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Optimal Portfolio and Consumption with Transaction Costs

Zheng Zhang
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Optimal Portfolio and Consumption with Transaction Costs

Zheng Zhang

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

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Morgantown, West Virginia
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Explicit solutions; Verification theorem; Viscosity solutions

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Abstract

Optimal Portfolio and Consumption with Transaction Costs

Zheng Zhang

In Chapter 1, we study optimal portfolio and consumption with both fixed and proportional transaction costs. For a power utility function we find an explicit solution to the HJB equation governing the no-transaction region. Based on the explicit solution, we formulate a combined stochastic and impulse control problem as a quasi-variational inequality and find the transaction regions, the no-transaction region, and the boundary curves separating them. We show that the explicit solution we find satisfies the verification theorem and it is also a viscosity solution for the quasi-variational inequality. We present numerical results where we compare the various cases of the fixed and proportional transaction costs.

In Chapter 2, we discuss the optimal portfolio and consumption on multiple risky assets with both fixed and proportional transaction costs. Explicit solutions to the corresponding HJB equations are provided. The explicit solutions are viscosity solutions. Numerical results for two risky assets and N risky assets are given.

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Dedication

To

my parents Mao Zhang and Yingjie Zheng

my husband Xiaofeng Wang

and

my daughter Jennifer Wang

Chapter 1

One Risky Asset Problem

1.1 Introduction

We study the optimal portfolio and consumption with both the proportional and fixed transaction costs in diffusion markets. In this chapter, we restrict our attention to the infinite horizon problem with one non-risky asset (money) and one risky asset (stock). The problem is well-known. We maximize the time integral of a discounted utility function. Using the dynamic programming principle, we derive the Hamilton-Jacobi-Bellman (HJB) equation for the value function. In this chapter as a utility function we use a power function $u(c) = \frac{c^\gamma(t)}{\gamma}$ ($0 < \gamma < 1$), where $c(t)$ is the consumption rate. We have found an explicit solution to the HJB equation. Using the explicit solution we are able to find the transaction and no-transaction regions, the transaction boundaries, and the optimal portfolio curves. Explicit solutions can be easily found for other typical utility functions such as exponential and log functions. The finite horizon problems and the case with multiple risky assets should be also treated. They will be discussed in the future publications.

Our first contribution in this chapter is the discovery of the explicit solutions to the corresponding HJB equations. To handle the transaction costs the intervention operator is introduced in [6, 14, 20]. By obtaining the explicit solutions, the intervention operator is reduced to a constrained optimization of the value functions. We accurately show the transaction regions, no-transaction region, and the transaction boundaries separating them both analytically and numerically for several different cases. Provides the existence of the explicit solutions, Verification Theorem and the Theorem of the viscosity solutions are proved. For the Verification Theorem, we show the function Φ which is defined in (1.29) satisfies (1.100), so it is the value function we seek. For the Theorem of the viscosity solutions, we prove the value function Φ is a viscosity solution for the quasi-variational inequality problem (1.30).

Our second contribution is the finding of the optimal portfolio curves and the power series solutions of the transaction boundaries curves. It is well-known that

the optimal allocation of wealth forms a Merton line in the Merton's formulation. However, if the transaction costs are assessed, we observe that the Merton line is no longer a straight line but a curve, and moreover there are two such curves if the proportional transaction costs are considered. We also observe that no-transaction region is no longer a wedge shaped region and it is a region bounded by two curves.

Our third contribution is the investors preference to the safe and risky assets. Introducing the power utility function, we let the value function be the sum of a power of the money and a power of the risky asset, which represent the investors preference to safe and risky assets. They also affect the convexity and concavity of the optimal portfolio and transaction curves. If the power of the money is greater than the power of the risky asset, the graphs of the optimal portfolio curves are concave on the xy -plane. This case is appropriate for conservative investors. On the other hand, if the power of the money is less than the power of the risky asset, the graphs of the optimal portfolio curves are convex on the xy -plane. This case is appropriate for risk-taking or entrepreneur investors.

Merton [17] is one of the earliest papers that study the optimal portfolio selection and consumption for individuals. Merton [18] extends his results from [17] for more general utility functions, price behavior assumptions, and for income generated also from noncapital gains sources. Magill and Constantinides [16] showed the importance of transaction costs, yet the wealth is treated as an independent variable. The idea of cumulative purchase and sale of the stock is introduced in Davis and Norman [8] and the money and the risky assets are distinguished. Shreve and Soner [23] considered the same problem and have shown that their value function is a viscosity solution. Their value function is basically the right hand side of (1.6). Shreve, Soner and Xu [24] studied the problem for bonds. To treat a fixed costs the quasi-variational inequalities and the associated impulse control are introduced in Eastham and Hastings [9] and refined in Korn [14]. Chancelier, Øksendal and Sulem [6] and Øksendal and Sulem [20] formulated the transaction costs problem as a combined stochastic control/impulse control problem, which in turn leads to a nonlinear quasi-variational Hamilton-Jacobi-Bellman inequality (QVHJBI). It is proved that the value function of the form (1.6) is the unique viscosity solution of the QVHJBI, and numerical results are presented. The viscosity solutions and their applications including the HJB equations are discussed in Fleming and Soner [10]. The optimal portfolio selection and consumption in jump diffusion markets is considered in Framstad, Øksendal and Sulem [11] and Øksendal and Sulem [21]. In Ait-Sahalia, Diaz and Hurd [1], the consumption-portfolio selection problem of an investor facing both Brownian and jump risks is analyzed, and the optimal portfolio in closed form is determined by orthogonal decompositions. A verification theorem for the quasivariational problem is proved in Brekke and Øksendal [5]. The proofs of the uniqueness for the viscosity solutions are provided in Akian, Menaldi and Sulem [2] and Akian, Sulem, and Taksar [3]. A comprehensive summary of the subject is discussed in Korn [13].

The outline of this chapter is as follows. In Section 1.2 we derive the HJB equation for the no-transaction region and define the intervention operator for

the transaction regions. Thereby, we formulate a quasi-variational inequality problem consisting of a stochastic and an impulse control problems. Explicit solutions to the HJB equation is obtained in Section 1.3. In Section 1.4, we study analytically the optimal portfolios and the boundaries between the transaction regions and no-transaction region. We prove the verification theorem for the quasi-variational problem in Section 1.5. This verifies that the explicit solution we obtained is a value function to the optimal portfolio and consumption problem. In Section 1.6, we show that the explicit solution for the HJB equation is also a viscosity solution to the quasi-variational problem. In Section 1.7, we provide the numerical results and compare them with the analytical results obtained in Section 1.4. Explicit solutions by using exponential and natural logarithm utility functions are given in Section 1.8.

1.2 Formulation of the problem

We formulate a stochastic and impulse control problem for an optimal portfolio and consumption. Let (Ω, F, P) be a probability space with $F(t)(t \geq 0)$ a given filtration. We define $X(t)(t \geq 0)$ as the amount of money invested in the money market (bank) and $Y(t)(t \geq 0)$ as the stochastic process of the amount of money invested in stock. We assume that the processes $X(t)$ and $Y(t)$ satisfy

$$dX(t) = (rX(t) - c(t))dt, \quad \tau_i \leq t \leq \tau_{i+1}, \quad (1.1)$$

$$dY(t) = \alpha Y(t)dt + \sigma Y(t)dW(t), \quad \tau_i \leq t \leq \tau_{i+1}, \quad (1.2)$$

$$X(\tau_{i+1}) = X(\tau_{i+1}^-) - k - \xi_{i+1} - \lambda|\xi_{i+1}|, \quad (1.3)$$

$$Y(\tau_{i+1}) = Y(\tau_{i+1}^-) + \xi_{i+1}, \quad (1.4)$$

where $W(t)$ is one dimensional F_t -Brownian motion. A constant $r > 0$ is the interest rate of the bank account and $c(t) \geq 0$ is the consumption rate from the bank account. Constants $\alpha > 0$ and $\sigma > 0$ are the mean rate of return and standard volatility of the stock respectively. We assume that $\alpha > r$. Here, $\{\tau_1, \tau_2, \tau_3, \dots\}$ satisfying $0 \leq \tau_1 < \tau_2 < \dots$ are F_t -stopping times at which the investor decides to change his portfolio, and $\{\xi_1, \xi_2, \xi_3, \dots\}$ are sizes of the transactions at these times. In each transaction a fixed transaction cost $k \geq 0$ and a proportional transaction cost $\lambda|\xi_i|$ are assessed, where λ ($0 \leq \lambda < 1$) is a rate for the proportional transaction cost. The control of the investor is a combination of a regular stochastic control $c(t)$ and an impulse control $\phi = (\tau_1, \tau_2, \dots; \xi_1, \xi_2, \dots)$. We consider the case where the shorts are not allowed. Therefore, we define a solvency region by

$$S_+ := [0, \infty) \times [0, \infty) \quad (1.5)$$

and study the problem in S_+ .

We consider the problem with the utility function $u(c) = \frac{c^\gamma(t)}{\gamma}$ where γ is a constant satisfying $0 < \gamma < 1$. The objective of the investor is to control $(c(t), \phi)$ so as to maximize the time integral of the discounted utility function for his life time consumption or to find the value function

$$\tilde{\Phi}(s, x, y) = \sup_{c, \phi} E \left[\int_0^\infty e^{-\delta(s+t)} \frac{c^\gamma(t)}{\gamma} dt \right], \quad (1.6)$$

where $\delta > 0$ is a constant discount rate and we assume

$$\alpha > \delta > r. \quad (1.7)$$

In other words, the discount rate (or inflation rate) is larger than the interest rate and smaller than the growth rate of the risky asset. Note that $1 - \gamma$ is called the relative risk aversion coefficient. In the infinite horizon problem we can simplify the problem in the following manner

$$\tilde{\Phi}(s, x, y) = e^{-\delta s} \sup_{c, \phi} E \left[\int_0^\infty e^{-\delta t} \frac{c^\gamma(t)}{\gamma} dt \right] = e^{-\delta s} \tilde{\Phi}(0, x, y) \triangleq e^{-\delta s} \Phi(x, y) \quad (1.8)$$

so that we seek a time independent value function $\Phi(x, y)$.

We derive the equations governing the value function in the rest of this section and in the next section define the explicit form of the value function. We need to distinguish the case where there are no transactions and the case where the transactions are taking place. First we discuss the no-transaction case. Denoting by $\tilde{v}(s, x, y) = e^{-\delta s} v(x, y)$ the value function for the no-transaction case and using the dynamical principle, we see that v satisfies the Hamilton-Jacobi-Bellman (HJB) equation given by

$$\max_c [-\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{c^\gamma}{\gamma}] = 0. \quad (1.9)$$

We define

$$L^c v(x, y) = -\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}. \quad (1.10)$$

Also, we define

$$Lv(x, y) = \sup_{c \geq 0} \{ L^c v(x, y) + \frac{c^\gamma}{\gamma} \}, \quad (x, y) \in S_+. \quad (1.11)$$

$$L_0 v(0, y) = -\delta v + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}, \quad (1.12)$$

$$L_1 v(x, 0) = \sup_{c \geq 0} \{ -\delta v + (rx - c) \frac{\partial v}{\partial x} + \frac{c^\gamma}{\gamma} \}. \quad (1.13)$$

To handle the transactions we introduce an intervention operator M defined by

$$Mv(x, y) = \sup_{\xi} \{v(x', y') \mid \xi \in R \setminus \{0\}, (x', y') \in S_+\}, \quad (1.14)$$

where

$$x' = x'(\xi) = x - k - \xi - \lambda|\xi|, \quad (1.15)$$

$$y' = y'(\xi) = y + \xi. \quad (1.16)$$

Here, k and λ are a fixed and a proportional transaction costs of trading, respectively and (1.15) gives a line of trading. This is a constrained optimization of the value function. We compare the current value v and the optimal value Mv after the trading. The region where $Mv \geq v$ is called the transaction region and the region where $Mv < v$ is called the no-transaction region and denoted by NT , i.e.,

$$NT = \{(x, y) \in S_+; v(x, y) > Mv(x, y)\}. \quad (1.17)$$

We also call the set of $(x, y) \in S_+$ at which $Mv(x, y) = v(x, y)$ the transaction boundaries.

