, v_1w_3, v_1w_4, v_1w_5, v_1w_6, v_2w_3, v_2w_4, v_2w_5, v_2w_6\}]$ is a spanning eulerian subgraph of G , contrary to (1). This completes the proof of Theorem 3.1.4. $\square$

In order to prove Corollary 3.1.6, we need the following lemmas.

Lemma 3.3.1 (Shao [38]) *Let G be a connected, essentially 3-edge-connected graph.*

- (i). *The core graph G_0 is uniquely defined, and $\kappa'(G_0) \geq 3$.*
- (ii). *If G_0 is supereulerian, then $L(G)$ is hamiltonian.*

Proof of Corollary 3.1.6. Let $L(G)$ be a 4-connected, essentially 6-connected line graph with $\delta(L(G)) \geq 7$. Then G is essentially 4-edge-connected with $\sigma(G) \geq 9$. Let G_0 be the core of G . Then G_0 is 3-edge-connected, and essentially 4-edge-connected. Thus, $D_3(G) = D_3(G_0)$.

Let $uv \in E(G_0)$ with $d_{G_0}(u) + d_{G_0}(v) \leq 8$. If $uv \notin E(G)$, then both u and v are adjacent to a vertex of degree 2 in G . Thus $d_{G_0}(u) = d_G(u) \geq 9 - 2 = 7$ and $d_{G_0}(v) = d_G(v) \geq 9 - 2 = 7$, and so $d_{G_0}(u) + d_{G_0}(v) \geq 14$, contrary to the assumption that $d_{G_0}(u) + d_{G_0}(v) \leq 8$. So $uv \in E(G)$ and $d_G(u) + d_G(v) \geq 9$. As $d_{G_0}(u) + d_{G_0}(v) \leq 8$, one of u or v (say u) must be adjacent to some vertex of degree 1 in G . Since $D_3(G) = D_3(G_0)$, we have $d_{G_0}(u) \geq 4$. As G_0 is 3-edge-connected and $d_{G_0}(u) + d_{G_0}(v) \leq 8$, we have $d_{G_0}(u) \leq 5$, and so $d_{G_0}(u) \in \{4, 5\}$. Since u is adjacent to some vertex of degree 1 in G and $\sigma(G) \geq 9$, u must be adjacent to at least three vertices of degree 1 in G . Thus $[u, G_0 - u]_{G_0}$ is an essential vertex cut of size 4 or 5 in $L(G)$, contrary to the hypothesis that $L(G)$ is essentially 6-connected. So $\sigma(G_0) \geq 9$. By Theorem 3.1.4, G_0 has a spanning eulerian subgraph. By Lemma 3.3.1, $L(G)$ is hamiltonian. $\square$

Chapter 4

4-connected line graph of a $P3D$ graph

4.1 Introduction

In [40], Thomassen conjectured that every 4-connected line graph is hamiltonian. In [31], Matthews and Sumner conjectured that every 4-connected claw-free graph is hamiltonian. Ryjáček [35] discovered the line graph closure of claw-free graphs, and showed that the above mentioned conjectures by Thomassen and by Matthews and Sumner are in fact equivalent. Recently, Ryjáček and Vrána further conjectured the following:

Conjecture 4.1.1 (*Ryjáček and Vrána, [37]*) *Every 4-connected claw-free graph is hamiltonian connected.*

In 1986, Zhan proved a sufficient condition involving the edge-connectivity $\kappa'(G)$ for $L(G)$ to be hamiltonian-connected.

Theorem 4.1.2 (*Zhan, [43]*) *If G is a 4-edge-connected graph, then the line graph $L(G)$ is hamiltonian connected.*

By the spanning tree packing theorem of Tutte and Nash-Williams, it is known that every 4-edge-connected graph has two edge-disjoint spanning trees. Thus the following extends Zhan's theorem.

Theorem 4.1.3 (*Catlin and Lai, Corollary 4A in [12]*) *If G has two edge-disjoint spanning trees, then the line graph $L(G)$ is hamiltonian-connected if and only if $\kappa(L(G)) \geq 3$.*

A number of former researches have also investigated the hamiltonicity of the line graph of claw-free graphs. For example, it has been shown that (Corollary 3.9, [14]) every 4-connected line graph of a claw free graph is hamiltonian. Kriesell presented a stronger result.

Theorem 4.1.4 (*Kriesell, [23]*) *Every 4-connected line graph of a claw free graph is hamiltonian connected.*

Ryjáček ([36]) introduced the almost claw free graphs. A graph G is **almost claw free** if the vertices that are centers of claws in G are independent and if the neighborhoods of the center of each claw in G is 2-dominated (having 2 vertices in the neighborhoods of the center adjacent to other neighbors). By definition, every claw free graph is an almost claw free graph.

Theorem 4.1.5 (*Corollary 1.5 in [27]*) *Every 4-connected line graph of an almost claw free graph is hamiltonian-connected.*

For vertices $x, y \in V(G)$ with $\text{dist}_G(x, y) = 2$, define

$$J_G(x, y) = \{u \in N_G(x) \cap N_G(y) : N_G[u] \subseteq N_G[x] \cup N_G[y]\},$$

and

$$\begin{aligned} J'_G(x, y) = & \{u \in N_G(x) \cap N_G(y) : \text{if } v \in N_G(u) - (N_G[x] \cup N_G[y]), \text{ then} \\ & N_G(x) \cup N_G(y) \cup N_G(u) - \{x, y, v\} \subseteq N_G(v)\}. \end{aligned}$$

Once again, when the graph G is understood from the context, the subscript G will be omitted.

Ainouche ([1]) introduced the quasi claw-free graphs. A graph G is **quasi claw-free** if $J_G(x, y) \neq \emptyset$ for any $x, y \in V(G)$ with $\text{dist}_G(x, y) = 2$. Broersma and Vumer [5] recently introduced the P_3 -dominating graphs. A graph G is **P_3 -dominating** (P3D for short) if $J_G(x, y) \cup J'(x, y) \neq \emptyset$ for any $x, y \in V(G)$ with $\text{dist}_G(x, y) = 2$. By definitions, every claw-free graph is quasi claw-free; and every quasi claw-free graph is P3D. A recent result also extends the above mentioned theorem by Kriesell.

Theorem 4.1.6 (*Theorem 1.1 in [28]*) *Every 4-connected line graph of a quasi claw free graph is hamiltonian-connected.*

The main purpose of this paper is to investigate whether these theorems can be extended to the line graphs of P_3 -dominating graphs. We obtained the following theorem.

Theorem 4.1.7 *Every 4-connected line graph of a P_3 -dominating graph is hamiltonian-connected.*

4.2 Preliminaries

A graph is 4-cycle-connected if for every pair of distinct edges $e', e'' \in E(G)$, G has a sequence of cycles $C_1, C_2, \dots, C_t$ ($t \geq 1$), referred as a **4-cycle** (e', e'') -**chain**, such that

- (4C1) $e' \in E(C_1)$ and $e'' \in E(C_t)$;
- (4C2) for each i with $1 \leq i \leq t$, $|E(C_i)| \leq 4$; and
- (4C3) if $t \geq 2$, then $\forall i$ with $1 \leq i \leq t-1$, $E(C_i) \cap E(C_{i+1}) \neq \emptyset$.

Lemma 4.2.1 *Suppose that G is a 4-cycle-connected graph. Then G is collapsible if and only if G has a nontrivial collapsible subgraph.*

Proof It suffices to prove that if G has a nontrivial collapsible subgraph H , then G is collapsible. Thus we assume that G has a nontrivial collapsible subgraph H . If H spans G , then $G/H \cong K_1$ is also collapsible. It follows by Theorem 1.2.1(i) that G is also collapsible. Thus we assume that H is a maximal collapsible subgraph of G , and that $V(G) - V(H) \neq \emptyset$. Since G is 4-cycle connected, G is also connected, and so there must be an edge $e' \in E(G)$ incident with a vertex in $V(H)$ and a vertex in $V(G) - V(H)$. Since H is nontrivial, $\exists e'' \in E(H)$. Since G is 4-cycle-connected, G has a 4-cycle (e', e'') -chain $C_1, C_2, \dots, C_t$, satisfying (4C1)-(4C3). If we choose e' and e'' so that t is smallest, then we may assume that $t = 1$ and e' and e'' are in a cycle C_1 of length at most 4. Since $e'' \in E(H)$, $(H \cup C_1)/H$ is a cycle of length at most 3. By Theorem 1.2.1(ii) and (i), $H \cup C_1$ is also collapsible, contrary to the assumption that H is a maximally collapsible subgraph of G . Thus we must have $G = H$, and so G is collapsible. $\square$

Corollary 4.2.2 *Let $n \geq 3$ be an integer. Every $K_{3,n}$ is collapsible.*

Proof Every $K_{3,n}$ is 4-cycle-connected and contains a collapsible subgraph $K_{3,3} - e$ (see Theorem 1.2.1(vii)), and so by Lemma 4.2.1, $K_{3,n}$ is collapsible. $\square$

We say that an edge $e \in E(G)$ is **subdivided** when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by that length 2 path is called **subdividing** e . For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e' , and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e'' . Then,

$$V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}.$$

The above definitions imply the following lemma.

Lemma 4.2.3 *For a graph G and edges $e', e'' \in E(G)$, each of the following holds.*

- (i) *if $G(e', e'')$ has a spanning $(v(e'), v(e''))$ -trail, then G has a spanning (e', e'') -trail.*
- (ii) *if $G(e', e'')$ has a dominating $(v(e'), v(e''))$ -trail, then G has a dominating (e', e'') -trail.*

Lemma 4.2.4 *Let G be a graph and $G' = G - D_1(G)$. If $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete, then*

- (i) G' is nontrivial and $\delta(G') \geq \kappa'(G') \geq 2$.
- (ii) G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$.
- (iii) for $v \in V(G)$ with $d_G(v) = 1$ or $d_G(v) = 2$, $N_G(v) \subseteq V(G_0)$.

Lemma 4.2.5 *Let G be a graph such that $\kappa(G) \geq 3$ and $L(G)$ is not complete and let G_0 be the core of G . If $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$ -trail for any $e', e'' \in E(G_0)$, then for any $e', e'' \in E(G)$, $G(e', e'')$ has a dominating $(v(e'), v(e''))$ -trail.*

Proof Let $e', e'' \in E(G)$. If $e' \in E(G_0)$, let $f' = e'$; if e' is incident with a vertex of degree 2, let f' be the corresponding new edge in G_0 ; if e' is incident to a vertex of degree 1, let f' be any edge in G_0 incident with the other vertex incident with e' . Similarly we define f'' . Then a spanning $(v(f'), v(f''))$ -trail in $G_0(f', f'')$ can be adjusted to a dominating $(v(e'), v(e''))$ -trail in G . $\square$

Lemma 4.2.6 *(Proposition 2.2 of [27]) Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'') -trail.*

Theorem 4.2.7 *(Lemma 3.1 of [27]) Let G be a 3-edge-connected graph without loops, $v, v_1, u_1, u_2 \in V(G)$ be such that $d_G(v_1) = 3$ and $N_G(v_1) = \{v, u_1, u_2\}$, and for an integer $k \geq 1$ let $X' = \{u_1u_2, u_1v_i, u_2v_i : 1 \leq i \leq k\}$ be an edge subset of G and $W = G[X']$. Then each of the following holds.*

- (i) *If $(G - vv_1)/W$ is nontrivial and $\tau((G - vv_1)/W) \geq 2$, then $\tau(G) \geq 2$.*
- (ii) *If $G/W = K_1$, then $\tau(G) \geq 2$.*

Theorem 4.2.8 *(Theorem 3.3 of [27]) Let G be a graph with $\kappa'(G) \geq 3$. If every 3-edge-cut of G has at least one edge in a 2-cycle or 3-cycle of G , then the graph $G(e', e'')$ is collapsible for any $e', e'' \in E(G)$.*

Figure 4.2.1

Figure 4.2.2

Lemma 4.2.9 *Let H_1 and H_2 be graphs depicted as Fig 4.2.1. Then both H_1 and H_2 are collapsible.*

Proof Note that H_1 contains a C_3 and a C_2 . Since $H_1/(C_2 \cup C_3) \cong C_3$, by Theorem 1.2.1 (i) and (ii), H_1 is collapsible.

Since $F(H_2) = 0$, by Theorem 1.2.4 (ii), H_2 is collapsible. $\square$

Lemma 4.2.10 *Let H_3 be a graph depicted as Fig 4.2.2. Then H_3 is collapsible.*

Proof Note that H_3 contains a C_3 . Note H_3/C_3 is isomorphic to $K_4 - e$, which is 2-edge-connected. Since $F(H_3/C_3) = 1$, by Theorem 1.2.4(ii), H_3/C_3 is collapsible. Thus by Theorem 1.2.1 (i) and (ii), H_3 is collapsible. $\square$

Figure 4.2.3

Lemma 4.2.11 For $\forall e, e' \in E(K_4)$, $K_4(e, e')$ is collapsible.

Proof If e and e' have a common endpoint (Fig 4.2.3 (a)), then $K_4(e, e')$ has a C_3 . $K_4(e, e')/C_3$ is isomorphic to $K_4 - e$, which is 2-edge-connected. Since $F(K_4(e, e')/C_3) = 1$, by Theorem 1.2.4(ii), $K_4(e, e')/C_3$ is collapsible. Thus by Theorem 1.2.1 (i) and (ii), $K_4(e, e')$ is collapsible. So we may assume that e_1 and e_2 is a matching of K_4 (Fig 4.2.3 (b)). Note that $C = v_1v_ev_2v_3$ is a 4-circuit of $K_4(e, e')$. Since $K_4(e, e')/\pi(C)$ has a spanning 3-circuit, by Theorem 1.2.1 (ii), the reduction of $K_4(e, e')/\pi(C)$ is K_1 , which means $K_4(e, e')/\pi(C)$ is collapsible. Hence, by Theorem 1.2.6 (i), $K_4(e, e')$ is collapsible. $\square$

Lemma 4.2.12 Let H_4 be a graph depicted as Fig 4.2.4. Then H_4 is collapsible.

