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Group Connectivity of Graphs

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Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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ABSTRACT

Group Connectivity of Graphs

Senmei Yao

Tutte introduced the theory of nowhere-zero flows and showed that a plane graph G has a face k -coloring if and only if G has a nowhere-zero A -flow, for any Abelian group A with $|A| \geq k$. In 1992 Jaeger et al [16] extended nowhere-zero flows to group connectivity of graphs: given an orientation D of a graph G , if for any $b: V(G) \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$, there always exists a map $f: E(G) \mapsto A - \{0\}$, such that at each $v \in V(G)$,

$$\sum_{e = vw \text{ is directed from } v \text{ to } w} f(e) - \sum_{e = uv \text{ is directed from } u \text{ to } v} f(e) = b(v)$$

in A , then G is A -connected. For a 2-edge-connected graph G , define $\Lambda_g(G) = \min\{k: \text{for any Abelian group } A \text{ with } |A| \geq k, G \text{ is } A\text{-connected}\}$.

Let $G_1 \otimes G_2$ and $G_1 \times G_2$ denote the strong and Cartesian product of two connected nontrivial graphs G_1 and G_2 . We prove that $\Lambda_g(G_1 \otimes G_2) \leq 4$, where equality holds if and only if both G_1 and G_2 are trees and $\min\{|V(G_1)|, |V(G_2)|\} = 2$; $\Lambda_g(G_1 \times G_2) \leq 5$, where equality holds if and only if both G_1 and G_2 are trees and either $G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, for $n, m \geq 2$ or $\min\{|V(G_1)|, |V(G_2)|\} = 2$. A similar result for the lexicographical product graphs is also obtained.

Let P denote a path in G , let $\beta_G(P)$ be the minimum length of a circuit containing P , and let $\beta_i(G)$ be the maximum of $\beta_G(P)$ over paths of length i in G . We show that $\Lambda_g(G) \leq \beta_i(G) + 1$ for any integer $i > 0$ and for any 2-connected graph G . Partial solutions toward determining the graphs for which equality holds were obtained by Fan et al. in [*J. Comb. Theory, Ser. B*, **98**(6) (2008), 1325-1336], among others. We completely determine all graphs G with $\Lambda_g(G) = \beta_2(G) + 1$.

Let $\mathbb{Z}_3$ denote the cyclic group of order 3. In [16], Jaeger et al conjectured that every 5-edge-connected graph is $\mathbb{Z}_3$ -connected. We proved the following:

- (i) Every 5-edge-connected graph is $\mathbb{Z}_3$ -connected if and only if every 5-edge-connected line graph is $\mathbb{Z}_3$ -connected.
- (ii) Every 6-edge-connected triangular line graph is $\mathbb{Z}_3$ -connected.
- (iii) Every 7-edge-connected triangular claw-free graph is $\mathbb{Z}_3$ -connected.

In particular, every 6-edge-connected triangular line graph and every 7-edge-connected triangular claw-free graph have a nowhere-zero 3-flow.

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DEDICATION

To

My Parents

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Chapter 1

Preliminaries

1.1 Introduction

Undefined terms and notations will follow Bondy and Murty [2]. And $\kappa(G)$, $\kappa'(G)$, and $\delta(G)$ denote the connectivity, edge-connectivity, and minimum degree of a graph G , respectively. Different from [2], a 2-regular nontrivial connected graph is called a **circuit**, and a circuit with k edges is referred as a k -circuit. A subgraph H of G is a **clique** if H is isomorphic to a complete graph. For an edge subset $J \subseteq E(G)$ of a graph G , we take the convention of using J to denote both the edge subset as well as the subgraph $G[J]$ induced by the edge set J . Throughout this paper A denotes an (additive) Abelian group with identity 0. and $A^* = A - \{0\}$.

Let $D = D(G)$ be an orientation of a graph G . If an edge $e \in E(G)$ is directed from a vertex u to a vertex v , then let $\text{tail}(e) = u$ and $\text{head}(e) = v$. For a vertex $v \in V(G)$, define

$$E_D^+(v) = \{e \in E(G) : v = \text{tail}(e)\}, \text{ and } E_D^-(v) = \{e \in E(G) : v = \text{head}(e)\}.$$

Following Jaeger et al [16], we define $F(G, A) = \{f | f: E(G) \mapsto A\}$ and $F^*(G, A) = \{f | f: E(G) \mapsto A^*\}$. For a function $f: E(G) \rightarrow A$, define $\partial f: V(G) \mapsto A$ by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where “ $\sum$ ” refers to the addition in A .

Assume that G has an orientation $D(G)$. A function $b: V(G) \mapsto A$ is called an **A -valued zero sum function** on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A -valued zero sum functions on G is denoted by $Z(G, A)$. A function $f \in F(G, A)$ is an **A -flow** of G if $\partial f(v) = 0$ for every vertex $v \in V(G)$. An A -flow f is a **nowhere-zero A -flow** (abbreviated as A -NZF) if $f \in F^*(G, A)$. For a $b \in Z(G, A)$, a function $f \in F^*(G, A)$ is a **nowhere-zero (A, b) -flow** (abbreviated as (A, b) -NZF) if $\partial f = b$. A graph G is **A -connected** if for all $b \in Z(G, A)$, G has an (A, b) -NZF.

Let $\langle A \rangle$ be the family of graphs that are A -connected. The **group connectivity number** of a graph G is defined as

$$\Lambda_g(G) = \min\{k: G \in \langle A \rangle \text{ for every Abelian group } A \text{ with } |A| \geq k\}.$$

The concept of group connectivity was first introduced by Jaeger, Linial, Payan, and Tarsi in [16] as a nonhomogeneous form of the nowhere-zero flow problem. The nowhere-zero flow problem was first introduced by Tutte [36] in his way to attack the 4-color-conjecture. Tutte left with several fascinating conjectures in this area, which have remained open as of today.

Conjecture 1.1.1 (*Tutte [36], [15]*)

- (i) Every graph G with $\kappa'(G) \geq 2$ has a nowhere-zero $\mathbb{Z}_5$ -flow.
- (ii) Every graph G with $\kappa'(G) \geq 2$ without a subgraph contractible to the Peterson graph admits a nowhere-zero $\mathbb{Z}_4$ -flow.
- (iii) Every graph G with $\kappa'(G) \geq 4$ admits a nowhere-zero $\mathbb{Z}_3$ -flow.

Jaeger et al made the following conjectures about group connectivity. The truth of these conjectures will imply the truth of Tutte's $\mathbb{Z}_5$ -flow conjecture and $\mathbb{Z}_3$ -flow conjecture, as indicated by Kochol [17].

Conjecture 1.1.2 (*Jaeger, Linial, Payan and Tarsi [16]*)

- (i) If G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 5$.
- (ii) If G is a 5-edge-connected graph, then $\Lambda_g(G) \leq 3$.
- (iii) There exists an integer $k \geq 5$ such that if $\kappa'(G) \geq k$, then $\Lambda_g(G) \leq 3$

While many have contributed to the literature of nowhere-zero flows, all these conjectures remain open. Several researchers have investigated the problem what kind of products graphs will have nowhere-zero A -flows when $|A|$ is small, as seen in [13], and [34] and [40].

In [38], Xu and Zhang proposed a triangulated version of the 3-flow conjecture. Let J_3 denote the family of all connected graphs such that $G \in J_3$ if and only if every edge of G lies in a K_3 of G . A graph in J_3 will also be referred as a J_3 **graph**.

Conjecture 1.1.3 (*Xu and Zhang, [38]*) If $\kappa'(G) \geq 4$ and if $G \in J_3$, then G has a 3-NZF.

Devos (Problem 1 in [28]) suggested that if $\kappa'(G) \geq 4$ and if $G \in J_3$, then $\Lambda_g(G) \leq 3$. But a counterexample to this stronger version was given in [28], where a modified version of the conjecture is proposed: If $\kappa'(G) \geq 5$ and if $G \in J_3$, then G has a 3-NZF.

There have been lots of researches conducted to attack Conjectures 1.1.1 and 1.1.2. See Jaeger [15] and Zhang [39] for literature surveys. Jaeger [14] was the first to show that every 2-edge-connected graph has an 8-NZF, and that every 4-edge-connected graph has a 4-NZF. Later

Seymour [33] proved that every 2-edge-connected graph has a 6-NZF. Jaeger, Linial, Payan and Tarsi [16] further showed that if G is a 3-edge-connected graph, then $\Lambda_g(G) \leq 6$. More recently, Sudakov [35] showed that almost every random graph with minimum degree at least 2 has a 3-NZF. As for highly connected graphs, Lai and Zhang [19] first proved that every $4\log_2 |V(G)|$ -edge-connected graph has a 3-NZF. More recently in [26], it is proved that every $3\log_2 |V(G)|$ -edge-connected graph is $\mathbb{Z}_3$ -connected.

1.2 Notation and Terminology

For $m, n \geq 1$, P_m is a path with m edges, K_m is a complete graph with m vertices, and $K_{m,n}$ is a complete bipartite graph with bipartition (X, Y) such that $|X| = m$ and $|Y| = n$.

Let H_1 and H_2 be two subgraphs of a graph G . We say that G is a **parallel connection** of H_1 and H_2 , denoted by $H_1 \oplus_2 H_2$, if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$. The edge $e \in E(H_1) \cap E(H_2)$ is usually referred as the **base edge**.

A **wheel** W_k is the graph obtained from a k -circuit by adding a new vertex **the center of the wheel**, and then by joining the center to every vertex of the k -circuit. A fan F_k is the graph obtained from W_k by deleting an edge not incident with the center. Note that F_2 is the 3-circuit, and W_3 is the complete graph K_4 . The family $\langle WF \rangle$ can now be recursively defined as follows:

(WF1) For all $k \geq 1$, and $n \geq 2$, $W_{2k+1}, F_n \in \langle WF \rangle$.

(WF2) If $G, H \in \langle WF \rangle$, then any parallel connection of G and H is also in $\langle WF \rangle$.

Graphs in $\langle WF \rangle$ are usually referred as WF -graphs. For an integer $k \geq 3$, graph G is **k -circuit connected** if for any pair of edges $e, e' \in E(G)$, G has a sequence of circuits $C_1, C_2, \dots, C_m$ such that $|E(C_i)| \leq k$, $(1 \leq i \leq m)$, $e \in E(C_1)$, $e' \in E(C_m)$ and $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, $(1 \leq i \leq m-1)$. The sequence $C_1, C_2, \dots, C_m$ is often referred as an (e, e') - **k -circuit-path**. A 3-circuit connected graph is also referred as a **triangularly connected** graph. By definition, every WF -graph is triangularly connected.

A graph G is **collapsible** if for every even subset $R \subseteq V(G)$, G has a subgraph Γ_R (called the **R -subgraph** of G) such that $G - E(\Gamma_R)$ is connected and R is the set of odd-degree vertices of Γ_R . The collection of all collapsible graphs is denoted by $\mathcal{CL}$. The following summarizes some useful result related to collapsible graphs.

A J_3 graph G is **triangularly connected** if for all $e, e' \in E(G)$, G has a sequence of circuits $C^1, C^2, \dots, C^m$ in G such that each of the following holds.

(TC1) $e \in E(C^1)$ and $e' \in E(C^m)$,

(TC2) for all $1 \leq i \leq m$, $|E(C^i)| \leq 3$, and

(TC3) for all $1 \leq i \leq m-1$, $|E(C^i) \cap E(C^{i+1})| > 0$.

The sequence $\{C^1, C^2, \dots, C^m\}$ will be referred as an (e, e') -**triangle-path** in G . Graphs in

$\langle WF \rangle$ are usually referred as WF -graphs. By definition, every WF -graph is triangularly connected.

We follow the notations in [18]. Let G be a graph with C_4 as a subgraph, and $\pi = \{X, Y\}$ the bipartition of $V(C_4)$ so that both X and Y are independent sets of C_4 . Let G/π denote the graph obtained from G by identifying all vertices of X to form a single vertex x , identifying all vertices of Y to form a single vertex y , and then joining x, y with a new edge $e_\pi = xy$, so that

$$E(G) - E(C_4) = E(G/\pi) - \{e_\pi\}.$$

Let P denote a path in G , let $\beta_G(P)$ be the minimum length of a circuit containing P , and let $\beta_i(G)$ be the maximum of $\beta_G(P)$ over paths of length i in G .

1.3 Known Results

Proposition 1.3.1 (*Proposition 3.2 of [21]*) *Let A be an Abelian group with $|A| \geq 3$. Then $\langle A \rangle$ satisfies each of the following:*

- (C1) $K_1 \in \langle A \rangle$,
- (C2) if $G \in \langle A \rangle$ and if $e \in E(G)$, then $G/e \in \langle A \rangle$,
- (C3) if H is a subgraph of G and if both $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.

Theorem 1.3.2 (*Proposition 2.2 of [16]*) *Let G be a connected graph and A be an Abelian group. The following are equivalent.*

- (i) $G \in \langle A \rangle$.
- (ii) For all $\bar{f} \in F(G, A)$, there exists $f \in F_0(G, A)$ such that for all $e \in E(G)$, $f(e) \neq \bar{f}(e)$.
- (iii) For all $b \in Z(G, A)$, and for all $\bar{f} \in F(G, A)$, there exists $f \in F(G, A)$ such that $\partial f = b$ and for all $e \in E(G)$, $f(e) \neq \bar{f}(e)$.

Lemma 1.3.3 *Let G be a graph and A be an Abelian group with $|A| \geq 3$, K_n a complete graph of order n , and let C_n denote the circuit on n vertices (also referred as an n -circuit).*

- (i) (*Lemma 2.1 of [23]*) *If for every edge e in a spanning tree of G , G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.*
- (ii) (*[16] and Lemma 3.3 of [21]*) $\Lambda_g(C_n) = n + 1$.
- (iii) (*Lemma 2.8 of [5], Lemma 2.6 of [9]*) *For any integer $k > 1$, $\Lambda_g(W_{2k}) = 3$.*
- (iv) (*Corollary 3.5 of [21]*) *Let $n \geq 5$ be an integer. Then $K_n \in \langle A \rangle$.*

Theorem 1.3.4 *Let G be a graph and H be a collapsible subgraph of G . Each of the following holds.*

- (i) (*Catlin, Theorem 3 and its Corollary of [4]*) *G is collapsible if and only if G/H is collapsible.*
- (ii) (*Catlin, Lemma 3 of [4]*) *If G is collapsible, then for any $e \in E(G)$, G/e is collapsible.*

(iii) (Catlin, [4] and Lemma 1 of [3]) Let $e \in E(K_{3,3})$. Then C_2 , C_3 , $K_{3,3}$ and $K_{3,3} - e$ are collapsible.

(iv) (Theorem 1.5, [20]) Let A be an Abelian group with $|A| = 4$. Then $H \in \langle A \rangle$.

Theorem 1.3.5 (Fan et al, [9]) Let G be a triangularly connected graph with $|V(G)| \geq 3$. Then

(i) G is $\mathbb{Z}_3$ -connected if and only if G contains a nontrivial $\mathbb{Z}_3$ -connected subgraph.

(ii) G is $\mathbb{Z}_3$ -connected if and only if $G \notin \langle WF \rangle$.

Theorem 1.3.6 (Fan et al, [9]) Let G be a triangularly connected graph with $|V(G)| \geq 2$. Each of the following holds.

(i) (Theorem 1.4 of [9]) G is $\mathbb{Z}_3$ -connected if and only if $G \notin \langle WF \rangle$.

(ii) (Lemma 2.4 of [9]) G is $\mathbb{Z}_3$ -connected if and only if G contains a nontrivial $\mathbb{Z}_3$ -connected subgraph.

The following is an immediate corollary of Theorem 1.3.6 and Lemma 1.3.3(ii) and (iii).

Corollary 1.3.7 If $G \in \langle WF \rangle$, then G does not contain any even wheel or 2-circuit.

Lemma 1.3.8 Let G be a connected graph, and let A be an Abelian group.

(i) (Lemma 2.1 of [23]) Let T be a connected spanning subgraph of G . If for each edge $e \in E(T)$, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.

(ii) $G \in \langle A \rangle$ if and only if every block of G is A -connected.

1.4 Main Results

The main results in the dissertation is as following.

1. (i) $\Lambda_g(G_1 \otimes G_2) \leq 4$, where equality holds if and only if both G_1 and G_2 are trees and $\min\{|V(G_1)|, |V(G_2)|\} = 2$.

(ii) $\Lambda_g(G_1[G_2]) \leq 4$, where equality holds if and only if both G_1 and G_2 are trees and $\min\{|V(G_1)|, |V(G_2)|\} = 2$.

(iii) $\Lambda_g(G_1 \times G_2) \leq 5$, where equality holds if and only if either $G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, for $n, m \geq 2$ or G_1 is a tree and $G_2 \cong K_2$.

2.(i) Every 5-edge-connected graph is $\mathbb{Z}_3$ -connected if and only if every 5-edge-connected line graph is $\mathbb{Z}_3$ -connected.

(ii) Every 6-edge-connected triangular line graph is $\mathbb{Z}_3$ -connected.

(iii) Every 7-edge-connected triangular claw-free graph is $\mathbb{Z}_3$ -connected.

(iv) In particular, every 6-edge-connected triangular line graph has a nowhere-zero 3-flow, and every 7-edge-connected triangular claw-free graph has a nowhere-zero 3-flow.

3. $\Lambda_g(G) \leq \beta_i(G) + 1$ for any integer $i > 0$ and for any 2-connected graph G . Partial solutions toward determining the graphs for which equality holds were obtained by Fan et al. in [*J. Comb. Theory, Ser. B*, **98**(6) (2008), 1325-1336], among others. In this paper, we completely determine all graphs G with $\Lambda_g(G) = \beta_2(G) + 1$.

Chapter 2

Group Connectivity in Products of Graphs

2.1 Introduction and Main Results

We consider finite graphs which may have multiple edges but no loops.