1.3 Solutions to HJB equation

We consider the equation (1.9) first. Concerning the solutions of (1.9), we have

Theorem 1.1. *A solution to (1.9) is given by*

$$v(x, y) = ax^\gamma + \frac{y^m}{m}, \quad (1.18)$$

where

$$a = \frac{1}{\gamma} \left[\frac{1}{1-\gamma} (\delta - r\gamma) \right]^{\gamma-1}, \quad (1.19)$$

and

$$m = \frac{(\frac{1}{2}\sigma^2 - \alpha) \pm \sqrt{(\frac{1}{2}\sigma^2 - \alpha)^2 + 2\sigma^2\delta}}{\sigma^2}. \quad (1.20)$$

Proof. Differentiating (1.9) with respect to c , we have $c^* = (\frac{\partial v}{\partial x})^{\frac{1}{\gamma-1}}$, and substituting c^* into (1.9), we obtain

$$-\delta v + rx \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{1-\gamma}{\gamma} \left(\frac{\partial v}{\partial x} \right)^{\frac{\gamma}{\gamma-1}} = 0. \quad (1.21)$$

Substituting the ansatz (1.18) in (1.21) and collecting the like terms, we obtain two equations

$$[-\delta a + ra\gamma + \frac{1-\gamma}{\gamma} a^{\frac{\gamma}{\gamma-1}} \gamma^{\frac{\gamma}{\gamma-1}}] x^\gamma = 0, \quad (1.22)$$

and

$$[\frac{1}{2} \sigma^2 m^2 + (\alpha - \frac{1}{2} \sigma^2) m - \delta] \frac{y^m}{m} = 0. \quad (1.23)$$

Since they hold for arbitrary nonnegative x and y , the coefficients must be zero. Therefore,

$$-\delta a + ra\gamma + \frac{1-\gamma}{\gamma} a^{\frac{\gamma}{\gamma-1}} \gamma^{\frac{\gamma}{\gamma-1}} = 0, \quad (1.24)$$

and

$$\frac{1}{2}\sigma^2 m^2 + (\alpha - \frac{1}{2}\sigma^2)m - \delta = 0. \quad (1.25)$$

It is easy to obtain (1.19) from (1.24). For m , solving (1.25), we have

$$m = \frac{(\frac{1}{2}\sigma^2 - \alpha) \pm \sqrt{(\frac{1}{2}\sigma^2 - \alpha)^2 + 2\sigma^2\delta}}{\sigma^2}.$$

Set $g(m) = \frac{1}{2}\sigma^2 m^2 + (\alpha - \frac{1}{2}\sigma^2)m - \delta$, and notice that

$$g(0) = -\delta < 0, \quad g(1) = \alpha - \delta > 0.$$

The intermediate value theorem implies that the positive m satisfies $0 < m < 1$. $\square$

Remark 1.2. Note that $v(x, y) = ax^\gamma + by^m$, where a is given by (1.19) and b is a constant satisfying $bm > 0$, is also a solution. The condition $bm > 0$ guarantees that the value function is concave. Therefore, following the same practice as for the utility function, we choose $b = \frac{1}{m}$. In what follows, we denote

$$m_{\pm} = \frac{(\frac{1}{2}\sigma^2 - \alpha) \pm \sqrt{(\frac{1}{2}\sigma^2 - \alpha)^2 + 2\sigma^2\delta}}{\sigma^2}.$$

Now let us look at the second part of equation (1.30). For a given $\zeta = (x, y) \in S_+$, in order to compute

$$Mv = \sup v(x', y') \quad (1.26)$$

with $x' = x - k - \xi - \lambda|\xi|$, $y' = y + \xi$, we need to find the value of ξ that will maximize the value function

$$v(x', y') = ax'^\gamma + \frac{y'^m}{m} = \begin{cases} a(x - k - \xi - \lambda\xi)^\gamma + \frac{(y+\xi)^m}{m} & (\xi > 0) \\ a(x - k - \xi + \lambda\xi)^\gamma + \frac{(y+\xi)^m}{m} & (\xi < 0). \end{cases} \quad (1.27)$$

Since we know an explicit solution in the no transaction region, the problem is reduced to finding

$$\max[v(x, y), Mv(x, y)] \quad (1.28)$$

for each $(x, y) \in S_+$.

Combining the no-transaction and transaction cases, we define

$$\Phi(x, y) = \max[v(x, y), Mv(x, y)] \quad (1.29)$$

as the value function. Therefore, we obtain a quasi-variational inequality problem

$$\max_{c, \phi} [L^c v + \frac{c^\gamma}{\gamma}, Mv - \Phi(x, y)] = 0 \quad (1.30)$$

with the understanding that if both $L^c v + \frac{1}{\gamma} c^\gamma$ and $Mv - \Phi(x, y)$ are equal to zero, we make trades and move to a new allocation. With this formulation we can handle both fixed and proportional transaction costs.

1.4 Analytical study of the boundaries between transaction regions and NT-region

We study the transaction regions, no-transaction region (NT-region), and the transaction boundaries separating them. First we explain various notations in Figure 1.1. The line L_1 is the line along which the risky asset is sold. The curve C_1 is the set of (x_0, y_0) at which the value function attains its maximum along the line L_1 for each (x_1, y_1) given on the left of C_1 . The curve B_1 is the set of $(\hat{x}, \hat{y})$ on the boundary between the sell region and the NT-region. Note that (x_0, y_0) is the intersection of curves C_1 and a contour curve of the value function denoted as VFC , while $(\hat{x}, \hat{y})$ is the intersection of the curves B_1 and VFC . We call B_1 the transaction boundary and call C_1 the optimal portfolio curve. Similarly, we define L_2 to be the line along which the risky asset is bought, and B_2 and C_2 are the corresponding transaction boundary and optimal portfolio curve, respectively. In what follows, we obtain the relations among $(\hat{x}, \hat{y})$ and (x_0, y_0) in selling the risky asset.

Since the value function is a constant along each VFC and both (x_0, y_0) and $(\hat{x}, \hat{y})$ are on the same VFC , we obtain

$$a\hat{x}^\gamma + \frac{\hat{y}^m}{m} = ax_0^\gamma + \frac{y_0^m}{m}. \quad (1.31)$$

Also, the fact that $(\hat{x} - k, \hat{y})$ is on the line L_1 implies

$$\hat{x} - k - x_0 + (1 - \lambda)(\hat{y} - y_0) = 0. \quad (1.32)$$

The tangent line to the curve VFC at (x_0, y_0) agrees with the line of trading L_1 . Therefore, equating the slope of the tangent line to VFC at (x_0, y_0) and the slope of the trading line, we obtain the optimal portfolio curve C_1

$$-\frac{1}{1 - \lambda} = -\frac{a\gamma x_0^{\gamma-1}}{y_0^{m-1}} \quad \text{or} \quad y_0^{1-m} = \frac{1}{a\gamma(1 - \lambda)} x_0^{1-\gamma}. \quad (1.33)$$

Now we have four variables $\hat{x}, \hat{y}, x_0, y_0$, and three equations (1.31), (1.32), and (1.33). We express $\hat{x}$ and $\hat{y}$ in terms of x_0, y_0 and k .

As a passing note, we see that the optimal portfolio curve C_2 is given by

$$y_0^{1-m} = \frac{1}{a\gamma(1 + \lambda)} x_0^{1-\gamma}. \quad (1.34)$$

Figure 1.1: Graphs for several curves.

Theorem 1.3. *There is a saddle node type bifurcation of the solution $(\hat{x}, \hat{y})$ from (x_0, y_0) as we increase k from zero. One of the bifurcating solutions $(\hat{x}, \hat{y})$ forms the transaction boundary and is given by the power series in $k^{\frac{1}{2}}$, i.e.,*

$$\hat{x} = x_0 + \sum_{n=1}^{\infty} a_n k^{\frac{n}{2}}, \quad \hat{y} = y_0 + \sum_{n=1}^{\infty} b_n k^{\frac{n}{2}}, \quad (1.35)$$

where the first few terms of a_n and b_n are given in (1.47) and (1.50).

Proof. We find the power series solution to

$$v(x, y) = a\hat{x}^\gamma + \frac{1}{m}\hat{y}^m = ax_0^\gamma + \frac{1}{m}y_0^m \quad (1.36)$$

and

$$(\hat{x} - k - x_0) + (1 - \lambda)(\hat{y} - y_0) = 0 \quad (1.37)$$

when

$$(1 - \lambda)a\gamma x_0^{\gamma-1} = y_0^{m-1}. \quad (1.38)$$

We assume that the power series is given by

$$x = x_0 + \sum_{n=1}^{\infty} a_n k^{np}, \quad y = y_0 + \sum_{n=1}^{\infty} b_n k^{np}. \quad (1.39)$$

We expand v about (x_0, y_0)

$$\begin{aligned} & v_x(x - x_0) + \frac{1}{2}v_{xx}(x - x_0)^2 + \frac{1}{3!}v_{xxx}(x - x_0)^3 + \cdots \\ & + v_y(y - y_0) + \frac{1}{2}v_{yy}(y - y_0)^2 + \frac{1}{3!}v_{yyy}(y - y_0)^3 + \cdots = 0, \end{aligned} \quad (1.40)$$

where the partial derivatives of v are evaluated at (x_0, y_0) . Then, from (1.39) and (1.40)

$$\begin{aligned} & v_x \left(\sum_{n=1}^{\infty} a_n k^{np} \right) + \frac{1}{2} v_{xx} \left(\sum_{n=1}^{\infty} a_n k^{np} \right)^2 + \frac{1}{3!} v_{xxx} \left(\sum_{n=1}^{\infty} a_n k^{np} \right)^3 + \dots \\ & + v_y \left(\sum_{n=1}^{\infty} b_n k^{np} \right) + \frac{1}{2} v_{yy} \left(\sum_{n=1}^{\infty} b_n k^{np} \right)^2 + \frac{1}{3!} v_{yyy} \left(\sum_{n=1}^{\infty} b_n k^{np} \right)^3 + \dots \\ & = 0. \end{aligned} \quad (1.41)$$

Also, from (1.37) and (1.39)

$$\left(\sum_{n=1}^{\infty} a_n k^{np} \right) + (1 - \lambda) \left(\sum_{n=1}^{\infty} b_n k^{np} \right) = k. \quad (1.42)$$

For $n = 1$ if $p = 1$, collecting the $O(k)$ terms in (1.41) and (1.42), we have

$$A \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} v_x & v_y \\ 1 & 1 - \lambda \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (1.43)$$

Because of (1.38) the matrix A is singular. Therefore, to have a solution, by the Fredholm alternative the right hand side must be orthogonal to the left nullspace of A . We take $[-1 \ v_x]$ to be the left nullspace. Then, multiplying $[-1 \ v_x]$ from the left, we see that $v_x = a\gamma x_0^{\gamma-1} = 0$. Therefore, p can not be one and the right hand side of (1.43) is zero. So, (a_1, b_1) are undetermined from $n = 1$.

For $n = 2$ if we choose $p = \frac{1}{2}$, from (1.41) and (1.42) we have

$$A \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} v_{xx} a_1^2 - \frac{1}{2} v_{yy} b_1^2 \\ 1 \end{bmatrix}. \quad (1.44)$$

Applying the same argument as $n = 1$, we have

$$\frac{1}{2} v_{xx} (1 - \lambda)^2 b_1^2 + \frac{1}{2} v_{yy} b_1^2 + v_x = 0 \quad (1.45)$$

and therefore,

$$b_1^2 = -\frac{2v_x}{v_{xx}(1 - \lambda)^2 + v_{yy}} = \frac{2}{(1 - \gamma)x_0^{-1}(1 - \lambda)^2 + (1 - m)(1 - \lambda)y_0^{-1}}. \quad (1.46)$$

Denote the denominator as D_1 . Then,

$$b_1 = \pm \sqrt{\frac{2}{D_1}}, \quad a_1 = \mp (1 - \lambda) \sqrt{\frac{2}{D_1}}. \quad (1.47)$$

For a_1 we choose the negative and positive sign for selling and buying, respectively, while for b_1 we choose the positive and negative sign for selling and buying, respectively. Since (a_1, b_1) are determinable for $p = \frac{1}{2}$, we choose $p = \frac{1}{2}$

and continue the process where at the n th step we determine (a_{n-1}, b_{n-1}) by the Fredholm alternative.

For $n = 3$, collecting the $O(k^{\frac{3}{2}})$ terms in (1.41) and (1.42), we have

$$A \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = - \begin{bmatrix} \frac{1}{2}v_{xx}(2a_1a_2) + \frac{1}{3!}v_{xxx}a_1^3 + \frac{1}{2}v_{yy}(2b_1b_2) + \frac{1}{3!}v_{yyy}b_1^3 \\ 0 \end{bmatrix}. \quad (1.48)$$

Using the Fredholm alternative and $a_2 + (1 - \lambda)b_2 = 1$, we have

$$\begin{bmatrix} v_{xx}a_1 & v_{yy}b_1 \\ 1 & (1 - \lambda) \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3!}(v_{xxx}a_1^3 + v_{yyy}b_1^3) \\ 1 \end{bmatrix}. \quad (1.49)$$

Denote the determinant of the matrix on the left hand side by

$$D_2 = (1 - \lambda)^2 a \gamma (1 - \gamma) x_0^{\gamma-2} + (1 - m) y_0^{m-2}$$

. Then,

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \frac{1}{D_2} \begin{bmatrix} \frac{1}{3!}(1 - \lambda)((1 - \lambda)v_{xxx}a_1^2 - v_{yyy}b_1^2) - v_{yy} \\ -\frac{1}{3!}(1 - \lambda)v_{xxx}a_1^2 + \frac{1}{3!}v_{yyy}b_1^2 - (1 - \lambda)v_{xx} \end{bmatrix}. \quad (1.50)$$

For $n = n$, collecting the $O(k^{\frac{n}{2}})$ terms in (1.41) and (1.42), we have

$$\begin{aligned} & A \begin{bmatrix} a_n \\ b_n \end{bmatrix} \\ &= - \begin{bmatrix} f(a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}) \\ 0 \end{bmatrix}, \end{aligned} \quad (1.51)$$

where

$$\begin{aligned} & f(a_1, a_2, \dots, a_{n-1}, b_1, b_2, \dots, b_{n-1}) \\ &= v_{xx}a_1a_{n-1} + v_{xx}a_2a_{n-2} + \frac{1}{2}v_{xxx}a_1^2a_{n-2} \\ & \quad + v_{yy}b_1b_{n-1} + \frac{1}{2}v_{yyy}b_1^2b_{n-2} + \dots \end{aligned} \quad (1.52)$$

Using $a_{n-1} + (1 - \lambda)b_{n-1} = 0$ and the Fredholm alternative, we obtain

$$B \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = - \begin{bmatrix} v_{xx}a_2a_{n-2} + \frac{1}{2}(v_{xxx}a_1^2a_{n-2} + v_{yyy}b_1^2b_{n-2}) + \dots \\ 0 \end{bmatrix}. \quad (1.53)$$