Proof Let K be the induced subgraph of $\{v_1, v_2, v_3, v_4\}$ in H_4 . By Theorem 1.2.4(ii), K is collapsible. Since $H_4/K \cong C_3$, by Theorem 1.2.1 (i) and (ii), H_4 is collapsible. $\square$

Figure 4.2.4

Figure 4.2.5

Lemma 4.2.13 *Let H_i ($5 \leq i \leq 12$) be a graph depicted as Fig 4.2.5. Each of the following holds.*

- (i) *Both H_5 and H_6 are collapsible.*
- (ii) *For $\forall X \subseteq E(H_7)$ with $|X| = 2$ and $e_0 \notin X$, $H_7(X)$ is collapsible.*
- (iii) *For $\forall X \subseteq E(H_8)$ with $|X| = 2$, $H_8(X)$ is collapsible.*
- (iv) *Both H_9 and H_{10} are collapsible.*
- (v) *Both H_{11} and H_{12} are collapsible.*

Proof (i) Note that H_5 contains a C_3 and $(H_5/C_3) \cong C_3$. Then by Theorem 1.2.1 (i) and (ii), H_5 is collapsible. Note that H_6 contains a C_2 and H_6/C_2 is collapsible by Theorem 1.2.4(ii). Then by Theorem 1.2.1 (i) and (ii), H_6 is collapsible.

(ii) Suppose there exists $X \subseteq E(H_7)$ with $|X| = 2$ and $e_0 \notin X$ such that $H_7(X)$ is not collapsible. Denote G_1 and G_2 to be the induced subgraph of $\{v_1, v_3, v_5, v_4\}$ and $\{v_1, v_2, v_5, v_4\}$, respectively. If $|X \cap E(G_1)| \leq 1$, then $G_1(X)$ is isomorphic to a contraction of H_5 , which is collapsible. Then $G_1(X)$ is collapsible. Note that $F(H_7(X)/G_1(X)) \leq 1$ and it can not be contracted to a K_2 . Thus by Theorem 1.2.4, $H_7(X)/G_1(X)$ is collapsible, and by Theorem 1.2.1 (i), $H_7(X)$ is collapsible, a contradiction. So $X \subseteq E(G_1)$. If $|X \cap \{v_1v_5, v_4v_5\}| \leq 1$, then $G_2(X)$ is isomorphic to a contraction of H_5 and by a similar argument, we could claim that $H_7(X)$ is collapsible, a contradiction. Thus $X = \{v_1v_5, v_4v_5\}$. Then by Lemma 4.2.12, $H_7(X)$ is collapsible, a contradiction.

(iii) For $\forall X \subseteq E(H_8)$ with $|X| = 2$, $F(H_8(X)) = 0$. Thus by Theorem 1.2.4(ii), $H_8(X)$ is collapsible.

(iv) can be viewed as special cases of (ii).

(v) Denote C to be the 4-circuit $v_1v_2v_3v_4$ in H_{11} . By (iii), $H_{11}/\pi(C)$ is collapsible, and by Theorem 1.2.6 (i), H_{11} is collapsible. Denote C' to be the 4-circuit $v_1v_2v_3v_4$ in H_{12} . Since $H_{12}/\pi(C') \cong H_{11}$ is collapsible, by Theorem 1.2.6 (i), H_{12} is collapsible. $\square$

Lemma 4.2.14 *Let H_i ($i = 13, 14, 15$) be a graph depicted as Fig 4.2.6. Then H_i is collapsible.*

Figure 4.2.6

Figure 4.2.7

Proof Note that $F(H_i) = 2$ for $i = 13, 14, 15$. By Theorem 1.2.4 (iii), H_i is collapsible.

□

Lemma 4.2.15 Let H_i ($i = 16, 17, 18$) be a graph depicted as Fig 4.2.7. Then H_i is collapsible.

Proof Note that $F(H_i) = 0$ for $i = 16, 17, 18$. By Theorem 1.2.4 (ii), H_i is collapsible.

□

Lemma 4.2.16 Let H_i ($19 \leq i \leq 22$) be a graph depicted as Fig 4.2.8. Each of the following holds.

(i) For $\forall X \subseteq E(H_{19})$ with $|X| = 2$, $|X \cap \{e_1, e_2, e_3, e_4\}| = 1$ and $|X \cap \{e_5, e_6\}| = 1$,

Figure 4.2.8

$H_{19}(X)$ is collapsible.

(ii) For $\forall X \subseteq E(H_{20})$ with $|X| = 2$, $|X \cap \{e_1, e_2, e_3\}| = 1$ and $|X \cap \{e_4, e_5\}| = 1$, $H_{20}(X)$ is collapsible.

(iii) For $\forall X \subseteq E(H_{21})$ with $|X| = 2$, $|X \cap \{e_1, e_2, e_3\}| = 1$ and $e_4 \in X$, $H_{21}(X)$ is collapsible.

(iv) $H_{22}(e_1, e_2)$ is collapsible.

Proof (i) Without loss of generality, we may assume that $e_4 \in X$. Note that e_4 and e_5 form a 2-circuit in H_{19} , denoted by C . Then $C(e_4)$ is a triangle in $H_{19}(X)$. Since $H_{19}(X)/C(e_4)$ can be viewed as a graph obtained by subdividing two edges of a K_4 , by Lemma 4.2.11, $H_{19}(X)/C(e_4)$ is collapsible. Thus by Theorem 1.2.1 (i), $H_{19}(X)$ is collapsible.

(ii) can be proved similarly.

(iii) Denoted by C to be the 2-circuit in H_{21} containing e_4 . Then $C(e_4)$ is a triangle in $H_{21}(X)$. Since 3-circuit is collapsible, the reduction of $H_{21}(X)/C(e_4)$ is K_1 , and $H_{21}(X)/C(e_4)$ is collapsible. Thus by Theorem 1.2.1 (i), $H_{21}(X)$ is collapsible.

(iv) $H_{22}(e_1, e_2)$ contains a subgraph isomorphic to H_5 and $H_{22}(e_1, e_2)/H_5$ is isomorphic to C_2 . Thus by Lemma 4.2.13(i) and Theorem 1.2.1 (i), $H_{22}(e_1, e_2)$ is collapsible. $\square$

4.3 Local Graphs of Small Degree Vertices in a P3D Graph

In this section, we shall investigate the local structure of vertices with degree at most 3 in a P_3 -dominating graph. Such analysis will be useful in the proof of the main result. Throughout this section, G denotes a connected P_3 -dominating graph. To avoid triviality, we assume that $|V(G)| \geq 2$. For $\forall u \in D_3(G)$, the **local subgraph** of u , denoted by $L(u)$, is a subgraph of G containing $E_G(u)$.

Lemma 4.3.1 *Let G be an essential 4-edge-connected P3D graph and $u \in D_3(G)$ with $N_G(u) = \{a_1, a_2, a_3\}$. If $N_G(u)$ is an independent set, then $L(u)$ must be one member of $\{L_1, L_2, L_3, L_4\}$, which are depicted as Fig 4.3.1.*

Proof Let $u \in D_3(G)$ be given. Suppose $N_G(u) = \{a_1, a_2, a_3\}$. There are two possible cases: (i) there exists $i, j \in \{1, 2, 3\}$ such that $J(a_i, a_j) = \emptyset$; (ii) or for $\forall i, j \in \{1, 2, 3\}$, $J(a_i, a_j) \neq \emptyset$

Case 1 $J(a_i, a_j) = \emptyset$ for some $i, j \in \{1, 2, 3\}$

Without loss of generality, we could assume that $J(a_1, a_3) = \emptyset$.

Claim 1 $u \in J'(a_1, a_3)$.

Suppose $u \notin J'(a_1, a_3)$. Since G is a P3D graph and $\text{dist}_G(a_1, a_3) = 2$, choose $x \in J'(a_1, a_3)$. Then by the definition of $J'(a_1, a_3)$, there exists $y \in N_G(x) - (N_G(a_1) \cup N_G(a_3))$

Figure 4.3.1

such that $N_G(a_1) \cup N_G(a_3) \cup N_G(x) - \{a_1, a_3, y\} \subseteq N_G(y)$. Therefore $u \in N_G(y)$, and then $y = a_2$. Thus $N_G(a_1) \cup N_G(a_3) - \{a_1, a_2, a_3\} \subseteq N_G(a_2)$, and $u \in J'(a_1, a_3)$, which contradicts to $u \notin J'(a_1, a_3)$. This verifies Claim 1.

Since G is essentially 4-edge-connected, $d_G(a_i) \geq 2$ ($i = 1, 2, 3$). Choose $x_1, x_2 \in N_G(a_1) - u$ and $y_1, y_2 \in N_G(a_3) - u$ such that $|\{x_1, x_2\} \cap \{y_1, y_2\}|$ as big as possible. Then $X = \{x_1, x_2, y_1, y_2\}$ is a collection of vertices of G .

Subcase 1.1 $|X| = 2$ (that is $|\{x_1, x_2\} \cap \{y_1, y_2\}| = 2$)

Then $\{x_1, x_2\} = \{y_1, y_2\}$ and $L(u) = L_1$.

Subcase 1.2 $|X| = 3$ (that is $|\{x_1, x_2\} \cap \{y_1, y_2\}| = 1$)

Assume that $\{x_1, x_2\} \cap \{y_1, y_2\} = x_2$. Then $L(u) = L_2$.

Subcase 1.3 $|X| = 4$ (that is $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$)

Claim 2 $x_1x_2 \in E(G)$ and $y_1y_2 \in E(G)$.

Note that, by the choice of X , $x_1, x_2 \notin N_G(a_3)$. It is sufficient to show that $x_1x_2 \in E(G)$. By way of contradiction, suppose $x_1x_2 \notin E(G)$. Since $\text{dist}_G(u, x_1) = 2$, $J(u, x_1) \cup J'(u, x_1) \neq \emptyset$, and $J(u, x_1) \cup J'(u, x_1) \subseteq N_G(u) \cap N_G(x_1) = \{a_1, a_2\}$. Assume that $a_i \in J(u, x_1) \cup J'(u, x_1)$ for some $i \in \{1, 2\}$. Since $x_2 \notin N_G(u) \cup N_G(x_1)$, $a_i \notin J(u, x_1)$. Thus $a_i \in J'(u, x_1)$ and $N_G(u) \cup N_G(x_1) \cup N_G(a_i) - \{u, x_1, a_i\} \subseteq N_G(x_2)$. Therefore $x_2a_3 \in E(G)$, which contradicts to $x_2 \notin N_G(a_3)$. This verifies Claim 2.

By Claim 2, when $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$, $L(u) = L_3$.

Case 2 for $\forall i, j \in \{1, 2, 3\}$, $J(a_i, a_j) \neq \emptyset$

Claim 3 $J(a_i, a_j) \cap J(a_i, a_k) = \emptyset$ where $\{i, j, k\} = \{1, 2, 3\}$.

By way of contradiction, we could assume that $x \in J(a_1, a_2) \cap J(a_1, a_3)$. Since $x \in J(a_1, a_2)$, $N_G(x) \subseteq N_G(a_1) \cup N_G(a_2)$. Since $x \in J(a_1, a_3)$, $xa_3 \in E(G)$. Thus $a_3 \in N_G(a_1) \cup N_G(a_2)$, which contradicts to $N_G(u)$ is an independent set. This verifies Claim 3.

By Claim 3, choose b_3, b_4 and b_5 such that $b_{i+j} \in J(a_i, a_j)$ and $b_{i+j}a_k \notin E(G)$ since $a_k \notin N_G(a_i) \cup N_G(a_j)$ ($\{i, j, k\} = \{1, 2, 3\}$).

Claim 4 $G[\{b_3, b_4, b_5\}]$ has at least two edges.

By way of contradiction, suppose $G[\{b_3, b_4, b_5\}]$ has at most one edge. Assume that $b_3b_4, b_3b_5 \notin E(G)$. Since $\text{dist}_G(u, b_3) = 2$, $J(u, b_3) \cup J'(u, b_3) \neq \emptyset$, and $J(u, b_3) \cup J'(u, b_3) \subseteq N_G(u) \cap N_G(b_3) = \{a_1, a_2\}$. Assume that $a_i \in J(u, b_3) \cup J'(u, b_3)$ and $\{i, j\} = \{1, 2\}$. Since $b_{i+3} \notin N_G(u) \cup N_G(b_3)$, $a_i \notin J(u, b_3)$. Thus $a_i \in J'(u, b_3)$, and then $N_G(u) \cup N_G(b_3) \cup N_G(a_i) - \{u, b_3, a_i\} \subseteq N_G(b_{i+3})$. Therefore $b_{i+3}a_j \in E(G)$, which contradicts to $b_4a_2, b_5a_1 \notin E(G)$. This verifies Claim 4.

By Claim 4, we could assume that $b_3b_4, b_4b_5 \in E(G)$. Then $L(u) = L_4$. $\square$

Note 1. In L_2 , $a_1x_3, a_3x_1 \notin E(G)$, otherwise, the local graph will be L_1 .

Lemma 4.3.2 Let $H \in \{L_1, L_2, L_3, L_4\}$ and let $X \subseteq E(H)$. Then each of the following holds.

- (i) If $|X| \leq 1$, then $H(X)$ is collapsible.
(ii) If $|X| \leq 2$, then either $H(X)$ is collapsible, or $H = L_2$.

Proof By Theorem 1.2.1 (vi), it is sufficient to show that $L_2(X)$ is collapsible for $\forall X \subseteq E(L_2)$ with $|X| = 1$, and $L_i(X)$ is collapsible for $\forall X \subseteq E(L_i)$ with $|X| = 2$ ($i = 1, 3, 4$).