For graph products, we adopt the notation in [10]. Let G_1, G_2 be two graphs. The **Cartesian product** graph $G = G_1 \times G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1 \text{ and } u_2v_2 \in E(G_2) \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)\}$. The **strong product** graph $G = G_1 \otimes G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1 \text{ and } u_2v_2 \in E(G_2), \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1), \text{ or both } u_1v_1 \in E(G_1) \text{ and } u_2v_2 \in E(G_2)\}$. And the **lexicographic product** (sometimes called composition, tensor or wreath product) $G = G_1[G_2]$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G_1), \text{ or } u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)\}$.

The purpose of this chapter is to determine the group connectivity for all strong product and lexicographical product graphs, and to investigate the group connectivity number of the Cartesian product graphs.

The following are immediate from the above definitions.

Proposition 2.1.1 *Each of the following holds.*

- (i) $G_1 \times G_2$ is a spanning subgraph of $G_1 \otimes G_2$, and $G_1 \otimes G_2$ is a spanning subgraph of $G_1[G_2]$.
- (ii) If $G_2 \cong K_m$ is a complete graph, then $G_1[K_m] = G_1 \otimes K_m$.

In this paper we will determine the group connectivity number of certain products of connected graphs by proving the following main results.

Theorem 2.1.2 $\Lambda_g(G_1 \otimes G_2) \leq 4$, where equality holds if and only if both G_1 and G_2 are trees and $\min\{|V(G_1)|, |V(G_2)|\} = 2$.

Corollary 2.1.3 $G_1 \otimes G_2$ has a nowhere-zero 3-flow if and only if either one of G_1 and G_2 is not a tree, or both G_1 and G_2 are trees with $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$.

Theorem 2.1.4 $\Lambda_g(G_1[G_2]) \leq 4$, where equality holds if and only if both G_1 and G_2 are trees and $\min\{|V(G_1)|, |V(G_2)|\} = 2$.

Corollary 2.1.5 $G_1[G_2]$ has a nowhere-zero 3-flow if and only if either one of G_1 and G_2 is not a tree, or both G_1 and G_2 are trees with $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$.

Theorem 2.1.6 $\Lambda_g(G_1 \times G_2) \leq 5$, where equality holds if and only if either $G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, for $n, m \geq 2$ or G_1 is a tree and $G_2 \cong K_2$.

2.2 Strong Products and Lexicographical Products

The following observation follows from the definition of strong product immediately.

$$G_1 \otimes G_2 \text{ is triangularly connected.} \quad (2.1)$$

Thus every edge of $G_1 \otimes G_2$ is contained in a 3-circuit. It follows by Lemma 1.3.6 and Proposition 1.3.7, that

$$\Lambda_g(G_1 \otimes G_2) \leq 4. \quad (2.2)$$

We shall prove Theorem 2.1.2 by proving each of the following lemmas.

Lemma 2.2.1 *Each of the following holds.*

- (i) $\Lambda_g(C_n \otimes K_2) = 3$ and $\Lambda_g(P_2 \otimes P_2) = 3$.
- (ii) If G_1 or G_2 contains a circuit, then $\Lambda_g(G_1 \otimes G_2) = 3$.
- (iii) If both G_1 and G_2 contains a path of length at least 2, then $\Lambda_g(G_1 \otimes G_2) = 3$.

Proof. (i) By (2.2), $\Lambda_g(C_n \otimes K_2) \leq 4$. By the definition of strong product, $C_n \otimes K_2$ is not a WF -graph. It follows by (2.1) and by Theorem 1.3.5 that $C_n \otimes K_2$ is $\mathbb{Z}_3$ -connected. Thus $\Lambda_g(C_n \otimes K_2) = 3$.

Let $G = P_2 \otimes P_2$ (see Figure 2.1). Then the subgraph G' induced by the vertex subset $\{x_0y_1, x_1y_0, x_1y_1, x_1y_2, x_2y_1\}$ is an even wheel. By Theorem 1.3.5(ii), $\Lambda_g(G') = 3$. And by Theorem 1.3.5(i), $\Lambda_g(G) = 3$.

(ii) and (iii) Both conclusions follow from (i) and from Theorem 1.3.5(i). ■

Lemma 2.2.2 *If G_1 is a tree, and $G_2 \cong K_2$, then each of the following holds.*

- (i) $G_1 \otimes G_2 \in \langle WF \rangle$.
- (ii) $\Lambda_g(G_1 \otimes G_2) = 4$.

Figure 2.1 $G = P_2 \otimes P_2$

(iii) If H is a nontrivial connected graph, then H has a nowhere-zero $\mathbb{Z}_3$ -flow if and only if $H \oplus_2 K_4$ has a nowhere-zero $\mathbb{Z}_3$ -flow.

(iv) $G_1 \otimes G_2$ does not have a nowhere-zero $\mathbb{Z}_3$ -flow.

Proof. (i) We first argue by induction on $|V(G_1)|$ to show that, under the assumption of this lemma, $G_1 \otimes G_2 \in \langle WF \rangle$. If $|V(G_1)| = 2$, then $G_1 \cong K_2$ as well, and so $G_1 \otimes G_2 \cong K_4 \in \langle WF \rangle$. Assume that for smaller values of $|V(G_1)|$, if G_1 is a tree, then $G_1 \otimes G_2 \in \langle WF \rangle$. Assume now $|V(G_1)| \geq 3$. Since G_1 is a tree, G_1 has an edge uv such that u has degree 1 in G_1 . It follows by assumption that $(G_1 - u) \otimes G_2 \in \langle WF \rangle$. By the definition of strong product, $G_1 \otimes G_2$ is a parallel connection of $(G_1 - u) \otimes G_2$ and $G_1[\{u, v\}] \otimes G_2 \cong K_4$. It follows by (WF2) in the definition of $\langle WF \rangle$ that $G_1 \otimes G_2 \in \langle WF \rangle$. Hence (i) holds by induction.

(ii) By (2.2), (i) and Theorem 1.3.5, (ii) follows as well.

(iii) Suppose that the K_4 , as a subgraph of $H \oplus_2 K_4$, has four vertices u_1, v_1, u_2, v_2 such that u_1, v_1 are the two vertices not in $V(H)$. If H has a nowhere-zero 3-flow, then extend the orientation of H to $H \oplus_2 K_4$ by orienting all edges incident with u_1 away from u_1 and all edges incident with v_1 into v_1 , and by extending the flow on H to $E(K_4) - E(H)$ taking a constant value 1. Then we obtain a nowhere-zero $\mathbb{Z}_3$ -flow of $H \oplus_2 K_4$. Conversely, if $H \oplus_2 K_4$ has a nowhere-zero $\mathbb{Z}_3$ -flow, then since both u_1 and v_1 are adjacent degree 3 vertices, the restriction of this $\mathbb{Z}_3$ -flow to $E(H)$ is also a nowhere-zero $\mathbb{Z}_3$ -flow of L .

(iv) This follows from (iii) and by induction on $|V(G_1)|$. ■

Proof of Theorem 2.1.2: By (2.2) and by Lemma 2.2.2, we may assume that $\Lambda_g(G_1 \otimes G_2) = 4$ to prove that both G_1 and G_2 are trees, and $\min\{|V(G_1)|, |V(G_2)|\} = 2$.

If G_1 or G_2 has a circuit, then by Lemma 2.2.1(ii), $G_1 \otimes G_2 \in \langle \mathbb{Z}_3 \rangle$. Hence we may assume that both G_1 and G_2 are trees. If $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$, then since G_1 and G_2 are connected, each of G_1 and G_2 contains a path of length 2. It follows by Lemma 2.2.1(i) that $G_1 \otimes G_2$

has a nontrivial subgraph in $\langle \mathbb{Z}_3 \rangle$, and so by (2.1) and by Theorem 1.3.5(i), $G_1 \otimes G_2 \in \langle \mathbb{Z}_3 \rangle$. Therefore, we must have $\min\{|V(G_1)|, |V(G_2)|\} = 2$. $\square$ **Proof of Corollary 2.1.3:** Since $G \in \langle \mathbb{Z}_3 \rangle$ implies that G has a nowhere-zero 3-flow, the sufficiency follows from Theorem 2.1.2. Conversely, if both G_1 and G_2 are trees, and $\min\{|V(G_1)|,$

$|V(G_2)|\} = 2$, then by Lemma 2.2.2(iv), $G_1 \otimes G_2$ does not have a nowhere-zero $\mathbb{Z}_3$ -flow. $\square$

Proof of Theorem 2.1.4: By Proposition 2.1.1(i), $G_1 \otimes G_2$ is a spanning subgraph of $G_1[G_2]$. By Theorem 2.1.2 and Lemma 1.3.6, $\Lambda_g(G_1[G_2]) \leq 4$. If $G_1 \otimes G_2 \in \langle \mathbb{Z}_3 \rangle$, then by Lemma 1.3.6, $G_1[G_2] \in \langle \mathbb{Z}_3 \rangle$ as well. If both G_1 and G_2 are trees, and $\min\{|V(G_1)|, |V(G_2)|\} = 2$, then by Proposition 2.1.1(ii), $G_1[G_2] = G_1 \otimes G_2$, and so by Theorem 2.1.2, $G_1[G_2] \notin \langle \mathbb{Z}_3 \rangle$. $\square$

Proof of Corollary 2.1.5: The proof is similar to that for Corollary 2.1.3, and so it is omitted. $\square$

2.3 Cartesian Products

For graph products, we adopt the notation in [10]. Let G_1, G_2 be two graphs. The **Cartesian product** graph $G = G_1 \times G_2$ is a graph with vertex set $V(G) = V(G_1) \times V(G_2)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1 \text{ and } u_2v_2 \in E(G_2) \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)\}$.

Then the following observation follows from the definition of Cartesian product immediately.

$$G_1 \times G_2 \text{ is 4-circuit connected.} \quad (2.3)$$

Thus every edge of $G_1 \times G_2$ is contained in a 4-circuit. It follows by Lemma 1.3.6 and Proposition 1.3.7 that

$$\Lambda_g(G_1 \times G_2) \leq 5. \quad (2.4)$$

Lemma 2.3.1 *Let G be 4-circuit connected and A be an Abelian group with $|A| = 4$. Each of the following holds.*

- (i) $G \in \langle A \rangle$ if and only if G has a nontrivial A -connected subgraph.
- (ii) $G \in \mathcal{CL}$ if and only if G has a nontrivial collapsible subgraph.
- (iii) If G has a nontrivial A -connected subgraph, then $\Lambda_g(G) \leq 4$; If G has a nontrivial collapsible subgraph, then G is collapsible and $\Lambda_g(G) \leq 4$.

Proof. (i) If $G \in \langle A \rangle$, then G is a nontrivial A -connected subgraph of G .

Conversely, let H be a nontrivial maximal A -connected subgraph of G . If $G = H$, then done. Assume that $H \neq G$. Since $|E(H)| \geq 1$, there is an edge $e_1 \in E(H)$. Since $E(G) - E(H) \neq \emptyset$, there is an edge $e_2 \in E(G) - E(H)$. By the definition of 4-circuit-connectedness, G has an (e_1, e_2) -4-circuit-path. By the choice of e_1 and e_2 , this 4-circuit-path has a circuit T with $|E(T)| \leq 4$ such that $T_1 = E(T) \cap E(H) \neq \emptyset$ and $T_2 = E(T) - T_1 \neq \emptyset$. By Proposition 1.3.7,

Figure 2.2 $G = H \oplus_2 C$

T/T_1 is A -connected. Let $H' = H \cup T$. Since $H'/H = T/T_1$ is A -connected, and since H is A -connected, it follows by Proposition 1.3.3(C3) that H' is A -connected, contrary to the maximality of H . Thus we must have $H = G$, and so G is A -connected. This, together with (2.4), implies that $\Lambda_g(G) \leq 4$.

(ii) If $G \in \langle A \rangle$, then G is a nontrivial collapsible subgraph of G .

Conversely, let H be a nontrivial maximal collapsible subgraph of G . If $G = H$, then done. Assume that $H \neq G$. Since $|E(H)| \geq 1$, there is an edge $e_1 \in E(H)$. Since $E(G) - E(H) \neq \emptyset$, there is an edge $e_2 \in E(G) - E(H)$. By the definition of 4-circuit-connectedness, G has an (e_1, e_2) -4-circuit-path. By the choice of e_1 and e_2 , this 4-circuit-path has a circuit T with $|E(T)| \leq 4$ such that $T_1 = E(T) \cap E(H) \neq \emptyset$ and $T_2 = E(T) - T_1 \neq \emptyset$. By Theorem 3.1.2(ii), T/T_1 is collapsible. Let $H' = H \cup T$. Since $H'/H = T/T_1$ is collapsible, and since H is collapsible, it follows by Theorem 3.1.2(i) that H' is collapsible, contrary to the maximality of H . Thus we must have $H = G$, and so G is collapsible.

(iii) This follows from (ii), (2.4) and Theorem 3.1.2(iv). ■

Lemma 2.3.2 *Let C be a 4-circuit and A be an Abelian group with $|A| = 4$. Let $G = H \oplus_2 C$. Then H is A -connected if and only if G is A -connected.*

Proof. Let $V(C) = \{v_1, v_2, v_3, v_4\}$, $E(C) = \{e_1, e_2, e_3, e_4\}$ and assume $V(H) \cap V(C) = \{v_1, v_4\}$ (see Figure 2.2). Let D be an orientation of G such that the edge $v_i v_{i+1}$ is directed from v_i from v_{i+1} , for $i = 1, 2, 3$.

If $H \in \langle A \rangle$, since G/H is a 3-circuit, by Proposition 1.3.7, $G/H \in \langle A \rangle$, then, by Proposition 1.3.3 (C3), $G \in \langle A \rangle$.

Conversely, if $G \in A$, since for any Abelian group A of order 4, either $A \cong \mathbb{Z}_4$ or $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we have the following two cases.

Case 1: $A \cong \mathbb{Z}_4$. Let $b \in Z(H, \mathbb{Z}_4)$. Define $b': V(G) \mapsto \mathbb{Z}_4$ to be

$$b'(v) = \begin{cases} 1, & \text{if } v = v_2, v_3, \\ b(v) + 1, & \text{if } v = v_1, v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $\sum_{v \in V(G)} b'(v) = \sum_{v \in V(H)} b(v) + 4 \equiv 0 \pmod{4}$. So $b' \in Z(G, \mathbb{Z}_4)$. By the definition of $\mathbb{Z}_4$ -connectedness, there is a $\mathbb{Z}_4$ -NZF f' of G such that $\partial f'(v) = b'(v)$, for any $v \in V(G)$. So $f'(e_3) = f'(e_2) + 1 = f'(e_1) + 2$. Therefore $\{f'(e_3), f'(e_2), f'(e_1)\} = \{1, 2, 3\}$. This concludes that $f'(e_1) = 1, f'(e_2) = 2$ and $f'(e_3) = 3$. Define $f: E(H) \mapsto \mathbb{Z}_4^*$ to be $f(e) = f'(e)$, for any $e \in E(H)$. Then

$$\partial f(v) = \begin{cases} \partial f'(v) - 1 = b'(v) - 1 = b(v), & \text{if } v = v_1, \\ \partial f'(v) + 3 = b'(v) + 3 = b(v) + 4 \equiv b(v), & \text{if } v = v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

That is, for any $v \in V(H)$, $\partial f(v) = b(v)$. Therefore by the definition of $\mathbb{Z}_4$ -connectedness, $H \in \langle \mathbb{Z}_4 \rangle$.

Case 2: $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let $b \in Z(H, \mathbb{Z}_2 \times \mathbb{Z}_2)$. Define $b': V(G) \mapsto \mathbb{Z}_2 \times \mathbb{Z}_2$ to be

$$b'(v) = \begin{cases} b(v) + (0, 1), & \text{if } v = v_1, \\ (1, 0), & \text{if } v = v_2, \\ (0, 1), & \text{if } v = v_3, \\ b(v) - (1, 0), & \text{if } v = v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $\sum_{v \in V(G)} b'(v) = \sum_{v \in V(H)} b(v) + 2(0, 1) = 0$. So $b' \in Z(G, \mathbb{Z}_4)$. By the definition of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -connectedness, there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -NZF f' of G such that $\partial f'(v) = b'(v)$, for any $v \in V(G)$. So $f'(e_3) = f'(e_2) + (0, 1) = f'(e_1) + (1, 1)$. Therefore $\{f'(e_3), f'(e_2), f'(e_1)\} = \{(0, 1), (1, 0), (1, 1)\}$. This concludes that $f'(e_1) = (0, 1), f'(e_2) = (1, 1)$ and $f'(e_3) = (1, 0)$. Let $f: E(H) \mapsto \mathbb{Z}_2 \times \mathbb{Z}_2^*$ be $f(e) = f'(e)$, for any $e \in E(H)$. Then

$$\partial f(v) = \begin{cases} \partial f'(v) - (0, 1) = b'(v) - (0, 1) = b(v), & \text{if } v = v_1, \\ \partial f'(v) + (1, 0) = b'(v) + (1, 0) = b(v), & \text{if } v = v_4, \\ b(v), & \text{otherwise.} \end{cases}$$

That is, for any $v \in V(H)$, $\partial f(v) = b(v)$. Therefore by the definition of $\mathbb{Z}_2 \times \mathbb{Z}_2$ -connectedness, $H \in \langle A \rangle$.

By Case 1 and Case 2, we prove that if $G \in \langle A \rangle$, then $H \in \langle A \rangle$. ■

Figure 2.3 $K_{1,3} \times K_{1,3}$

Lemma 2.3.3 *Each of the following holds.*

- (i) *Let G be a tree. Then $\Lambda_g(G \times K_2) = 5$.*
- (ii) *Let $m \geq 2, n \geq 2$. Then $\Lambda_g(K_{1,m} \times K_{1,n}) = 5$.*

Proof. (i) If we can prove that $G \times K_2$ is not $\mathbb{Z}_4$ -connected, then, by (2.4), $\Lambda_g(G \times K_2) = 5$. We will prove by induction on $|V(G)|$ that $G \times K_2$ is not $\mathbb{Z}_4$ -connected.