Since the determinant of B is nonzero, the procedure can be continued. $\square$

Theorem 1.4. *The transaction boundaries are concave downward if $m < \gamma$, and are concave upward if $m > \gamma$.*

Proof. We prove the case where $m < \gamma$. The case where $m > \gamma$ is proved in a similar manner. Substituting (1.33) to (1.31) and (1.32), we obtain

$$R_1(\hat{x}, x_0, \hat{y}) = a\hat{x}^\gamma + \frac{\hat{y}^m}{m} - ax_0^\gamma - \frac{[(1-\lambda)a\gamma]^{\frac{m}{m-1}} \cdot x_0^{\frac{m(\gamma-1)}{m-1}}}{m}, \quad (1.54)$$

and

$$R_2(\hat{x}, x_0, \hat{y}) = (\hat{x} - k - x_0) + (1-\lambda)\{\hat{y} - [(1-\lambda)a\gamma]^{\frac{1}{m-1}} \cdot x_0^{\frac{\gamma-1}{m-1}}\}. \quad (1.55)$$

We regard x_0 and $\hat{y}$ are functions of $\hat{x}$ and differentiate (1.54) and (1.55) with respect to $\hat{x}$. Then

$$a\gamma\hat{x}^{\gamma-1} + \hat{y}^{m-1} \cdot \frac{d\hat{y}}{d\hat{x}} - A_1 \cdot \frac{dx_0}{d\hat{x}} = 0, \quad (1.56)$$

and

$$1 + (1-\lambda)\frac{d\hat{y}}{d\hat{x}} - A_2 \cdot \frac{dx_0}{d\hat{x}} = 0, \quad (1.57)$$

where

$$A_1 = a\gamma x_0^{\gamma-1} + \frac{[(1-\lambda)a\gamma]^{\frac{m}{m-1}}}{m} \cdot \frac{m(\gamma-1)}{m-1} \cdot x_0^{[\frac{m(\gamma-1)}{m-1}-1]}, \quad (1.58)$$

$$A_2 = 1 + (1-\lambda)[(1-\lambda)a\gamma]^{\frac{1}{m-1}} \cdot \frac{\gamma-1}{m-1} \cdot x_0^{[\frac{\gamma-1}{m-1}-1]}. \quad (1.59)$$

Eliminating $\frac{dx_0}{d\hat{x}}$ from (1.56) and (1.57), and using the relation $A_1 = a\gamma x_0^{\gamma-1} \cdot A_2$, we obtain

$$\frac{d\hat{y}}{d\hat{x}} = \frac{a\gamma x_0^{\gamma-1} - a\gamma\hat{x}^{\gamma-1}}{\hat{y}^{m-1} - y_0^{m-1}}. \quad (1.60)$$

This yields

$$\frac{d^2\hat{y}}{d\hat{x}^2} = \frac{1}{(\hat{y}^{m-1} - y_0^{m-1})^2} \{T_1 + T_2 + T_3 + T_4\}, \quad (1.61)$$

where

$$T_1 = a\gamma(\gamma-1)x_0^{\gamma-2} \frac{1}{A_2} [\hat{y}^{m-1} - (1-\lambda)a\gamma\hat{x}^{\gamma-1}], \quad (1.62)$$

$$T_2 = -a\gamma(\gamma-1)\hat{x}^{\gamma-2} \cdot (\hat{y}^{m-1} - y_0^{m-1}), \quad (1.63)$$

$$T_3 = -(m-1) \frac{\hat{y}^{m-2}}{\hat{y}^{m-1} - y_0^{m-1}} (a\gamma x_0^{\gamma-1} - a\gamma\hat{x}^{\gamma-1})^2, \quad (1.64)$$

and

$$T_4 = a\gamma(\gamma-1) \frac{x_0^{\gamma-1} - \hat{x}^{\gamma-1}}{\hat{y}^{m-1} - y_0^{m-1}} [(1-\lambda)a\gamma] x_0^{\gamma-2} \cdot \frac{1}{A_2} [\hat{y}^{m-1} - (1-\lambda)a\gamma\hat{x}^{\gamma-1}]. \quad (1.65)$$

Since $(\hat{x}, \hat{y})$ is on B_1 and (x_0, y_0) is on C_1 , $\hat{x} < x_0$, $\hat{y} > y_0$. Therefore, since $0 < \gamma < 1$ and $0 < m < 1$, we see that

$$\hat{x}^{\gamma-2} > x_0^{\gamma-2}, \quad (1.66)$$

and

$$\hat{y}^{m-1} < y_0^{m-1}. \quad (1.67)$$

Using $\frac{1}{A_2} < 1$, we estimate $T_1 + T_4$ as follows.

$$\begin{aligned} T_1 + T_4 &= a\gamma(\gamma-1)x_0^{\gamma-2} \frac{1}{A_2} \cdot \frac{[\hat{y}^{m-1} - (1-\lambda)a\gamma\hat{x}^{\gamma-1}]^2}{\hat{y}^{m-1} - y_0^{m-1}} \\ &< a\gamma(\gamma-1)x_0^{\gamma-2} \cdot \frac{[\hat{y}^{m-1} - (1-\lambda)a\gamma\hat{x}^{\gamma-1}]^2}{\hat{y}^{m-1} - y_0^{m-1}} \\ &< a\gamma(\gamma-1)\hat{x}^{\gamma-2} \cdot \frac{[\hat{y}^{m-1} - (1-\lambda)a\gamma x_0^{\gamma-1}]^2}{\hat{y}^{m-1} - y_0^{m-1}} \\ &= |T_2|. \end{aligned} \quad (1.68)$$

Then, since $T_2 < 0$, $T_3 < 0$, and $0 < T_1 + T_4 < |T_2|$, $T_1 + T_2 + T_3 + T_4 < 0$. Therefore,

$$\frac{d^2 \hat{y}}{d\hat{x}^2} < 0.$$

□

In what follows, we denote the transaction boudaries B_1 and B_2 by $y = h_s(x)$ and $y = h_b(x)$, respectively. After B_2 reaches the x -axis, $h_b(x)$ is continued as zero till the origin.

1.5 Verification Theorem

To show the verification theorem it is necessary to study the processes X , Y , and the discounted utility between the two consecutive transaction times τ_i^+ and τ_{i+1}^- . We start from X .

Lemma 1.5. *The amount of money and the discounted utility of money decrease between the two consecutive transaction times.*

Proof. Since

$$c^* = (a\gamma)^{\frac{1}{\gamma-1}} x = \frac{(\delta - r\gamma)}{(1-\gamma)} x, \quad (1.69)$$

for $\tau_i^+ < t < \tau_{i+1}^-$,

$$dX(t) = (rX(t) - c(t))dt = \left(r - \frac{(\delta - r\gamma)}{(1-\gamma)}\right)X(t)dt = \frac{(r - \delta)}{(1-\gamma)}X(t)dt. \quad (1.70)$$

By assumption (1.7), (1.70) implies that the amount of money decreases for $\tau_i^+ < t < \tau_{i+1}^-$. For the discounted utility of money we have

$$d(e^{-\delta t} a \cdot X^\gamma(t)) = -\frac{(\delta - r\gamma)}{(1 - \gamma)} e^{-\delta t} a \cdot X^\gamma(t) dt, \quad \tau_i^+ < t < \tau_{i+1}^-. \quad (1.71)$$

Integrating in t from τ_i^+ to τ_{i+1}^- , we obtain

$$\begin{aligned} & e^{-\delta \tau_{i+1}^-} a \cdot X^\gamma(\tau_{i+1}^-) - e^{-\delta \tau_i^+} a \cdot X^\gamma(\tau_i^+) \\ &= e^{-\delta \tau_i^+} a \cdot X^\gamma(\tau_i^+) [e^{-\frac{(\delta - r\gamma)}{(1 - \gamma)} (\tau_{i+1}^- - \tau_i^+)} - 1]. \end{aligned} \quad (1.72)$$

Since $\delta - r\gamma > 0$, the right hand side is negative. $\square$

Next consider Y . We have

Lemma 1.6. *The discounted utility of risky asset satisfies*

$$e^{-\delta \tau_{i+1}^-} \frac{Y^m(\tau_{i+1}^-)}{m} - e^{-\delta \tau_i^+} \frac{Y^m(\tau_i^+)}{m} = e^{-\delta \tau_i^+} \frac{Y^m(\tau_i^+)}{m} [e^{M_i} - 1], \quad (1.73)$$

where

$$M_i = \{[m(\alpha - \frac{1}{2}\sigma^2) - \delta](\tau_{i+1}^- - \tau_i^+) + m\sigma[W(\tau_{i+1}^-) - W(\tau_i^+)]\}. \quad (1.74)$$

Furthermore, the expectation of e^{M_i} is one.

Proof. Since Y follows the geometric Brownian motion for $\tau_i^+ \leq x < \tau_{i+1}^-$, $Y(\tau_{i+1}^-)$ is expressed as

$$Y(\tau_{i+1}^-) = Y(\tau_i^+) e^{\{(\alpha - \frac{1}{2}\sigma^2)(\tau_{i+1}^- - \tau_i^+) + \sigma[W(\tau_{i+1}^-) - W(\tau_i^+)]\}}, \quad i = 0, 1, 2, \dots, N. \quad (1.75)$$

Also, the Ito-Doeblin formula implies that the discounted value function of the risky asset is given by

$$\begin{aligned} & e^{-\delta \tau_{i+1}^-} \frac{Y^m(\tau_{i+1}^-)}{m} \\ &= e^{-\delta \tau_i^+} \frac{Y^m(\tau_i^+)}{m} e^{\{[m(\alpha - \frac{1}{2}\sigma^2) - \delta](\tau_{i+1}^- - \tau_i^+) + m\sigma[W(\tau_{i+1}^-) - W(\tau_i^+)]\}}. \end{aligned} \quad (1.76)$$

Therefore, we obtain (1.73).

The expectation of e^{M_i} is given by

$$E\{e^{M_i}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\{m(\alpha - \frac{1}{2}\sigma^2) - \delta\}(\tau_{i+1}^- - \tau_i^+) + m\sigma\sqrt{\tau_{i+1}^- - \tau_i^+} z_i - \frac{1}{2} z_i^2} dz_i. \quad (1.77)$$

By (1.23) the exponent of integrand is simplified as follows

$$\begin{aligned} & [m(\alpha - \frac{1}{2}\sigma^2) - \delta](\tau_{i+1}^- - \tau_i^+) + m\sigma\sqrt{\tau_{i+1}^- - \tau_i^+} z_i - \frac{1}{2} z_i^2 \\ &= [m(\alpha - \frac{1}{2}\sigma^2) - \delta + \frac{1}{2}m^2\sigma^2](\tau_{i+1}^- - \tau_i^+) - \frac{1}{2}(z_i - m\sigma\sqrt{\tau_{i+1}^- - \tau_i^+})^2 \\ &= -\frac{1}{2}(z_i - m\sigma\sqrt{\tau_{i+1}^- - \tau_i^+})^2, \end{aligned} \quad (1.78)$$

and consequently we have

$$E\{e^{M_i}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z_i - m\sigma\sqrt{\tau_{i+1}^- - \tau_i^+})^2} dz_i = 1. \quad (1.79)$$

□

We need to know an estimate for the probability of staying in the no-transaction region or reaching the transaction boundary at τ_{i+1}^- starting from $(X(\tau_i^+), Y(\tau_i^+))$. From (1.75), define

$$D(\tau_i^+, Y(\tau_{i+1}^-)) = \frac{\log \frac{Y(\tau_{i+1}^-)}{Y(\tau_i^+)} - (\alpha - \frac{1}{2}\sigma^2)(\tau_{i+1}^- - \tau_i^+)}{\sigma\sqrt{\tau_{i+1}^- - \tau_i^+}}. \quad (1.80)$$

Lemma 1.7. *The probability P_s of staying in the no-transaction region at τ_{i+1}^- is estimated as*

$$P_s \leq \frac{1}{\sqrt{2\pi}} \int_{D(\tau_i^+, h_b(X(\tau_{i+1}^-)))}^{D(\tau_i^+, h_s(X(\tau_{i+1}^-)))} e^{-\frac{1}{2}z_i^2} dz_i. \quad (1.81)$$

Also, the probability P_o of leaving the no-transaction region by τ_{i+1}^- is estimated as

$$P_o \leq \frac{2}{\sqrt{2\pi}} \int_{D(\tau_i^+, h_s(X(\tau_{i+1}^-)))}^{\infty} e^{-\frac{1}{2}z_i^2} dz_i + \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{D(\tau_i^+, h_b(X(\tau_i^+)))} e^{-\frac{1}{2}z_i^2} dz_i. \quad (1.82)$$

Furthermore, $P_o \rightarrow 0$ as $\tau_{i+1}^- \rightarrow \tau_i^+$ and this implies that the probability of immediate transaction after a transaction is zero.

Proof. To stay in the no-transaction region till τ_{i+1}^- , Z_i must satisfy

$$D(\tau_i^+, h_b(X(\tau_{i+1}^-))) < Z_i < D(\tau_i^+, h_s(X(\tau_{i+1}^-))). \quad (1.83)$$

Since Z_i follows the standard normal distribution, this implies (1.81). The probability of Z_i being the outside of the no-transaction region at τ_{i+1}^- is

$$\begin{aligned} & P\{Z_i \geq D(\tau_i^+, h_s(X(\tau_{i+1}^-))) \text{ or } Z_i \leq D(\tau_i^+, h_b(X(\tau_i^+)))\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{D(\tau_i^+, h_s(X(\tau_{i+1}^-)))}^{\infty} e^{-\frac{1}{2}z_i^2} dz_i \\ & \quad + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{D(\tau_i^+, h_b(X(\tau_i^+)))} e^{-\frac{1}{2}z_i^2} dz_i. \end{aligned} \quad (1.84)$$

By the reflection principle, the probability of leaving the no-transaction region by τ_{i+1}^- is less than the twice of the above probability. Therefore, we obtain (1.82).