Let $X_1 \subseteq E(L_1)$ with $|X_1| = 2$. Since $L_1 \cong K_{3,3}$, by symmetry, we can assume that $X_1 = \{x_1y_1, x_1y_2\}$ or $\{x_1y_1, x_2y_2\}$. Let $C_1 = \begin{cases} x_2y_2x_3y_3, & \text{if } X_1 = \{x_1y_1, x_1y_2\} \\ x_2y_1x_3y_3, & \text{if } X_1 = \{x_1y_1, x_2y_2\} \end{cases}$ be a 4-circuit of $L_1(X_1)$. Then $L_1(X_1)/\pi(C_1)$ is isomorphic to either H_1 or H_2 of Fig 3.1, which are collapsible by Lemma 4.2.9. Thus, by Theorem 1.2.6 (i), $L_1(X_1)$ is collapsible.

Let $X_2 \subseteq E(L_2)$ with $|X_2| = 1$. Suppose $X_2 = \{e\}$. Note that both $L_2 - x_1$ and $L_2 - y_1$ are isomorphic to $K_{3,3} - e$, which is collapsible by Theorem 1.2.1 (vii). If $e \in E_{L_2}(x_1)$, then both $L_2 - x_1$ and $L_2(e)/(L_2 - x_1) \cong C_3$ are collapsible. Thus by Theorem 1.2.1 (i), $L_2(e)$ is collapsible. By symmetry, $L_2(e)$ is collapsible if $e \in E_{L_2}(y_1)$. Therefore, $e \in E(L_2) \subseteq (E_{L_2}(x_1) \cup E_{L_2}(y_1))$, and by symmetry, we may assume that $e = ua_1$ or ua_2 . Let $C_2 = a_1x_1a_2x_2$ be a 4-circuit of $L_2(e)$. Then $L_2(e)/\pi(C_2)$ is isomorphic to H_3 of Fig 3.2, which are collapsible by Lemma 4.2.10. Thus, by Theorem 1.2.6 (i), $L_2(e)$ is collapsible.

Let $X_3 \subseteq E(L_3)$ with $|X_3| = 2$. Denote $V_1 = \{u, a_1, a_2, x_1, x_2\}$ and $V_2 = \{u, a_2, a_3, y_1, y_2\}$. Denote that $G_i = L_3[V_i]$ ($i = 1, 2$). Denote that $X_3^{(i)} = X \cap E(G_i)$ for $i = 1, 2$. Without loss of generality, we assume that $|X_3^{(1)}| \leq |X_3^{(2)}|$. Then $|X_3^{(1)}| \leq 1$ and $G_1(X_3^{(1)})$ is isomorphic to either H_4 of Fig 3.4 or $K_4(e')$. By Lemma 4.2.11, Lemma 4.2.12 and Theorem 1.2.1 (vi), $G_1(X_3^{(1)})$ is collapsible. Note that $L_3(X_3)/G_1(X_3^{(1)})$ is isomorphic to either $K_4(e')$ or $K_4(e', e'')$. By Lemma 4.2.11 and Theorem 1.2.1 (vi), $L_3(X_3)/G_1(X_3^{(1)})$ is also collapsible. Thus by Theorem 1.2.1 (i), $L_3(X_3)$ is collapsible.

Let $X_4 \subseteq E(L_4)$ with $|X_4| = 2$. Denote $C_4 = ua_1b_4a_3$ and $C'_4 = a_2b_3b_4b_5$, which are two 4-circuits of L_4 . If $X_4 \cap E(C_4) = \emptyset$, then $L_4(X_4)/\pi(C_4)$ is isomorphic to $H_7(e', e'')$ (Fig 4.2.5) where $e_0 \notin \{e', e''\}$. By Lemma 4.2.13 (ii), $L_4(X_4)/\pi(C_4)$ is collapsible, and then by Theorem 1.2.6 (i), $L_4(X_4)$ is collapsible. So we may assume that $X_4 \cap E(C_4) \neq \emptyset$. Similarly, $X_4 \cap E(C'_4) \neq \emptyset$. Denote $G_4^1 = L_4[\{u, a_1, a_2, b_3, b_4\}]$ and $G_4^2 = L_4[\{u, a_2, a_3, b_4, b_5\}]$. Note

Figure 4.3.2

that $G_4^1 \cong H_5$ (Fig 4.2.5), which is collapsible by Lemma 4.2.13 (i). If $X \cap E(G_4^1) = \emptyset$, then $L_4(X)/G_4^1$ is isomorphic to $H_8(e', e'')$ of Fig 4.2.5, which is collapsible by Lemma 4.2.13 (iii). Thus by Theorem 1.2.1 (i), $L_4(X_4)$ is collapsible. So we may assume that $X_4 \cap E(G_4^1) \neq \emptyset$. Similarly, $X_4 \cap E(G_4^2) \neq \emptyset$. Hence, by symmetry, we may assume that $|X \cap \{ua_1, a_1b_4\}| = |X \cap \{a_2b_5, b_4b_5\}| = 1$.

Suppose $X_4 = \{e', e''\}$ where $e' \in \{ua_1, a_1b_4\}$ and $e'' \in \{a_2b_5, b_4b_5\}$. Denote $J_1 = a_1b_3b_4$ and $J_2 = a_3b_4b_5$, which are two triangle of L_4 . If $X_4 \cap J_1 = \emptyset$, then $L_4(X_4)/J_1$ is isomorphic to either H_9 or H_{10} (Fig 4.2.5), which are collapsible by Lemma 4.2.13 (iv). Therefore $X_4 \cap J_1 \neq \emptyset$ and then $e' = a_1b_4$. Similarly, $X_4 \cap J_2 \neq \emptyset$ and $e'' = b_4b_5$. Hence $X_4 = \{a_1b_4, b_4b_5\}$ and $L_4(X_4)$ is isomorphic to H_{12} (Fig 4.2.5), which is collapsible by Lemma 4.2.13 (v). $\square$

Lemma 4.3.3 *Let $J_1 = L_2 \cup x_1x_2$ be the graph depicted in Fig 4.3.2. For any $X \subseteq E(J_1)$ with $|X| = 2$, $J_1(X)$ is not reduced.*

Proof Suppose $X = \{e_1, e_2\}$ is a subset of $E(J_1)$ such that $J_1(X)$ is reduced. Since $L_2 \subseteq J_1$, by Lemma 4.3.2 (i), $X \subseteq E(L_2)$ and $x_1x_2 \notin X$. Since $C_3 = a_1x_1x_2$ is collapsible, $X \cap \{a_1x_1, a_1x_2\} \neq \emptyset$. Since $C_3 = a_2x_1x_2$ is collapsible, $X \cap \{a_2x_1, a_2x_2\} \neq \emptyset$. Thus we may assume that $e_1 \in \{a_1x_1, a_1x_2\}$ and $e_2 \in \{a_2x_1, a_2x_2\}$. Since $J_1 - x_1 \cong K_{3,3} - e$, which is collapsible by Theorem 1.2.1 (vii), $X - N_{J_1}(x_1) \neq \emptyset$. Thus $X = \{a_1x_1, a_2x_2\}$

Figure 4.3.3

or $\{a_1x_2, a_2x_1\}$ or $\{a_1x_2, a_2x_2\}$. Denote C to be the 4-circuit $ua_2x_3a_3$ in $J_1(X)$. Since $J_1(X)/\pi(C)$ is isomorphic to a member in $\{H_{13}, H_{14}, H_{15}\}$, by Lemma 4.2.14 and Theorem 1.2.6 (i), $J_1(X)$ is collapsible, a contradiction to the fact that $J_1(X)$ is reduced. $\square$

Lemma 4.3.4 *Let $J_2 = L_2 \cup x_1x_3$ be the graph depicted in Fig 4.3.3. For any $X \subseteq E(J_2)$ with $|X| = 2$, $J_2(X)$ is not reduced.*

Proof Suppose $X = \{e_1, e_2\}$ is a subset of $E(J_2)$ such that $J_2(X)$ is reduced.

Since $L_2 \subseteq J_2$, by Lemma 4.3.2 (i), $X \subseteq E(L_2)$ and $x_1x_3 \notin X$. Since $C_3 = a_2x_1x_3$ is collapsible, $X \cap \{a_2x_1, a_2x_3\} \neq \emptyset$. By symmetry, without loss of generality, we may assume that $e_1 = a_2x_1$. Since $J_2 - x_1 \cong K_{3,3} - e$, which is collapsible by Theorem 1.2.1 (vii), $X - N_{J_2}(x_1) \neq \emptyset$. Thus $e_2 \in E(J_2) - N_{J_2}(x_1)$. Let $C =$

$$\begin{cases} ua_2x_3a_3, & \text{if } e_2 \in \{ua_1, a_1x_2, a_2x_2, a_3x_2\} \\ a_2x_2a_3x_3, & \text{if } e_2 \in \{ua_2, ua_3\} \\ ua_1x_2a_2, & \text{if } e_2 \in \{a_2x_3, a_3x_3\} \end{cases}$$

be a 4-circuit of J_2 with $E(C) \cap X = \emptyset$.

Then $J_2/\pi(C)$ is isomorphic to a member of $\{H_{16}, H_{17}, H_{18}\}$ (Fig 4.2.7), and by Lemma 4.2.15, $F(J_2/\pi(C)) = 0$. Thus $F(J_2(X)/\pi(C)) \leq 2$. Note that $J_2(X)/\pi(C)$ can not be contracted to K_2 or a member in $\{K_{2,t} : t \geq 1\}$. Hence by Theorem 1.2.4 (iii) and Theorem 1.2.6 (i), $J_2(X)/\pi(C)$ is collapsible and $J_2(X)$ is collapsible, a contradiction to the fact that $J_2(X)$ is reduced. $\square$

Figure 4.3.4

Lemma 4.3.5 *Let J_3 be the graph depicted in Fig 4.3.4. If $J_3(X)$ is reduced for some $X \subseteq E(J_3)$ with $|X| = 2$, then $X = \{ua_2, ua_3\}$ or $\{a_1x_1, a_1x_2\}$.*

Proof Note that J_3 has an automorphism (Fig 4.3.4). Suppose there exists $X \subseteq E(J_3)$ with $|X| = 2$, such that X is neither $\{ua_2, ua_3\}$ nor $\{a_1x_1, a_1x_2\}$, and $J_3(X)$ is reduced.

Denote $E_1 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $E_2 = \{e_7, e_8\}$. Since $L_2 \subseteq J_3$, by Lemma 4.3.2 (i), $X \subseteq E(L_2)$ and $N_{J_3}(y) \cap X = \emptyset$. By the automorphism of J_3 , $N_{J_3}(x_3) \cap X = \emptyset$. Thus $X \subseteq E_1 \cup E_2$. Since $J_3 - \{x_1, y\} \cong K_{3,3} - e$, which is collapsible by Theorem 1.2.1 (vii), $X - E_2 \neq \emptyset$. So $|X \cap E_1| \geq 1$.

Case 1 $|X \cap E_1| = 2$, which means $X \subseteq E_1$.

Since $X \neq E_2$, $X \neq \{e_3, e_6\}$ by the automorphism. Also by the automorphism, $J_3(e_1, e_2) \cong J_3(e_1, e_4)$ and $J_3(e_2, e_5) \cong J_3(e_4, e_5)$. So we may assume that $X \neq \{e_1, e_4\}$, $\{e_4, e_5\}$.

Let

$$C = \begin{cases} a_2x_2a_3x_3, & \text{if } X \in \{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}\} \\ ua_2x_3a_3, & \text{if } X \in \{\{e_1, e_5\}, \{e_1, e_6\}, \{e_4, e_6\}\} \\ a_1x_1yx_2, & \text{if } X \in \{\{e_2, e_5\}, \{e_3, e_5\}, \{e_5, e_6\}\} \\ a_1x_1a_2x_2, & \text{if } X = \{e_2, e_6\} \end{cases}$$

be a 4-circuit of J_3 with $E(C) \cap X = \emptyset$. Then $J_3/\pi(C)$ is isomorphic to a member of $\{H_{19}, H_{20}, H_{21}, H_{22}\}$ (Fig 4.2.8). By Lemma 4.2.16 (i)-(iv), $J_3(X)/\pi(C)$ is collapsible. Hence by Theorem 1.2.6 (i), $J_3(X)$ is collapsible, a contradiction to the fact that $J_3(X)$ is reduced.

Case 2 $|X \cap E_1| = 1$.

In this case $|X \cap E_1| = 1$ and $|X \cap E_2| = 1$. If $(X \cap E_1) \subseteq \{e_1, e_2, e_4, e_5\}$, then by the automorphism, it could be solved similarly by case 1. So we assume that $(X \cap E_1) \subseteq \{e_3, e_6\}$. Also by the automorphism, $J_3(e_3, e_8) \cong J_3(e_6, e_7)$. So we may assume that $X \neq \{e_6, e_7\}$.

Let

$$C = \begin{cases} a_2x_2a_3x_3, & \text{if } X \in \{\{e_3, e_7\}, \{e_3, e_8\}\} \\ ua_2x_3a_3, & \text{if } X = \{e_6, e_8\} \end{cases}$$

be a 4-circuit of J_3 with $E(C) \cap X = \emptyset$. Then $J_3/\pi(C)$ is isomorphic to either $\{H_{19}$ or H_{20} (Fig 4.2.8). By Lemma 4.2.16 (i) and (ii), $J_3(X)/\pi(C)$ is collapsible. Hence by Theorem 1.2.6 (i), $J_3(X)$ is collapsible, a contradiction to the fact that $J_3(X)$ is reduced. $\square$

Lemma 4.3.6 *Let J_4 be the graph depicted in Fig 4.3.5 . Then $J_4 - \{v_1, v_2\} = L_2$. For $\forall X \subseteq E(L_2)$ with $|X| = 2$, $J_4(X)$ is not reduced.*

Proof Suppose there exists $X \subseteq E(L_2)$ with $|X| = 2$ such that $J_4(X)$ is reduced. Since $(L_2 - x_1) \cong K_{3,3} - e$ is collapsible, $|X \cap \{a_1x_1, a_2x_1\}| \leq 1$. Let

$$C = \begin{cases} zyx_1a_2, & \text{if } a_1x_1 \in X \\ zyx_1a_1, & \text{if } a_2x_1 \in X \\ zyx_1a_1, & \text{if } \{a_1x_1, a_2x_1\} \cap X = \emptyset \end{cases}$$

Figure 4.3.5

If $|X \cap \{a_1x_1, a_2x_1\}| = 1$ and assume that $X - \{a_1x_1, a_2x_1\} = e$, then $J_4(X)/\pi(C)$ contains a triangle C_3 , and $(J_4(X)/\pi(C))/C_3$ is isomorphic to $J'_4(e)$, which is depicted in Fig 4.5.3 (b). Note that J'_4 is a contraction of L_2 . By Lemma 4.3.2 (i) and Theorem 1.2.1 (vi), $L_2(e)$ is collapsible and $J'_4(e)$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $J_4(X)/\pi(C)$ is collapsible and $J_4(X)$ is collapsible, a contradiction.