If $|V(G)| = 2$, then $G \cong K_2$, and so $G \times K_2 \cong C_4$. By Proposition 1.3.7, $G \times K_2$ is not $\mathbb{Z}_4$ -connected. Assume that for smaller values of $|V(G)|$, if G is a tree, then $G \times K_2$ is not $\mathbb{Z}_4$ -connected. Assume now $|V(G)| \geq 3$. Since G is a tree, G has an edge uv such that u has degree 1 in G . It follows by assumption that $(G - u) \times K_2$ is not $\mathbb{Z}_4$ -connected. By the definition of Cartesian product, $G \times K_2$ is a parallel connection of $(G - u) \times K_2$ and $G[\{u, v\}] \times K_2 \cong C_4$. It follows by Lemma 2.3.2 that $G \times K_2$ is not $\mathbb{Z}_4$ -connected. This completes the proof of (i).

(ii) By (2.4), if we can prove that $K_{1,m} \times K_{1,n}$ is not $\mathbb{Z}_4$ -connected, then $\Lambda_g(K_{1,m} \times K_{1,n}) = 5$. By contradiction, we assume that $K_{1,m} \times K_{1,n}$ is $\mathbb{Z}_4$ -connected.

Suppose $V(K_{1,m}) = \{x_0, x_1, \dots, x_m\}$ with $d_{K_{1,m}}(x_0) = m$ and $V(K_{1,n}) = \{y_0, y_1, \dots, y_n\}$ with $d_{K_{1,n}}(y_0) = n$. Let $I = \{1, 2, \dots, m\}$, $I_0 = \{0, 1, 2, \dots, m\}$, $J = \{1, 2, \dots, n\}$ and $J_0 = \{0, 1, 2, \dots, n\}$. By the definition of Cartesian product, $V(K_{1,m} \times K_{1,n}) = \{v_{ij} = x_i y_j : \text{for } i \in I_0 \text{ and } j \in J_0\}$. Let $E_1 = \{v_{0j} v_{ij} : \text{for } i \in I, j \in J_0\}$ and $E_2 = \{v_{i0} v_{ij} : \text{for } i \in I_0, j \in J\}$. Then $E(K_{1,m} \times K_{1,n}) = E_1 \cup E_2$ and v_{ij} has degree 2, for $i \in I$ and $j \in J$.

Let D be an orientation of $K_{1,m} \times K_{1,n}$ such that $v_{0j} v_{ij} \in E_1$ is directed from v_{0j} to v_{ij} ; $v_{i0} v_{ij} \in E_2$ is directed from v_{ij} to v_{i0} (see $K_{1,3} \times K_{1,3}$ in Figure 2.3).

Let $b: V(K_{1,m} \times K_{1,n}) \mapsto \mathbb{Z}_4$ such that

$$b(v) = \begin{cases} 1, & \text{if } v = v_{01}, \\ 3, & \text{if } v = v_{10}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $b \in Z(K_{1,m} \times K_{1,n}, \mathbb{Z}_4)$.

Let $\bar{f}: K_{1,m} \times K_{1,n} \mapsto \mathbb{Z}_4$ such that

$$\bar{f}(e) = \begin{cases} 1, & \text{if } e = v_{01}v_{00}, v_{ij}v_{i0}, \text{ for } i \in I, j \in J, \\ 3, & \text{if } e = v_{10}v_{00}, v_{ij}v_{0j}, \text{ for } i \in I, j \in J, \\ 2, & \text{otherwise.} \end{cases}$$

By Proposition 1.3.1, there is an $f \in F(K_{1,m} \times K_{1,n}, \mathbb{Z}_4)$ such that $\partial f = b$, and $\bar{f}(e) \neq f(e)$, for any $e \in E(K_{1,m} \times K_{1,n})$. For v_{ij} , where $i \in I$ and $j \in J$, since $b(v_{ij}) = f(v_{ij}v_{i0}) - f(v_{ij}v_{0j}) = 0$, $f(v_{ij}v_{i0}) = f(v_{ij}v_{0j})$. Together with $f(v_{ij}v_{i0}) \neq \bar{f}(v_{ij}v_{i0}) = 1$ and $f(v_{ij}v_{0j}) \neq \bar{f}(v_{ij}v_{0j}) = 3$, we have

$$f(v_{ij}v_{i0}), f(v_{ij}v_{0j}) \in \{0, 2\}, \text{ for } i \in I \text{ and } j \in J. \quad (2.5)$$

For vertex v_{i0} , $i = 2, \dots, m$, since $b(v_{i0}) = 0 = -\sum_{j=1}^n f(v_{ij}v_{i0}) - f(v_{i0}v_{00})$, $f(v_{i0}v_{00}) = -\sum_{j=1}^n f(v_{ij}v_{i0})$. By (2.5), $f(v_{i0}v_{00}) \in \{0, 2\}$, and since $f(v_{i0}v_{00}) \neq \bar{f}(v_{i0}v_{00}) = 2$, $f(v_{i0}v_{i0}) = 0$, for $i = 2, \dots, m$. By the similar argument, $f(v_{00}v_{0j}) = 0$, for $j = 2, \dots, n$. That is

$$f(v_{i0}v_{00}) = 0, \text{ for } i = 2, \dots, m; \quad f(v_{00}v_{0j}) = 0, \text{ for } j = 2, \dots, n. \quad (2.6)$$

Since $b(v_{00}) = 0$, by (2.6), $f(v_{10}v_{00}) = f(v_{01}v_{00})$. And $f(v_{10}v_{00}) \neq \bar{f}(v_{10}v_{00}) = 3$, $f(v_{01}v_{00}) \neq \bar{f}(v_{01}v_{00}) = 1$, so $f(v_{10}v_{00}), f(v_{01}v_{00}) \in \{0, 2\}$. For vertex v_{10} , $b(v_{10}) = 3 = -\sum_{j=1}^n f(v_{1j}v_{10}) - f(v_{10}v_{00})$. But by (2.5) and (2.6), $-\sum_{j=1}^n f(v_{1j}v_{10}) - f(v_{10}v_{00}) \in \{0, 2\}$. This is a contradiction. Therefore $K_{1,m} \times K_{1,n}$ is not $\mathbb{Z}_4$ -connected.

Thus by (2.4), $\Lambda_g(K_{1,m} \times K_{1,n}) = 5$. ■

Lemma 2.3.4 *Each of the following holds.*

(i) $\Lambda_g(P_2 \times P_3) \leq 4$.

(ii) Let $n \geq 3$. Then $\Lambda_g(C_n \times K_2) \leq 4$.

(iii) If one of G_1 and G_2 is not a tree, then $\Lambda_g(G_1 \times G_2) \leq 4$.

Proof. (i) We label most of the vertices of $P_2 \times P_3$ as in Figure 2.4 a. Let $\pi_1 = \langle \{v_3, v'_3\}, \{v_4, v'_4\} \rangle$ and $H_1 = (P_2 \times P_3)/\pi_1$ (see Figure 2.4 b). Let $\pi_2 = \langle \{v_1, v'_1\}, \{v_2, v'_2\} \rangle$ and $H_2 = H_1/\pi_2$ (see Figure 2.5 a). Let $\pi_3 = \langle \{v'_3, v''_3\}, \{v'_4, v''_4\} \rangle$ and $H_3 = H_2/\pi_3$ (see Figure 2.5 b). If we redraw

Figure 2.4

Figure 2.5

H_3 (see Figure 2.5 c), then $H_3 \cong K_{3,3} - e$. By Theorem 3.1.2(iii), $H_3 \in \mathcal{CL}$. It follows by Theorem 3.1.3 that $H_2 \in \mathcal{CL}$. Similarly by Theorem 3.1.3, $H_1 \in \mathcal{CL}$ and $P_2 \times P_3 \in \mathcal{CL}$. Then by Theorem 3.1.2(iv) and (2.4), $\Lambda_g(P_2 \times P_3) \leq 4$.

(ii) First we will prove by induction on n that $C_n \times K_2 \in \mathcal{CL}$. When $n = 3$, by Proposition 3.1.2(iii), $C_3 \in \mathcal{CL}$, and by Lemma 2.3.1(iii), $C_3 \times K_2 \in \mathcal{CL}$. When $n = 4$, let $C = v_1 v_2 v_3 v_4$ be a 4-circuit in $C_4 \times K_2$ (see Figure 2.6 a). Let $\pi = \langle \{v_1, v_3\}, \{v_2, v_4\} \rangle$ and $G' = (C_4 \times K_2)/\pi$ (see Figure 2.6 b). If we redraw G' (see Figure 2.6 c), then $G' \cong K_{3,3}$. It follows by Proposition 3.1.2(iii) that G' is collapsible. Therefore by Lemma 3.1.3, $C_4 \times K_2 \in \mathcal{CL}$.

For a fixed $n > 4$, assume that for any $m < n$, $C_m \times K_2 \in \mathcal{CL}$. Let $C = v_1 v_2 v'_1 v'_2$ and $C' = v'_1 v_2 v''_1 v''_2$ be two 4-circuits contained in $C_n \times K_2$ (see Figure 2.7 a). Let $\pi_1 = \langle \{v_1, v'_1\}, \{v_2, v'_2\} \rangle$ and $\pi_2 = \langle \{v_1, v''_1\}, \{v_2, v''_2\} \rangle$. Let $G' = (C_n \times K_2)/\pi_1$ (see Figure 2.7 a) and $G'' = (C_n \times K_2)/\pi_1/\pi_2$ (see Figure 2.7 b). Then by assumption $(C_n \times K_2)/\pi_1/\pi_2 \cong C_{n-2} \times K_2 \in \mathcal{CL}$ (see Figure 2.7 c). By Lemma 3.1.3, $(C_n \times K_2)/\pi_1 \in \mathcal{CL}$. And by Lemma 3.1.3 again, $(C_n \times K_2) \in \mathcal{CL}$. Thus $C_n \times K_2 \in \mathcal{CL}$, for $n \geq 3$. It follows by (2.4) and Theorem 3.1.2 (iv) that $\Lambda_g(C_n \times K_2) \leq 4$, for $n \geq 3$.

(iii) Suppose G_1 is not a tree, then there is a circuit $C_n \subseteq G_1$, where $n \geq 3$. Therefore $G_1 \times G_2$

Figure 2.6

Figure 2.7

contains a nontrivial collapsible subgraph $H \cong C_n \times K_2$. It follows by Theorem 2.3.1(iii) that $\Lambda_g(G_1 \times G_2) \leq 4$. ■

Proof of Theorem 2.1.6: By (2.4) and by Lemma 2.3.3, we may assume that $\Lambda_g(G_1 \otimes G_2) = 5$ to prove that either $G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, where $n, m \geq 2$ or G_1 is a tree and $G_2 \cong K_2$.

If G_1 or G_2 has a circuit, then by Lemma 2.3.4(iii), $\Lambda_g(G_1 \otimes G_2) \leq 4$. Hence we may assume that both G_1 and G_2 are trees.

Case 1: If $\min\{|V(G_1)|, |V(G_2)|\} = 2$, assume $V(G_2) = 2$. Since G_2 is connected, $G_2 \cong K_2$, by Lemma 2.3.3(ii), $\Lambda_g(G_1 \times G_2) = 5$.

Case 2: If $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$, then since G_1 and G_2 are connected, both G_1 and G_2 contain a path of length 2. If one of G_1 and G_2 contains a path of length 3, then it follows by Lemma 2.3.4(i) that $G_1 \otimes G_2$ has a nontrivial subgraph $H \cong P_2 \times P_3$ with $\Lambda_g(H) \leq 4$, and so by (2.1) and by Theorem 2.3.1(iii), $\Lambda_g(G_1 \otimes G_2) \leq 4$. Therefore G_1 and G_2 contains only paths with length 2. So $G_1 \cong K_{1,m}$ and $G_2 \cong K_{1,n}$, for $m, n \geq 2$. $\square$

Chapter 3

Group Connectivity in Line Graphs

3.1 Introduction

Given an $f \in F(G, A)$ and a subset $X \in E(G)$, $f|_X$ denotes the **restriction** of f to X . For $b \in Z(G, A)$, a graph G is (A, b) -**extensible from** v , if for all $f': E(v) \mapsto A^*$ satisfying $\partial f'(v) = b(v)$, there exists an $f \in F^*(G, A)$ with $\partial f = b$ such that $f|_{E(v)} = f'$. If for any $b \in Z(G, A)$, G is (A, b) -extensible from v , then G is called **A -extensible from** v . By definition, if G is A -extensible from v , then $G \in \langle A \rangle$.

Figure 3.1: Reduction in Lemma 3.1.1

Figure 3.2: Reduction in Lemma 3.1.1

Lemma 3.1.1 (Lemma 2.3, [27]) Let G be a graph and $H \cong K_4$ be a subgraph of G and

$v \in V(H)$ (see Figure 3.1(a) and Figure 3.2(a)). If $d_G(v) = 6$ and if G has another subgraph $H' \cong K_4$ such that $V(H) \cap V(H') = \{v\}$, $N_H(v) = \{x_1, x_2, x_3\}$ and $N_{H'}(v) = \{y_1, y_2, y_3\}$, then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 (as depicted in Figure 3.1(b)), and by first deleting x_3v_1, y_3v_2 and then contracting v_1x_1, v_2y_1 (depicted in Figure 3.1(c)); and if $d_G(v) > 6$, then let G_v be the graph obtained from G by splitting the vertex $v \in V(G)$ into v_1, v_2 , deleting the edge v_1x_3 , and then contracting v_1x_1 (depicted in Figure 3.2(c)).

(i) If $G_v \in \langle \mathbb{Z}_3 \rangle$, then $G \in \langle \mathbb{Z}_3 \rangle$.

(ii) If for some $u \in V(G) - v$, G_v is $\mathbb{Z}_3$ -extensible from u , then G is also $\mathbb{Z}_3$ -extensible from u .

Proof. The proof for (i) is given in [27]. The proof for (ii) is similar to that for (i) and so omitted. ■

Definition 3.1.2 Suppose that $N_G(v) = \{v_1, v_2, \dots, v_n\}$, and let $Y = \{vv_1, vv_2\}$. As in [28], define $G_{[v, Y]}$ to be the graph obtained from $G - \{vv_1, vv_2\}$ by adding a new edge that joins v_1 and v_2 .

Lemma 3.1.3 (Lemma 6, [28]): For any Abelian group A and $b \in Z(G, A)$, if $G_{[v, Y]}$ has an (A, b) -NZF, then G has an (A, b) -NZF. Moreover, if $G_{[v, Y]}$ is A -extensible from a vertex u with $u \neq v$, then G is also A -extensible from u .

Lemma 3.1.4 (Lemma 7, [28]): Let A be an Abelian group, G be a graph and $H \in \langle A \rangle$ be a subgraph of G . We define $G^* = G/H$ and denote by v_H the vertex in G^* onto which H is contracted. For any $b \in Z(G, A)$, define $b': V(G^*) \mapsto A$ by $b'(v_H) = \sum_{u \in V(H)} b(u)$ and $b'(v) = b(v)$ for $v \neq v_H$. If G^* admits an (A, b') -NZF f^* , then f^* can be extended to an (A, b) -NZF of G .

3.2 Ryjáček ([32]) Closure of a Claw-free Graph

We shall follow [6] to define a line graph. The **line graph** of a graph G , denoted by $L(G)$, has $E(G)$ as its vertex set, where for an integer $k \in \{0, 1, 2\}$, two vertices in $L(G)$ are joined by k edges in $L(G)$ if and only if the corresponding edges in G are sharing k common vertices in G . In other words, if e_1 and e_2 are adjacent but not parallel in G , then e_1 and e_2 are joined by one edge in $L(G)$; if e_1 and e_2 are parallel edges in G , then e_1 and e_2 are joined by two (parallel) edges in $L(G)$. Note that our definition for line is slightly different from the one defined in [2] (called an edge graph there). But when G is a simple graph, both definitions are the same. The main reason for us to adopt this definition in [6] instead of the traditional definition of a line graph is explained in the introduction section of [27].

For an integer $i > 0$ and for a graph G , define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

A vertex $v \in V(G)$ is **locally connected** if $G[N_G(v)]$ is connected. A graph G is **claw-free** if G does not have an induced subgraph isomorphic to $K_{1,3}$. It is well known ([1], [31]) that every line graph is a claw-free graph.

Following the definition of Ryjáček ([32]), a graph H is the **closure** of a claw-free graph G , denoted by $H = cl(G)$, if

(CL1) there is a sequence of graphs $G_1, \dots, G_t$ such that $G_1 = G, G_t = H, V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \cup \{uv : u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected non-complete $G_i[N_{G_i}(x_i)]$, for $i = 1, \dots, t-1$, and

(CL2) No vertex of H has a connected non-complete neighborhood.

Lemma 3.2.1 *Let G be a claw-free graph.*

(i) *For any $v \in V(G)$, either $G[N_G(v)]$ is an edge disjointed union of two cliques or v is a locally connected vertex.*

(ii) *If v is a locally connected vertex of G , then $G[N_G[v]]$ is triangularly connected.*

Proof. (i) follows from the definition of claw-free graphs immediately.