Since $Y(\tau_i^+) < h_s(X(\tau_i^+))$, $D(\tau_i^+, h_s(X(\tau_{i+1}^-))) \rightarrow \infty$ as $\tau_{i+1}^- \rightarrow \tau_i^+$. Similarly, $D(\tau_i^+, h_b(X(\tau_i^+))) \rightarrow -\infty$ as $\tau_{i+1}^- \rightarrow \tau_i^+$. Therefore, $P_o \rightarrow 0$ as $\tau_{i+1}^- \rightarrow \tau_i^+$. □

Remark 1.8. The probability of immediate transaction after a transaction is zero can be shown also by the continuity of the Brownian motion.

Now we show that the expectation of the discounted value function along the sequence $\{\tau_1, \tau_2, \tau_3, \dots\}$ of stopping times satisfies

Theorem 1.9. *We have*

$$E[e^{-\delta\tau_{N+1}^-} \Phi(X(\tau_{N+1}^-), Y(\tau_{N+1}^-))] \rightarrow 0 \text{ as } N \rightarrow \infty, \quad (1.85)$$

where N is a positive integer.

Proof. Adding (1.72) and (1.73) for $i = 0, 1, 2, \dots, N$ and using the relation

$$v(\tau_i^-) = [a \cdot X^\gamma(\tau_i^-) + \frac{Y^m(\tau_i^-)}{m}] \geq [a \cdot X^\gamma(\tau_i^+) + \frac{Y^m(\tau_i^+)}{m}] = v(\tau_i^+), \quad (1.86)$$

we obtain

$$\begin{aligned} & e^{-\delta\tau_{N+1}^-} \cdot v(X(\tau_{N+1}^-), Y(\tau_{N+1}^-)) \\ = & e^{-\delta\tau_0^+} \cdot v(X(\tau_0^+), Y(\tau_0^+)) \\ & + \sum_{i=1}^N [e^{-\delta\tau_i^+} v(\tau_i^+) - e^{-\delta\tau_i^-} v(\tau_i^-)] \\ & + \sum_{i=0}^N e^{-\delta\tau_i^+} a \cdot X^\gamma(\tau_i^+) [e^{-\frac{(\delta-r\gamma)}{(1-\gamma)}(\tau_{i+1}^- - \tau_i^+)} - 1] \\ & + \sum_{i=0}^N e^{-\delta\tau_i^+} \frac{Y^m(\tau_i^+)}{m} [e^{M_i} - 1]. \end{aligned} \quad (1.87)$$

Taking the expectation leads to

$$\begin{aligned} & E\{e^{-\delta\tau_{N+1}^-} \cdot v(X(\tau_{N+1}^-), Y(\tau_{N+1}^-))\} \\ = & E\{e^{-\delta\tau_0^+} \cdot v(X(\tau_0^+), Y(\tau_0^+))\} + \sum_{i=1}^N E[e^{-\delta\tau_i}(v(\tau_i^+) - v(\tau_i^-))] \\ & + \sum_{i=0}^N E\{e^{-\delta\tau_i^+} a \cdot X^\gamma(\tau_i^+) [e^{-\frac{(\delta-r\gamma)}{(1-\gamma)}(\tau_{i+1}^- - \tau_i^+)} - 1]\} + \sum_{i=0}^N E\{e^{-\delta\tau_i^+} \frac{Y^m(\tau_i^+)}{m}\} \\ & \cdot [E\{e^{M_i} \mid Y(t) \text{ leaves or stays in NT region at } t = \tau_{N+1}^-\} - 1]. \end{aligned} \quad (1.88)$$

We are assuming that a given sequence $\{\tau_1, \tau_2, \tau_3, \dots\}$ satisfying $0 \leq \tau_1 < \tau_2 < \dots$ is a sequence of the F_t -stopping times at which the investor decides to change his portfolio. Since the investor can change the portfolio either on the transaction boundaries or inside the no-transaction region, we have

$$\begin{aligned}
& E\{e^{M_i} \mid Y(t) \text{ leaves or stays in NT region at } t = \tau_{N+1}^-\} \\
& \leq \frac{1}{\sqrt{2\pi}} \int_{D(\tau_i^+, h_b(X(\tau_{i+1}^-))}^{D(\tau_i^+, h_s(X(\tau_{i+1}^-))} e^{-\frac{1}{2}(z_i - m\sigma\sqrt{\tau_{i+1}^- - \tau_i^+})^2} dz_i \\
& = \frac{1}{\sqrt{2\pi}} \int_{G(\tau_i^+, h_b(X(\tau_{i+1}^-))}^{G(\tau_i^+, h_s(X(\tau_{i+1}^-))} e^{-\frac{1}{2}z_i^2} dz_i,
\end{aligned} \tag{1.89}$$

where

$$G(\tau_i^+, Y) = \frac{\log \frac{Y}{Y(\tau_i^+)} - [\alpha + (m - \frac{1}{2})\sigma^2](\tau_{i+1}^- - \tau_i^+)}{\sigma\sqrt{\tau_{i+1}^- - \tau_i^+}}. \tag{1.90}$$

Since $E\{e^{M_i}\} = 1$, this implies that the last term in (1.88) is negative.

If the number of transactions is finite, there is a time $\tau_{\max}$ beyond which there is no transaction. Then, since the money decreases if there is no transaction, both the stock and money will be confined in the region given by $0 \leq X \leq X(\tau_{\max})$ and $h_b(X) \leq Y \leq h_s(X)$. Therefore, in this case

$$\lim_{t \rightarrow \infty} e^{-\delta t} \Phi(X(t), Y(t)) = 0. \tag{1.91}$$

Now suppose we make transactions infinitely many times. Lemma 1.7 implies that there is no accumulation point in the sequence $\{\tau_1, \tau_2, \tau_3, \dots\}$. Therefore, τ_N approaches infinity almost surely if the number of transactions are infinite. We show (1.85) by contradiction. Assume that there exists $\varepsilon > 0$ such that for every N there exists $i \geq N$ such that $E[e^{-\delta\tau_i^+} a \cdot X^\gamma(\tau_i^+)] \geq \varepsilon$ or $E[e^{-\delta\tau_i^+} \frac{Y^m(\tau_i^+)}{m}] \geq \varepsilon$. Then, since the terms on the right hand side of (1.88) are negative except the first term, it will be negative. This is a contradiction since the left hand side is positive. Therefore, for every $\varepsilon > 0$, there exists N such that for every $i \geq N$ $E[e^{-\delta\tau_i^+} a \cdot X^\gamma(\tau_i^+)] < \varepsilon$ and $E[e^{-\delta\tau_i^+} \frac{Y^m(\tau_i^+)}{m}] < \varepsilon$. This means that

$$\lim_{i \rightarrow \infty} E[e^{-\delta\tau_i^+} a \cdot X^\gamma(\tau_i^+)] = \lim_{i \rightarrow \infty} E[e^{-\delta\tau_i^+} \frac{Y^m(\tau_i^+)}{m}] = 0. \tag{1.92}$$

From (1.72) and (1.73)

$$E[e^{-\delta\tau_{i+1}^-} a \cdot X^\gamma(\tau_{i+1}^-)] < E[e^{-\delta\tau_i^+} a \cdot X^\gamma(\tau_i^+)], \tag{1.93}$$

$$E[e^{-\delta\tau_{i+1}^-} \frac{Y^m(\tau_{i+1}^-)}{m}] < E[e^{-\delta\tau_i^+} \frac{Y^m(\tau_i^+)}{m}]. \tag{1.94}$$

Therefore,

$$\lim_{N \rightarrow \infty} E[e^{-\delta\tau_{N+1}^-} \Phi(X(\tau_{N+1}^-), Y(\tau_{N+1}^-))] = 0. \tag{1.95}$$

We have shown that for every sequence $\{\tau_1, \tau_2, \tau_3, \dots\}$ of stopping times which approach ∞ , the pathwise expectation $E[e^{-\delta\tau_{N+1}^-} v(X(\tau_{N+1}^-), Y(\tau_{N+1}^-))]$ approaches zero. $\square$

Figure 1.2: Trading details

Before we prove the verification theorem it is instructive to explain what typically happens with our formulation. Let (x_i^-, y_i^-) be the coordinates of the portfolio at time τ_i^- on the curve B_1 , where x_i^- and y_i^- represent the amount of money in the bank account and stock at time τ_i^- (i is a positive integer), respectively. Then we will sell the stock along the line L_i . The portfolio point will arrive the optimal point (x_{0i}^+, y_{0i}^+) , which is a point on the optimal portfolio curve C_1 . Since (x_{0i}^+, y_{0i}^+) is in the NT-region, it will move to another position depending on the stochastic processes given in equations (1.1) and (1.2). From Lemma 1.5, the amount of money in the bank account decreases between any two consecutive transaction times, so the portfolio point can only move to the left. In addition, the stock follows the Geometric Brownian motion, and this will make the portfolio point either go up or go down. In comprehensive, the portfolio point will either move to the left up or move to the left down along a path of random work, until it reach either curve B_1 or curve B_2 at time τ_{i+1}^- . If it reaches the curve B_1 , we will sell the stock again by repeating the procedure provided above. If it reaches the curve B_2 , we will buy the stock by using the similar ideas. The tradings for the case where (x_i^-, y_i^-) are the coordinates of the portfolio at time τ_i^- on the curve B_2 or in the NT-region are similar. Graph that describes the movement of a portfolio point is shown in Figure 1.2.

We now verify that the solution to the HJB equation is a value function for the quasi-variational inequality problem (1.30).

Theorem 1.10. (*Verification Theorem*) For $v(x, y) = ax^\gamma + \frac{y^m}{m}$ define

$$c^* = \left(\frac{\partial v}{\partial x} \right)^{\frac{1}{\gamma-1}} \quad (1.96)$$

and also define the impulse control

$$\phi^* := (\tau_1^*, \tau_2^*, \dots; \xi_1^*, \xi_2^*, \dots) \quad (1.97)$$

to be the stopping times and transaction sizes with $\tau_0^* = 0$ and given inductively by

$$\tau_{k+1}^* = \inf\{t > \tau_k^* \mid (X(t), Y(t)) \notin NT\}, \quad (1.98)$$

$$\xi_{k+1}^* = \xi(X(\tau_{k+1}^{*-}), Y(\tau_{k+1}^{*-})). \quad (1.99)$$

Then, Φ satisfies

$$\Phi(x, y) = \tilde{\Phi}(0, x, y) = \sup_{c, \phi} E\left[\int_0^\infty e^{-\delta t} \frac{c^\gamma(x)}{\gamma} dt\right] \quad (1.100)$$

and $w^* = (c^*, \phi^*)$ is optimal.

Proof. Choose $c = c^*$ and fix a point $q = (t, x, y)$. For $i = 0, 1, 2, \dots, N$, by the Itô-Doeblin formula,

$$\begin{aligned} & e^{-\delta \tau_{i+1}^{*-}} v(X(\tau_{i+1}^{*-}), Y(\tau_{i+1}^{*-})) \\ = & e^{-\delta \tau_i^*} v(X(\tau_i^*), Y(\tau_i^*)) + \int_{\tau_i^*}^{\tau_{i+1}^*} e^{-\delta t} L^{c^*} v(X(t), Y(t)) dt \\ & + \int_{\tau_i^*}^{\tau_{i+1}^*} e^{-\delta t} \sigma Y(t) dW(t), \end{aligned} \quad (1.101)$$

where

$$L^{c^*} v(x, y) = -\delta v + (rx - c^*) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}. \quad (1.102)$$

Since

$$e^{-\delta t} L^{c^*} v(X(t), Y(t)) = -e^{-\delta t} \frac{c^{*\gamma}(X(t))}{\gamma}, \quad (1.103)$$

substituting this in (1.101) and rearranging the terms, we have

$$\begin{aligned} & e^{-\delta \tau_i^*} v(X(\tau_i^*), Y(\tau_i^*)) \\ = & e^{-\delta \tau_{i+1}^{*-}} v(X(\tau_{i+1}^{*-}), Y(\tau_{i+1}^{*-})) + \int_{\tau_i^*}^{\tau_{i+1}^*} e^{-\delta t} \frac{c^{*\gamma}(X(t))}{\gamma} dt \\ & - \int_{\tau_i^*}^{\tau_{i+1}^*} e^{-\delta t} \sigma Y(t) dW(t). \end{aligned} \quad (1.104)$$

Summing (1.104) for $i = 0$ to $i = N$ and taking the expectation, we obtain

$$\begin{aligned} & v(x, y) + \sum_{i=1}^N E[e^{-\delta \tau_i^*} v(X(\tau_i^*), Y(\tau_i^*)) - e^{-\delta \tau_i^{*-}} v(X(\tau_i^{*-}), Y(\tau_i^{*-}))] \\ = & E[e^{-\delta \tau_{N+1}^{*-}} v(X(\tau_{N+1}^{*-}), Y(\tau_{N+1}^{*-}))] + E\left[\int_0^{\tau_{N+1}^*} e^{-\delta t} \frac{c^{*\gamma}(X(t))}{\gamma} dt\right]. \end{aligned} \quad (1.105)$$

If the investor make transactions on the transaction boundaries,

$$v(X(\tau_i^*), Y(\tau_i^*)) = Mv(X(\tau_i^{*-}), X(\tau_i^{*-})). \quad (1.106)$$

Therefore, we obtain

$$v(x, y) = E[e^{-\delta\tau_{N+1}^*} v(X(\tau_{N+1}^{*-}), Y(\tau_{N+1}^{*-}))] + E\left[\int_0^{\tau_{N+1}^*} e^{-\delta t} \frac{c^*(X(t))}{\gamma} dt\right]. \quad (1.107)$$

Let $N \rightarrow \infty$ and use Theorem 1.9. Then, we have

$$\Phi(x, y) = E\left[\int_0^\infty e^{-\delta t} \frac{c^*(X(t))}{\gamma} dt\right], \quad (1.108)$$

and this gives us

$$\Phi(x, y) = \sup_{c, \phi} E\left[\int_0^\infty e^{-\delta t} \frac{c^*(X(t))}{\gamma} dt\right] = \tilde{\Phi}(0, x, y). \quad (1.109)$$

Thus, (1.100) satisfies for all $(x, y) \in S_+$ and w^* is optimal. $\square$

Remark 1.11. Usually, the condition

$$\lim_{N \rightarrow \infty} E[e^{-\delta\tau_{N+1}^*} v(X(\tau_{N+1}^{*-}), Y(\tau_{N+1}^{*-}))] = 0$$

is assumed to hold or given as the terminal condition.