So we may assume that $\{a_1x_1, a_2x_1\} \cap X = \emptyset$. Let $V_1 = \{z, y, a_1, a_2, x_1, u\}$ and $V_2 = \{z, y, a_1, a_2, x_1, x_2\}$. Since $J_4[V_1] \cong K_{3,3} - e$, $X \cap \{ua_1, ua_2\} \neq \emptyset$. Similarly, since $J_4[V_2] \cong K_{3,3} - e$, $X \cap \{a_1x_2, a_2x_2\} \neq \emptyset$. Thus $|X \cap \{ua_1, ua_2\}| = |X \cap \{a_1x_2, a_2x_2\}| = 1$. Assume that $e' = X \cap \{a_1x_2, a_2x_2\}$. In $J_4(X)/\pi(C)$, there is a 2-circuit C_2 , and $(J_4(X)/\pi(C))/C_2$ is isomorphic to $J'_4(X)$. Note that the triangle ua_1a_2 of $J'_4(X)$ becomes a 4-circuit in $J'_4(X)$, which we may denote by C' . Then $J'_4(X)/\pi(C')$ is isomorphic to either $J''_4(e')$ or $J'''_4(e')$. Since $F(J''_4) = F(J'''_4) = 1$, $F(J''_4(e')) = F(J'''_4(e')) = 2$. By Theorem 1.2.4 (iii), both $J''_4(e')$ and $J'''_4(e')$ are collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $J'_4(X)/\pi(C')$, $J_4(X)/\pi(C)$ and $J_4(X)$ are collapsible, a contradiction to the fact that $J_4(X)$ is reduced. $\square$

Figure 4.3.6

Lemma 4.3.7 In L_2 , let $X \subseteq E(L_2) - N(u)$ with $|X| = 2$. Then each of the following holds.

- (i) If $L_2(X)$ is reduced, then $X \in \left\{ \{a_1x_1, a_3x_3\}, \{a_1x_1, a_2x_3\}, \{a_1x_2, a_2x_1\}, \{a_1x_1, a_1x_2\}, \{a_1x_2, a_2x_2\}, \{a_1x_2, a_3x_2\}, \{a_2x_1, a_2x_3\} \right\}$.
- (ii) Let $J_5 = L_2 \cup a_1ya_3$. If $J_5(X)$ is reduced, then $X \in \left\{ \{a_1x_2, a_2x_2\}, \{a_1x_2, a_3x_2\} \right\}$.
- (iii) Let $J_3 = L_2 \cup x_1yx_2$ (Fig 4.3.4). If $J_3(X)$ is reduced, then $X = \{a_1x_1, a_1x_2\}$.

Proof (i) Considering symmetry of L_2 , by way of contradiction, we may assume that there exists $X \in \left\{ \{a_1x_1, a_2x_2\}, \{a_1x_1, a_3x_2\}, \{a_2x_1, a_3x_2\}, \{a_1x_1, a_2x_1\}, \{a_2x_1, a_2x_2\} \right\}$ such that $L_2(X)$ is not reduced. Let C be the 4-circuit $ua_2x_3a_3$ in L_2 . If $X = \{a_1x_1, a_2x_2\}$, then $(L_2 - x_1) (\cong K_{3,3} - e)$ is a collapsible subgraph of $L_2(X)$, a contradiction. So we may assume that $X \neq \{a_1x_1, a_2x_2\}$. Then in $L_2/\pi(C)$, denote C' to be the 2-circuit a_2x_2 , which becomes a 3-circuit $C'(X)$ in $L_2(X)/\pi(C)$ (Fig 4.3.6). Then $(L_2(X)/\pi(C))/C'(X) (\cong H_5)$ is collapsible by Lemma 4.2.13 (i). Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L_2(X)/\pi(C)$ is collapsible and $L_2(X)$ is collapsible, a contradiction.

- (ii) By way of contradiction, suppose there exists $X \notin \left\{ \{a_1x_2, a_2x_2\}, \{a_1x_2, a_3x_2\} \right\}$ such that $J_5(X)$ is reduced. By (i), $X \in \left\{ \{a_1x_1, a_3x_3\}, \{a_1x_1, a_2x_3\}, \{a_1x_2, a_2x_1\}, \{a_1x_1, a_1x_2\}, \right.$

Figure 4.3.7

$$\{a_2x_1, a_2x_3\}\}.$$

Let

$$C = \begin{cases} ua_1x_2a_2, & \text{if } X \in \left\{ \{a_1x_1, a_3x_3\}, \{a_1x_1, a_2x_3\}, \{a_2x_1, a_2x_3\} \right\} \\ a_2x_2a_3x_3, & \text{if } X \in \left\{ \{a_1x_2, a_2x_1\}, \{a_1x_1, a_1x_2\} \right\} \end{cases}$$

Denote $J'_5 = J_5/\pi(C)$ which is depicted as Fig 4.3.7 (b) (if $C = ua_1x_2a_2$) or Fig 4.3.7 (c) (if $C = a_2x_2a_3x_3$). Note that $E(C) \cap X = \emptyset$, and then C is also a 4-circuit in $J_5(X)$. Thus $J_5(X)/\pi(C) = J'_5(X)$ Let

$$V_0 = \begin{cases} \{u, a_1, y, a_3\}, & \text{if } C = ua_1x_2a_2 \\ \{u, a_1, y, a_2\}, & \text{if } C = a_2x_2a_3x_3 \end{cases}$$

Then the induced subgraph of V_0 in J'_5 , denoted by H , is collapsible. Denote $J''_5 = J'_5/H$, which is depicted as Fig 4.3.7 (d). Note that $X \cap E(H) = \emptyset$. Then $J'_5(X)/H = J''_5(X)$,

which is a collapsible depicted as Fig 4.3.7 (e). Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $J'_5(X)$ is collapsible and $J_5(X)$ is collapsible, a contradiction.

(iii) By Lemma 4.3.5, $X = \{a_1x_1, a_1x_2\}$. $\square$

4.4 Hamiltonian Connectedness of P_3 -dominating Graphs

In order to prove Theorem 1.7, we prove the following theorem first.

Theorem 4.4.1 *Let G be a 3-edge-connected, essentially 4-edge-connected graph. For $\forall u \in D_3(G)$, if $N_G(u)$ is an independent set, then u lies in a local subgraph $L(u)$ isomorphic to a contraction of a member in $\{L_1, L_2, L_3, L_4\}$ (Fig 4.4.1). Then for any $X \subseteq E(G)$ with $|X| \leq 2$, $G(X)$ is collapsible.*

Proof For $\forall u \in D_3(G)$, if $N_G(u)$ is not an independent set, then we assume its local subgraph $L(u)$ is isomorphic to L_5 (Fig 4.4.1). By contradiction, we assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \quad (4.1)$$

Then for some edge set X with $|X| \leq 2$,

$$G(X) \text{ is not collapsible.} \quad (4.2)$$

By Theorem 1.2.1 (vi), we may assume that for $\forall u \in D_3(G)$, the local subgraph $L(u)$ is isomorphic to a member in $\{L_1, L_2, L_3, L_4, L_5\}$

Claim 1 $G(X)$ can not be contracted to a member in $\{K_2\} \cup \{K_{2,t} : t \geq 1\}$.

Otherwise, it will contradict to the facts that $\kappa'(G) \geq 3$ and $D_2(G(X)) \leq 2$.

Claim 2 $D_3(G) \neq \emptyset$.

Figure 4.4.1

Suppose $D_3(G) = \emptyset$. Since G is 3-edge-connected and essentially 4-edge-connected, every 3-edge-cut of G (if it exists) must be peripheral. Since $D_3(G) = \emptyset$, G is 4-edge-connected. By Theorem 1.2.2 (i), G has 2 edge-disjoint spanning trees and $F(G) = 0$. Then $F(G(X)) \leq 2$. By (4.2) and Theorem 1.2.4 (iii), $G(X)$ can be contracted to a member in $\{K_2\} \cup \{K_{2,t} : t \geq 1\}$, which contradicts Claim 1. This verifies Claim 2.

Claim 3 $G(X)$ is reduced.

By way of contradiction, let H be a collapsible subgraph of $G(X)$. Denote $X_1 = X \cap E(H)$ and $X_2 = X - X_1$. Let $H_0 = H/X_1$ and $G' = G/H_0$. Note that G' is a graph satisfying the hypothesis and $|V(G')| < |V(G)|$. Thus by (4.1), $G'(X_2)$ is collapsible. Since $G'(X_2) = G(X)/H$, by Theorem 1.2.1 (i), $G(X)$ is collapsible, a contradiction to (4.2). This verifies Claim 2.

Claim 4 $|D_3(G)| \geq 6$.

Since $\delta(G) \geq \kappa'(G) = 3$, $\delta(G(X)) \geq 2$ and $D_2(G(X)) = |X| \leq 2$. By Claim 1 and Theorem 1.2.4 (iii), $F(G) \geq 3$. By Claim 3 and Theorem 1.2.4 (i), $F(G(X)) =$

$2|V(G(X))| - |E(G(X))| - 2$. Then

$$\begin{aligned} 6 &\leq 2F(G(X)) = 4|V(G(X))| - 2|E(G(X))| - 4 \\ &= 4(|V(G)| + |X|) - 2(|E(G)| + |X|) - 4 \\ &= 4|V(G)| - 2|E(G)| + 2|X| - 4. \end{aligned} \quad (4.3)$$

Since $\delta(G) \geq 3$, we have

$$2|E(G)| \geq 4(|V(G) - D_3(G)|) + 3|D_3(G)| = 4|V(G)| - |D_3(G)|. \quad (4.4)$$

Thus, by (4.4) and (4.4)

$$|D_3(G)| \geq 10 - 2|X| \geq 10 - 4 = 6.$$

This verifies Claim 4.

Denote $D'_3(G) = \{u \in D_3(G) : L(u) \cong L_i, 1 \leq i \leq 4\}$. By Theorem 4.2.8, $D'_3(G) \neq \emptyset$. Denote $D''_3(G) = D_3(G) - D'_3(G) = \{u \in D_3(G) : L(u) \cong L_5\}$.

Claim 5 $|X| = 2$.

Suppose, by contradiction, that $|X| \leq 1$. Since $D'_3(G) \neq \emptyset$, choose $u \in D'_3(G)$, whose local subgraph $L(u)$ is isomorphic to a member in $\{L_1, L_2, L_3, L_4\}$. By Lemma 4.3.2 (i), $G(X)$ is not reduced, which contradicts to Claim 3. This verifies Claim 5.

Claim 6 For $\forall v \in D'_3(G)$, $L(u) \cong L_2$ and $X \subseteq E(L(u))$.

Claim 6 follows by Lemma 4.3.2 and Claim 3.

Claim 7 For $\forall v \in D'_3(G)$, $L(u)$ is an induced subgraph of G .

Choose $u \in D'_3(G)$, whose local graph $L(u_0) \cong L_2$. By Lemma 4.3.3 and Lemma 4.3.4, $\{x_1, x_2, x_3\}$ is an independent set. By Note 1 of Lemma 4.3.1, $a_1x_3, a_3x_1 \notin E(G)$. Thus $L(u_0)$ is a reduced subgraph of G . This verifies Claim 7.

Choose $u_0 \in D'_3(G)$, whose local graph $L(u_0) \cong L_2$ is shown in Fig 4.4.2 (a).

Claim 8 $a_1, a_3 \notin D_3(G)$.

Figure 4.4.2

By way of contradiction, we assume that $a_1 \in D_3(G)$. Since $N_G(a_1) = \{u, x_1, x_2\}$ is an independent set, $L(a_1) \cong L_2$ (Fig 4.4.2 (b)). Denote $H = L(u_0) \cup L(a_1)$ (Fig 4.4.2 (c)), which is isomorphic to J_3 (Fig 4.3.5). By Claim 3 and Lemma 4.3.5, $X = \{ua_2, ua_3\}$ or $\{a_1x_1, a_1x_2\}$. By the automorphism of J_3 , $H(ua_2, ua_3) \cong H(a_1x_1, a_1x_2)$. Thus without loss of generality, we may assume that $X = \{ua_2, ua_3\}$. If $a_3 \in D_3(G)$, then by a similar argument, $L(a_3) \cong L_2$ and $H' = L(u_0) \cup L(a_3)$ (Fig 4.4.2 (d)) is isomorphic to J_3 . By Lemma 4.3.5, $H'(X)$ is not reduced, a contradiction to Claim 3. Thus we may assume that $a_3 \notin D_3(G)$. Since $|D_3(G)| \geq 6$, choose $v \in D_3(G) - \{u_0, a_1\}$. Since $v \notin N_G(u_0) \cup \{x_2\}$, u_0 is not adjacent to v . If $L(v) \cong L_5$, then $X \cap E(L(v)) = \emptyset$, and $L(v)$ is not reduced, which contradicts to Claim 3. So $L(v) \cong L_2$. By Claim 6, $X \subseteq E(L(v))$. If $N_G(v) = N_G(u_0)$, then $(L(u_0) - u_0) \cup G[E_G(v)] (\cong L_2)$ is a collapsible subgraph of $G(X)$, a contradiction. So we may assume that $N_G(v) \neq N_G(u_0)$. Thus u_0 is of degree 2 in $L(v)$ (Fig 4.4.2 (e)). Thus $(L(v) - X) (\cong K_{3,3} - e)$ is a collapsible subgraph of $G(X)$, a contradiction to Claim 3. Hence $a_1 \notin D_3(G)$, and similarly $a_3 \notin D_3(G)$. This verifies Claim 8.