(ii) Let $e = xy, e' = uw \in E(G[N_G[v]])$, where $y, w \in N_G(v)$ and e and e' are not contained in the same triangle. Since v is locally connected, there is a path $P = v_1v_2\dots v_s$ joining $y = v_1$ and $w = v_s$, where $v_i \in N_G(v)$, for $i = 2, \dots, s-1$, in such a way that if $x \neq v$, then $x = v_2$, and if $u \neq v$, then $u = v_{s-1}$. Since $vv_i \in E(G)$, and since e is in the 3-circuit $G[\{v, v_1, v_2\}]$ and e' is in the 3-circuit $G[\{v, v_{s-1}, v_s\}]$, the 3-circuits $G[\{v, v_i, v_{i+1}\}]$, $1 \leq i \leq s-1$, is an (e, e') -triangle-path. Therefore $G[N_G(v)]$ is triangularly connected. ■

Theorem 3.2.2 *The following are equivalent.*

(i) *Every 5-edge-connected graph is $\mathbb{Z}_3$ -connected.*

(ii) *Every 5-edge-connected line graph is $\mathbb{Z}_3$ -connected.*

Proof. As (i) trivially implies (ii), it suffices to show that (ii) implies (i). Let G be a graph with $\kappa'(G) \geq 5$ and let $S(G)$, the **subdivided graph** of G , be the graph obtained from G by replacing each edge $e = uv$ of G by a 2-path $uv_e v$, where v_e is a new vertex. Let e' be the edge in $L(S(G))$ that has uv_e and $v_e v$ as its ends, and let $E' = \{e' \in E(L(S(G))) | e \in E(G)\}$. It then follows that $L(S(G))/[E(L(S(G))) - E'] = G$. (See Claims 1 and 2 within the proof of Theorem 3.4 in [6]). Moreover, If $\kappa'(G) \geq 5$, then $\kappa'(L(S(G))) \geq 5$, and so $L(S(G)) \in \langle \mathbb{Z}_3 \rangle$ follows by (ii). As $L(S(G))/[E(L(S(G))) - E'] = G$, by Proposition 1.3.1(C2), $G \in \langle \mathbb{Z}_3 \rangle$, and so (i) must hold. ■

Figure 3.3 The graph L_1 in Lemma 3.2.5

Theorem 3.2.3 *Let A be an Abelian group with $|A| \geq 4$ and G be a claw-free graph with $\delta(G) \geq 3$. Each of the following holds:*

- (i) *Suppose that a vertex $v \in V(G)$ is locally connected, and $x, y \in N_G(v)$ are not adjacent. If $G + xy$ is A -connected, then G is A -connected.*
- (ii) *If $cl(G)$ is A -connected, then G is A -connected.*

Proof. By the definition of the closure of a claw-free graph, $cl(G)$ contains G as a spanning connected subgraph. Thus Theorem 3.2.3(ii) follows from Theorem 3.2.3(i) and Lemma 1.3.3(i). Therefore, it suffices to prove Theorem 3.2.3(i).

Let G be a claw-free graph and let $v \in V(G)$ be a locally connected vertex. By Lemma 3.2.1(ii), every edge in the graph $G[N_G[v]]$ lies in a 3-circuit. As $|A| \geq 4$, by Lemma 1.3.3(ii) with $n = 3$, every edge of $G[N_G[v]]$ lies in an A -connected subgraph of $G[N_G[v]]$. It follows by Lemma 1.3.3(i) that $G[N_G[v]] \in \langle A \rangle$. Let $G' = G + xy$. Then $G'[N_{G'}[v]] = G[N_G[v]] + xy$. As $G[N_G[v]] \in \langle A \rangle$, it follows by Lemma 1.3.3(i) that $G'[N_{G'}[v]] \in \langle A \rangle$. Hence if $G' \in \langle A \rangle$, then by Proposition 1.3.1(C2), $G'/G'[N_{G'}[v]] \in \langle A \rangle$. As $G/G[N_G[v]] = G'/G'[N_{G'}[v]] \in \langle A \rangle$, and as $G[N_G[v]] \in \langle A \rangle$, it follows by Proposition 1.3.1(C3) that $G \in \langle A \rangle$. ■

Lemma 3.2.4 *Let G be a claw-free graph with $\delta(G) \geq 3$ and $v \in V(G)$ be locally connected. Then $G[N_G(v)]$ has a Hamilton path.*

Proof. Arguing by contradiction, we assume that $G[N_G(v)]$ does not have a Hamilton path. As every connected graph on three vertices has a Hamilton path, we assume $d_G(v) \geq 4$.

Let $P = x_1x_2\dots x_p$ be a longest path in $G[N_G(v)]$. As $V(P) \neq N_G(v)$, we can pick $x \in N_G(v) - V(P)$. As P is longest, $xx_1, xx_p \notin E(G)$. Since $G[\{x, x_1, x_p, v\}] \not\cong K_{1,3}$, we must have $x_1x_p \in E(G)$. Since $G[N_G(v)]$ is connected, $G[N_G(v)]$ has a path P' from x to a vertex $x_{i_0} \in V(P)$, internally disjoint from $V(P)$. It follows that $xP'x_{i_0}x_{i_0+1}\dots x_px_1x_2\dots x_{i_0-1}$ is a longer path, contrary to the assumption that P is a longest path in $G[N_G(v)]$. ■

Lemma 3.2.5 *Let G be a claw-free graph with $\delta(G) \geq 6$ and $v \in V(G)$ be a locally connected vertex. Each of the following holds.*

- (i) If $d_G(v) \geq 6$ and if $G[N_G[v]] \in \langle WF \rangle$, then $G[N_G[v]]$ contains the graph L_1 depicted in Figure 3 as an induced subgraph. Moreover, if $d_G(v) = 6$, then $G[N_G[v]] = L_1$.
- (ii) If $d_G(v) \geq 7$, then $G[N_G[v]]$ is $\mathbb{Z}_3$ -connected.

Proof. (i) Suppose $d_G(v) = m \geq 6$. By Lemma 3.2.4, $G[N_G(v)]$ has a path $P = v_1v_2\dots v_m$, where $v_i \in N_G(v)$, $1 \leq i \leq m$.

We claim that $G[N_G[v]]$ has a K_4 with $v \in V(K_4)$. If not, then $L = G[\{v, v_1, v_3, v_5\}] \not\cong K_4$, and so both $v_1v_3 \notin E(G)$ and $v_3v_5 \notin E(G)$. Since $G[\{v, v_1, v_3, v_5\}] \not\cong K_{1,3}$, we must have $v_1v_5 \in E(G)$. Similarly, $v_2v_6 \in E(G)$ as $G[\{v, v_2, v_4, v_6\}] \not\cong K_4$. It follows that $G[\{v, v_1, v_2, v_5, v_6\}]$ consists a W_4 , contrary to Corollary 1.3.7 as $G[N_G[v]] \in \langle WF \rangle$. Thus $G[N_G[v]]$ must have a K_4 .

Let $H_1 \cong K_4$ be a subgraph of $G[N_G[v]]$ with $v \in V(H_1)$. Let $W = N_G(v) - V(H_1)$. Note that for all $w \in W$, if w is adjacent to two vertices in $V(H_1) - \{v\}$, then $W_4 \subseteq G[V(H_1) \cup \{w\}]$, contrary to Corollary 1.3.7. Since $|W| \geq 3$, and since every $w \in W$ is adjacent to at most one vertex in $V(H_1)$, it follows from the fact that P is a Hamilton path that there must be $x, y, z \in W$ such that $xz, yz \in E(G)$. Let $V(H_1) - \{v\} = \{u_1, u_2, u_3\}$. With these notations, we further claim that $K_3 \subseteq G[W]$.

Assume that $G[W]$ contains no K_3 's. Then $xy \notin E(G)$. Since for all $u_i \in V(H_1) - \{v\}$, $G[\{v, x, y, u_i\}] \not\cong K_{1,3}$, u_i must be adjacent to x or y . Hence we may assume that there are two u_i 's, say u_1, u_2 , that are adjacent to a same vertex in $\{x, y\}$, say x . It follows that $G[\{v, u_1, u_2, u_3, x\}]$ contains a W_4 , contrary to Corollary 1.3.7. Thus we must have both $G[\{x, y, z\}] \cong K_3$ and $G[\{v, x, y, z\}] \cong K_4$. Let $H_2 = G[\{v, x, y, z\}]$.

Now assume that $d_G(v) = 6$, and so $N_G(v) = V(H_1) \cup W$. Since v is locally connected, $G[N_G(v)]$ has an edge e , say $e = u_1x$, joining H_1 and H_2 . Let $G' = G[E(H_1) \cup E(H_2) \cup \{e\}]$. Then $G' \subseteq G[N_G[v]]$. By the definition of $\langle WF \rangle$, $G' \in \langle WF \rangle$. Let $e' \in E(G[N_G[v]]) - E(G')$. If e and e' are not adjacent, say $e' = u_2y$, then $W_4 \subseteq G[\{v, u_1, u_2, x, y\}]$; if e and e' are adjacent, say $e' = u_2x$, then $W_4 \subseteq G[\{v, u_1, u_2, u_3, x\}]$, contrary to Corollary 1.3.7 in either case. Thus we must have $G[N_G[v]] = G'$, as desired.

(ii) By contradiction, assume that $G[N_G[v]] \notin \langle \mathbb{Z}_3 \rangle$. By Lemma 2.2.1(ii), $G[N_G[v]]$ is triangularly connected. By Theorem 1.3.6, $G[N_G[v]] \in \langle WF \rangle$.

By (i), $G[N_G[v]]$ contains a subgraph L_1 as depicted in Figure 3. Define H_1 and H_2 as the two 4-cliques above in $G[N_G[v]]$ with $V(H_1) \cap V(H_2) = \{v\}$, and let $W' = N_G(v) - (V(H_1) \cup V(H_2))$. Again since $G[N_G[v]]$ contains no W_4 , every vertex $w' \in W'$ is adjacent to at most one vertex in $V(H_i)$, $i \in \{1, 2\}$. It follows that $G[N_G[v]]$ contains an induced subgraph $G[\{v, w', z_1, z_2\}] \cong K_{1,3}$, for some $z_i \in V(H_i) - \{v\}$, ($1 \leq i \leq 2$), contrary to the assumption that G is claw-free. Thus $G[N_G[v]]$ must be $\mathbb{Z}_3$ -connected if $d_G(v) \geq 7$. ■

Theorem 3.2.6 *Let G be a claw-free graph with $\delta(G) \geq 7$. If $cl(G) \in \langle \mathbb{Z}_3 \rangle$, then $G \in \langle \mathbb{Z}_3 \rangle$.*

Proof. For any locally connected $v \in V(G)$ with $d_G(v) \geq 7$, by Lemma 3.2.5(ii), $G[N_G[v]]$ is $\mathbb{Z}_3$ -connected. Let $H_1, \dots, H_m$ be all the maximal $\mathbb{Z}_3$ -connected subgraphs of G . Suppose $G_1 = G, G_2, \dots, G_m, G_{m+1}$ is a sequence of graphs such that, for $i = 1, 2, 3, \dots, m$, $G_{i+1} = G_i/H_i$. Suppose $G'_1 = cl(G), G'_2, \dots, G'_m, G'_{m+1}$ is a sequence of graphs such that, for $i = 1, 2, 3, \dots, m$, $G'_{i+1} = G'_i/H'_i$, where H'_i is the subgraph induced by $V(H_i)$ in $cl(G)$. Note that $H_i \subseteq H'_i$.

Now we claim that $G'_{m+1} = G_{m+1}$. By the construction of G_m and G'_m , we have $V(G'_{m+1}) = V(G_{m+1})$ and $E(G_{m+1}) \subset E(G'_{m+1})$. We only need to show $E(G'_{m+1}) \subset E(G_{m+1})$. Let $e \in E(G'_{m+1})$ and $e \notin E(G_{m+1})$. Assume $e = v_1v_2$ in $cl(G)$. By the definition of closure, there is a locally connected vertex $v \in V(G)$ such that $v_1, v_2 \in N_G(v)$ and v_1 and v_2 are not adjacent. By Lemma 3.2.5(ii) $G[N_G[v]]$ is $\mathbb{Z}_3$ -connected, then $G[N[v]]$ will be contained in some H_i , and $e \in E(H'_i)$, contrary to the fact that $e \in G'_{m+1}$.

Therefore $G_{m+1} = G'_{m+1}$. Since $cl(G) = G'_1 \in \langle \mathbb{Z}_4 \rangle$, by Proposition 1.3.1(C2) $G'_2 \in \langle \mathbb{Z}_4 \rangle$. Inductively, we conclude that $G'_i \in \langle \mathbb{Z}_4 \rangle$, $1 \leq i \leq m+1$. It follows that $G_{m+1} = G'_{m+1} \in \langle \mathbb{Z}_4 \rangle$. Since $H_m \in \langle \mathbb{Z}_4 \rangle$, by Proposition 1.3.1(C3) $G_m \in \langle \mathbb{Z}_4 \rangle$. Inductively, we conclude that $G_i \in \langle \mathbb{Z}_4 \rangle$, $1 \leq i \leq m-1$. In particular, $G = G_1 \in \langle \mathbb{Z}_4 \rangle$. ■

Chapter 4

Group Connectivity of J_3 Line Graphs and J_3 claw-free Graphs

4.1 Introduction

An edge cut X of G is **essential** if $G - X$ has at least two nontrivial components. For any integer $k > 0$, a graph is **essentially k -edge-connected** if G has no essential edge cut X with $|X| < k$. By this definition, if a graph G is k -edge-connected, then G is also essentially k -edge-connected. An edge cut X of G is a **cyclical edge cut** if neither side of $G - X$ is acyclic; G is **cyclically k -edge-connected** if G has no cyclical edge cut of size less than k .

By the definition of a line graph, for all $v \in V(G)$, $E(v)$ induce a complete subgraph H_v in $L(G)$. When $u, v \in V(G)$ with $u \neq v$, if G is simple, then H_v and H_u are edge disjoint complete subgraphs of $L(G)$. Such an observation motivates the following definition.

For a connected graph G , a partition $(E_1, E_2, \dots, E_k)$ of $E(G)$ is a **clique partition** of G if $G[E_i]$ is spanned by a maximal complete subgraph of G for each $i \in \{1, 2, \dots, k\}$. Furthermore, $(E_1, E_2, \dots, E_k)$ is a **(≥ 3) -clique partition** of G , if for each $i \in \{1, 2, \dots, k\}$, $G[E_i]$ is spanned by a K_{n_i} with $n_i \geq 3$; and a **(K_3, K_4) -partition** if for each $i \in \{1, 2, \dots, k\}$, $G[E_i]$ is spanned by a maximal subgraph of G isomorphic to a K_3 or a K_4 . Note that if G is simple, and if $(E_1, E_2, \dots, E_k)$ of $E(G)$ is a clique partition of G , then $|V(G[E_i]) \cap V(G[E_j])| \leq 1$ where $i \neq j$ and $i, j \in \{1, 2, \dots, k\}$. By the definition of a line graph, every J_3 line graph must have a (≥ 3) -clique partition. By Proposition 1.3.1 and Lemma 1.3.3(iv), it suffices to study the $\mathbb{Z}_3$ -connectedness of graphs with a (K_3, K_4) -partition.

For an integer $m > 0$, mK_2 denotes the graph with 2 vertices and m parallel edges. Define $\mathcal{F}_0 = \{G: G \text{ has a } (K_3, K_4)\text{-partition}\}$, and $\mathcal{F}$ to be the family of graphs such that $G \in \mathcal{F}$ if and only if either $G \in \mathcal{F}_0$, or G is obtained from a member $G' \in \mathcal{F}_0$ by contracting some edges in $E(G')$.

Figure 4.1 H_0, H_1, H_2, H_3, H_4

Let $H_1 \cong K_4$ and H_0, H_2, H_3 be contractions of H_1 , where $H_0 = 4K_2$. Let $H_4 \cong 2K_2$ be the graph obtained from K_3 by contracting an edge (see Figure 4.1 for H_i , $0 \leq i \leq 4$). Then for every graph $G \in \mathcal{F}$, $E(G)$ is partitioned into $E_1, E_2, \dots, E_k$, such that $G[E_j] \in \{H_0, H_1, H_2, H_3, K_3, H_4\}$, for $j = 1, 2, \dots, k$.

4.2 Main Results

Theorem 4.2.1 *Each of the following holds.*

- (i) *Every 6-edge-connected J_3 line graph is $\mathbb{Z}_3$ -connected.*
- (ii) *Every 7-edge-connected J_3 claw-free graph is $\mathbb{Z}_3$ -connected.*

We shall prove the following stronger result, which implies Theorem 2.3.1.

Theorem 4.2.2 *Let $G \in \mathcal{F}$ be an essentially 6-edge-connected graph with $|D_3(G) \cup D_4(G) \cup D_5(G)| \leq 1$. Each of the following holds.*

- (i) *For any $u \in D_6(G) \cup D_7(G) \cup D_8(G)$, G is $\mathbb{Z}_3$ -extensible from u .*
- (ii) *If $D_6(G) \cup D_7(G) \cup D_8(G) = \emptyset$, then G is $\mathbb{Z}_3$ -connected.*

Assuming the truth of Theorem 4.2.2, we can derive the following results. A graph G is **$\mathbb{Z}_3$ -reduced** if G does not have a nontrivial subgraph in $\langle \mathbb{Z}_3 \rangle$.

Theorem 4.2.3 *Every 6-edge-connected graph with a (≥ 3) -clique partition is $\mathbb{Z}_3$ -connected.*

Proof. Let G be a counterexample with $|V(G)|$ minimized. As the theorem holds trivially if $|V(G)| \leq 6$, we assume that $|V(G)| \geq 7$. By the minimality of G , G is $\mathbb{Z}_3$ -reduced. By Lemma 1.3.3 (iv), G must have a (K_3, K_4) -partition, and so $G \in \mathcal{F}$. Thus $G \in \langle \mathbb{Z}_3 \rangle$ by Theorem 4.2.2. ■

Proof of Theorem 2.3.1 (i) Let G be a 6-edge-connected J_3 line graph. By the definition of a line graph, and since G is a J_3 graph, G is a 6-edge-connected graph with a (≥ 3) -clique partition. It follows by Theorem 4.2.3 that G is $\mathbb{Z}_3$ -connected.

(ii) Let G be a 7-edge-connected J_3 claw-free graph, and let $cl(G)$ be its closure. Then $cl(G)$ is a 7-edge-connected J_3 line graph. By Theorem 4.2.1(i), $cl(G)$ is $\mathbb{Z}_3$ -connected. By Theorem 3.2.6, G is $\mathbb{Z}_3$ -connected. This completes the proof of Theorem 4.2.1. $\blacksquare$

4.3 The Proof of Theorem 4.2.2

Throughout this section, for a graph G and for $W \subseteq E(G)$, any map $g: W \mapsto \mathbb{Z}_3$ is viewed as a map $g: E(G) \mapsto \mathbb{Z}_3$ such that $g(e) = 0$, for all $e \in E(G) - W$.

By contradiction, assume that there exists a graph $G \in \mathcal{F}$ such that

$$G \text{ is a counterexample to Theorem 4.2.2 with } |V(G)| + |E(G)| \text{ minimized.} \quad (4.1)$$

Thus either

$$D_6(G) \cup D_7(G) \cup D_8(G) = \emptyset, \text{ and } G \notin \langle \mathbb{Z}_3 \rangle, \quad (4.2)$$

or

$$\exists u \in D_6(G) \cup D_7(G) \cup D_8(G) \text{ such that } G \text{ is not } \mathbb{Z}_3\text{-extensible from } u. \quad (4.3)$$

For a graph Γ , let $N(\Gamma) = |V(\Gamma)| + |E(\Gamma)|$. We have the following claims.