1.6 Viscosity Solutions

It has been shown that the value function $\Phi(x, y) = \sup_{c, \phi} E\left[\int_0^\infty e^{-\delta t} \frac{c^*(X(t))}{\gamma} dt\right]$ is a viscosity solution [23, 24]. Since we know that (1.18) in Theorem 1.1 is an explicit solution to the HJB equation and we define the value function Φ which is given in (1.29), it is interesting to prove that Φ is a viscosity solution for the quasi-variational inequality problem (1.30).

Definition 1.12. Suppose G is a topological space, x_0 is a point in G and $\psi : G \rightarrow R$ is a function.

(1) We say that φ is upper semi-continuous (usc) at x_0 if $\limsup_{x \rightarrow x_0} \varphi(x) \leq \varphi(x_0)$.

The function φ is called upper semi-continuous if it is upper semi-continuous at every point of G .

(2) Similarly, we say that φ is lower semi-continuous (lsc) at x_0 if $\liminf_{x \rightarrow x_0} \varphi(x) \geq \varphi(x_0)$. The function φ is called lower semi-continuous if it is lower semi-continuous at every point of G .

(3) We let $USC(G)$ and $LSC(G)$ denote the set of *usc* functions and *lsc* functions on G , respectively.

Definition 1.13. Let S_+^0 be the interior of S_+ and $\zeta = (x, y)$. We define F as follows.

$$F(D^2v(\zeta), Dv(\zeta), v(\zeta), \zeta) = \begin{cases} \max_{c, \phi} \{-\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{c^\gamma}{\gamma} \\ \quad, Mv - \Phi\} \text{ if } \zeta \in S_+^0, \\ \max_{c, \phi} \{-\delta v + (rx - c) \frac{\partial v}{\partial x} + \frac{c^\gamma}{\gamma}, Mv - \Phi\} \\ \quad \text{if } y = 0 \text{ and } x \in (0, \infty), \\ \max_{\phi} \{-\delta v + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2}, Mv - \Phi\} \\ \quad \text{if } x = 0 \text{ and } y \in (0, \infty). \end{cases} \quad (1.110)$$

(1) A function $\psi \in USC(S_+)$ is a viscosity subsolution of

$$F(D^2v(\zeta), Dv(\zeta), v(\zeta), \zeta) = 0 \quad (1.111)$$

if for every $\zeta_0 \in S_+$ and for every function $f \in C^2(S_+)$ satisfying $f \geq \psi$ on S_+ and $f(\zeta_0) = \psi(\zeta_0)$, we have

$$F(D^2f(\zeta_0), Df(\zeta_0), f(\zeta_0), \zeta_0) \geq 0. \quad (1.112)$$

(2) A function $\psi \in LSC(S_+)$ is a viscosity supersolution of (1.111) if for every $\zeta_0 \in S_+$ and for every function $f \in C^2(S_+)$ satisfying $f \leq \psi$ on S_+ and $f(\zeta_0) = \psi(\zeta_0)$, we have

$$F(D^2f(\zeta_0), Df(\zeta_0), f(\zeta_0), \zeta_0) \leq 0. \quad (1.113)$$

(3) A function $\psi : S_+ \rightarrow R$ is a viscosity solution of (1.111) if ψ is locally bounded and it is a viscosity sub and super solution of (1.111).

Theorem 1.14. *The value function Φ is a viscosity solution of (1.111).*

Proof. (1) For $\zeta \in S_+^0$, we prove that $\Phi(x, y) = \max[v(x, y), Mv(x, y)]$ is a viscosity subsolution of the first case in (1.110). It is easy to see that v is C^2 . Consider f be a C^2 function in a neighborhood of S_+^0 and let $\zeta_0 = (x_0, y_0) \in S_+^0$ be such that $f \geq \Phi$ on S_+^0 and $f(\zeta_0) = \Phi(\zeta_0)$. We show two cases separately.

Case (i). $v(\zeta_0) \leq Mv(\zeta_0)$. Then, $f(\zeta_0) = \Phi(\zeta_0) = \max[v(\zeta_0), Mv(\zeta_0)] = Mv(\zeta_0)$. Since $Mv(\zeta_0) = v(x', y') \leq f(x', y') \leq Mf(\zeta_0)$, we have

$$\begin{aligned} & \max_{c, \phi} \left\{ [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{c^\gamma}{\gamma}] (\zeta_0), \right. \\ & \quad \left. [Mf - \max(Mf, f)](\zeta_0) \right\} \\ & \geq [Mf - \max(Mf, f)](\zeta_0) \\ & = [Mf - \max(Mf, Mv)](\zeta_0) \geq 0. \end{aligned} \quad (1.114)$$

So (1.112) holds.

Case (ii). $v(\zeta_0) > Mv(\zeta_0)$. We need to prove

$$\max_c [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{c^\gamma}{\gamma}] (\zeta_0) \geq 0. \quad (1.115)$$

Since $f \geq \Phi \geq v$ on S_+^0 and $f(\zeta_0) = \Phi(\zeta_0) = v(\zeta_0)$, $f - v$ has a local minimum at ζ_0 . Therefore, $\frac{\partial f}{\partial x}(\zeta_0) = \frac{\partial v}{\partial x}(\zeta_0)$, $\frac{\partial f}{\partial y}(\zeta_0) = \frac{\partial v}{\partial y}(\zeta_0)$ and $\frac{\partial^2 f}{\partial y^2}(\zeta_0) \geq \frac{\partial^2 v}{\partial y^2}(\zeta_0)$ [7]. This implies

$$\begin{aligned} & \max_c [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{c^\gamma}{\gamma}] (\zeta_0) \\ & \geq \max_c [-\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + \frac{c^\gamma}{\gamma}] (\zeta_0). \end{aligned} \quad (1.116)$$

We choose $c^* = \left(\frac{\partial v}{\partial x}\right)^{\frac{1}{\gamma-1}}$ and substitute $v(x, y) = ax^\gamma + \frac{y^m}{m}$ into the right hand side of (1.116). Then, by Theorem 1.1 we see that the right hand side of (1.116) is zero.

The proofs for the cases where $x = 0$ or $y = 0$ are similar.

(2) For $\zeta \in S_+^0$, we prove that $\Phi(x, y) = \max[v(x, y), Mv(x, y)]$ is a viscosity supersolution of the first case in (1.110). So we let f be a C^2 function in a neighborhood of S_+^0 and we let $\zeta_0 \in S_+^0$ such that $f \leq \Phi = \max(v, Mv)$ on S_+^0 and $f(\zeta_0) = \Phi(\zeta_0) = \max[v(\zeta_0), Mv(\zeta_0)]$. We want to show (1.113).

Since $[Mf - \max(Mf, f)](\zeta_0) \leq 0$, we only need to prove

$$\max_c [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{c^\gamma}{\gamma}] (\zeta_0) \leq 0. \quad (1.117)$$

For $\varepsilon > 0$ and $\hat{w} = \hat{w}_{\varepsilon, c}$ be an admissible control beginning with a constant consumption rate $c > 0$ and no transactions up to the first time τ_ε at which the stochastic process exists from

$$K_\varepsilon = \{(s, x, y); |(s, x, y) - (0, x_0, y_0)| < \varepsilon\} \cap \tilde{S}_+, \quad (1.118)$$

where $\zeta_0 = (x_0, y_0)$ and $\tilde{S}_+ = R^+ \times S_+$. Choose $\zeta_n \in K_\varepsilon$ such that $\zeta_n \rightarrow \zeta_0$ and $\Phi(\zeta_n) \rightarrow \Phi(\zeta_0)$ as $n \rightarrow \infty$. By using Itô-Doeblin formula and the definition of expectation, we obtain

$$\begin{aligned} \Phi(\zeta_n) & \geq E^{0, \zeta_n} [e^{-\delta \tau_\varepsilon} \Phi(X^{\hat{w}}(\tau_\varepsilon), Y^{\hat{w}}(\tau_\varepsilon))] + E^{0, \zeta_n} \left[\int_0^{\tau_\varepsilon} e^{-\delta t} \frac{c^\gamma(X(t))}{\gamma} dt \right] \\ & \geq E^{0, \zeta_n} [e^{-\delta \tau_\varepsilon} f(X^{\hat{w}}(\tau_\varepsilon), Y^{\hat{w}}(\tau_\varepsilon))] + E^{0, \zeta_n} \left[\int_0^{\tau_\varepsilon} e^{-\delta t} \frac{c^\gamma(X(t))}{\gamma} dt \right] \\ & = f(\zeta_n) + E^{0, \zeta_n} \left[\int_0^{\tau_\varepsilon} e^{-\delta t} L^c f(X^{\hat{w}}(t), Y^{\hat{w}}(t)) dt \right] + E^{0, \zeta_n} \left[\int_0^{\tau_\varepsilon} e^{-\delta t} \frac{c^\gamma(X(t))}{\gamma} dt \right] \end{aligned} \quad (1.119)$$

where $L^c f(x, y) = -\delta f + (rx - c)\frac{\partial f}{\partial x} + \alpha y\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 f}{\partial y^2}$ and for all n . From (1.119),

$$E^{0, \zeta_n} \left[\int_0^{\tau_\varepsilon} e^{-\delta t} \{L^c f(X^{\widehat{w}}(t), Y^{\widehat{w}}(t)) + \frac{c^\gamma(X(t))}{\gamma}\} dt \right] \leq \Phi(\zeta_n) - f(\zeta_n) \quad (1.120)$$

for all n . Taking the limit as $n \rightarrow \infty$, we obtain

$$E^{0, \zeta_0} \left[\int_0^{\tau_\varepsilon} H(t) dt \right] \leq 0, \quad (1.121)$$

where

$$H(t) = e^{-\delta t} (L^c f(X^{\widehat{w}}(t), Y^{\widehat{w}}(t)) + \frac{c^\gamma(X(t))}{\gamma}). \quad (1.122)$$

By dividing the left-hand side by $E^{0, \zeta_0}[\tau_\varepsilon]$ we get

$$\frac{E^{0, \zeta_0} \left[\int_0^{\tau_\varepsilon} H(t) dt \right]}{E^{0, \zeta_0}[\tau_\varepsilon]} = \frac{E^{0, \zeta_0} \left[\int_0^{\tau_\varepsilon} [H(t) - H(0)] dt \right] + H(0) E^{0, \zeta_0}[\tau_\varepsilon]}{E^{0, \zeta_0}[\tau_\varepsilon]} \rightarrow H(0) \quad (1.123)$$

as $\varepsilon \rightarrow 0$, since $H(t)$ is continuous at $t = 0$. Thus, we have

$$H(0) = [-\delta f + (rx - c)\frac{\partial f}{\partial x} + \alpha y\frac{\partial f}{\partial y} + \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 f}{\partial y^2} + \frac{c^\gamma}{\gamma}](\zeta_0) \leq 0$$

for all $c \geq 0$ such that $\widehat{w}_{\varepsilon, c}$ is admissible for ε small enough [20].

The proofs for the cases where $x = 0$ or $y = 0$ are similar.

(3) (1) and (2) imply that Φ is a viscosity solution of (1.111). $\square$

1.7 Numerical results

In this section we examine numerically what we discussed in the previous sections. One goal is to find numerically the transaction and no-transaction regions as well as the optimal portfolio and transaction boundaries for various cases of the parameters. Another goal is to compare the approximate transaction boundaries obtained in Section 1.4 and the one obtained numerically.

As the parameters we consider γ , m and k , λ . The parameters γ and m represent the investors preference to safe and risky assets. We observed that γ and m affect the convexity and concavity of the optimal portfolio and transaction curves. If $\gamma > m$, the graphs of the optimal portfolio curves are concave in the xy -plane. This shows that as the wealth $x + y$ increases the investor allocates more wealth to the safe asset, indicating that the case where $\gamma > m$ is appropriate for conservative investors. On the other hand, if $\gamma < m$, the graphs of the optimal portfolio curves are convex in the xy -plane. This shows that as the wealth $x + y$ increases the investor allocates more wealth to the risky asset, indicating that the case where $\gamma < m$ is appropriate for risk-taking or entrepreneur investors. In Subsection 1.7.1 we examine the case where $\gamma > m$.