Claim 9 $x_1, x_3 \notin D'_3(G)$.

Figure 4.4.3

By way of contradiction, we assume that $x_1 \in D'_3(G)$. Then $L(a_1) \cong L_2$, and we may assume that $N_G(x_1) = \{a_1, a_2, y\}$. If $yx_2 \notin E(G)$, then by the structure of L_2 , there exists $z \in (N_G(a_1) \cap N_G(a_2) \cap N_G(y) - E(L(u_0)))$, and the graph is isomorphic to J_4 in Fig 4.3.5. By Lemma 4.3.6, $J_4(X)$ is not reduced, a contradiction to Claim 3. Thus $yx_2 \in E(G)$ and there are two possible cases for $L(x_1)$: Fig 4.4.3 (a) and (b).

Suppose $L(a_1)$ is shown as Fig 4.4.3 (a). Denote $H = L(u_0) \cup L(a_1)$ (Fig 4.4.3 (a)). Note that $H - z$ is isomorphic to J_3 . By Claim 3 and Lemma 4.3.5, $X = \{ua_2, ua_3\}$ or $\{a_1x_1, a_1x_2\}$. Since $(L(x_1) - u_0) \cong K_{3,3} - e$ is a collapsible subgraph in G , $X \cap E((L(x_1) - u_0)) \neq \emptyset$. Thus $X = \{a_1x_1, a_1x_2\}$. Note that H has an automorphism (Fig 4.4.3 (a)), and $H - x_3 \cong J_3$. Thus by Lemma 4.3.5, $(H - x_3)(X)$ is not reduced, a contradiction to Claim 3. Similarly, if $L(a_1)$ is shown as Fig 4.4.3 (b), then $X = \{a_1x_1, a_1x_2\}$ and $(H - x_3)(X)$ is not reduced, a contradiction. This verifies Claim 9.

Claim 10 $D'_3(G) \subseteq \{u_0, x_2\}$.

Figure 4.4.4

Suppose there exists $u_1 \in D'_3(G) - \{u_0, x_2\}$ such that $L(u_0) \cong L(u_2) \cong L_2$ (Fig 4.4.4). By Claim 8, Claim 9 and $u_1 \neq x_2$,

$$u_1 \notin V(L(u_0)), \text{ and similarly, } u_0 \notin V(L(u_1)). \quad (4.5)$$

Since $u_1 \neq x_2$,

$$N_G(u_0) \neq N_G(u_1). \quad (4.6)$$

By Claim 8,

$$u_1 \notin N_G(a_1) \cap N_G(a_2), \text{ and similarly, } u_1 \notin N_G(a_2) \cap N_G(a_3). \quad (4.7)$$

By Claim 3, Claim 6 and Lemma 4.3.7 (i), in $L(u_0)$,

$$X \in \left\{ \{a_1x_1, a_3x_3\}, \{a_1x_1, a_2x_3\}, \{a_1x_2, a_2x_1\}, \{a_1x_1, a_1x_2\}, \right. \\ \left. \{a_1x_2, a_2x_2\}, \{a_1x_2, a_3x_2\}, \{a_2x_1, a_2x_3\} \right\},$$

and in $L(u_1)$,

$$X \in \left\{ \{a'_1x'_1, a'_3x'_3\}, \{a'_1x'_1, a'_2x'_3\}, \{a'_1x'_2, a'_2x'_1\}, \{a'_1x'_1, a'_1x'_2\}, \right. \\ \left. \{a'_1x'_2, a'_2x'_2\}, \{a'_1x'_2, a'_3x'_2\}, \{a'_2x'_1, a'_2x'_3\} \right\}.$$

Case 1 $X = \{a_1x_1, a_3x_3\}$

Since $a_1x_3, a_3x_1 \notin E(G)$, $X = \{a'_1x'_1, a'_3x'_3\}$ in $L(u_1)$. By Lemma 4.3.7 (ii), a_1 and a_3 can not have a common neighbor outside $L(u_0)$. Thus $\{a_1, a_3\} = \{x'_1, x'_3\}$. Without

Figure 4.4.5

loss of generality, we may assume that $a_1 = x'_1$ and $a_3 = x'_3$. Then $x_1 = a'_1$ and $x_3 = a'_3$ (Fig 4.4.5 (a)). Since a'_2 is a common neighbor of $x'_1(a_1)$ and $x'_3(a_3)$, by Lemma 4.3.7 (ii), $a'_2 \in V(L(u_0))$ and then $x_2 = a'_2$ (Fig 4.4.5 (b)). By Lemma 4.3.7 (iii), $(L(u_0) + \{x_1u_1, x_2u_1\})(X)$ is not reduced, which contradicts to Claim 3.

Case 2 $X = \{a_1x_1, a_2x_3\}$

Since $a_1x_3 \notin E(G)$ and $a_2x_1 \in E(G)$, $X = \{a'_1x'_1, a'_2x'_3\}$ in $L(u_1)$. By (4.7), $\{a_1, a_2\} = \{x'_1, x'_3\}$, and then $\{x_1, x_3\} = \{a'_1, a'_2\}$. Since $a_1x_3 \notin E(G)$, $x_3 = a'_1$ and $x_1 = a'_2$. And then $a_1 = x'_3$ and $a_2 = x'_1$. Denote $L = L(u_0) + \{x_1u_1, x_3u_1\}$ and $L(X)$ is depicted as Fig 4.4.6 (a). Denote C to be the 4-circuit $u_0a_1x_2a_2$ in $L(X)$, and denote $L'(X) = L(X)/\pi(C)$ (Fig 4.4.6 (b)). Denote C' to be the 2-circuit u_0a_2 in $L'(X)$, and denote $L''(X) = L'(X)/C'$ (Fig 4.4.6 (c)). Denote C'' to be the 4-circuit $a_1u_0v_{a_2x_3}x_3$ in $L''(X)$. Since $L''(X)/\pi(C'') \cong H_5$ (Fig 4.4.6 (b)) is collapsible by Lemma 4.2.13 (i), by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L''(X)$, $L'(X)$ and $L(X)$ are collapsible, a contradiction to Claim 3.

Case 3 $X = \{a_1x_2, a_2x_1\}$

Since $a_1x_1, a_2x_2 \in E(G)$, $X = \{a'_1x'_2, a'_2x'_1\}$ in $L(u_1)$. By (4.7), $\{a_1, a_2\} = \{x'_1, x'_2\}$, and then $\{x_1, x_2\} = \{a'_1, a'_2\}$ (Fig 4.4.7). Let $L = L(u_0) + \{x_1u_1, x_2u_1\}$. By Lemma 4.3.7 (iii), $H(X)$ is not reduced, a contradiction to Claim 3.

Case 4 $X = \{a_1x_1, a_1x_2\}$

Figure 4.4.6

Figure 4.4.7

Figure 4.4.8

By the structure of L_2 , either $\{x_1, x_2\} \subseteq N_G(u_1)$ or $a_1 \in N_G(u_1)$. If $\{x_1, x_2\} \subseteq N_G(u_1)$, then by the possible cases for X in $L(u_1)$, $a_1 = x'_2$. Since $a_1x_3 \neq E(G)$, $x_3 \notin N_G(u_1)$. Assume that $N_G(u_1) = \{x_1, x_2, a'_i\}$ and denote $L = L(u_0) + \{x_1u_1, x_2u_i, a'_iu_1, a_1a'_i\}$. $L(X)$ is depicted as Fig 4.4.8 (a). Denote C to be the 4-circuit $u_0a_2x_2a_3$ in $L(X)$, and denote $L'(X) = L(X)/\pi(C)$ (Fig 4.4.8 (a)). Denote C' to be the 2-circuit a_2x_3 in $L'(X)$, and denote $L''(X) = L'(X)/C'$. Denote H to be the induced subgraph of $\{u_0, a_1, v_{a_1x_2}, a'_i, u_1\}$ in $L''(X)$. Note that $H \cong H_5$ is collapsible by Lemma 4.2.13 (i). Since 3-circuit is collapsible, the reduction of $L''(X)/H$ is K_1 , which means $L''(X)/H$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L''(X)$, $L(X)$ and $L(X)$ are all collapsible, a contradiction to Claim 3.

So we may assume that $a_1 \in N_G(u_1)$. By (4.7) and Lemma 4.3.7 (i), $a_2, a_3 \notin N_G(u_1)$. We may assume that $a_1 = a'_i$, and $N_G(u_1) = \{a'_i, a'_j, a'_k\}$ where $\{i, j, k\} = \{1, 2, 3\}$ (Fig 4.4.8 (b)). By the structure of L_2 , $x_1, x_2 \in \{x'_1, x'_2, x'_3\}$.

If $x_1 = x'_2$, then $a'_jx_1, a'_kx_1 \in E(G)$. By the structure of L_2 , x_2 is adjacent to one of a'_j and a'_k . Without loss of generality, we may assume that $a'_kx_2 \in E(G)$. Denote $L = L(u_0) + \{u_1a_1, u_1a'_j, u_1a'_k, a'_jx_1, a'_kx_1, a'_kx_2\} - X$ (Fig 4.4.8 (c)). Note that L is a subgraph of $G(X)$. Denote C to be the 4-circuit $a_2x_2a_3x_3$ in L , and denote $L' = L/\pi(C)$. Denote C' to be the 2-circuit u_0a_2 in L' , and denote $L'' = L'/C'$. Denote C'' to be the 4-circuit $u_0x_2a'_kx_1$ in L'' . Since $L''/\pi(C'') \cong H_5$, by Lemma 4.2.13 (i), $L''/\pi(C'')$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), L'' , L' and L are all collapsible, a contradiction to Claim 3.

If $x_2 = x'_2$, then $a'_jx_2, a'_kx_2 \in E(G)$. By the structure of L_2 , x_1 is adjacent to one of a'_j and a'_k . Without loss of generality, we may assume that $a'_jx_1 \in E(G)$. Denote $L = L(u_0) + \{u_1a_1, u_1a'_j, u_1a'_k, a'_jx_2, a'_kx_2, a'_jx_1\} - X$ (Fig 4.4.8 (d)). Note that L is a subgraph of $G(X)$. Denote C to be the 4-circuit $a_2x_2a_3x_3$ in L , and denote $L' = L/\pi(C)$. Denote C' to be the 2-circuit u_0a_2 in L' , and denote $L'' = L'/C'$. Denote C'' to be the 4-circuit $u_0x_2a'_jx_1$ in L'' . Since $L''/\pi(C'') \cong H_5$, by Lemma 4.2.13 (i), $L''/\pi(C'')$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), L'' , L' and L are all collapsible, a contradiction to Claim 3.

Thus $\{x_1, x_2\} = \{x'_1, x'_3\}$, and then $a_1 = a'_2$. By the structure of L_2 , without loss of generality, we may assume that $a'_j x_1, a'_k x_2 \in E(G)$. Then $L(u_0) \cup L(u_1)$ is depicted as Fig 4.4.8 (e). Denote $L = L(u_0) \cup L(u_1) - X$. Denote C to be the 4-circuit $u_1 a_1 x'_2 a'_j$ in L , and denote $L' = L/\pi(C)$. Denote C' to be the 2-circuit $u_1 a'_k$ in L' , and denote $L'' = L'/C'$. Since $L''/\pi(C'') \cong L(u_0)(a_1 x_2)$, by Lemma 4.3.2 (i), $L''/\pi(C'')$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), L'' , L' and L are all collapsible, a contradiction to Claim 3.

Case 5 $X = \{a_1 x_2, a_2 x_2\}$

By (4.7), $\{a_1, a_2\} \subseteq \{x'_1, x'_2, x'_3\}$ and $x_2 \in N_G(u_1)$. By Lemma 4.3.7 (ii), $x_1, x_3 \notin N_G(u_1)$. Assume that $x_2 = a'_i$ and $N_G(u_1) = \{a'_i, a'_j, a'_k\}$ where $\{i, j, k\} = \{1, 2, 3\}$ (Fig 4.4.9 (a)).

If $a_1 = x'_2$, then $a'_j a_1, a'_k a_1 \in E(G)$. By the structure of L_2 , a_2 is adjacent to one of a'_j and a'_k . Without loss of generality, we may assume that $a'_k a_2 \in E(G)$. Denote $L = L(u_0) + \{u_1 x_2, u_1 a'_j, u_1 a'_k, a'_j a_1, a'_k a_1, a'_k a_2\}$ and $L(X)$ is depicted as Fig 4.4.9 (b)). Denote C to be the 4-circuit $u_0 a_2 x_3 a_2$ in $L(X)$, and denote $L'(X) = L(X)/\pi(C)$. Denote H to be the induced subgraph of $\{a_2, v_{a_2 x_2}, x_2, a'_k, u_1\}$ in $L'(X)$, and denote $L''(X) = L'/H$. Denote C' to be the 3-circuit $a_1 a_2 v_{a_1 x_2}$ in $L''(X)$. Since 2-circuit is collapsible, the reduction of $L''(X)/C'$ is K_1 , which means $L''(X)/C'$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L''(X)$, $L'(X)$ and $L(X)$ are all collapsible, a contradiction to Claim 3.

If $a_2 = x'_2$, then $a'_j a_2, a'_k a_2 \in E(G)$. By the structure of L_2 , a_1 is adjacent to one of a'_j and a'_k . Without loss of generality, we may assume that $a'_j a_1 \in E(G)$. Denote $L = L(u_0) + \{u_1 x_2, u_1 a'_j, u_1 a'_k, a'_j a_2, a'_k a_2, a'_j a_1\}$ (Fig 4.4.9 (c)). Denote C to be the 4-circuit $a_2 a'_j u_1 a'_k$ in $L(X)$, and denote $L'(X) = L(X)/\pi(C)$. Denote C' to be the 3-circuit $a_2 x_2 v_{a_2 x_2}$ in $L'(X)$, and denote $L''(X) = L'/C'$. Denote H to be induced subgraph of $\{u_0, a_1, v_{a_1 x_2}, a_2, a_3\}$ in $L''(X)$. Note that $H \cong H_5$ is collapsible by Lemma 4.2.13 (i). Since 2-circuit is collapsible, the reduction of $L''(X)/H$ is K_1 , which means $L''(X)/H$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L''(X)$, $L'(X)$ and $L(X)$ are all collapsible, a contradiction to Claim 3.