Claim 1: If (4.1) holds, then G is $\mathbb{Z}_3$ -reduced; if (4.2) holds, then $G - u$ is $\mathbb{Z}_3$ -reduced.

Assume (4.3) holds. Suppose $G - u$ has a nontrivial subgraph H with $H \in \langle \mathbb{Z}_4 \rangle$. Since $G \in \mathcal{F}$, $G/H \in \mathcal{F}$. As H is nontrivial, $N(G/H) < N(G)$. Since G is essentially 6-edge-connected, G/H is also essentially 6-edge connected. By (4.1), G/H satisfies (i). It follows by Lemma 3.1.4 that G is A -extensible from u , contrary to (4.1). The proof for the case when (4.2) holds is similar. This proves Claim 1.

By Lemma 1.3.3(ii) and Proposition 1.3.1, any $\mathbb{Z}_3$ -reduced graph does not have H_0, H_2, H_3 and H_4 as a subgraph. Thus by Claim 1,

$$G \text{ (when (4.2) holds) or } G - u \text{ (when (4.3) holds) does not have } H_0, H_2, H_3, \text{ or } H_4 \text{ as a subgraph.} \quad (4.4)$$

Claim 2: G is cyclically 9-edge-connected.

Suppose that G has a minimal cyclical edge-cut X with $|X| < 9$. Let G_1 and G_2 be the two components of $G - X$. Since G is essentially 6-edge connected and since both G_1 and G_2 are nontrivial, we have $6 \leq |X| \leq 8$. Let v_{G_i} be the new vertex in G/G_i onto which G_i is contracted, for $i = 1, 2$. Then

$$E_{G/G_1}(v_{G_1}) = E_{G/G_2}(v_{G_2}) = X.$$

Case 1: (4.2) holds.

Let $b \in Z(G, \mathbb{Z}_3)$. Define $b_2: V(G/G_2) \mapsto \mathbb{Z}_3$ by

$$b_2(v) = \begin{cases} \sum_{z \in V(G_2)} b(z), & \text{if } v = v_{G_2} \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_2 \in Z(G/G_2, \mathbb{Z}_3)$ as $b \in Z(G, \mathbb{Z}_3)$. By (4.1) and since $N(G/G_2) < N(G)$, G/G_2 has a $(\mathbb{Z}_3, b)$ -NZF f_2 . Now define $b_1: V(G/G_1) \mapsto \mathbb{Z}_3$ by

$$b_1(v) = \begin{cases} \sum_{z \in V(G_1)} b(z), & \text{if } v = v_{G_1} \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_1 \in Z(G/G_1, \mathbb{Z}_3)$ as $b \in Z(G, \mathbb{Z}_3)$. Define $f'_0 = f_2|_X: X \mapsto \mathbb{Z}_3^*$. Then

$$\partial f'_0(v_{G_1}) = -\partial f_2(v_{G_2}) = -b_2(v_{G_2}) = -\sum_{z \in V(G_2)} b(z) = \sum_{z \in V(G_1)} b(z) = b_1(v_{G_1}).$$

Since $6 \leq d_{G/G_1}(v_{G_1}) \leq 8$, and by (4.1), G/G_1 is $\mathbb{Z}_3$ -extensible from v_{G_1} . Therefore there is a $(\mathbb{Z}_3, b)$ -NZF f_1 of G/G_1 such that $f_1|_X = f'_0 = f_2|_X$. Then $f = f_1 + f_2 - f_2|_X$ is a $(\mathbb{Z}_3, b)$ -NZF of G , contrary to (4.1).

Case 2: (4.3) holds.

Let $b \in Z(G, \mathbb{Z}_3)$. Assume $u \in V(G_1)$ and $f_0: E(u) \mapsto \mathbb{Z}_3^*$ such that $\partial f_0(u) = b(u)$.

Define $b_2: V(G/G_2) \mapsto \mathbb{Z}_3$ by

$$b_2(v) = \begin{cases} \sum_{z \in V(G_2)} b(z), & \text{if } v = v_{G_2}, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_2 \in Z(G/G_2, \mathbb{Z}_3)$ as $b \in Z(G, \mathbb{Z}_3)$. By (4.1) and since $N(G/G_2) < N(G)$, G/G_2 is $\mathbb{Z}_3$ -extensible from u , and so G/G_2 has a $(\mathbb{Z}_3, b)$ -NZF f_2 such that $f_2|_{E(u)} = f_0$.

Now define $b_1: V(G/G_1) \mapsto \mathbb{Z}_3$ by

$$b_1(v) = \begin{cases} \sum_{z \in V(G_1)} b(z), & \text{if } v = v_{G_1}, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b_1 \in Z(G/G_1, \mathbb{Z}_3)$ as $b \in Z(G, \mathbb{Z}_3)$. For v_{G_1} , define $f'_0 = f_2|_X: X \mapsto \mathbb{Z}_3^*$. Then

$$\partial f'_0(v_{G_1}) = -\partial f_2(v_{G_2}) = -b_2(v_{G_2}) = -\sum_{z \in V(G_2)} b(z) = \sum_{z \in V(G_1)} b(z) = b_1(v_{G_1}).$$

By (4.1), by $N(G/G_1) < N(G)$, and since $6 \leq d_{G/G_1}(v_{G_1}) \leq 8$, G/G_1 is $\mathbb{Z}_3$ -extensible from v_{G_1} . Therefore G/G_1 has a $(\mathbb{Z}_3, b_1)$ -NZF f_1 satisfying $f_1|_X = f'_0 = f_2|_X$. Thus $f = f_1 + f_2 - f_2|_X$ is a $(\mathbb{Z}_3, b)$ -NZF of G such that $f|_{E(u)} = f_2|_{E(u)} = f_0$, contrary to (4.1). This proves Claim 2.

Let $\mathcal{H} = \{H_0, H_1, H_2, H_3, K_3, H_4\}$. For a graph $G \in \mathcal{F}$, a subgraph $H \subseteq G$ is $\mathcal{H}$ -**maximal** if $H \in \{H_0, H_1, H_2, H_3, K_3, H_4\}$ and H is not properly contained in another subgraph of G that

is also a member in $\{H_0, H_1, H_2, H_3, K_3, H_4\}$. By the definition of $\mathcal{F}$, if $G \in \mathcal{F}$, then every edge must be in an $\mathcal{H}$ -maximal subgraph of G .

Claim 3: $D_3(G) \cup D_4(G) \cup D_5(G) \neq \emptyset$.

By contradiction, assume that

$$D_3(G) \cup D_4(G) \cup D_5(G) = \emptyset. \quad (4.5)$$

Let $v \in V(G)$ such that if (4.3) holds, then choose v so that u and v are not in the same $\mathcal{H}$ -maximal subgraph of G . Thus $d_G(v) \geq 6$. Since $G \in \mathcal{F}$ and by (4.4), v must be in an $\mathcal{H}$ -maximal subgraph H of G such that $H \in \{K_3, K_4\}$.

Case 1: Suppose $v \in V(H)$ where $H \cong K_4$ with $V(H) = \{v, x_1, x_2, x_3\}$. Let G_v be the graph as defined in Lemma 3.1.1, and we shall use the notations in Figures 3.1 and 3.2.

By the definition of G_v , $N(G_v) < N(G)$ and $G_v \in \mathcal{F}$. If G_v is essentially 6-edge-connected, then by (4.1), G_v satisfies (i) or (ii). By Lemma 3.1.1, G satisfies (i) or (ii) respectively, contrary to (4.1).

Thus G_v has a minimal essential edge cut X with $|X| < 6$. Let G_1, G_2 be the two components of $G - X$. Since G is essentially 6-edge-connected, $\{x_1, x_2, x_3\}$ and $N_G(v) - \{x_1, x_2, x_3\}$ must be in distinct components of $G_v - X$. By the assumption that $G \in \mathcal{F}$ and by (4.4), neither G_1 nor G_2 is acyclic. It follows that in G , $X \cup \{vx_1, vx_2, vx_3\}$ is a cyclical edge-cut with at most 8 edges, contrary to Claim 2. This precludes Case 1 of Claim 3.

Case 2: Suppose $v \in V(H)$ where $H \cong K_3$ with $V(H) = \{v, v_1, v_2\}$. Let $Y = \{vv_1, vv_2\}$ and $G_{[v,Y]}$ be the graph defined in Definition 3.1.2. Then $N(G_{[v,Y]}) < N(G)$. By the choice of H , $G_{[v,Y]} \in \mathcal{F}$. If $G_{[v,Y]}$ is essentially 6-edge-connected, then by (4.1), $G_{[v,Y]}$ satisfies (i) or (ii). By Lemma 3.1.3 G satisfies (i) or (ii) respectively, contrary to (4.1).

Thus $G_{[v,Y]}$ must have a minimal essential edge cut X with $|X| < 6$. Let G_1, G_2 be the two components of $G_{[v,Y]} - X$. Using the notation in Definition 3.1.2, since G is essentially 6-edge-connected, v and $\{v_1, v_2\}$ must be separated by X in $G_{[v,Y]}$. We may assume that $\{v_1, v_2\} \subseteq V(G_1)$ and $N_G[v] - \{v_1, v_2\} \subseteq V(G_2)$. Note that $G_1[\{v_1, v_2\}]$ is a 2-circuit, and by (4.4) and since $d_G(v) \geq 6$, G_2 cannot be acyclic. It follows that $X \cup \{vv_1, vv_2\}$ is a cyclical 7-edge-cut of G , contrary to Claim 2. This precludes Case 2 of Claim 3, and completes the proof for Claim 3.

Claim 4: $\kappa(G) \geq 2$.

By contradiction, assume that G has two subgraphs G_1, G_2 with $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{w\}$. Without loss of generality, if (4.3) holds, we may further assume that $u \in V(G_1)$. By (4.1), $G_2 \in \langle \mathbb{Z}_4 \rangle$, contrary to Claim 1. This proves Claim 4.

Figure 4.2: Case 1a in the proof of Theorem 2.3.2

By Claim 3, we assume that

$$D_3(G) \cup D_4(G) \cup D_5(G) = \{v_0\}.$$

Let $b \in Z(G, \mathbb{Z}_3)$ and $f_0: E(u) \mapsto \mathbb{Z}_3^*$ be such that $\partial f_0(u) = b(u)$. Without loss of generality, we assume that all edges in $E_G(u)$ are oriented away from u .

In the rest of the proof, we shall assume the existence of $u \in D_6(G) \cup D_7(G) \cup D_8(G)$ to prove that G is $\mathbb{Z}_3$ -extensible from u . We shall also show that no matter whether the degree of v_0 in G is 3, 4 or 5, a contradiction will be obtained. The proof for the case when $D_6(G) \cup D_7(G) \cup D_8(G) = \emptyset$ is similar.

By (4.3), in each of the cases below, we always assume that there exists a $b \in Z(G, \mathbb{Z}_3)$ and an $f_0: E_G(u) \mapsto \mathbb{Z}_3^*$ with $\partial f_0(u) = b(u)$, such that Theorem 4.2.2 (i) fails.

Case 1: $v_0 \in D_3(G)$.

Since $v_0 \in D_3(G)$, G has an $\mathcal{H}$ -maximal subgraph H with $v_0 \in V(H)$. By Claim 4 and by $v_0 \in D_3(G)$, $H \in \{H_1, H_2\}$. By (4.4), if $H = H_2$, then u must be the degree 4 vertex in H_2 .

Case 1a: $H \cong H_2$.

Denote $V(H) = \{v_0, u, v_1\}$ where $u \in D_4(H)$ and $G_{v_0} = G/\{v_0v_1\}$ (see Figure 4.2). Then $N(G_{v_0}) < N(G)$. Since $G \in \mathcal{F}$ and since G is essentially 6-edge-connected, $G_{v_0} \in \mathcal{F}$ and G_{v_0} is essentially 6-edge connected. By (4.1), G_{v_0} satisfies (i).

Define $b': V(G_{v_0}) \mapsto \mathbb{Z}_3$ by

$$b'(v) = \begin{cases} b(v_0) + b(v_1), & \text{if } v = v_1 \\ b(v), & \text{otherwise.} \end{cases}$$

As $\sum_{v \in V(G_0)} b'(v) = \sum_{v \in V(G)} b(v) = 0$, $b' \in Z(G_{v_0}, \mathbb{Z}_3)$. Since G_{v_0} is $\mathbb{Z}_3$ -extensible from u , $\exists f' \in F^*(G_{v_0}, \mathbb{Z}_3)$ such that $\partial f' = b'$ and $f'|_{E(u)} = f_0$. Assume that the edge v_0v_1 is oriented from v_0 to v_1 . Define $f: E(G) \mapsto \mathbb{Z}_3^*$ by

$$f(e) = \begin{cases} b(v_0) + f'(e_1) + f'(e_2), & \text{if } e = v_0v_1 \\ f'(e), & \text{otherwise.} \end{cases}$$

Figure 4.3: Case 1b in the proof of Theorem 4.2.2

Then for all $v \in V(G)$,

$$\partial f(v) = \begin{cases} b(v_0) + f'(e_1) + f'(e_2) - f'(e_1) - f'(e_2) = b(v_0) & \text{if } v = v_0, \\ (b'(v_1) + f'(e_1) + f'(e_2)) - (b(v_0) + f'(e_1) + f'(e_2)) = b(v_1) & \text{if } v = v_1, \\ b'(v) = b(v), & \text{otherwise.} \end{cases}$$

It follows that $\partial f = b$, and $f|_{E(u)} = f'|_{E(u)} = f_0$. Therefore G is $\mathbb{Z}_3$ -extensible from u , contrary to (4.1). This completes the proof for Case 1a.

Case 1b: $H = H_1 \cong K_4$ and $u \in V(H)$.

Let $V(H) = \{v_0, u, v_2, v_3\}$. Define G_{v_0} to be the graph obtained from $G - v_0v_2$ by replacing uv_0v_3 by one edge e_0 (see Figure 4.3). Then $N(G_{v_0}) < N(G)$.

Suppose that G_{v_0} has an essential edge-cut X with $|X| < 6$. Since G is essentially 6-edge-connected, X must separate v_0 and v_2 . It follows by (4.4) that $X \cup \{v_0v_2\}$ is a cyclical edge-cut of G with $|X \cup \{v_0v_2\}| \leq 6$, contrary to Claim 2. Thus G_{v_0} is essentially 6-edge-connected and so by (4.1),

$$G_{v_0} \text{ is } \mathbb{Z}_3\text{-extensible from } u. \quad (4.6)$$

We shall show that f_0 can be extended to $f \in F^*(G, \mathbb{Z}_3)$ to find a contradiction to (4.1).

Case 1b1: $b(v_0) = 0$. Define $b': V(G_{v_0}) \mapsto \mathbb{Z}_3$ by

$$b'(v) = \begin{cases} b(v_2) - f_0(uv_0), & \text{if } v = v_2, \\ b(v_3) + f_0(uv_0), & \text{if } v = v_3, \\ b(v), & \text{otherwise} \end{cases}$$

Since $\sum_{v \in V(G_{v_0})} b'(v) = \sum_{v \in V(G)} b(v) = 0$, $b' \in Z(G_{v_0}, \mathbb{Z}_3)$. By (4.6), $\exists f' \in F^*(G_{v_0}, \mathbb{Z}_3)$ such that $\partial f' = b'$, and $f'|_{E(u)} = f_0$. Assume that v_0v_2 is oriented from v_0 to v_2 and v_0v_3 is oriented from v_0 to v_3 . Define $f: E(G) \mapsto \mathbb{Z}_3$ by

$$f(e) = \begin{cases} f'(uv_0), & \text{if } e = v_0u, \\ -f'(uv_0), & \text{if } e = v_0v_2, \\ 2f'(uv_0), & \text{if } e = v_0v_3, \\ f'(e), & \text{otherwise.} \end{cases}$$

Since $f' \in F^*(G_{v_0}, \mathbb{Z}_3)$, $f \in F^*(G, \mathbb{Z}_3)$. For each $v \in V(G)$,

$$\partial f(v) = \begin{cases} 2f'(uv_0) - f'(uv_0) - f'(uv_0) = 0 = b(v_0), & \text{if } v = v_0, \\ \partial f'(v_2) - (-f'(uv_0)) = b'(v_2) + f'(uv_0) = b(v_2), & \text{if } v = v_2, \\ b'(v_3) + f'(uv_0) - 2f'(uv_0) = b(v_3), & \text{if } v = v_3, \\ \partial f'(v) = b'(v) = b(v), & \text{otherwise.} \end{cases}$$

Thus $\partial f = b$ and $f|_{E(u)} = f'|_{E(u)} = f_0$. Hence G is $\mathbb{Z}_3$ -extensible from u , contrary to (4.1).

Case 1b2: $b(v_0) \neq 0$.

Define $b': V(G_{v_0}) \mapsto \mathbb{Z}_3$ by

$$b'(v) = \begin{cases} b(v_2) + b(v_0), & \text{if } v = v_2, \\ b(v), & \text{otherwise.} \end{cases}$$

Then $b' \in Z(G_{v_0}, \mathbb{Z}_3)$. By (4.6), G_{v_0} has an $f': E(G_{v_0}) \mapsto \mathbb{Z}_3^*$ such that $\partial f' = b'$ and $f'|_{E(u)} = f_0$. Assume that v_0v_2 and v_0v_3 are oriented away from v_0 . Define $f: E(G) \mapsto \mathbb{Z}_3^*$ by

$$f(e) = \begin{cases} b(v_0), & \text{if } e = v_0v_2, \\ f'(v_0u), & \text{if } e = v_0u, v_0v_3, \\ f'(e), & \text{otherwise.} \end{cases}$$

Since $f' \in F^*(G_{v_0}, \mathbb{Z}_3)$ and since $b(v_0) \neq 0$, $f \in F^*(G, \mathbb{Z}_3)$. For each $v \in V(G)$,

$$\partial f(v) = \begin{cases} b(v_0) + f'(v_0u) - f'(v_0u) = b(v_0), & \text{if } v = v_0, \\ \partial f'(v_2) - b(v_0) = b'(v_2) - b(v_0) = b(v_2), & \text{if } v = v_2, \\ \partial f'(v) = b'(v) = b(v), & \text{otherwise.} \end{cases}$$

Therefore $\partial f = b$ and $f|_{E(u)} = f'|_{E(u)} = f_0$. Thus G is $\mathbb{Z}_3$ -extensible from u , contrary to (4.1).