Figure 1.3: (Left) The value function $v(x, y) = ax^\gamma + \frac{y^m}{m}$. (Right) The contour curves for the value function $v(x, y) = ax^\gamma + \frac{y^m}{m}$.

The graphs and explanations of the four different cases of k and λ are presented. The case where $\gamma < m$ will be discussed in Subsection 1.7.2.

Concerning the approximate transaction boundaries, in Subsection 1.7.3 we examine how the approximation depends on k and (x_0, y_0) . We observe that the approximations are good for smaller values of k .

1.7.1 The case when $\gamma > m$.

For the numerical calculation, we choose $\lambda = 0.1$, $k = 0.03$ if $(k \neq 0)$, $\alpha = 0.11$, $\sigma = 0.25$, $r = 0.07$, $\delta = 0.09$, $\gamma = 0.90$. Then, m is given by $m = 0.8537$. In this case the allocation of wealth to the risky asset decreases as the wealth increases. This case may represent a conservative investor who would allocate the smaller portion of his wealth to the risky asset as his wealth increases. The graphs of the value function $v(x, y) = ax^\gamma + \frac{y^m}{m}$ and its contour curves are given in Figure 1.3. We use Matlab for programming. For each given x and y which represent the amount of money in the bank account and stock, we use the Matlab function “fminbnd” to compute the transaction size ξ so that the value function attains the maximum value along the line of trading. Then compare the values of $v(x, y)$ and $v(x', y') = Mv(x, y)$, where M is the intervention operator defined in (2.47). If $v(x', y') \geq v(x, y)$, we trade the stock. Otherwise, we do not trade. Depending on the values of k and λ , we discuss the problem (1.30) numerically in Subsections 1.7.1.1 to 1.7.1.3.

1.7.1.1 The case when $k > 0$, $\lambda > 0$.

This is the case where both fixed and proportional transaction costs are present. We have $x' = x - k - \xi - \lambda\xi$ if $\xi > 0$ and $x' = x - k - \xi + \lambda\xi$ if $\xi < 0$. We first explain various notations in Figure 1.4. It is easily seen that the first quadrant has been divided into three regions. The shaded region in the left side is the sell region, the shaded region in the right side is the buy region, and the middle region, which is bounded between the green curve and red curve,

is the no-transaction (NT) region. Suppose (x_1, y_1) is the current position for the portfolio in the sell region, where x_1 and y_1 represent the amount of money in the bank account and stock, respectively. Then, the line L_1 starting from (x_1, y_1) is the line along which the stock is exchanged to the money and is given by

$$x_{01} = x_1 - k - \xi + \lambda\xi, \quad y_{01} = y_1 + \xi, \quad \xi < 0, \quad (1.124)$$

or

$$x_{01} + (1 - \lambda)y_{01} = x_1 + (1 - \lambda)y_1 - k, \quad x_{01} > x_1. \quad (1.125)$$

The distance between (x_1, y_1) and $P_1(x_{01}, y_{01})$ is the transaction size $|\xi_1|$, i.e., we need to sell $|\xi_1|$ ($\xi_1 < 0$) shares of stock. Similarly, suppose (x_2, y_2) is the current position in the buy region. Then, the line L_2 is the line starting from (x_2, y_2) along which the money is exchanged to the stock and is given by

$$x_{02} = x_2 - k - \xi - \lambda\xi, \quad y_{02} = y_2 + \xi, \quad \xi > 0, \quad (1.126)$$

or

$$x_{02} + (1 + \lambda)y_{02} = x_2 + (1 + \lambda)y_2 - k, \quad x_{02} < x_2. \quad (1.127)$$

The distance between (x_2, y_2) and $P_2(x_{02}, y_{02})$ is the transaction size $|\xi_2|$, i.e., we need to buy $|\xi_2|$ ($\xi_2 > 0$) shares of stock.

As explained in Section 1.4 the optimal portfolio curve C_1 is the set of (x_{01}, y_{01}) at which the value function attains its maximum along the line L_1 for each (x_1, y_1) given in the sell region. Similarly, the optimal portfolio curve C_2 is the set of (x_{02}, y_{02}) at which the value function attains its maximum along the line L_2 for each (x_2, y_2) given in the buy region.

The curve B_1 is the transaction boundary between the sell and no-transaction region and in Figure 1.4 it is the boundary between the left shaded region and the middle region. The approximate curve for B_1 obtained in Theorem 1.3 is the green line in Figure 1.4. Similarly, the curve B_2 is the transaction boundary between the buy and no-transaction region and in Figure 1.4 it is the boundary between the right shaded region and the middle region. The approximate curve for B_2 obtained in Theorem 1.3 is the red line in Figure 1.4. For small values of k the approximate transaction boundaries agree well with the boundary curves B_1 and B_2 obtained numerically.

1.7.1.2 The case when $k > 0$, $\lambda = 0$.

This is the case when there is a fixed transaction cost $k > 0$ after each trades, but without proportional transaction costs, i.e. $\lambda = 0$. In Figure 1.5 the curve C_1 (or C_2) gives the curve of the optimal portfolio for a given wealth level (x_0, y_0) on the sell (or buy) region, and the blank region in the first quadrant is the NT region. Compare Figure 1.5 with Figure 1.4. We find that the curves of the optimal portfolios C_1 and C_2 overlap as we see the curves in (1.33) and (1.34) are identical if $\lambda = 0$.

Figure 1.4: Transaction regions and NT region when $k > 0$, $\lambda > 0$ and $m = 0.8537$.

Figure 1.5: Transaction regions and NT region when $k > 0$, $\lambda = 0$ and $m = 0.8537$.

Figure 1.6: Transaction regions and NT region when $k = 0$, $\lambda = 0$, $m = 0.8537$.

1.7.1.3 The cases when $k = 0$.

The cases when $k = 0$ are for the sake of comparison with the cases when $k > 0$. Graphs when $k = 0$ are shown in Figure 1.6 and 1.7. There are two cases. In Figure 1.6, the case where both k and λ are zero are treated. The curves C_1 and C_2 overlap and they give the allocation in the optimal portfolio for a given wealth level $w = x + y$. They play the similar role to the “Merton line” introduced in Davis and Norman [8]. We should note that in the Merton’s formulation the consumption is a function of wealth and in the current formulation it is a function of money. Also, in the Merton’s formulation the independent variable is the wealth and the relative weight of the money and the risky asset is a control variable. On the other hand both the money and the risky asset are independent variables in this formulation. These differences in the formulation may be the reasons why “Merton line” is no longer a straight line in the current formulation.

If $\lambda > 0$, C_1 and C_2 no longer overlap. On the other hand, they coincide with the transaction boundaries.

1.7.2 The case when $m > \gamma$.

As an example we choose $\lambda = 0.1$, $k = 0.03$ (if $k \neq 0$), $\alpha = 0.11$, $\sigma = 0.28$, $r = 0.07$, $\delta = 0.1$, $\gamma = 0.9$, then $m = 0.9318$. In this case the allocation of wealth to the risky asset increases as the wealth increases. This case may represent a risk-taking investor who is willing to allocate the larger portion of his wealth to

Figure 1.7: Transaction regions and NT region when $k = 0$, $\lambda > 0$, $m = 0.8537$.

the risky asset as his wealth increases. The case when $k > 0$, $\lambda = 0$ is shown in Figures 1.8 and the case when $k > 0$, $\lambda > 0$ is shown in Figure 1.9. B_1 (or B_2) is the boundary between the sell region (or buy region) and NT region. The curve in the red (or green) is the approximate boundary. For a small k the approximate transaction boundary and the corresponding numerical boundary are in good agreement. We see clearly that all of the curves C_1 , C_2 , B_1 and B_2 are concave upward curves.

1.7.3 The approximate transaction boundaries

We take the first three terms in the power series approximations obtained in Theorem 1.3 and examine the effects of k in the approximation. We use the first three terms in the power of k , *i.e.*,

$$x = x_0 + \sum_{n=1}^2 a_n k^{n/2}, \quad y = y_0 + \sum_{n=1}^2 b_n k^{n/2}, \quad (1.128)$$

where the coefficients a_i and b_i ($i = 1, 2$) are given in (1.47) and (1.50). Except for k we use the same parameters as in Subsection 1.7.2. The points $(\hat{x}, \hat{y})$ on the transaction curve B_1 depend on the fixed transaction cost k and the points (x_0, y_0) on the optimal portfolio curve C_1 . Similar situation applies for the points on the transaction curve B_2 . The results are reported in Figures 1.10 to 1.12. We observe that the curve B_1 (or B_2) separates away from curve C_1 (or C_2) as k increases. We also see that both the sell and buy regions recede as

Figure 1.8: Transaction regions and NT region when $k > 0$, $\lambda = 0$ and $m = 0.9318$.

Figure 1.9: Transaction regions and NT region when $k > 0$, $\lambda > 0$ and $m = 0.9318$.

Figure 1.10: Transaction regions and NT region when $k = 0.1$, $\lambda > 0$ and $m = 0.9318$.

k increases. The approximations (1.128) deteriorate as k increases because we take only three terms for the approximations.

1.8 Explicit solutions for exponential and natural logarithm utility functions

1.8.1 Exponential utility function

Theorem 1.15. *Suppose that the utility function is an exponential function given by*

$$u(c) = 1 - e^{-\gamma c}, \quad (1.129)$$

where $\gamma > 0$, then an explicit solution for the value function which satisfies

$$\max_c [-\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + u(c)] = 0 \quad (1.130)$$

is given by

$$v(x, y) = -\frac{1}{r} e^{1 - \frac{\delta}{r} - r\gamma x} + \frac{y^m}{m} + \frac{1}{\delta}. \quad (1.131)$$

Proof. Differentiating (1.130) with respect to c and solving for c , we have

$$c^* = -\frac{1}{\gamma} \ln\left(\frac{1}{\gamma} \frac{\partial v}{\partial x}\right). \quad (1.132)$$

Figure 1.11: Transaction regions and NT region when $k = 0.3$, $\lambda > 0$ and $m = 0.9318$.

Figure 1.12: Transaction regions and NT region when $k = 0.5$, $\lambda > 0$ and $m = 0.9318$.

We assume that the value function is given by

$$v(x, y) = ae^{bx} + \frac{y^m}{m} + \beta, \quad (1.133)$$

where a , b , and β are constants. Substituting (1.133) in (1.130) with c^* given in (1.132) and collecting the like terms, we obtain

$$-rabxe^{bx} + \frac{ab^2}{\gamma}xe^{bx} = 0 \quad (1.134)$$

$$-\delta ae^{bx} + \frac{ab}{\gamma} \ln\left(\frac{ab}{\gamma}\right)e^{bx} - \frac{ab}{\gamma}e^{bx} = 0, \quad (1.135)$$

$$-\delta\beta + 1 = 0. \quad (1.136)$$

Therefore, $a = -\frac{1}{r}e^{(1-\frac{\delta}{r})}$, $b = -r\gamma$ and $\beta = \frac{1}{\delta}$. Also, we may obtain $c^* = rx - \frac{1}{\gamma}(1 - \frac{\delta}{r})$. Since the value of m is exactly the same as in Theorem 1.1, we obtain (1.131). $\square$

Figure 1.13 shows the transaction regions and NT region for the exponential utility function (1.129).

Figure 1.13: Transaction regions and NT region when $u(c) = 1 - e^{-\gamma c}$, $k > 0$, $\lambda > 0$ and $m = 0.9342$.

1.8.2 Natural Logarithm utility function

Theorem 1.16. Suppose that the utility function is a natural logarithm function given by

$$u(c) = \ln c, \quad (1.137)$$

then an explicit solution for the value function which satisfies

$$\max_c [-\delta v + (rx - c) \frac{\partial v}{\partial x} + \alpha y \frac{\partial v}{\partial y} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + u(c)] = 0 \quad (1.138)$$

is given by

$$v(x, y) = \frac{1}{\delta} \ln x + \frac{y^m}{m} + \frac{1}{\delta} \left(\frac{r - \delta}{\delta} + \ln \delta \right). \quad (1.139)$$

Proof. Differentiating (1.138) with respect to c and solving for c , we have

$$c^* = \left(\frac{\partial v}{\partial x} \right)^{-1}. \quad (1.140)$$

We assume that the value function is given by

$$v(x, y) = a \ln x + \frac{y^m}{m} + b, \quad (1.141)$$

where a and b are constants. Substituting (1.141) in (1.138) with c^* given in (1.140) and collecting the like terms, we obtain

$$-\delta a \ln x + \ln x = 0, \quad (1.142)$$

$$-\delta b + rx \frac{a}{x} - \frac{x}{a} \frac{a}{x} - \ln a = 0. \quad (1.143)$$

Therefore, $a = \frac{1}{\delta}$ and $b = \frac{1}{\delta} \left(\frac{r - \delta}{\delta} + \ln \delta \right)$, and since the value of m is exactly the same as in Theorem 1.1, we obtain (1.139). $\square$

Remark 1.17. Notice that $c^* = \delta x$ from equation (1.140). This indicates that we always have to consume a proportional amount of money from the bank account.

Figure 1.14 shows the transaction regions and NT region for the natural logarithm utility function (1.137).

Chapter 2

Multiple Risky Assets Problem

2.1 Introduction

This chapter describes the optimal portfolio and consumption on multiple assets with transaction costs. We derive the explicit solutions to the HJB equations. We discuss the case with two risky assets and multiple risky assets. We maximize the time integral of the discounted utility function. Using the dynamic programming principle, we derive the Hamilton-Jacobi-Bellman (HJB) equation for the value function. Explicit solutions are found for the typical utility function which is a power function $u(c) = \frac{c^\gamma(t)}{\gamma}$ ($0 < \gamma < 1$), where $c(t)$ is the consumption rate.