Figure 4.4.9

Thus $\{a_1, a_2\} = \{x'_1, x'_3\}$ and then $x_2 = a'_2$. By the structure of L_2 , without loss of generality, we may assume that $a'_j a_1, a'_k a_2 \in E(G)$.

If x'_2 is in $L(u_0)$, then $x'_2 = x_3$ and $a'_j a_3, a'_k a_3 \in E(G)$. Denote $L = L(u_0) + \{u_1 x_2, u_1 a'_j, u_1 a'_k, a'_j a_1, a'_j a_3, a'_k a_2, a'_k a_3\} - X$ (Fig 4.4.9 (d)). Note that L is a subgraph of $G(X)$. Denote C to be the 4-circuit $u_0 a_2 x_3 a_3$ in L , and denote $L' = L/\pi(C)$. Since L' is a 4-circuit-connected graph containing a 2-circuit, by Lemma 4.2.1, L' is collapsible. Thus, by Theorem 1.2.6 (i), L is collapsible, a contradiction to Claim 3.

So we may assume that x'_2 is not in $L(u_0)$. Denote $L = L(u_0) \cup L(u_1) - X$ (Fig 4.4.9 (e)). Denote C to be the 4-circuit $u_0 a_2 x_3 a_3$ in L , and denote $L' = L/\pi(C)$. Denote C' to be the 4-circuit $a'_j u_1 x_2 x'_2$ in L' , and denote $L'' = L'/\pi(C')$. Denote H to be induced subgraph of $\{a_2, x_2, u_1, a'_k\}$ in L'' . Note that H is collapsible by Lemma 4.2.1. Since 3-circuit is collapsible, the reduction of L''/H is K_1 , which means L''/H is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), L'' , L' and L are all collapsible, a contradiction to Claim 3.

Case 6 $X = \{a_1 x_2, a_3 x_2\}$

We may assume that in $L(u_1)$, X is either $\{a'_1 x'_2, a'_3 x'_2\}$ or $\{a'_2 x'_1, a'_2 x'_3\}$, otherwise we may use a similar argument to previous cases.

If $X = \{a'_1 x'_2, a'_3 x'_2\}$, then $a_1 = a'_1$, $a_3 = a'_3$ and $x_2 = x'_2$. By (4.6), $a_2 \neq a'_2$ and a'_2 is not in $L(u_0)$. By Lemma 4.3.7 (iii), $x_1 a'_2, x_3 a'_2 \notin E(G)$. Thus x'_1 and x'_3 are not in $L(u_0)$. Denote $L = L(u_0) \cup L(u_1)$ and $L(X)$ is depicted as Fig 4.4.10 (a). Denote C to be the 4-circuit $u_0 a_2 x_3 a_3$ in $L(X)$, and denote $L'(X) = L(X)/\pi(C)$. Denote C' to be the 3-circuit $a_2 x_2 v_{a_3 x_2}$ in $L'(X)$, and denote $L''(X) = L'/C'$. Denote H to be induced subgraph of $\{u_1, a_1, v_{a_1 x_2}, a_2, a'_2\}$ in $L''(X)$. Note that $H \cong H_5$ is collapsible by Lemma 4.2.13 (i). Since 2-circuit is collapsible, the reduction of $L''(X)/H$ is K_1 , which means $L''(X)/H$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L''(X)$, $L'(X)$ and $L(X)$ are all collapsible, a contradiction to Claim 3.

Thus we may assume that $X = \{a'_2 x'_1, a'_2 x'_3\}$. Then $a_1 = x'_1$, $a_3 = x'_3$ and $x_2 = a'_2$. By Lemma 4.3.7 (iii), $x_1 u_1, x_3 u_1 \notin E(G)$. Thus a'_1 and a'_3 are not in $L(u_0)$.

Figure 4.4.10

If x'_2 is in $L(u_0)$, then $x'_2 = a_2$ (Fig 4.4.10 (b)). Denote $L = (L(u_0) - \{x_2, x_3\}) + \{a_1a'_1, a_2a'_1, a_2a'_3, a_3a'_3, a'_1u_1, a'_3u_1\}$. Note that $L \subseteq G(X)$. Denote C to be the 4-circuit $a_2a'_1u_1a'_3$ in L . Since $L/\pi(C) \cong K_{3,3}-e$ is collapsible, by Theorem 1.2.6 (i), L is collapsible, a contradiction to Claim 3.

So we may assume that x'_2 is not in $L(u_0)$ (Fig 4.4.10 (c)). Denote $L = L(u_0) \cup L(u_1) - X$. Note that $L \subseteq G(X)$. Denote C to be the 4-circuit $u_0a_2x_3a_3$ in L . Since $L/\pi(C)$ to L' in Fig 4.4.9 (e), then $L/\pi(C)$ is collapsible. Thus, by Theorem 1.2.6 (i), L is collapsible, a contradiction to Claim 3.

Case 7 $X = \{a_2x_1, a_2x_3\}$

We may assume that in $L(u_1)$, $X = \{a'_2x'_1, a'_2x'_3\}$, otherwise we may use a similar argument to previous cases. Then $a_2 = a'_2$, $x_1 = x'_1$ and $x_3 = x'_3$. By (4.7), $a_1, a_3 \notin N_G(u_1)$. Thus a'_1 and a'_3 are not in $L(u_0)$. By Lemma 4.3.7 (iii), $x_2a'_1, x_2a'_3 \notin E(G)$. Thus x'_2 is not in $L(u_0)$. Denote $L = L(u_0) \cup L(u_1)$ and $L(X)$ is depicted as Fig 4.4.11. Denote C to be the 4-circuit $u_0a_2x_2a_3$ in $L(X)$ and denote $L'(X) = L(X)/\pi(C)$. Denote C' to be the 3-circuit $a_2x_3v_{a_2x_3}$ in $L'(X)$ and denote $L''(X) = L'(X)/C'$. let $H = L''(X) - \{u_0, a_1\}$. Note that $H \cong L(u_0)(a_2x_1)/a_2x_3$. By Lemma 4.3.2 (i) and Theorem 1.2.1 (vi), $L(u_0)(a_2x_1)/a_2x_3$ is collapsible, and then H is collapsible. Since 3-circuit is collapsible, the reduction of $L''(X)/H$ is K_1 , which means $L''(X)/H$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L''(X)$, $L'(X)$ and $L(X)$ are all collapsible, a contradiction to Claim 3.

This verifies Claim 10.

Claim 11 For $\forall v \in D_3''(G)$, $|E(L(v)) \cap E(L(u_0))| = |E(L(v)) \cap X| = 1$. Moreover, $E(L(v)) \cap E_G(u_0) = \emptyset$.

Choose $v \in D_3''(G)$. Since $L(v)$ is a 3-circuit, by Claim 3, $|E(C_3(v)) \cap X| \geq 1$. By Claim 6, $(E(C_3(v)) \cap X) \subseteq (E(L(v)) \cap E(L(u_0)))$. By Claim 7, $V(L(v)) \not\subseteq V(L(u_0))$, and then $|E(L(v)) \cap E(L(u_0))| \leq 1$. Thus $|E(L(v)) \cap E(L(u_0))| = |E(L(v)) \cap X| = 1$. Since all neighbors of u_0 are in $L(u_0)$, $E(L(v)) \cap E_G(u_0) = \emptyset$. This verifies Claim 11.

Claim 12 For $\forall v \in D_3''(G) - V(L(u_0))$, v has exact one neighbor not in $L(u_0)$.

Figure 4.4.11

Figure 4.4.12

Figure 4.4.13

By Claim 9, for $\forall v \in D_3''(G) - V(L(u_0))$, $|N_G(v) \cap V(L(u_0))| \geq 2$. Suppose there exists $v \in D_3''(G) - V(L(u_0))$ such that $N_G(v) = \{z_1, z_2, z_3\} \subseteq V(L(u_0))$. Denote $L = L(u_0) \cup E_G(v)$ and $L(X)$ is depicted as Fig 4.4.12. Note that $L(v) = vz_1z_2$ and $X = \{z_1z_2, e'\}$. Denote C to be the 4-circuit $vz_1vz_1z_2z_2$ in $L(X)$ and denote $L'(X) = L(X)/\pi(C)$. Denote $H = L'(X) - v$. Then $L''(X) = L'(X)/H$ is a 2-circuit, which is collapsible. Note that $H \cong L(u_0)(e')/z_1z_2$. By Lemma 4.3.2 (i) and Theorem 1.2.1 (vi), $(L(u_0)(e')/z_1z_2)$ is collapsible, and then H is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L'(X)$ is collapsible and $L(X)$ is collapsible, a contradiction to Claim 3. This verifies Claim 12.

Claim 13 $D_3''(G) - V(L(u_0))$ is an independent set.

By way of contradiction, suppose $v, v' \in D_3''(G) - V(L(u_0))$ such that $vv' \in E(G)$. Let $N_G(v) = \{z_1, z_2, v'\}$ where $L(v) = vz_1z_2$ and $X = \{z_1z_2, e'\}$. If $L(v') \cap X = \{z_1z_2\}$, then $vv'z_1$ is a triangle in $G(X)$, a contradiction to Claim 3. Thus $L(v') \cap X = \{e'\}$. Denote $L = L(u_0) \cup E_G(v) \cup E_G(v')$ and $L(X)$ is depicted as Fig 4.4.13. Denote C to be the 4-circuit $vz_1vz_1z_2z_2$ in $L(X)$ and denote $L'(X) = L(X)/\pi(C)$. Denote $H = L'(X) - \{v, v'\}$ and denote $L''(X) = L'(X)/H$. Since $L''(X)$ has a spanning 3-circuit, the reduction of $L''(X)$ is K_1 , and then $L''(X)$ is collapsible. Note that $H \cong L(u_0)(e')/z_1z_2$. By Lemma 4.3.2 (i) and Theorem 1.2.1 (vi), $(L(u_0)(e')/z_1z_2)$ is collapsible, and then H is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $L'(X)$ is collapsible and $L(X)$ is collapsible, a contradiction to Claim 3. This verifies Claim 13.

Figure 4.4.14

Claim 14 For $\forall e \in X$, $|\{v \in D_3''(G) - V(L(u_0)) : e \in L(v)\}| \leq 1$. That is, every edge in X is in at most one triangle.

Suppose that $X = \{e, e'\}$. We may assume that $\{v \in D_3''(G) - V(L(u_0)) : e \in L(v)\} = \{y_1, y_2, \dots, y_k\}$ and $\{v \in D_3''(G) - V(L(u_0)) : e' \in L(v)\} = \{z_1, z_2, \dots, z_j\}$. By way of contradiction, we may assume that $k \geq 2$. Assume that r_1, r_2 are the endpoints of e .

Let $L = L(u_0) \cup (\cup_{v \in D_3''(G)} L(v))$. Note that $D_3(G) \subseteq V(L)$ and $X \subseteq E(L)$. Then $G(X)/L$ is 4-edge-connected. By Claim 12, let w_i be the neighbor of y_i ($i = 1, 2$) that is outside $L(u_0)$. Denote that $e_i = y_i w_i$ for $i = 1, 2$. By Claim 13, w_1 and w_2 are not in L (Fig 4.4.14).

Denote C to be the 4-circuit $y_1 r_1 y_2 r_2$ in $G(X)$. Denote $G'(X) = G(X)/\pi(C)$ and denote $L'(X) = L(X)/\pi(C)$, which is a subgraph of $G'(X)$. Denote $H = L'(X) - D_3''(G)$. Note that $H \cong L(u_0)(e')/e$. By Lemma 4.3.2 (i) and Theorem 1.2.1 (vi), H is collapsible. Denote $G''(X) = G'(X)/H$ and $L''(X) = L'(X)/H$, which is a subgraph of $G''(X)$. Denote $H' = L''(X) - y_1$. Since 2-circuit is collapsible, H' is collapsible. Denote $G'''(X) = G''(X)/H'$ and $L'''(X) = L''(X)/H'$, which is isomorphic to K_2 . Since $G'''(X)/L'''(X)$

Figure 4.4.15

is isomorphic to $G(X)/L$, $G'''(X)/L'''(X)$ is 4-edge-connected. By Theorem 1.2.2 (i), $G'''(X)/L'''(X) - \{e_1, e_2\}$ has two edge-disjoint spanning trees T_1 and T_2 . Then $T_1 + e_1$ and $T_2 + e_2$ are two edge-disjoint spanning trees of $G'''(X)$. Thus $F(G'''(X)) = 0$ and by Theorem 1.2.4 (ii), $G'''(X)$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $G''(X)$, $G'(X)$ and $G(X)$ are all collapsible, a contradiction to Claim 3. This verifies Claim 14.