Case 1c: $H = H_1 \cong K_4$ and $u \notin V(H)$.

Let $V(H) = \{v_0, v_1, v_2, v_3\}$. Then $d_G(v_i) \geq 6$ for $i = 1, 2, 3$. Let G_{v_1} be the graph obtained from G by first splitting the vertex $v_1 \in V(G)$ into v_1, v'_1 (where v'_1 is adjacent to v_0, v_2, v_3), deleting the edge v'_1v_2 , and then contracting v'_1v_3 (see Figure 4.4). As before, if G_{v_1} has an essential edge cut X with $|X| < 6$, then X must separate v_1 and $\{v_0, v_2, v_3\}$, and so $X \cup \{v_1v_0, v_1v_2, v_1v_3\}$ is a cyclical edge cut of G . It follows by Claim 2 that G_{v_1} is essentially 6-edge-connected.

Let $L' = G_{v_1}[\{v_0, v_2, v_3\}]$. As L' is the three vertex graph with four edges, $L' \in \langle \mathbb{Z}_4 \rangle$. Let $G' = G_{v_1}/L'$ with a new vertex $v_{L'}$. Define $b_1: V(G_{v_1}) \mapsto \mathbb{Z}_3$ such that $b_1(v) = b(v)$, for all $v \in V(G_{v_1})$. As $b \in Z(G, \mathbb{Z}_3)$, $b_1 \in Z(G_{v_1}, \mathbb{Z}_3)$. Define $b': V(G') \mapsto \mathbb{Z}_3$ to be

$$b'(v) = \begin{cases} b_1(v_0) + b_1(v_2) + b_1(v_3), & \text{if } v = v_{L'}, \\ b_1(v), & \text{otherwise.} \end{cases}$$

Figure 4.4: Case 1c in the proof of Theorem 2.3.2

Figure 4.5: Case 2a

Then as $b_1 \in Z(G_{v_1}, \mathbb{Z}_3)$, $b' \in Z(G', \mathbb{Z}_3)$.

As G_{v_1} is essentially 6-edge-connected, so is G' . By (4.1), G' satisfies (i). For any $(\mathbb{Z}_3, b')$ -NZF f' of G' , by Lemma 3.1.4, f' can be extended to a $(\mathbb{Z}_3, b_1)$ -NZF f_1 of G_{v_1} , and by Lemma 3.1.3, f_1 can be extended to a $(\mathbb{Z}_3, b)$ -NZF f of G . Therefore G satisfies (i), a contrary to (4.1).

Case 2: $v_0 \in D_4(G)$.

Since $G \in \mathcal{F}$, either G has two $\mathcal{H}$ -maximal subgraphs H', H'' isomorphic to K_3 , with $v_0 \in V(H') \cap V(H'')$, or G has an $\mathcal{H}$ -maximal subgraph $H \cong H_2$ with $v_0 \in V(H)$, as by Claim 4, $H \cong H_0$ is impossible.

Case 2a: Suppose $v_0 \in V(H') \cap V(H'')$ for two maximal subgraph $H' \cong H'' \cong K_3$ (see Figure 4.5).

Let $N_G(v_0) = \{v_1, v_2, v_3, v_4\}$. Without loss of generality, we may assume that $V(H'') = \{v_0, v_3, v_4\}$ and $u \notin V(H'')$. Let $Y = \{v_4v_0, v_4v_3\}$ and define $G_{[v_4, Y]}$ as in Definition 3.1.2. Denote the two parallel edges joining v_0 and v_3 by e_1, e_2 . Let $G_{v_4} = G_{[v_4, Y]} / \{e_1, e_2\}$. Then $N(G_{v_4}) < N(G)$. As before, if G_{v_4} has an essential edge cut X with $|X| < 6$, then X must separate v_4 and v_0 in G_{v_4} , and so $X \cup \{v_4v_0, v_4v_3\}$ is a cyclical edge cut of G . It follows by Claim 2 that G_{v_4} is essentially 6-edge-connected. By (4.1), G_{v_4} satisfies Theorem 4.2.2(i). By Lemma 3.1.3, G also satisfies Theorem 4.2.2(i), contrary to (4.1).

Figure 4.6: $(H', H'') = (K_3, H_3)$ in Case 3

Case 2b: Suppose v_0 is contained in a subgraph $H \cong H_2$.

Since $G \in \mathcal{F}$, $d_G(v_0) = d_H(v_0) = 4$, G must have a 2-circuit which does not contain u as a vertex, contrary to (4.4). This precludes Case 2.

Case 3: $v_0 \in D_5(G)$.

Since $G \in \mathcal{F}$, by the definition of $\mathcal{F}$, G must have two $\mathcal{H}$ -maximal subgraphs H', H'' such that $H' \in \{K_3, H_4\}$ and $H'' \in \{H_1, H_2, H_3\}$ with $v_0 \in V(H') \cap D_3(H'')$. By (4.4), H' and H'' cannot both have multiple edges, and so

$$(H', H'') \in \{(K_3, H_1), (H_4, H_1), (K_3, H_2), (K_3, H_3)\}. \quad (4.7)$$

If $(H', H'') = (K_3, H_3)$, (see Figure 4.6), then let $V(K_3) = \{v_0, v_1, v_2\}$ and $V(H_3) = \{v_0, v_3\}$. By (4.4), $u = v_3$. Let $V_1 = \{v_0, u\}$, $V_2 = V(G) - V_1$, and W be the set of edges with one end in V_1 and the other in V_2 . Since $d_G(u) \leq 8$, $|W| \leq 2 + d_G(u) - 3 < 8$, and so X is a cyclical edge cut of G with at most seven edges, contrary to Claim 2.

Assume that $(H', H'') = (K_3, H_1)$. Let $V(K_3) = \{v_0, v_1, v_2\}$, and define $Y = \{v_0v_1, v_0v_2\}$. Define $G_{[v_0, Y]}$ as in Definition 3.1.2. Then $N(G_{[v_0, Y]}) < N(G)$. If $G_{[v_0, Y]}$ has an essential edge cut X with $|X| < 6$, then X must separate $V(K_3) - \{v_0\}$ and $V(H_1) - \{v_0\}$ in $G_{[v_0, Y]}$, and so $X \cup \{v_0v_1, v_0v_2\}$ is a cyclical edge cut of G . It follows by Claim 2 that $G_{[v_0, Y]}$ is essentially 6-edge-connected. By (4.1), $G_{[v_0, Y]}$ satisfies (i). By Lemma 3.1.4 G also satisfies (i) of Theorem 4.2.2, contrary to (4.1).

Next, we assume that $(H', H'') = (H_4, H_1)$. Then by (4.4), we denote $V(H_1) = \{v_0, z_1, z_2, z_3\}$ and $V(H_4) = \{v_0, u\}$ (see Figure 4.7). Let G_{z_1} be the graph obtained from G by first splitting the vertex $z_1 \in V(G)$ into z_1, z'_1 (where z'_1 is adjacent to v_0, z_2, z_3), deleting the edge z'_1z_2 , and the contracting z'_1z_3 . If G_{z_1} has an essential edge cut X with $|X| < 6$, then X must separate z_1 and v_0, z_2, z_3 in G_{z_1} , and so $X \cup \{z_1v_0, z_1z_2, z_1z_3\}$ is a cyclical edge cut of G . It follows by Claim 2 that G_{z_1} is essentially 6-edge-connected. Let $L' = G_{z_1}[\{v_0, z_2, z_3\}]$. As L' is the 3 vertex

Figure 4.7: $(H', H'') = (H_4, H_1)$ in Case 3

Figure 4.8: $(H', H'') = (K_3, H_2)$ in Case 3

graph with 4 edges, $L' \in \langle \mathbb{Z}_4 \rangle$. Let $G' = G_{z_1}/L'$. As G_{z_1} is essentially 6-edge-connected, so is G' . By (4.1), G' satisfies (i). By Lemma 3.1.4, G_{z_1} satisfies (i). It follows by Lemma 3.1.1 that G satisfies (i), a contrary to (4.1).

Therefore, we must have $(H', H'') = (K_3, H_2)$. Since $v_0 \in V(H') \cap V(H'')$, we may assume that $V(H') = \{v_0, v_1, v_2\}$. By (4.4), u must be the only vertex of degree four in H'' . Let e_1 and e_2 denote the two parallel edges joining v_0 and u (see Figure 4.8).

Note that $d_G(v_1) \geq 6$. Let $Y = \{v_1 v_0, v_1 v_2\}$. Define $G_{[v_1, Y]}$ as in Definition 3.1.2. By the definition of $\mathcal{F}$, $G_{[v_1, Y]} \in \mathcal{F}$. If $G_{[v_1, Y]}$ has an essential edge cut X with $|X| < 6$, then X must separate v_1 and v_0 (see Figure 4.7) in $G_{[v_1, Y]}$, and so $X \cup \{v_1 v_0, v_1 v_2\}$ is a cyclical edge cut of G . It follows by Claim 2 that $G_{[v_1, Y]}$ is essentially 6-edge-connected.

Let $L' = G_{[v_1, Y]}[\{v_0, v_2\}]$, which is a 2-circuit, and so $L' \in \langle \mathbb{Z}_4 \rangle$. Let $G' = G_{[v_1, Y]}/L'$. As

$G_{[v_1, Y]}$ is essentially 6-edge-connected, so is G' . By (4.1), G' satisfies (i). By Lemma 3.1.4, $G_{[v_1, Y]}$ satisfies (i). It follows by Lemma 3.1.3 that G satisfies (i), contrary to (4.1). This completes the proof for Case 3.

As all the cases lead to contradictions, the theorem is established.

Chapter 5

An Upper Bound on Group Connectivity

5.1 Introduction

Graphs in this chapter are finite and connected, with parallel edges permitted.

It is known that if G is a loopless plane graph without cut edges and with geometric dual G^* , then G has a mapping $f: E(G) \mapsto A^*$ with $\partial f = 0$ if and only if $\chi(G^*) = |A|$ (Tutte [36]). In addition, $\Lambda_g(G) = \chi_g(G^*)$ ([16], also see Theorem 3.6 of [7]).

Brooks' Theorem states that equality in the trivial bound $\chi(G) \leq \Delta(G) + 1$ hold among connected graphs if and only if G is an odd circuit or a complete graph. The group coloring analogue (Theorem 4.2 of [30]) implies that if G is a 2-edge-connected plane graph, then $\Lambda_g(G) \leq \Delta(G^*) + 1$. The fact that edges incident with a vertex in G^* induce a circuit in G motivates us to consider the problem of using certain circuit lengths of G to describe best possible upper bounds for $\Lambda_g(G)$ for general 2-edge-connected graphs that may not be planar. An objective of this paper is to seek best possible upper bounds on $\Lambda_g(G)$ with such a feature.

Let P denote a path in G , and let $\beta_G(P)$ be the minimum length of a circuit containing P . For a positive integer i , let $\beta_i(G)$ be the maximum of $\beta_G(P)$ over paths of length i in G . By this definition, we have

$$\text{girth}(G) \leq \beta_1(G) \leq \beta_2(G) \leq \cdots \leq \beta_i(G) \leq \beta_{i+1}(G) \leq \cdots \leq \beta_{c(G)}(G) = c(G). \quad (5.1)$$

Let H_1 and H_2 be two subgraphs of a graph G . We say that G is a *parallel connection* of H_1 and H_2 , if $E(H_1) \cup E(H_2) = E(G)$, $|V(H_1) \cap V(H_2)| = 2$ and $|E(H_1) \cap E(H_2)| = 1$.

In Section 2, we shall show that for any positive integer i with $i \geq 1$,

$$\Lambda_g(G) \leq \beta_i(G) + 1. \quad (5.2)$$

Determining exactly when the equality $\Lambda_g(G) = \beta_1(G) + 1$ holds seems to be difficult. When $\beta_1(G) = 3$, Fan et al. [9] solved a special case of this problem by showing that if in G every pair of edges are connected by a sequence of mutually intersecting circuits of length at most 3, then $\Lambda_g(G) = \beta_1(G) + 1$ if and only if G can be constructed from odd wheels and K_3 by a finite number of parallel connections. Xu and Zhang [38] conjectured a weaker version of Tutte's 3-flow conjecture (see [15] and [39]): if G is 4-edge-connected and $\beta_1(G) = 3$, then there exists $f: E(G) \mapsto \mathbb{Z}_3^*$ with $\partial f = 0$. It was further conjectured by DeVos ([8], [28]) that every 4-edge-connected graph G with $\beta_1(G) = 3$ satisfies $\Lambda_g(G) \leq 3$. This stronger conjecture was disproved in [28]. As of today, it is not known (see [28]) whether every 5-edge-connected graph G with $\beta_1(G) = 3$ satisfies $\Lambda_g(G) \leq 3$. See a recent survey [25] for more in the literature.

The main purpose of this chapter is to prove the inequality (5.2) and to determine for $i = 2$ the graphs such that equality holds in (5.2). To describe the main result of this paper, we need to introduce some notation.

Theorem 5.1.1 *If G is a 2-connected graph, then*

$$\Lambda_g(G) \leq \beta_2(G) + 1, \quad (5.3)$$

where equality holds in (5.3) if and only if $G \in \{C_k: k \geq 2\} \cup \{K_{2,t}^{1/m}: m \geq 1, t \geq 3\} \cup \{K_4^{1/k}: k \geq 1\}$.

Corollary 5.1.2 *If G is a 2-connected graph, then*

$$\Lambda_g(G) \leq c(G) + 1, \quad (5.4)$$

where equality holds if and only if either $c(G)$ is odd and G is an odd circuit, or $c(G)$ is even and G is isomorphic to a $K_{2,t}^{1/(c(G)/2)}$, for some t .

Corollary 5.1.3 *If G is a 2-edge-connected graph, then (5.4) holds, with equality if and only if each of the following holds:*

- (i) *G has at least one block B such that either $c(G)$ is odd and B is an odd circuit of length $c(G)$, or $c(G)$ is even and B is isomorphic to a $K_{2,t}^{1/(c(G)/2)}$, for some t .*
- (ii) *Every block H of G is either a subgraph with $\Lambda_g(H) \leq c(G)$, or $c(G)$ is odd and H is a circuit of length $c(G)$, or $c(G)$ is even and H is isomorphic to a $K_{2,t}^{1/(c(G)/2)}$, for some integer $t \geq 2$.*

Jaeger et al. [16] showed that if G is 3-edge-connected, then $\Lambda_g(G) \leq 6$, which extends Seymour's famous 6-flow theorem from [33]. Thus it is clear that when $\beta_2(G) \geq 6$, all the extremal graphs in Theorem 5.1.1 will have 2-edge cuts. In Section 2, we investigate some preliminary properties of $\beta_2(G)$, which lead to a proof for (5.3), and present the extremal graphs

in Theorem 5.1.1, as well as the proofs for Corollaries 5.1.2 and 5.1.3. In Section 3, we complete the characterization of the extremal graphs in Theorem 5.1.1. We make some remarks on the applications of Theorem 5.1.1 in the last section.

Let G be a graph and H be a subgraph of G . Following Seymour [33], we define the k -closure of H in G , denoted $cl_k(H)$, to be $H \cup C^1 \cup C^2 \cup \dots$, where $C^1, C^2, \dots$ are circuits of G such that $|E(C^i) - (E(H) \cup_{j=1}^{i-1} E(C^j))| \leq k$.

Corollary 5.1.4 *Let G be a graph and H be a subgraph of G with $cl_k(H) = G$. Let A be an Abelian group with $|A| \geq k + 1$. If H is A -connected, then G is also A -connected. In particular, if $\Lambda_g(H) \leq k + 1$ and if $cl_k(H) = G$, then $\Lambda_g(G) \leq k + 1$.*

Proof: Suppose that $H \cup C^1 \cup \dots \cup C^m = G$. We argue by induction on m to show that for any A with $|A| \geq \Lambda_g(H)$, $G \in \langle A \rangle$. Since $|A| \geq \Lambda_g(H)$, this holds if $m = 0$. Now assume that $m \geq 1$. Let $H' = H \cup C^1 \cup \dots \cup C^{m-1}$. By the induction hypothesis, $H' \in \langle A \rangle$. By the definition of k -closure, every circuit of $G/H' = C^m / (C^m \cap H')$ has length at most k , where $k < |A|$. By Lemmas 1.3.3 (ii) and 1.3.7, $G/H' \in \langle A \rangle$. It follows by Proposition 1.3.1(C3) that $G \in \langle A \rangle$. ■

Lemma 5.1.5 *For any graph G with $\kappa'(G) \geq 2$,*

$$\Lambda_g(G) \leq \max\{\text{girth}(G) + 1, \beta_1(G)\} \leq \max\{\text{girth}(G) + 1, \beta_2(G)\} \leq \beta_2(G) + 1. \quad (5.5)$$

Proof: By (5.1), $\text{girth}(G) \leq \beta_1(G)$. If $\beta_1(G) = \text{girth}(G)$, then for every edge $e \in E(G)$, G has a circuit C_e with length $\text{girth}(G)$ and with $e \in E(C_e)$. By Lemma 1.3.3 (ii), for any Abelian group A with $|A| \geq \text{girth}(G) + 1$, $C_e \in \langle A \rangle$. By Lemma 1.3.7, $G \in \langle A \rangle$, and so $\Lambda_g(G) \leq \text{girth}(G) + 1 = \beta_1(G) + 1$. Hence we may assume $\beta_1(G) > \text{girth}(G)$. Let C^0 be a circuit in G with $|E(C^0)| = \text{girth}(G)$. By Lemma 1.3.3 (ii), $\Lambda_g(C^0) = \text{girth}(G) + 1 \leq \beta_1(G)$. By the definition of $\beta_1(G)$ and by $\kappa'(G) \geq 2$, $cl_{\beta_1(G)-1}(C^0) = G$. By Corollary 5.1.4, $\Lambda_g(G) \leq \beta_1(G)$. Hence we proved the first inequality of (5.5). The second inequality of (5.5) follows from (5.1). ■

Thus (5.2) and (5.3), now follow from (5.1) and (5.5). Let $\mathcal{E}$ denote the set of all 2-connected graphs satisfying equality in (5.3), and define

$$\mathcal{E}_k = \{G \in \mathcal{E} : \text{girth}(G) = \beta_2(G) = k\}. \quad (5.6)$$

By Lemma 1.3.3 (ii), $C_k \in \mathcal{E}_k$. We next show that two other classes of graphs are also in $\mathcal{E}_k$.