Our contributions in this chapter are to distinguish and treat the money and the risky assets as multiple independent variables. The optimal consumption depends on the money but not on the risky assets. We find the explicit solutions to the HJB equations. Thanks to explicit solutions the intervention operator is reduced to a constrained optimization of the value functions. The theorem for the Viscosity solutions are proved in easier way also because of knowing the explicit solutions. Using the explicit solutions we are able to find the transaction and no-transaction regions, and the optimal portfolio curves for the two risky assets case. Also, we are able to discuss the algorithms for trading N risky assets.

Merton [17] derive the optimality equations for a multiasset problem when the rate of returns are generated by a Wiener Brownian-motion process. Muthuraman and Kumar [19] provide a computational study of the problem of optimal allocating wealth among multiple stocks and a bank account, to maximize the infinite horizon discounted utility of consumption. Liu [15] shows that when asset returns are uncorrelated, the optimal investment policy is to keep the dollar amount invested in each risky asset between two constant levels and upon reaching either of these thresholds, to trade to the corresponding optimal targets. As

the portfolio under consideration consists of multiple risky assets, which makes the numerical methods formidable, Atkinson and Ingpochai [4] apply perturbation analyses and also allow the risky assets to have correlation between their price processes. Inequalities for Hadamard product are given and proved in Jia [12].

The outline of this chapter is as follows. The mathematical formulation is given in section 2.2. In section 2.3, we discuss the transaction costs model with both proportional and fixed transaction costs, and obtain the explicit solution for the HJB equation. In section 2.4, we prove the explicit solutions for the HJB equation are viscosity solutions. Numerical results for two risky assets and N risky assets as long as the algorithm for trading N risky assets are discussed in section 2.5.

2.2 Formulation of the problem

Let (Ω, F, P) be a probability space with $F(t) (t \geq 0)$ a given filtration. We define $S_0(t) (t \geq 0)$ as the amount of money invested in the money market (bank) and $S(t) = [S_1(t), S_2(t), \dots, S_N(t)] (t \geq 0)$ as the stochastic process of the amount of money invested in the N stocks. We assume that the processes $S_0(t)$ and $S(t)$ satisfy

$$dS_0(t) = (rS_0(t) - c(t))dt \quad \tau_l \leq t \leq \tau_{l+1}, \quad (2.1)$$

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t) \sum_{j=1}^N \sigma_{ij} dW_j \quad \tau_l \leq t \leq \tau_{l+1} \quad (i = 1, 2, \dots, N), \quad (2.2)$$

$$S_0(\tau_{l+1}) = S_0(\tau_{l+1}^-) - k\tilde{p} - \sum_{i=1}^N \xi_{i,l+1} - \lambda \sum_{i=1}^N |\xi_{i,l+1}| \quad (i = 1, 2, \dots, N), \quad (2.3)$$

$$S_i(\tau_{l+1}) = S_i(\tau_{l+1}^-) + \xi_{i,l+1} \quad (i = 1, 2, \dots, N), \quad (2.4)$$

$$S_0(0^-) = x, \quad (2.5)$$

$$S_i(0^-) = y_i, \quad (2.6)$$

where $\xi_{i,l+1} (i = 1, 2, \dots, N)$ are sizes of transactions for all N stocks at time τ_{l+1}^- and $\tilde{p}$ is the number of how many $\xi_{i,l+1} (i = 1, 2, \dots, N)$ which are not 0. The control of the investor is a combination of a regular stochastic control $c(t)$ and an impulse control

$$\phi = (\tau_1, \tau_2, \dots; \xi_{1,1}, \xi_{1,2}, \dots, \xi_{1,N}, \xi_{2,1}, \xi_{2,2}, \dots, \xi_{2,N}, \dots, \xi_{N,1}, \xi_{N,2}, \dots, \xi_{N,N}). \quad (2.7)$$

We consider the case where the shorts are not allowed. Therefore, we define a solvency region by

$$S_+ := [0, \infty) \times [0, \infty) \times [0, \infty) \times \dots \times [0, \infty). \quad (2.8)$$

We consider the problem using the utility function $u(c) = \frac{c^\gamma(t)}{\gamma}$ where $\gamma \in \Gamma := \{\gamma \in R : 0 < \gamma < 1\}$. The objective of the investor is to maximize the time integral of the discounted utility function for his life time consumption or to find the value function

$$\tilde{\Phi}(s, x, y) = \sup_{c, \phi} E \left[\int_0^\infty e^{-\delta(s+t)} \frac{c^\gamma(t)}{\gamma} dt \right], \quad (2.9)$$

where $\delta > 0$ is a constant discount rate and $y = (y_1, y_2, \dots, y_N)$. We assume that $\alpha > \delta$. For the case $\alpha < \delta$, in particular $\alpha < 0$, this means stocks will go down and it is better to sell the stocks. Note that $1 - \gamma$ is called the relative risk aversion coefficient. In the infinite horizon problem we can simplify the problem in the following manner

$$\tilde{\Phi}(s, x, y) = e^{-\delta s} \sup_{c, \phi} E \left[\int_0^\infty e^{-\delta t} \frac{c^\gamma(t)}{\gamma} dt \right] = e^{-\delta s} \tilde{\Phi}(0, x, y) \triangleq e^{-\delta s} \Phi(x, y), \quad (2.10)$$

so that we seek a time independent value function $\Phi(x, y)$.

Similarly as we did for the one risky asset, we need to distinguish the case where there are no transactions and the case where the transactions are taking place. First we discuss the no-transaction case. Let $\tilde{v}(s, x, y) = e^{-\delta s} v(x, y)$ be the value function for the no-transaction case and using the dynamical principle, we see that v satisfies the Hamilton-Jacobi-Bellman (HJB) equation given by

$$\max_c [-\delta v + (rx - c) \frac{\partial v}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}] = 0. \quad (2.11)$$

We define

$$L^c v(x, y) = -\delta v + (rx - c) \frac{\partial v}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j}. \quad (2.12)$$

Also, we define

$$Lv(x, y) = \sup_{c \geq 0} \{L^c v(x, y) + \frac{c^\gamma}{\gamma}\}, \quad (x, y) \in S_+, \quad (2.13)$$

$$L_0 v(0, y) = -\delta v + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j}. \quad (2.14)$$

On the boundaries where y_i ($i = 1, 2, \dots, N$) could be 0, the situation where several y_i s are zero and the rest of y_i s are non-zero may take place.

Let $v : S_+ \rightarrow R^+$ be a measurable function. The intervention operator M is defined by

$$Mv(x, y) = \sup_{\xi} \{v(x', y') : \xi \in R \setminus \{0\}, (x', y') \in S_+\}, \quad (2.15)$$

where

$$x'(\xi) = x - k\tilde{p} - \sum_{i=1}^N \xi_i - \lambda \sum_{i=1}^N |\xi_i|, \quad (2.16)$$

here $\tilde{p}$ is the number of nonzero transactions, and

$$y'(\xi) = y + \xi. \quad (2.17)$$

Here, k and λ are a fixed transaction cost and a proportional transaction cost of trading respectively. We compare the current value v and the optimal value Mv after the trading. The region where $Mv \geq v$ is called the transaction region and the region where $Mv < v$ is called the no-transaction region and denoted by NT . We also call the set of $(x, y) \in S_+$ at which $Mv(x, y) = v(x, y)$ the transaction boundaries.

2.3 Solutions to HJB equation

We consider the equation (2.11) first. Concerning the solutions of (2.11), we have

Theorem 2.1. *A solution to (2.11) is given by*

$$\begin{aligned} v(x, y_1, y_2, \dots, y_N) = & ax^\gamma + \sum_{i=1}^N b_i y_i^{m_i} + \sum_{i_1, i_2=1, i_1 \neq i_2}^N b_{i_1 i_2} (y_{i_1} y_{i_2})^{m_{i_1 i_2}} \\ & + \sum_{i_1, i_2, i_3=1, i_1 \neq i_2 \neq i_3}^N b_{i_1 i_2 i_3} (y_{i_1} y_{i_2} y_{i_3})^{m_{i_1 i_2 i_3}} \\ & + \sum_{i_1, i_2, \dots, i_k=1, i_1 \neq i_2 \neq \dots \neq i_k}^N b_{i_1 i_2 \dots i_k} (y_{i_1} y_{i_2} \dots y_{i_k})^{m_{i_1 i_2 \dots i_k}} \\ & + \dots + \sum_{i_1, i_2, \dots, i_N=1, i_1 \neq i_2 \neq \dots \neq i_N}^N b_{i_1 i_2 \dots i_N} (y_{i_1} y_{i_2} \dots y_{i_N})^{m_{i_1 i_2 \dots i_N}}, \end{aligned} \quad (2.18)$$

where

$$a = \frac{1}{\gamma} \left[\frac{1}{1-\gamma} (\delta - r\gamma) \right]^{\gamma-1}, \quad (2.19)$$

and

$$\begin{aligned} & m_{i_1 i_2 \dots i_k} \\ = & \frac{(\frac{1}{2} \sum_{i=1}^k \sigma_{ii} - \sum_{i=1}^k \alpha_i) \pm \sqrt{(\sum_{i=1}^k \alpha_i - \frac{1}{2} \sum_{i=1}^k \sigma_{ii})^2 + 2\delta \sum_{p,q=1}^k \sigma_{i_p i_q}}}{\sum_{p,q=1}^k \sigma_{i_p i_q}}, \\ & (k = 1, 2, \dots, N). \end{aligned} \quad (2.20)$$

The $b_{i_1 i_2 \dots i_k}$ ($k = 1, 2, \dots, N$) in (2.18) are positive real numbers.

Proof. Differentiating (2.11) with respect to c , we have $c^* = \left(\frac{\partial v}{\partial x}\right)^{\frac{1}{\gamma-1}}$. Substituting c^* into (2.11), we obtain

$$-\delta v + rx \frac{\partial v}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j} + \frac{1-\gamma}{\gamma} \left(\frac{\partial v}{\partial x}\right)^{\frac{\gamma}{\gamma-1}} = 0. \quad (2.21)$$

Substituting the ansatz (2.18) into (2.21) and collecting the like terms, we get two equations

$$[-\delta a + ra\gamma + \frac{1-\gamma}{\gamma} a^{\frac{\gamma}{\gamma-1}} \gamma^{\frac{\gamma}{\gamma-1}}] x^\gamma = 0, \quad (2.22)$$

and

$$-\delta v + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j} = 0. \quad (2.23)$$

Since each y terms in equation (2.18) satisfies (2.23), from equations (2.22) and (2.23) we get (2.19) and (2.20). $\square$

Remark 2.2. A more general form for the explicit solution to (2.11) is: the y terms in equation (2.18) have the forms of $y_{i_1}^{m_{i_1}} y_{i_2}^{m_{i_2}} \dots y_{i_k}^{m_{i_k}}$ ($m_{i_1} \geq 0, m_{i_2} \geq 0, \dots, m_{i_k} \geq 0$).

Since we know an explicit solution in the no transaction region, the problem is reduced to finding $\max[v(x, y), Mv(x, y)]$ for each $(x, y) \in S_+$. Combining the no-transaction and transaction cases, we define

$$\Phi(x, y) = \max[v(x, y), Mv(x, y)] \quad (2.24)$$

as the value function. Therefore, we obtain a quasi-variational inequality problem

$$\max_{c, \phi} \left\{ L^c v + \frac{c^\gamma}{\gamma}, Mv - \Phi(x, y) \right\} = 0 \quad (2.25)$$

with the understanding that if both $L^c v + \frac{1}{\gamma} c^\gamma$ and $Mv - \Phi(x, y)$ are equal to zero, we make trades and move to a new allocation. With this formulation we can handle both fixed and proportional transaction costs.

Lemma 2.3. (*Concavity of the value function*) A simplified explicit solution satisfies (2.11) is given by

$$v(x, y_1, y_2, \dots, y_N) = ax^\gamma + \sum_{i=1}^N b_i y_i^{m_i} + \sum_{i,j=1, i \neq j}^N b_{ij} (y_i y_j)^{m_{ij}}. \quad (2.26)$$

The value function given by (2.26) is concave if $0 < m_i < 1$, $0 < m_{ij} < 0.5$ for $i, j = 1, 2, \dots, N$, ($i \neq j$).