By Claim 8 and Claim 11, for $\forall e \in X \cap E_G(u_0)$, e is not in $L(v)$ for any $v \in D_3''(G)$; by Claim 8 and Claim 14, for $\forall e \in X - E_G(u_0)$, e is in at most one triangle, which contains

Figure 4.4.16

at most two elements of $D_3''(G)$. Thus by Claim 4,

$$\begin{aligned}
 6 &\leq |D_3(G)| = |D_3'(G)| + |D_3''(G)| \\
 &\leq |\{u_0, x_2\}| + 2|X - E_G(u_0)| \\
 &\leq |\{u_0, x_2\}| + 2|X| = 6
 \end{aligned}$$

So all equalities must hold, that is $D_3'(G) = \{u_0, x_2\}$ and $X \cap E_G(u_0) = \emptyset$. Since $L(x_2) = L(u_2)$, by Claim 11, $X \cap E_G(x_2) = \emptyset$. Thus $X \subseteq \{a_1x_1, a_2x_1, a_2x_3, a_3x_3\}$. Since $X \not\subseteq \{\{a_1x_1, a_2x_1\}, \{a_2x_3, a_3x_3\}\}$, $|X \cap \{a_1x_1, a_2x_1\}| = |X \cap \{a_2x_3, a_3x_3\}| = 1$. Since each edge e in X is in a unique triangle, which contains two elements of $D_3''(G)$, then $D_3''(G) = \{x_1, x_3, v_1, v_2\}$ where $L(v_1) = L(x_1)$ and $L(v_2) = L(x_3)$. Denote $L(D_3) = L(u_0) \cup (\cup_{v \in D_3''(G)} L(v))$. Then G/L is 4-edge-connected. By Claim 11 - 14, $L(D_3)$ has the three possible cases depicted as Fig 4.4.15.

If $X = \{a_1x_1, a_3x_3\}$ (Fig 4.4.15 (a), also Fig 4.4.16), Denote C to be the 4-circuit $a_1v_1x_1v_{a_1x_1}$ in $G(X)$. Denote $G'(X) = G(X)/\pi(C)$ and denote $L'(X) = L(X)/\pi(C)$, which is a subgraph of $G'(X)$. Denote $H = L'(X) - \{v_1, v_2\}$. Note that $H \cong L(u_0)(a_3x_3)/a_1x_1$. By Lemma 4.3.2 (i) and Theorem 1.2.1 (vi), H is collapsible. Denote $G'''(X) = G'(X)/H$ and $L''(X) = L'(X)/H$, which is a subgraph of $G'''(X)$. Denote $H' = L''(X) - y_1$. S-

ince 2-circuit is collapsible, H' is collapsible. Denote C' to be the 2-circuit $v_{L(u_0)}v_2$ in $G''(X)$. Denote $G'''(X) = G''(X)/C'$ and $L'''(X) = L''(X)/C'$, which is isomorphic to K_2 . Since $G'''(X)/L'''(X)$ is isomorphic to $G(X)/L$, $G'''(X)/L'''(X)$ is 4-edge-connected. By Theorem 1.2.2 (i), $G'''(X)/L'''(X) - \{e_1, e_2\}$ has two edge-disjoint spanning trees T_1 and T_2 . Then $T_1 + e_1$ and $T_2 + e_2$ are two edge-disjoint spanning trees of $G'''(X)$. Thus $F(G'''(X)) = 0$ and by Theorem 1.2.4 (ii), $G'''(X)$ is collapsible. Thus by Theorem 1.2.1 (i) and Theorem 1.2.6 (i), $G''(X)$, $G'(X)$ and $G(X)$ are all collapsible, a contradiction to Claim 3.

If $X = \{a_1x_1, a_2x_3\}$ or $\{a_2x_1, a_2x_3\}$, then by a similar argument we could claim that $G(X)$ is collapsible, a contradiction.

This complete the proof of Theorem 4.4.1. $\square$

Proof of Theorem 4.1.7. Assume that $L(G)$ is not complete. By Theorem 1.1.4 (i), $\kappa'(G_0) \geq 3$. Since $L(G)$ is 4-connected, G_0 is essentially 4-edge-connected. By Lemma 4.3.1, for $\forall u \in D_3(G_0)$, u lies in a local subgraph $L(u)$ either isomorphic to a contraction of a member in $\{L_1, L_2, L_3, L_4\}$ or L_5 . By Theorem 4.4.1 and Theorem 1.2.1(iii), $G_0(e', e'')$ has a spanning $(v(e'), v(e''))$ -trail for any pair of vertices $e', e'' \in V(G_0)$. Then by Lemma 4.2.5, $G(e', e'')$ has a dominating $(v(e'), v(e''))$ -trail, for any $e', e'' \in E(G)$. By Lemma 4.2.3 and Lemma 4.2.6, Theorem 4.1.7 is proved. $\square$

Chapter 5

Group Colorability of Multigraphs

5.1 Introduction

Graphs considered in this chapter are multigraphs that are finite and loopless. A nontrivial 2-regular connected graph is called a **circuit**. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We define an equivalence relation " $\sim$ " on $E(G)$ such that $e_1 \sim e_2$ if $e_1 = e_2$, or if e_1 and e_2 form a circuit in G . Edges in the same equivalence class are **parallel edges**, and they have the same endpoints. For $u, v \in V(G)$, let $m(u, v)$ denote the number of parallel edges between u and v . The **multiplicity** of G , denoted by $M(G)$, is the maximum size of an equivalence class. The **simplification** of G , denoted by G_0 , is the simple graph obtained by replacing each equivalence class by a single edge. For a graph H and a positive integer k , we define kH to be the graph obtained by replacing each edge of H by a class of k parallel edges.

Group coloring was first introduced by Jeager et al. [21]. He introduced a concept of group connectivity as a generalization of nowhere-zero flows. He also introduced group coloring as a dual concept to group connectivity.

An **orientation** D of G is a map $t_D : E(G) \rightarrow V(G)$ such that $t_D(e)$ is a vertex incident with e in G , and the edge e is oriented from $t_D(e)$ to the other endpoint. The

graph G under the orientation D is sometimes denoted by $D(G)$. In $D(G)$, an oriented edge is called an **arc**, and we write $e = uv$ to mean $t_D(e) = u$.

Let Γ be a nontrivial group and let $F(G, \Gamma)$ be the set of all functions $f : E(G) \rightarrow \Gamma$. For a function $f \in F(G, \Gamma)$, a (Γ, f) -coloring of $D(G)$ is a function $c : V(G) \rightarrow \Gamma$ such that $c(u)c(v)^{-1} \neq f(e)$ when uv is an arc; $D(G)$ is **Γ -colorable** if and only if for any $f \in F(G, \Gamma)$ there exists a (Γ, f) -coloring. It is known ([21]) that whether G is Γ -colorable is independent of the choice of the orientation. Therefore, we say simply that G is Γ -colorable when some orientation of G is Γ -colorable. The **group chromatic number** of a graph G , denoted by $\chi_g(G)$, is defined to be the minimum positive integer m for which G is Γ -colorable for every group Γ of order at least m .

For a subgraph H of G , (G, H) is said to be **Γ -extendible** if for any $f \in F(G, \Gamma)$ and any $(\Gamma, f|_{E(H)})$ -coloring c' of H , there is a (Γ, f) -coloring c of G such that $c|_{E(H)} = c'$. The coloring c is then called an **extension** of c' .

Prior research on group chromatic number was restricted to simple graphs, and considered only Abelian groups in the definition of $\chi_g(G)$. The following results were under the assumptions that the groups involved are Abelian groups. However, they remain valid without this assumption.

Theorem 5.1.1 (*Lai and Zhang, [29]*) *For any connected simple graph G ,*

$$\chi_g(G) \leq \Delta(G) + 1$$

with equality if and only if G is a cycle or G is complete.

Theorem 5.1.2 *Let G be a simple graph.*

(i) (*Lai and Zhang, [30]*) *If G is K_5 -minor free, then $\chi_g(G) \leq 5$.*

(ii) (*Lai and Li, [25]*) *If G is $K_{3,3}$ -minor free, then $\chi_g(G) \leq 5$.*

The bound of Theorem 5.1.2 (ii) is sharp. A 3-colorable simple planar graph with $\chi_g(G) = 5$ is constructed in [22].

Note that although group coloring is a generalization of vertex coloring, they have different behaviors: for any bipartite graph G , $\chi(G) = 2$ while $\chi_g(G)$ can be arbitrary large. Lai and Zhang [29] showed that for any complete bipartite graph $K_{m,n}$ with $n \geq m$, $\chi_g(K_{m,n}) = m + 1$. Moreover, group coloring of multigraphs is different from that of simple graphs. Consider K_2 , for an example. By Brooks' Theorem and TM5.1.1, $\chi(K_2) = \chi_g(K_2) = 2$ while $\chi_g(mK_2) = m + 1$ for $m \in \mathbf{Z}^+$ by the definition of $\chi_g(G)$.

The main purpose of this note is to extend Theorem 5.1.1 to multigraphs.

Theorem 5.1.3 *For any connected multigraph G ,*

$$\chi_g(G) \leq \Delta(G) + 1$$

where equality holds if and only if $G = kC_n$ or kK_n for some positive integer k and n .

In order to prove our theorem, we will first show some properties of group coloring and then introduce the degeneration method. Finally we will give the upper bound of $\chi_g(G)$ for multigraphs.

5.2 Properties of Group Coloring

In [29], Lai and Zhang proved the following properties of χ_g . Although they assumed that graphs are simple and groups are Abelian groups, these properties still hold without such assumptions, as their proofs did not utilize the property that the binary operations of the involved groups are commutative.

Proposition 5.2.1 *(Lai and Zhang, [29]) For any simple graph G , $\chi_g(G) = 2$ if and only if G is a forest.*

Proposition 5.2.2 *(Lai and Zhang, [29]) If G is a multigraph and Γ is a group, then each of the following holds:*

- (i) If G is Γ -colorable under an orientation D , then G is Γ -colorable under every orientation of G .
- (ii) G is Γ -colorable if and only if each block of G is Γ -colorable.

By Proposition 5.2.2 (i), $\chi_g(G)$ is not dependent on the choice of orientations of G . Furthermore, Proposition 5.2.2 (ii) shows that it suffices to consider 2-connected graphs in our proofs.

Proposition 5.2.3 (*Lai and Zhang, [29]*) *If G is a multigraph and Γ is a group, then each of the following holds:*

- (i) Let $H \subseteq G$. If (G, H) is Γ -extendible and H is Γ -colorable, then G is Γ -colorable.
- (ii) Let $H_2 \subseteq H_1 \subseteq G$. If (G, H_1) and (H_1, H_2) are Γ -extendible, then (G, H_2) is also Γ -extendible.

Let Γ and Γ' be two groups and let $\varphi : \Gamma \rightarrow \Gamma'$ be a homomorphism. Then $\text{im}(\varphi)$, the image of Γ under φ , is a subgroup of Γ' .

Proposition 5.2.4 (*Lai and Zhang, [29]*) *Let $\varphi : \Gamma \rightarrow \Gamma'$ be a homomorphism and G be a multigraph. If G is $\text{im}(\varphi)$ -colorable, then G is also Γ -colorable.*

Note the fact that: if N is a normal subgroup of Γ , then the function $\pi : \Gamma \rightarrow \Gamma/N$ (Γ/N is called the quotient group of Γ) defined by $\pi(a) = aN$ is a homomorphism of Γ onto Γ/N . Also note the fact that if Γ and Γ' are two finite cyclic groups with orders m and n , respectively, then there exists a homomorphism of Γ onto Γ' if and only if $n|m$. Thus, by Proposition 5.2.4, we have the following result.

Corollary 5.2.5 (*Lai and Zhang, [29]*) *Let G be a multigraph. Then each of the following holds:*

(i) Let Γ be a group and N be any subgroup of Γ . If G is Γ/N -colorable, then G is Γ -colorable.

(ii) For any positive integers k and n , if G is Z_n -colorable, then G is also Z_{kn} -colorable.

Lemma 5.2.6 *If G is a multigraph and $k \in \mathbf{Z}^+$, then $\chi_g(kG) \geq k(\chi_g(G) - 1) + 1$.*

Proof. By the definition of $\chi_g(G)$, there exist a group Γ of order $\chi_g(G) - 1$ and a function $f \in F(G, \Gamma)$ such that there are no (Γ, f) -colorings of G . Let $\Gamma' = \Gamma \times \mathbf{Z}_k$ (where $|\Gamma'| = k(\chi_g(G) - 1)$). Define $f' \in F(kG, \Gamma')$ by $f(e_i) = (f(e), i)$ for $1 \leq i \leq k$ where $\{e_1, e_2, \dots, e_k\}$ are the k parallel edges in $E(kG)$ corresponding to $e \in E(G)$. There are no (Γ', f') -colorings of kG , since any (Γ', f') -coloring of kG can give rise to a (Γ, f) -coloring of G . Therefore, kG is not Γ' -colorable, and $\chi_g(kG) \geq k(\chi_g(G) - 1) + 1$. $\square$

5.3 Degeneration Method

Theorem 5.3.1 *Suppose that $V(G)$ can be linearly ordered as $v_1, v_2, \dots, v_n$ such that $d_{G_i}(v_i) \leq k$ ($i = 1, 2, \dots, n$), where $G_i = G[\{v_1, v_2, \dots, v_i\}]$. Then for any group Γ of order at least $k + 1$, (G_{i+1}, G_i) ($i = 1, 2, \dots, n - 1$) is Γ -extendible and so G is Γ -colorable.*

Such a list of $V(G)$ in Theorem 2.3.1 is called **k -degenerated**. An immediate corollary of Theorem 2.3.1 is given below.

Corollary 5.3.2 *(Lai and Zhang, [29]) If G is a multigraph, then each of the following holds:*

(i) $\chi_g(G) \leq \max_{H \subseteq G} \{\delta(H)\} + 1$.

(ii) $\chi_g(G) \leq \Delta(G) + 1$.

Since every nontrivial simple graph without a subdivision of K_4 has a vertex of degree at most 2, by Proposition 5.2.1 and Theorem 2.3.1, we have the following result:

Corollary 5.3.3 (*Lai and Zhang, [29]*) *Let G be a nontrivial simple graph without subdivision of K_4 . Then $\chi_g(G) = 3$ if and only if G has a cycle.*

5.4 Group Chromatic Number of Multigraphs

Brooks proved that for any connected graph G , $\chi(G) \leq \Delta(G) + 1$ where equality holds if and only if G is an odd cycle or a complete graph. Lai and Zhang [29] proved Theorem 5.1.1 as a strengthening of Brooks' Theorem for the group chromatic number of simple graphs. In this section, we shall extend Theorem 5.1.1 to Theorem 5.1.3. We start with two lemmas.