Lemma 5.1.6 *Let $t \geq 2$ and $m \geq 1$ be integers. If $G \cong K_{2,t}^{1/m}$, then $\Lambda_g(G) = \beta_2(G) + 1$.*

Proof: The lemma holds trivially for $m = 1$, and so we assume that $m \geq 2$. Let the two (nonadjacent) vertices of degree t in G be w_1 and w_2 ; and let the vertices of degree 2 in $V(G) - \{w_1, w_2\}$ be v_j^i , for $1 \leq j \leq m - 1$ and $1 \leq i \leq t$, such that for each i with $1 \leq i \leq t$,

$$w_1, v_1^i, v_2^i, \dots, v_{m-1}^i, w_2$$

is a directed path under a fixed orientation D of G . Note that in this case, $\beta_2(G) = 2m$, and by (5.5), $\Lambda_g(G) \leq 2m + 1$. To prove that $\Lambda_g(G) = 2m + 1$, it suffices to show that G is not $\langle \mathbb{Z}_4 \rangle_{2m}$ -connected.

We shall apply the equivalence between Theorem 1.3.2(i) and (iii) to prove that G is not $\langle \mathbb{Z}_4 \rangle_{2m}$ -connected. Let $A = \langle \mathbb{Z}_4 \rangle_{2m}$, and let $A_e = \{\bar{2}, \bar{4}, \dots, \overline{2m-2}\} \subset \langle \mathbb{Z}_4 \rangle_{2m}$. We shall assume that G is A -connected to show that either of the the following two cases will lead to a contradiction.

Case 1: $t = 2k$. Choose $b: V(G) \mapsto A$ to be the mapping given by

$$b(z) = \begin{cases} \bar{0} & \text{if } z \notin \{w_1, w_2\} \\ \bar{1} & \text{if } z = w_1 \\ -\bar{1} & \text{if } z = w_2. \end{cases}$$

Note that $b \in Z(G, A)$. Choose $\bar{f}: E(G) \mapsto A$ by, for each i with $1 \leq i \leq t$,

$$\bar{f}(e) = \begin{cases} \bar{1} & \text{if } e = (w_1, v_1^i) \\ \bar{3} & \text{if } e = (v_1^i, v_2^i) \\ \vdots & \vdots \\ \overline{2j+1} & \text{if } e = (v_j^i, v_{j+1}^i) \\ \vdots & \vdots \\ \overline{2m-1} & \text{if } e = (v_{m-1}^i, w_2). \end{cases}$$

Since G is A -connected, by Theorem 1.3.2(iii) there must be a function $f \in F(G, A)$ such that $\partial f = b$ and such that $f(e) \neq \bar{f}(e)$ for $e \in E(G)$. For each i with $1 \leq i \leq t$, let $x_i = f(w_1, v_1^i)$. Since $b(z) = 0$ for $z = v_j^i$, and since the path $w_1, v_1^i, v_2^i, \dots, v_{m-1}^i, w_2$ is a directed path, $f(v_j^i, v_{j+1}^i) = x_i$ for $1 \leq j \leq m-2$ and $f(v_{m-1}^i, w_2) = x_i$. By the choice of $\bar{f}$, since $f - \bar{f} \in F^*(G, A)$, we must have $x_i \in A_e$. It follows by $\partial f(w_1) = b(w_1)$ that $1 \equiv \sum_{i=1}^t x_i \pmod{2m}$. This implies that the sum of certain even numbers can be equal to an odd number, leading to a contradiction.

Case 2: $t = 2k + 1$. Choose $b: V(G) \mapsto A$ to be the mapping given by $b(z) = 0$, for all $z \in V(G)$,

so $b \in Z(G, A)$. Choose $\bar{f}: E(G) \mapsto A$ by defining, for each i with $2 \leq i \leq t$,

$$\bar{f}(e) = \begin{cases} \bar{1} & \text{if } e = (w_1, v_1^i) \\ \bar{3} & \text{if } e = (v_1^i, v_2^i) \\ \vdots & \vdots \\ \overline{2j+1} & \text{if } e = (v_j^i, v_{j+1}^i) \\ \vdots & \vdots \\ \overline{2m-1} & \text{if } e = (v_{m-1}^i, w_2). \end{cases}$$

and,

$$\bar{f}(e) = \begin{cases} \bar{0} & \text{if } e = (w_1, v_1^1) \\ \bar{2} & \text{if } e = (v_1^1, v_2^1) \\ \vdots & \vdots \\ \overline{2j} & \text{if } e = (v_j^1, v_{j+1}^1) \\ \vdots & \vdots \\ \overline{2m-2} & \text{if } e = (v_{m-1}^1, w_2). \end{cases}$$

Since G is A -connected, by Theorem 1.3.2(iii) there must be a function $f \in F(G, A)$ such that $\partial f = b$ and such that $f(e) \neq \bar{f}(e)$ for $e \in E(G)$. For each i with $1 \leq i \leq t$, let $x_i = f(w_1, v_1^i)$. Since $b(z) = 0$ for $z = v_j^i$, and since the path $w_1, v_1^i, v_2^i, \dots, v_{m-1}^i, w_2$ is a directed path, $f(v_j^i, v_{j+1}^i) = x_i$ for $1 \leq j \leq m-2$ and $f(v_{m-1}^i, w_2) = x_i$. By the choice of $\bar{f}$, since $f - \bar{f} \in F^*(G, A)$, we must have $x_i \in A_e$ for $i > 1$, and $x_1 \in A - A_e$. It follows by $\partial f(w_1) = b(w_1)$ that $0 \equiv \sum_{i=1}^t x_i \pmod{2m}$. This implies that the sum of certain even numbers plus one odd number can be equal to an even number, leading to a contradiction.

These contradictions establish the validity of the lemma. ■

Lemma 5.1.7 *If k is a positive integer, then $\Lambda_g(K_4^{1/k}) = \beta_2(K_4^{1/k}) + 1 = 3k + 1$.*

Proof: By the definition of $K_4^{1/k}$, $\beta_2(K_4^{1/k}) = 3k$. By (5.5), it suffices to prove that $K_4^{1/k} \notin \langle \mathbb{Z}_3 \times \langle \mathbb{Z}_4 \rangle_k \rangle$. Denote the four vertices of degree 3 in $K_4^{1/k}$ by v_1, v_2, v_3, v_4 and orient the edges as shown in Figure 5.1.

Figure 5.1 Oriented $K_4^{1/k}$, each line representing a path of k edges.

Let $P^{(v_i, v_j)}$ denote the directed (v_i, v_j) -path whose internal vertices have degree 2, and label these paths by

$$P_1 = P^{(v_1, v_4)}, P_2 = P^{(v_2, v_4)}, P_3 = P^{(v_3, v_4)}, P_4 = P^{(v_2, v_3)}, P_5 = P^{(v_3, v_1)}, \text{ and } P_6 = P^{(v_1, v_2)}.$$

Let $\bar{f} \in F(K_4^{1/k}, \mathbb{Z}_3 \times \langle \mathbb{Z}_4 \rangle_k)$ be a function such that for each $P^{(v_i, v_j)}$, $\bar{f}: E(P^{(v_i, v_j)}) \rightarrow \{\bar{0}\} \times \langle \mathbb{Z}_4 \rangle_k$ is surjective. We argue by contradiction and assume that there exists an A -flow f such that $f(e) \neq \bar{f}(e)$ for any $e \in E(K_4^{1/k})$. Since $\partial f = 0$, f must have the same value on every edge in $E(P^{(v_i, v_j)})$. For $1 \leq j \leq 6$, let (x_j, y_j) denote the common value of f on the edges of P_j , where $x_i \in \mathbb{Z}_3, y_i \in \mathbb{Z}_k$. Then we have $x_i \neq \bar{0}$ for $1 \leq i \leq 6$ and $x_1 + x_2 + x_3 = \bar{0}$. Hence $x_1 = x_2 = x_3 = a \in \mathbb{Z}_3$, where either $a = \bar{1}$ or $a = -\bar{1}$. On the other hand, $x_5 = x_4 + a, x_6 = x_4 - a$, so $\bar{0} \in \mathbb{Z}_3 = \{x_4, x_5, x_6\}$. The contradiction completes the proof. ■

The above results show that these three classes of graphs are extremal cases of Theorem 5.1.1 when equality in (5.3) holds. We shall prove that they are the only extremal graphs, mainly in the next section.

Lemma 5.1.8 *Let G be a 2-connected graph. Each of the following holds.*

- (i) *If $\Delta(G) = 2$, then $\Lambda_g(G) = \beta_2(G) + 1$ if and only if $G \cong C_m$ for some integer $m \geq 2$.*
- (ii) *If $\Delta(G) \geq 3$ and if G is not simple, then $\Lambda_g(G) = \beta_2(G) + 1$ if and only if $G = K_{2,s}^1$ for some integer $s \geq 3$.*
- (iii) *Let G be a graph with $\text{girth}(G) = k \geq 3$ and let C^1, C^2 be two distinct k -circuits in G . If C^1 and C^2 have at least one common edge, then the intersection of C^1 and C^2 must be a path of length at most $k/2$.*

Proof: (i) follows from Lemma 1.3.3 (ii). Suppose that G has parallel edges. By (5.5), $\beta_2(G) = \text{girth}(G) = 2$. Thus (ii) follows from Lemma 5.1.6. Part (iii) follows from the assumption that $\text{girth}(G) = k$. ■

Proof of Corollary 5.1.2: By (5.3), $\Lambda_g(G) \leq \beta_2(G) + 1 \leq c(G) + 1$. By (5.5), when the equality holds in (5.4), we must have $\text{girth}(G) = \beta_2(G) = c(G)$. As $g(K_4^{1/k}) < c(K_4^{1/k})$, Corollary 5.1.2 follows from Theorem 5.1.1. ■

Proof of Corollary 5.1.3: Let $H_1, H_2, \dots, H_s$ be the blocks of G . If $\Lambda_g(G) = c(G) + 1$, then by Lemma 1.3.8, some H_i has its group connectivity number equal to $c(G) + 1$, in which case (4) implies $c(H_i) = c(G)$. Without loss of generality, and by Theorem 5.1.1, we may assume that $\Lambda_g(H_i) = c(G) + 1$ for $H_i \in \{H_1, \dots, H_{s'}\}$ and $\Lambda_g(H_i) \leq c(G)$ for $H_i \in \{H_{s'+1}, \dots, H_s\}$. Thus Corollary 5.1.3 follows from Corollary 5.1.2. ■

5.2 Characterization of the Extremal Graphs

By Lemma 5.1.8, it suffices to characterize the extremal graphs for 2-connected simple graphs G with $\text{girth}(G) \geq 3$ and $\Delta(G) \geq 3$. Moreover, the intersection of any two circuits in G has at most $\lfloor k/2 \rfloor$ edges.

Define $C^2(k, l)$, where $1 \leq l \leq k/2$, to be the union of two k -circuits whose intersection is a path of length l ; and $C^3(k, l_1, l_2, l_3)$, where $1 \leq l_1, l_2, l_3 \leq k/2$ and $l_1 + l_2 + l_3 \leq k$, to be the union of three k -circuits among which the intersection of any two circuits is a path of length l_1, l_2, l_3 , respectively. See Figure 5.2 for examples of these graphs.

Figure 5.2 $C^2(k, l)$ and $C^3(k, l_1, l_2, l_3)$.

Lemma 5.2.1 *If l_1, l_2, l_3 are not identically equal, then $\Lambda_g(C^3(k, l_1, l_2, l_3)) \leq k$.*

Proof: Let A be a group of order at least k . Without loss of generality, we assume that $l_1 < l_3$. Let $H = C^3(k, l_1, l_2, l_3)$ be annotated and oriented as in Figure 5.3. We shall adopt the same notation as in the proof of Lemma 5.1.7 and denote $P^{(v_i, v_j)}$ to be the undirected (v_i, v_j) -path of which all the internal vertices have degree 2. Let $\bar{f} \in F(H, A)$. We shall construct an A -flow f such that $f(e) \neq \bar{f}(e)$ for any e in H . Thus by Theorem 1.3.2, H is A -connected for any A with $|A| \geq k$.

Figure 5.3 Oriented $C^3(k, l_1, l_2, l_3)$.

Denote $\bar{f}(P^{(v_1, v_4)}) = \{\bar{a}_1, \dots, \bar{a}_{l_1}\}$, $\bar{f}(P^{(v_2, v_4)}) = \{\bar{b}_1, \dots, \bar{b}_{l_2}\}$, $\bar{f}(P^{(v_3, v_4)}) = \{\bar{c}_1, \dots, \bar{c}_{l_3}\}$, $\bar{f}(P^{(v_1, v_2)}) = \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}$, $\bar{f}(P^{(v_2, v_3)}) = \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}$, and $\bar{f}(P^{(v_3, v_1)}) = \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}\}$.

Claim 1 There exist $x, y, z \in A$ satisfying each of the following:

- (i) $y \in A - \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}$;
- (ii) $x \in A - \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}, y - \bar{b}_1, \dots, y - \bar{b}_{l_2}\}$;
- (iii) $z \in A - \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}, x - \bar{a}_1, \dots, x - \bar{a}_{l_1}\}$.

Since $|A| \geq k \geq l_1 + l_2 + l_3$, $|A - \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}| \geq k - l_1 \geq l_2 + l_3$. An element y satisfying (i) has at least $l_2 + l_3$ choices. Since $l_1 < l_3$, we have $l_2 + l_3 > l_1 + l_2$. If

$$|\{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}| < k - l_1 - l_2,$$

then there exists $y \in A - \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}$ such that

$$|\{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}, y - \bar{b}_1, \dots, y - \bar{b}_{l_2}\}| < k - l_1, \quad (5.7)$$

and so an x satisfying (ii) can also be chosen. If $|\{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}| = k - l_1 - l_2$ and $y_1 - \bar{b}_1, \dots, y_{l_2+l_3} - \bar{b}_1$ are $(l_2 + l_3)$ distinct elements, then $y_i - \bar{b}_1 \in \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}$ for some i , and so (5.7) holds as well. Hence we can find x and y satisfying both (i) and (ii) in either case.

With a similar argument, for a given y , either $|\{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}\}| < k - l_1$ or we can choose x such that $x - \bar{a}_1 \in \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}\}$. In either case, $|\{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}, y + \bar{c}_1, \dots, y + \bar{c}_{l_3}, x - \bar{a}_1, \dots, x - \bar{a}_{l_1}\}| < k$ and so there must be at least a z satisfying (iii). This proves Claim 1.

By Claim 1, there exist $x, y, z \in A$ satisfying Claim 1 (i)-(iii). Set $a = x - z$, $b = y - x$, and $c = z - y$. We define $f: E(H) \mapsto A$ such that $f(P^{(v_1, v_4)}) = \{a\}$, $f(P^{(v_2, v_4)}) = \{b\}$, $f(P^{(v_3, v_4)}) = \{c\}$, $f(P^{(v_1, v_2)}) = \{x\}$, $f(P^{(v_2, v_3)}) = \{y\}$, and $f(P^{(v_3, v_1)}) = \{z\}$. Note that f defines an A -flow on H . Moreover, by Claim 1 (ii) and (iii),

$$a \notin \{\bar{a}_1, \dots, \bar{a}_{l_1}\}, \quad b \notin \{\bar{b}_1, \dots, \bar{b}_{l_2}\}, \quad c \notin \{\bar{c}_1, \dots, \bar{c}_{l_3}\},$$

and

$$x \notin \{\bar{x}_1, \dots, \bar{x}_{k-l_1-l_2}\}, \quad y \notin \{\bar{y}_1, \dots, \bar{y}_{k-l_2-l_3}\}, \quad \text{and} \quad z \notin \{\bar{z}_1, \dots, \bar{z}_{k-l_1-l_3}\}.$$

Hence $f(e) \neq \bar{f}(e)$ for any e in H . This completes the proof. ■

Figure 5.4

Lemma 5.2.2 Let $K_3 \times K_2$ denote the Cartesian product (see Page 30 of [2]) of the complete graphs K_3 and K_2 . For integers k, l with $k \geq 3l > 0$, let H denote the subdivided $K_3 \times K_2$ as depicted in Figure 5.4(a), where the integer i_e on each edge e of $K_3 \times K_2$ indicates that the edge e is subdivided into a path of i_e edges. If $k > 3l$, then H is A -connected for any A with $|A| \geq k$.

Proof: Let A be an Abelian group of order at least k , and assume that H is oriented as in Figure 5.4(b). Let $\bar{f} \in F(H, A)$. We shall adopt the convention so that the path labelled with $\gamma \in \{\alpha, \beta, \xi, \delta, \epsilon, \phi, \psi, \zeta, \omega\}$ in Figure 5.4(b) is denoted as P_γ , and the values that $\bar{f}$ has assigned on the edges in P_γ are denoted as $\bar{\gamma}_1, \bar{\gamma}_2, \dots$. We will show $\Lambda_g(H) \leq k$ by applying the equivalence between Theorem 1.3.2(iii) and Theorem 1.3.2(i) again.

Claim 2 There exist $x, y, z, w \in A$ satisfying each of the following:

- (i) $x \in A - \{\bar{\phi}_1, \dots, \bar{\phi}_{k-3l}\}$,
- (ii) $y \in A - \{\bar{\psi}_1, \dots, \bar{\psi}_{k-2l}, x - \bar{\delta}_1, \dots, x - \bar{\delta}_l\}$,
- (iii) $z \in A - \{\bar{\zeta}_1, \dots, \bar{\zeta}_{k-3l}, y - \bar{\epsilon}_1, \dots, y - \bar{\epsilon}_l, x - \bar{\xi}_1, \dots, x - \bar{\xi}_l\}$,
- (iv) $w \in A - \{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}, x - \bar{\alpha}_1, \dots, x - \bar{\alpha}_l, z + \bar{\beta}_1, \dots, z + \bar{\beta}_l\}$.