Lemma 5.4.1 *Let Γ be a group. If S_1 and S_2 are subsets of Γ such that $|S_2| > |S_1|$, then there exist x and y in S_2 such that $S_1x \neq S_1y$.*

Proof. We argue by contradiction and assume that $\forall x, y \in S_2, S_1x = S_1y$. Let $|S_1| = m$. Since $|S_2| > |S_1|$, we can pick $m + 1$ distinct elements $b_1, b_2, \dots, b_{m+1}$ from S_2 . If $1 \leq i < j \leq m + 1$, then $S_1b_i = S_1b_j$. Fix $a \in S_1$. Since $ab_i \in S_1b_i = S_1b_1$ ($1 \leq i \leq m + 1$), $\{ab_i : 1 \leq i \leq m + 1\} \subseteq S_1b_1$. Since $m = |S_1b_1| \geq |\{ab_i : 1 \leq i \leq m + 1\}|$, there exist $1 \leq k < l \leq m + 1$ such that $ab_k = ab_l$. Hence $b_k = b_l$, contrary to the assumption that $b_1, b_2, \dots, b_{m+1}$ are distinct. $\square$

Lemma 5.4.2 *If G is a 2-connected graph whose simplification is neither a cycle nor a complete graph, then there exist three vertices v_1, v_2, v_n in G such that both of the following holds:*

- (i) $v_1v_n, v_2v_n \in E(G)$ and $v_1v_2 \notin E(G)$, and
- (ii) $G - \{v_1, v_2\}$ is connected.

Proof. An ordered triple (v_1, v_2, v_n) is called "good" if it satisfies both (i) and (ii). We argue by contradiction and assume that G is a counterexample. So there are no good ordered triples in $V(G)$. Since G and its simplification G_0 have the same vertex connectivity, without loss of generality, we may assume that G is simple. Let $C = v_1v_2 \dots v_k$ be a longest circuit in G .

If C is not a Hamilton circuit of G , let H be one component of $G - C$. Let $V_1 = \{v_i \in V(C) : v_iu \in E(G), u \in V(H)\}$. Since G is 2-connected, then $|V_1| \geq 2$. We can assume that $v_1 \in V_1$, and $v_1u_1 \in E(G)$ where $u_1 \in V(H)$. Then $v_2 \notin V_1$ by the choice of C . Therefore (u_1, v_2, v_1) is good, a contradiction to the choice of G .

So C must be a Hamilton circuit of G . Since $G = G_0$ is not a cycle, C must have a chord, say v_1v_i . Then $v_2v_k \in E(G)$, otherwise (v_2, v_k, v_1) is a good triple. Since (v_1, v_3, v_2) is not a good triple, $v_1v_3 \in E(G)$. Inductively, since (v_1, v_j, v_{j-1}) is not a good triple (for $j = 4, 5, \dots, i-1$), $v_1v_j \in E(G)$. Therefore v_1 is adjacent to all vertices of $\{v_2, v_3, \dots, v_i\}$. Similarly, v_1 is adjacent to all vertices of $\{v_{i+1}, v_{i+2}, \dots, v_k\}$. That is v_1 is adjacent to every other vertex of G . Since $G = G_0$ is not a complete graph, there exist v_{i_1} and v_{i_2} such that $v_{i_1}v_{i_2} \notin E(G)$. Thus (v_{i_1}, v_{i_2}, v_1) is a good triple, a contradiction to the choice of G . $\square$

Proof of Theorem 5.1.3 By Corollary 5.3.2 (ii), we only need to show that when the equality holds, G must be either a kC_n or a kK_n for some positive integer k and n . Let G be a multigraph such that $\chi_g(G) = \Delta(G) + 1$ and $n = |V(G)|$. By Proposition 5.2.2 (ii), we may assume that G is 2-connected.

Claim 1 G is regular.

If G not regular, then $\max_{H \subseteq G} \{\delta(H)\} \leq \Delta(G) - 1$. Therefore, by Corollary 5.3.2 (i), $\chi_g(G) \leq \max_{H \subseteq G} \{\delta(H)\} + 1 \leq \Delta(G)$, a contradiction to $\chi_g(G) = \Delta(G) + 1$.

Claim 2 G_0 is either a cycle or a complete graph.

Suppose that G_0 is neither a cycle nor a complete graph. Since $\chi_g(G) = \Delta(G) + 1$, there exist a group Γ of order $\Delta(G)$ and a function $f \in F(G, \Gamma)$ such that G has no (Γ, f) -colorings.

By Lemma 5.4.2, G has three vertices v_1, v_2, v_n such that both $v_1v_n, v_2v_n \in E(G)$ and $v_1v_2 \notin E(G)$, and such that $G - \{v_1, v_2\}$ is connected. Now we arrange the vertices of $G - \{v_1, v_2\}$ in non-increasing order of their distance from v_n , say $v_3, \dots, v_n$. Then the list $\{v_1, v_2, \dots, v_n\}$ is such that each vertex other than v_n is adjacent to at least one vertex following it. Thus each vertex other than v_n is adjacent to at most $\Delta(G) - 1$ vertices preceding it.

Let D be an orientation such that every arc between v_i and v_j is directed from v_j to v_i if $i < j$ and from v_i to v_j otherwise. Define a map $c : V(G) \rightarrow \Gamma$ as follows. For $i = 1, 2$, let e_i denote an arc from v_n to v_i . Choose $a_1, a_2 \in \Gamma$ such that $f(e_1)a_1 = f(e_2)a_2$. Define $c(v_i) = a_i$ ($i = 1, 2$). For v_j ($3 \leq j \leq n$), let $A_j = \{f(e)c(v_i) : e \in E[v_j, v_i] \text{ and } i = 1, 2, \dots, j-1\}$. If $3 \leq j < n$, then $|A_j| \leq d_{G[\{v_1, v_2, \dots, v_j\}]}(v_j) \leq \Delta(G) - 1$ and $\Gamma - A_j \neq \emptyset$; if $j = n$, then $\Gamma - A_n \neq \emptyset$ since $f(e_1)a_1 = f(e_2)a_2$. Hence we can choose $c(v_j) \in \Gamma - A_j$ ($3 \leq j \leq n$), so that c is a (Γ, f) -coloring of G , contrary to the assumption that G has no (Γ, f) -colorings.

Claim 3 If $G = kC_n$ or kK_n , then $\chi_g(G) = \Delta(G) + 1$.

If $G = kC_n$, then by Corollary 5.3.2 (ii) and Lemma 5.2.6, $\Delta(G) + 1 \geq \chi_g(G) \geq k(3 - 1) + 1 = \Delta(G) + 1$, and so $\chi_g(G) = \Delta(G) + 1$. Similarly, if $G = kK_n$, then $\chi_g(G) = \Delta(G) + 1$.

Claim 4 If $G_0 = C_n$, then $G = kC_n$, where $k = M(G)$.

Assume that $G \neq kC_n$. By Claim 1, G is regular. It follows that G must satisfy both $G_0 = C_{2k} = u_1v_1u_2v_2 \dots u_kv_k$, and $m(u_iv_i) = a$, $m(v_iu_{i+1}) = b$ with $a \neq b$, where $i = 1, 2, \dots, k$ and where the subscripts are taken modulo k . Without loss of generality, assume $a > b$. Let D be an orientation of G such that every arc $e \in E[u_i, v_i]$ is directed from u_i to v_i and every arc $e \in E[v_i, u_{i+1}]$ is directed from v_i to u_{i+1} ($i = 1, 2, \dots, k$, where subscripts are taken modulo k).

Since $\chi_g(G) = \Delta(G) + 1$, there exist a group Γ of order $\Delta(G) = a + b$ and a function $f \in F(G, \Gamma)$ such that G has no (Γ, f) -colorings. Let $H = G[\{u_1, v_1, u_2, v_2, \dots, u_{k-1}, v_{k-1}\}]$. Since H_0 is a path, it follows by Corollary 5.3.2 that $\chi_g(H) \leq \Delta(H) = a + b$ and then H has a $(\Gamma, f|_H)$ -coloring c' . Let $c : V(G) \rightarrow \Gamma$ be a function where $c|_{V(H)} = c'$.

Let $\Gamma_1 = \Gamma \setminus \{f(e) : e \in E[v_{k-1}, u_k]\}$ and $\Gamma_2 = \{f(e) : e \in E[v_k, u_1]\}$. Pick $y_0 \in \Gamma \setminus \{f(e) : e \in E[u_k, v_k]\}$. Since $|\{y_0^{-1}x^{-1}c(v_{k-1})c(u_1)^{-1} : x \in \Gamma_1\}| = |\Gamma_1| \geq a > b \geq |\Gamma_2|$, choose $x_0 \in \Gamma_1$ such that $y_0^{-1}x_0^{-1}c(v_{k-1})c(u_1)^{-1} \notin \Gamma_2$. Let $c(u_k) = x_0^{-1}c(v_{k-1})$ and $c(v_k) = y_0^{-1}c(u_k)$. Now c is a (Γ, f) -coloring of G , contrary to the assumption that G has no (Γ, f) -colorings. This completes the proof of Claim 4.

Claim 5 *If $G_0 = K_n$, then $G = kK_n$, where $k = M(G)$.*

Assume that $G \neq kK_n$ and $n \geq 4$. Since $\chi_g(G) = \Delta(G) + 1$, there exist a group Γ of order $\Delta(G)$ and a function $f \in F(G, \Gamma)$ such that there are no (Γ, f) -colorings of G . By Claims 1, G is regular. It follows that there exist $u, v_1, v_2 \in V(G)$ with $m(uv_1) = a$, $m(uv_2) = b$ and $m(v_1v_2) = d$ such that $a < b$. Let $H = G - \{u, v_1, v_2\}$. Let D be such an orientation that arcs in $E[\{u, v_1, v_2\}, V(H)]$ are all directed into H ; arcs in $E[u, v_i]$ are all directed from u to v_i ($i = 1, 2$) and arcs in $E[v_2, v_1]$ are all directed from v_2 to v_1 .

Since H is not regular, it follows by Corollary ??(i) that $\chi_g(H) \leq \Delta(H) \leq \Delta(G)$. Thus H has a $(\Gamma, f|_H)$ -coloring c . For any $v \in \{v_1, v_2, u\}$, define $A_v = \Gamma \setminus \{f(e)c(x) : x \in V(H), e \in E[v, x]\}$. Since $|\Gamma| = \Delta(G)$, $|A_{v_1}| \geq a + d$, $|A_{v_2}| \geq b + d$, and $|A_u| \geq a + b$. Taking a subset if needed, we may assume that $|A_{v_1}| = a + d$, $|A_{v_2}| = b + d$, and $|A_u| = a + b$. Since $|\{f(e) : e \in E[v_2, v_1]\}| \leq d$ and $|A_{v_1}| = a + d$, it follows by Lemma 5.4.1 that there

exist $x_1, x_2 \in A_{v_1}$ such that $\{f(e)x_1 : e \in E[v_2, v_1]\} \neq \{f(e)x_2 : e \in E[v_2, v_1]\}$.

Let c_1 be an extension of c on $G[V(H) \cup v_1]$ such that $c_1(v_1) = x_1$. For any $v \in \{v_2, u\}$, define $A'_v = A_v \setminus \{f(e)x_1 : e \in E[v, v_1]\}$. Note that $|A'_{v_2}| \geq b$ and $|A'_u| \geq b$. If $|A'_{v_2}| > b$, then choose $c_1(u) \in A'_u$ and $c_1(v_2) \in A'_{v_2} \setminus \{f(e)^{-1}c_1(u)\}$, and now c_1 is a (Γ, f) -coloring of G , contrary to the assumption that G has no (Γ, f) -colorings. Thus

$$|A'_{v_2}| = b, \text{ and similarly } |A'_u| = b. \quad (5.1)$$

Assume that there is a $z \in A'_u$ such that $\{f(e)^{-1}z : e \in E[u, v_2]\} \neq A'_{v_2}$. Since $|A'_{v_2}| = b = m(uv_2) \geq |\{f(e)^{-1}z : e \in E[u, v_2]\}|$, we can pick $y \in A'_{v_2} \setminus \{f(e)^{-1}z : e \in E[u, v_2]\}$ and extend c_1 to a map $c_2 : V(G) \mapsto \Gamma$ by assigning $c_2(v_2) = y$ and $c_2(u) = z$. By the choices of y and z , it is routine to verify that c_2 is indeed a (Γ, f) -coloring of G , contrary to the assumption that G has no (Γ, f) -colorings. Hence we may assume that

$$\forall z \in A'_u, \{f(e)^{-1}z : e \in E[u, v_2]\} = A'_{v_2}. \quad (5.2)$$

Let c_0 be an extension of c on G such that $c_0(v_1) = x_2$. For any $v \in \{v_2, u\}$, define $A''_v = A_v \setminus \{f(e)x_2 : e \in E[v, v_1]\}$. By the choice of x_1 and x_2 , $A'_{v_2} \neq A''_{v_2}$. So we can pick $y_0 \in A''_{v_2} \setminus A'_{v_2}$. As G has no (Γ, f) -colorings, it follows by a similar argument to conclude (1) that we must also have $|A''_{v_2}| = |A''_u| = b$.

Since $|A_u| = a + b \leq 2b$, $A''_u \cap A'_u \neq \emptyset$. Take $z_0 \in A''_u \cap A'_u$. Define $c_2(v_2) = y_0$ and $c_2(u) = z_0$. By (2) and by the fact that $y_0 \notin A'_{v_2}$, it is routine to verify that c_0 is indeed a (Γ, f) -coloring of G , contrary to the assumption that G has no (Γ, f) -colorings. This completes the proof of Claim 5.

After we have established these claims, it is straightforward to see that Theorem 5.1.3 now follows from Claims 3, 4 and 5. $\square$

Since $\Delta(G) \leq M(G)\Delta(G_0)$, Corollary 3.3 below follows from Theorem 5.1.3 immediately.

Corollary 5.4.3 *For any graph G , $\chi_g(G) \leq M(G)\Delta(G_0) + 1$ with equality if and only if $G = M(G)C_n$ or $G = M(G)K_n$.*

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