The following observations are straightforward.

$$\forall x \in A, |\{\bar{\psi}_1, \dots, \bar{\psi}_{k-2l}, x - \bar{\delta}_1, \dots, x - \bar{\delta}_l\}| \leq k - l < k,$$

and

$$\forall x, y \in A, |\{\bar{\zeta}_1, \dots, \bar{\zeta}_{k-3l}, y - \bar{\epsilon}_1, \dots, y - \bar{\epsilon}_l, x - \bar{\xi}_1, \dots, x - \bar{\xi}_l\}| \leq k - l < k.$$

Denote $A - \{\bar{\phi}_1, \dots, \bar{\phi}_{k-3l}\} = \{x_1, \dots, x_t\}$, where $t \geq 3l$. If $|\{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}\}| < k - 2l$, pick any $x \in \{x_1, \dots, x_t\}$. Otherwise $|\{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}\}| = k - 2l > k - 3l$, and so there exists an i such that $x_i - \bar{\alpha}_1 \in \{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}\}$. Hence we can pick $x = x_i$. Thus in either case, there always exists an x so that $|\{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}, x - \bar{\alpha}_1, \dots, x - \bar{\alpha}_l\}| < k - l$.

After x has been chosen, pick any $y \in A - \{\bar{\psi}_1, \dots, \bar{\psi}_{k-2l}, x - \bar{\delta}_1, \dots, x - \bar{\delta}_l\}$; and pick any $z \in A - \{\bar{\zeta}_1, \dots, \bar{\zeta}_{k-3l}, y - \bar{\epsilon}_1, \dots, y - \bar{\epsilon}_l, x - \bar{\xi}_1, \dots, x - \bar{\xi}_l\}$. By the choice of x , $|\{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}, x - \bar{\alpha}_1, \dots, x - \bar{\alpha}_l, z + \bar{\beta}_1, \dots, z + \bar{\beta}_l\}| < k$. Hence there exists $w \in A - \{\bar{\omega}_1, \dots, \bar{\omega}_{k-2l}, x - \bar{\alpha}_1, \dots, x - \bar{\alpha}_l, z + \bar{\beta}_1, \dots, z + \bar{\beta}_l\}$. This proves Claim 2.

By Claim 2, there exist $x, y, z, w \in A$ satisfying Claim 2 (i)-(iv). Set $a = x - w$, $b = w - z$, $c = x - z$, $d = x - y$, and $e = y - z$. Define $f: E(H) \mapsto A$ in such a way that f takes a constant value $f(\gamma)$ on every edge of P_γ , for all $\gamma \in \{\alpha, \beta, \xi, \delta, \epsilon, \phi, \psi, \zeta, \omega\}$, as follows:

$$f = \begin{pmatrix} \alpha & \beta & \xi & \delta & \epsilon & \phi & \psi & \zeta & \omega \\ a & b & c & d & e & x & y & z & w \end{pmatrix}.$$

For notational convenience, we also view f as a bijection from $\{\alpha, \beta, \xi, \delta, \epsilon, \phi, \psi, \zeta, \omega\}$ onto $\{a, b, c, d, e, x, y, z, w\}$.

As a mapping $f: E(H) \mapsto A$ under the indicated orientation in Figure 3.3(b), f defines an A -flow on H . Moreover, by Claim 2 (i)-(iv), for all $\gamma \in \{\alpha, \beta, \xi, \delta, \epsilon\}$, $f(\gamma) \notin \{\bar{\gamma}_i : i = 1, 2, \dots, l\}$ and $y \notin \{\bar{\psi}_i : 1 \leq i \leq k - 2l\}$, $w \notin \{\bar{\omega}_i : 1 \leq i \leq k - 2l\}$, $z \notin \{\bar{\zeta}_i : 1 \leq i \leq k - 3l\}$, and $x \notin \{\bar{\phi}_i : 1 \leq i \leq k - 3l\}$. Hence $f(e) \neq \bar{f}(e)$, for any $e \in E(H)$. This completes the proof. ■

From now on in this section, we assume that

$$G \in \mathcal{E}_k \text{ and } \Delta(G) \geq 3. \quad (5.8)$$

By (5.6), $\text{girth}(G) = \beta_2(G) = k$ and $\Lambda_g(G) = \beta_2(G) + 1$. We shall prove that either $G = K_{2, \Delta(G)}^{1/(k/2)}$ and k is even, or $G = K_4^{1/(k/3)}$ and $k \equiv 0 \pmod{3}$.

Let v be a vertex of degree at least 3 in G and let e_1, e_2, e_3 be three edges incident with v . By the definition of β_2 , there exists a k -circuit C^{ij} such that $e_i, e_j \in C^{ij}$ for any $1 \leq i < j \leq 3$. Let U denote the union of the C^{ij} 's. Then U is either $K_{2,3}^{1/(k/2)}$ or $C^3(k, l_1, l_2, l_3)$ for some l_1, l_2, l_3 .

Lemma 5.2.3 *If G satisfies (5.8) and contains $C^3(k, l, l, l)$, then $l = k/3$.*

Proof: By the definition of $C^3(k, l, l, l)$, we have $1 \leq l \leq k/3$. We argue by contradiction and assume that $l < k/3$ and that G contains $C^3(k, l, l, l)$ as depicted and annotated in Figure 5.5, where v_1, v_2 are two neighbors of v . Let C^1 and C^2 denote the two k -circuits containing v in this subgraph. Now consider the adjacent edges v_1v, v_2v , and let C be a k -circuit in G containing these two edges. Note that $C^1 \cup C^2 \cup C$ is a subgraph of G that is not $K_{2,3}^{1/(k/2)}$. Since C^1 and C^2 intersect in a path of length $l < k/2$, by Lemma 5.2.1 it must also be a $C^3(k, l, l, l)$. Hence G contains a subgraph H as depicted in Figure 5.4(a). Since $\beta_2(H) = k$, it follows by Lemma 5.2.2 that H is A -connected for any A with $|A| \geq k$. By the definition of $\beta_2(G)$, if $H \neq G$, then any edge $e \in E(H)$ adjacent to an edge in $E(G) - E(H)$ must be in a circuit of length at most k , and so by the 2-edge-connectedness of G , the closure $cl_{\beta_2(G)-1}(H) = G$. It follows by

Corollary 5.1.4 that $\Lambda_g(G) \leq k$, contrary to the assumption that $\Lambda_g(G) = \beta_2(G) + 1$. Hence we must have $k = 3l$. ■

Figure 5.5

Lemma 5.2.4 *If G satisfies (5.8) and contains $C^2(k, l)$ with $1 \leq l < k/2$, then $l = k/3$ and G contains $C^3(k, l, l, l) = K_4^{1/l}$.*

Proof: Let C^1 and C^2 be two k -circuits in G that intersect in a path of length l . Let v be an endpoint of the intersection path, and let v' and v'' be the two neighbors of v , with $v' \in C^1 - C^2$ and $v'' \in C^2 - C^1$. Let C be a k -circuit containing vv' and vv'' . Since $l < k/2$ and $C \neq C^1 \triangle C^2$, $C^1 \cup C^2 \cup C$ is isomorphic $C^3(k, l, m, n)$ for some m, n . By Lemma 5.2.1, $l = m = n$, so G contains a $C^3(k, l, l, l)$. By Lemma 5.2.3, $l = k/3$. ■

Lemma 5.2.5 *If G satisfies (5.8), then each of the following holds:*

- (i) *If $k \equiv 0 \pmod{3}$, and if C^1 and C^2 are two k -circuits in G which intersect in a path of length $k/3$, then any internal vertex in this path has degree 2 in G .*
- (ii) *Suppose that k is even and that any two circuits of G intersect in a path of length either 0 or $k/2$. If G contains $H = K_{2,3}^{1/(k/2)}$ as a subgraph, and u and v are the two degree 3 vertices, then $d_G(w) = 2$ for all $w \in V(H) - \{u, v\}$.*

Proof: (i) Let $P = C^1 \cap C^2$, and let u and v be the endpoints of P . If P has an internal vertex w with $d_G(w) > 2$, then there exists $e, e' \in E(G)$ incident with w such that $e \in E(C^1) \cap E(C^2)$ and $e' \notin E(C^1) \cap E(C^2)$. By definition of β_2 , G has a k -circuit C containing both e and e' . Since $e \in E(C^1) \cap E(C^2)$, by Lemmas 5.1.8 and 5.2.4, C and C^i intersect in a path of length at least $k/3$ for $i = 1, 2$. However, since w is an internal vertex of P , this is not possible. The contradiction proves (i).

(ii) Denote the three paths in H by $P^i, 1 \leq i \leq 3$, respectively. Let w be an internal vertex of P^i . Arguing similarly as in (i), we conclude that $d_G(w) = 2$. ■

Corollary 5.2.6 *If G satisfies (5.8) and contains a subgraph $H = K_n^{1/(k/3)}$ for some $n \geq 4$ and $k \equiv 0 \pmod{3}$, then for any $v \in V(H)$ with $d_H(v) = 2$, $d_G(v) = 2$.*

Proof: This follows from Lemma 5.2.5(i). ■

Lemma 5.2.7 *If l is a positive integer, then $L_g(K_5^{1/(l)}) \leq 3l$.*

The proof of Lemma 5.2.7 use Proposition 5.2.8 below. Let H be a graph and let $v \in V(H)$ with $d = d(v) \geq 4$. Denote $N(v) = \{v_1, \dots, v_d\}$ and denote $e_i = vv_i$. Define $H_{ij} = H - \{e_i, e_j\} + v_i v_j$.

Proposition 5.2.8 *(Lemma 3.1 (i) of [21]) If $H_{ij} \in \langle A \rangle$ for some $i \neq j$, then $H \in \langle A \rangle$.*

Proof of Lemma 5.2.7: Let A be an Abelian group of order at least $3l$. Let $H = K_5^{1/l}$ and let v be a vertex of degree 4 in H . Denote the 4 neighbors of v by v_1, v_2, v_3, v_4 . Let $H_{12} = H - \{vv_1, vv_2\} + v_1 v_2$. Let C be the $3l$ -circuit containing vv_1, vv_2 and $C' = C - \{vv_1, vv_2\} + v_1 v_2$. Now C' has length $3l - 1$ and thus is A -connected. Also $cl_{2l}(C') = H_{12}$ and thus $H_{12} \in \langle A \rangle$. By Proposition 5.2.8, $H \in \langle A \rangle$. ■

Lemma 5.2.9 *If G satisfies (5.8) and contains a $K_4^{1/(k/3)}$, then $G = K_4^{1/(k/3)}$.*

Proof: Let $H = K_4^{1/(k/3)}$ be a subgraph of G . Denote the four vertices of degree 3 in H by v_1, v_2, v_3, v_4 , and refer to all other vertices in H as internal vertices of H . For $1 \leq i < j \leq 4$, let P^{ij} denote the (v_i, v_j) -path in H of length $k/3$, and for $1 \leq i < j < s \leq 4$ let C^{ijs} denote the k -circuit in H containing v_i, v_j, v_s .

Assume that $G \neq H$. By Corollary 5.2.6, every internal vertex of H has degree 2 in G . Since $G \neq H$ and $\kappa'(G) \geq 2$, we may assume that v_4 is incident with an edge $e \notin E(H)$. Let e_i be the edge incident with v_4 in P^{i4} for $1 \leq i \leq 3$. Now e is adjacent with e_i , for $1 \leq i \leq 3$. Let C^i be a k -circuit in G containing e, e_i .

We claim that C^1 must intersect C^{124} in a path of $k/3$. Otherwise, by Lemma 5.2.4, C^1 and C^{124} intersect in a path of length $k/2$. This marks some internal vertex in P^{12} incident with an edge that is not in H . This leads to a contradiction. Hence C^1 intersects C^{124} exactly in P^{14} . Similarly, $C^1 \cap C^{134} = P^{14}$, and so $C^1 \cap H = P^{14}$. With similar arguments, $C^2 \cap H = P^{24}$ and $C^3 \cap H = P^{34}$.

Note that $C^1 \neq C^2$ and $C^1 \cap C^2 = e$. Apply Lemma 5.2.1 to $C^1 \cup C^2 \cup C^{124}$ to see that C^1 and C^2 must intersect in a path of $k/3$ containing e . Let u be the vertex in the path of $C^1 - P^{14}$ such that the two paths from v_1 to u and from v_4 to u have the same length in C^1 . Let P denote the path in C^1 from u to v_4 containing e , so $C^1 \cap C^2 = P$. Similarly $C^1 \cap C^3 = P = C^2 \cap C^3$.

Let $L = H \cup C^1 \cup C^2 \cup C^3$. It follows by Lemma 5.2.7 that $L = K_5^{1/(k/3)}$ and that $\Lambda_g(L) \leq k$. By repeating applications of Lemma 1.3.3 (ii), $\Lambda_g(G/L) \leq k$. By Proposition 1.3.1(C3), $\Lambda_g(G) \leq k$, contrary to (5.8). This contradiction proves that $G = H$. ■

Lemma 5.2.10 *Suppose G satisfies (5.8). If any two circuits of G intersect in a path of length either 0 or $k/2$, then $G = K_{2,\Delta(G)}^{1/(k/2)}$.*

Proof: Let $v \in V(G)$ and $d(v) = \Delta(G)$. Denote the edges incident with v by $e_1, \dots, e_{\Delta(G)}$. Let C^{12} be a k -circuit containing e_1 and e_2 , and denote the vertex in C^{12} by u such that either of the two (u, v) -paths in C^{12} has the length $k/2$. Let P^1 denote the (u, v) -path in C^{12} containing e_1 . Let C^{1m} be the k -circuit containing e_1 and e_m , where $m \geq 3$. By assumption, $|C^{12} \cap C^{1m}| = k/2$. Hence $C^{12} \cap C^{1m} = P^1$. Let $H = C^{12} \cup \dots \cup C^{\Delta(G)}$. Note that $H = K_{2,\Delta(G)}^{1/(k/2)}$ with u and v as the two common ends of all the paths. By Lemma 5.2.5(ii), the internal vertices of the paths have degree 2 in G . Moreover, $d_H(u) = d_H(v) = d_G(v) = \Delta(G) \geq d_G(u)$. Hence $G = H$ as $\kappa'(G) \geq 2$. ■

Proof of Theorem 5.1.1: By (5.5), $\Lambda_g(G) \leq \beta_2(G) + 1$. By Lemmas 1.3.3(ii), 5.1.6 and 5.1.7, for $G \in \{C_k: k \geq 2\} \cup \{K_{2,t}^{1/m}: m \geq 1, t \geq 3\} \cup \{K_4^{1/k}: k \geq 1\}$, we have $\Lambda_g(G) = \beta_2(G) + 1$.

Conversely, suppose that $\Lambda_g(G) = \beta_2(G) + 1$. If $\Delta(G) = 2$, then Theorem 5.1.1 follows from Lemma 5.1.8. Assume that $\Delta(G) \geq 3$. By Lemmas 5.2.3, 5.2.4, 5.2.9 and 5.2.10, either k is even and $G = K_{2,\Delta(G)}^{1/(k/2)}$, or $k \equiv 0 \pmod{3}$ and $G = K_4^{1/(k/3)}$. This completes the proof of Theorem 5.1.1. ■

5.3 Applications

We have seen that Theorem 5.1.1 can be applied to obtain Corollaries 5.1.2 and 5.1.3. In this section, we shall present additional evidence that Theorem 5.1.1 can be applied to study the group connectivity of certain families of graphs. For subgraphs H_1 and H_2 of a graph G , define

$$H_1 \triangle H_2 = H_1 \cup H_2 - E(H_1) \cap E(H_2).$$

Lemma 5.3.1 *If a graph G has $\Lambda_g(G) \geq m+1$ with $|V(G)| + |E(G)|$ minimized, then G contains no nontrivial subgraph H such that $\Lambda_g(H) \leq m$.*

Proof: If G has a nontrivial subgraph H with $\Lambda_g(H) \leq m$, then $|V(G/H)| + |E(G/H)| < |V(G)| + |E(G)|$, we have $\Lambda_g(G/H) \leq m$. Since $\Lambda_g(H) \leq m$, by Proposition 1.3.1(C3), $\Lambda_g(G) \leq m$, contrary to the assumption of the lemma. ■

Corollary 5.3.2 *(Theorem 3.1, [29]) If G is a 2-edge-connected loopless graph with diameter at most 2, then $\Lambda_g(G) \leq 6$, where equality holds if and only if G is a 5-circuit.*

Proof: By contradiction, assume that G is a counterexample with $|V(G)| + |E(G)|$ minimized. The diameter of G is at most 2, but $\Lambda_g(G) \geq 7$. By the definition of $\beta_2(G)$, G has a 2-path P_2

with $\beta_2(P_2) = \beta_2(G)$, and hence G has a circuit C containing P_2 with $|E(C)| = \beta_2(P_2)$. Let $V(P_2) = \{v_1, v_2, v_3\}$ and $V(C) = \{v_1, v_2, \dots, v_m\}$. By Theorem 5.1.1, $m \geq 6$.

Since the diameter of the graph G is at most 2, G has a (v_1, v_4) -path P' with $|E(P')| \leq 2$. Assume $P' = v_1 v' v_4$. Since $m \geq 6$, P' is not a path on C . Let P denote a (v_1, v_4) -path on C with $|E(P)| = 3$. Since P' and P are both (v_1, v_4) -paths, $P' \Delta P$ contains a circuit C' whose length is at most $|E(P')| + |E(P)| = 5$. By Lemma 1.3.3 (ii), G has a nontrivial subgraph C' with $\Lambda_g(C') \leq 6$, contrary to Lemma 5.3.1 with $m = 6$. ■

A argument similar to the proof above can also be employed to prove the following.

Corollary 5.3.3 *If G is a 2-edge-connected loopless graph with diameter at most m , where $m \geq 3$, then $\Lambda_g(G) \leq 2m + 2$.*

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