Proof. When $0 < \gamma < 1$ and $0 < m_i < 1$, $f(x) = x^\gamma$ and $g_i(y_i) = y_i^{m_i}$ are all concave functions for $i = 1, 2, \dots, N$. In order to prove the value function given by (2.26) is a concave function, next we show when $0 < m_{ij} < 0.5$,

$h_{ij}(y_i, y_j) = b_{ij}(y_i y_j)^{m_{ij}}$ are concave functions for $i, j = 1, 2, \dots, N$, ($i \neq j$). The Hessian of $-\sum_{i,j=1, i \neq j}^N b_{ij}(y_i y_j)^{m_{ij}}$ is given below,

$$H(y_i, y_j) = \begin{pmatrix} -b_{ij}m_{ij}(m_{ij}-1)y_i^{m_{ij}-2}y_j^{m_{ij}} & -b_{ij}m_{ij}^2y_i^{m_{ij}-1}y_j^{m_{ij}-1} \\ -b_{ij}m_{ij}^2y_i^{m_{ij}-1}y_j^{m_{ij}-1} & -b_{ij}m_{ij}(m_{ij}-1)y_i^{m_{ij}}y_j^{m_{ij}-2} \end{pmatrix} \quad (2.27)$$

The determinant of H is calculated, and

$$\begin{aligned} & |H(y_i, y_j)| \\ &= \begin{vmatrix} -b_{ij}m_{ij}(m_{ij}-1)y_i^{m_{ij}-2} \cdot y_j^{m_{ij}} & -b_{ij}m_{ij}^2y_i^{m_{ij}-1} \cdot y_j^{m_{ij}-1} \\ -b_{ij}m_{ij}^2y_i^{m_{ij}-1} \cdot y_j^{m_{ij}-1} & -b_{ij}m_{ij}(m_{ij}-1)y_i^{m_{ij}} \cdot y_j^{m_{ij}-2} \end{vmatrix} \\ &= (b_{ij}m_{ij})^2 y_i^{2m_{ij}-2} \cdot y_j^{2m_{ij}-2} \cdot \begin{vmatrix} m_{ij}-1 & m_{ij} \\ m_{ij} & m_{ij}-1 \end{vmatrix} \\ &= (b_{ij}m_{ij})^2 y_i^{2m_{ij}-2} \cdot y_j^{2m_{ij}-2} \cdot (1-2m_{ij}) > 0 \end{aligned} \quad (2.28)$$

if $m_{ij} < 0.5$, ($i \neq j$). Thus, the value function given by (2.26) is concave if $0 < m_i < 1$, $0 < m_{ij} < 0.5$ for $i, j = 1, 2, \dots, N$, ($i \neq j$). $\square$

2.4 Viscosity Solution

Since we know that (2.18) in Theorem 2.1 is an explicit solution to the HJB equation and we define the value function Φ which is given in (2.24), it is interesting to prove that Φ is a viscosity solution for the quasi-variational inequality problem given in (2.25).

Definition 2.4. Suppose G is a topological space, x_0 is a point in G and $\psi : G \rightarrow R$ is a function.

(1) We say that φ is upper semi-continuous (usc) at x_0 if $\limsup_{x \rightarrow x_0} \varphi(x) \leq \varphi(x_0)$.

The function φ is called upper semi-continuous if it is upper semi-continuous at every point of G .

(2) Similarly, we say that φ is lower semi-continuous (lsc) at x_0 if $\liminf_{x \rightarrow x_0} \varphi(x) \geq \varphi(x_0)$. The function φ is called lower semi-continuous if it is lower semi-continuous at every point of G .

(3) We let $USC(G)$ and $LSC(G)$ denote the set of *usc* functions and *lsc* functions on G , respectively.

Definition 2.5. Let S_+^0 be the interior of S_+ and $\zeta = (x, y_1, y_2, \dots, y_N)$. We

define F as follows.

$$\begin{aligned}
& F(D^2v(\zeta), Dv(\zeta), v(\zeta), \zeta) \\
& = \begin{cases} \max_{c, \phi} \{-\delta v + (rx - c) \frac{\partial v}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}, \\ Mv - \Phi\}, \text{ if } \zeta \in S_+^0; \\ \max_{c, \phi} \{-\delta v + (rx - c) \frac{\partial v}{\partial x} + \sum_{j=1, j \neq i_1, \dots, j \neq i_k}^N \alpha_j y_j \frac{\partial v}{\partial y_j} \\ + \frac{1}{2} \sum_{j=1, j \neq i_1, \dots, j \neq i_k}^N \sigma_{jl} y_j y_l \frac{\partial^2 v}{\partial y_j \partial y_l} + \frac{c^\gamma}{\gamma}, Mv - \Phi\}, \\ \text{if } y_{i_1} = 0 \& y_{i_2} = 0 \& \dots \& y_{i_k} = 0, \text{ and } x \in (0, \infty); \\ \max_{\phi} \{-\delta v + \sum_{j=1, j \neq i_1, \dots, j \neq i_k}^N \alpha_j y_j \frac{\partial v}{\partial y_j} \\ + \frac{1}{2} \sum_{j=1, j \neq i_1, \dots, j \neq i_k}^N \sigma_{jl} y_j y_l \frac{\partial^2 v}{\partial y_j \partial y_l}, Mv - \Phi\}, \\ \text{if } x = 0 \& y_{i_1} = 0 \& y_{i_2} = 0 \& \dots \& y_{i_k} = 0 \\ \text{and } (y_{i_{k+1}}, y_{i_{k+2}}, \dots, y_N) \in (0, \infty) \times (0, \infty) \times \dots \times (0, \infty). \end{cases} \quad (2.29)
\end{aligned}$$

(1) A function $\psi \in USC(S_+)$ is a viscosity subsolution of

$$F(D^2v(\zeta), Dv(\zeta), v(\zeta), \zeta) = 0 \quad (2.30)$$

if for every $\zeta_0 = (x_0, y_{01}, y_{02}, \dots, y_{0N}) \in S_+$ and for every function $f \in C^2(S_+)$ satisfying $f \geq \psi$ on S_+ and $f(\zeta_0) = \psi(\zeta_0)$, we have

$$F(D^2f(\zeta_0), Df(\zeta_0), f(\zeta_0), \zeta_0) \geq 0. \quad (2.31)$$

(2) A function $\psi \in LSC(S_+)$ is a viscosity supersolution of (2.30) if for every $\zeta_0 \in S_+$ and for every function $f \in C^2(S_+)$ satisfying $f \leq \psi$ on S_+ and $f(\zeta_0) = \psi(\zeta_0)$, we have

$$F(D^2f(\zeta_0), Df(\zeta_0), f(\zeta_0), \zeta_0) \leq 0. \quad (2.32)$$

(3) A function $\psi : S_+ \rightarrow R$ is a viscosity solution of (2.30) if ψ is locally bounded and it is a viscosity sub and super solution of (2.30).

Lemma 2.6. *The variance and covariance matrix*

$$\sum = (\sigma_{ij}) = Cov[S_i, S_j] \quad i, j = 1, 2, \dots, N, \quad (2.33)$$

of the N risky assets $S = (S_1, S_2, \dots, S_N)$, is positive semidefinite.

Proof. For any vector $z \in R^n$, and the symmetry of $\sum$, we obtain

$$\begin{aligned}
z^T \sum z &= \sum_{i=1}^N \sum_{j=1}^N (\sigma_{ij} z_i z_j) \\
&= \sum_{i=1}^N \sum_{j=1}^N (\text{Cov}[S_i, S_j] z_i z_j) \\
&= \sum_{i=1}^N \sum_{j=1}^N (E[(S_i - E[S_i])(S_j - E[S_j])] z_i z_j) \\
&= E \left[\sum_{i=1}^N \sum_{j=1}^N (S_i - E[S_i])(S_j - E[S_j]) z_i z_j \right] \\
&= E[(S - E[S])^T z]^2 \geq 0.
\end{aligned} \tag{2.34}$$

□

Lemma 2.7. *The Hadamard product matrix*

$$A \star B = (a_{ij})_{N \times N} \star (b_{ij})_{N \times N} = (a_{ij} \star b_{ij})_{N \times N}, \tag{2.35}$$

is positive semidefinite, if both of matrices $A = (a_{ij})_{N \times N}$ and $B = (b_{ij})_{N \times N}$ are positive semidefinite.

Proof. Let $\text{rank}(B) = \tilde{r}$, so there exists a matrix $G_{r \times N}$ such that $B = G^T G$. If we denote $G^T = (g_{ij})_{N \times r}$, then $b_{ij} = \sum_{l=1}^{\tilde{r}} g_{il} g_{jl}$. For any vector $z \in R^n$,

$$\begin{aligned}
z^T (A \star B) z &= \sum_{i=1}^N \sum_{j=1}^N z_i a_{ij} b_{ij} z_j \\
&= \sum_{i=1}^N \sum_{j=1}^N z_i a_{ij} \left(\sum_{l=1}^{\tilde{r}} g_{il} g_{jl} \right) z_j \\
&= \sum_{l=1}^{\tilde{r}} \left[\sum_{i=1}^N \sum_{j=1}^N a_{ij} (z_i g_{il}) (z_j g_{jl}) \right] \geq 0,
\end{aligned} \tag{2.36}$$

since matrix A is positive semidefinite [12].

□

Theorem 2.8. *The value function Φ is a viscosity solution of (2.30).*

Proof. (1) For $\zeta = (x, y_1, y_2, \dots, y_N) \in S_+^0$, we prove that $\Phi(\zeta) = \max[v(\zeta), Mv(\zeta)]$ is a viscosity subsolution of the first case in (2.29). It is easy to see that v is C^2 . Consider that f is a C^2 function in a neighborhood of S_+^0 and let $\zeta_0 = (x_0, y_{01}, y_{02}, \dots, y_{0N}) \in S_+^0$ be such that $f \geq \Phi$ on S_+^0 and $f(\zeta_0) = \Phi(\zeta_0)$. We show two cases separately.

Case (i). $v(\zeta_0) \leq Mv(\zeta_0)$. Then, $f(\zeta_0) = \Phi(\zeta_0) = \max[v(\zeta_0), Mv(\zeta_0)] = Mv(\zeta_0)$. Since

$$Mv(\zeta_0) = v(x', y'_1, y'_2, \dots, y'_N) \leq f(x', y'_1, y'_2, \dots, y'_N) \leq Mf(\zeta_0),$$

we have

$$\begin{aligned} & \max_{c, \phi} \{ [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}] (\zeta_0), \\ & [Mf - \max(Mf, f)](\zeta_0) \} \\ \geq & [Mf - \max(Mf, f)](\zeta_0) \\ = & [Mf - \max(Mf, Mv)](\zeta_0) \geq 0. \end{aligned} \quad (2.37)$$

So (2.31) holds.

Case (ii). $v(\zeta_0) > Mv(\zeta_0)$. We need to prove

$$\begin{aligned} & \max_c [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}] (\zeta_0) \\ \geq & 0. \end{aligned} \quad (2.38)$$

Since $f \geq \Phi \geq v$ on S_+^0 and $f(\zeta_0) = \Phi(\zeta_0) = v(\zeta_0)$, $f - v$ has a local minimum at ζ_0 . Therefore, $\frac{\partial f}{\partial x}(\zeta_0) = \frac{\partial v}{\partial x}(\zeta_0)$, $\frac{\partial f}{\partial y_i}(\zeta_0) = \frac{\partial v}{\partial y_i}(\zeta_0)$, ($i = 1, 2, \dots, N$) and the matrix $[\frac{\partial^2(f-v)}{\partial y_i \partial y_j}]_{N \times N}$ is positive semidefinite [7]. By using Lemma 2.7, $\sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} \geq \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j}$ because both of the matrices $(\sigma_{ij})_{N \times N}$ and $[\frac{\partial^2(f-v)}{\partial y_i \partial y_j}]_{N \times N}$ are positive semidefinite. This implies

$$\begin{aligned} & \max_c [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial f}{\partial y_i} \\ & + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}] (\zeta_0) \\ \geq & \max_c [-\delta v + (rx - c) \frac{\partial v}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial v}{\partial y_i} \\ & + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 v}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}] (\zeta_0). \end{aligned} \quad (2.39)$$

We choose $c^* = (\frac{\partial v}{\partial x})^{\frac{1}{\gamma-1}}$ and substitute the value function (2.18) into the right hand side of (2.39). By Theorem 2.1, the right hand side of (2.39) is zero.

The proofs for the cases where $x = 0$ & $y_{i_1} = 0$ & $y_{i_2} = 0$ & ... & $y_{i_k} = 0$ or $y_{i_1} = 0$ & $y_{i_2} = 0$ & ... & $y_{i_k} = 0$ are similar.

(2) For $\zeta \in S_+^0$, we prove that $\Phi(\zeta) = \max[v(\zeta), Mv(\zeta)]$ is a viscosity supersolution of the first case in (2.29). So we let f be a C^2 function in a neighbourhood of S_+^0 and we let $\zeta_0 \in S_+^0$ such that $f \leq \Phi$ on S_+^0 and $f(\zeta_0) = \Phi(\zeta_0)$. We want to show (2.32).

Since $[Mf - \max(Mf, f)](\zeta_0) \leq 0$, we only need to prove

$$\max_c [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}] (\zeta_0) \leq 0. \quad (2.40)$$

For $\bar{\varepsilon} > 0$ and $\bar{w} = \bar{w}_{\bar{\varepsilon}, c}$ be an admissible control beginning with a constant consumption rate $c > 0$ and no transactions up to the first time $\bar{\tau}_{\bar{\varepsilon}}$ at which the stochastic process exists from

$$\begin{aligned} & \bar{K}_{\bar{\varepsilon}} \\ &= \{ (s, x, y_1, y_2, \dots, y_N); |(s, x, y_1, y_2, \dots, y_N) - (0, x_0, y_{01}, y_{02}, \dots, y_{0N})| < \bar{\varepsilon} \} \\ & \cap \bar{S}_+, \end{aligned} \quad (2.41)$$

where $\zeta_0 = (x_0, y_{01}, y_{02}, \dots, y_{0N})$ and $\bar{S}_+ = R^+ \times S_+$. Choose $\zeta_n \in \bar{K}_{\bar{\varepsilon}}$ such that $\zeta_n \rightarrow \zeta_0$ and $\Phi(\zeta_n) \rightarrow \Phi(\zeta_0)$ as $n \rightarrow \infty$. By using Itô-Doeblin formula and the definition of expectation, we obtain

$$\begin{aligned} & \Phi(\zeta_n) \\ & \geq E^{0, \zeta_n} [e^{-\delta \bar{\tau}_{\bar{\varepsilon}}} \Phi(X^{\bar{w}}(\bar{\tau}_{\bar{\varepsilon}}), Y_1^{\bar{w}}(\bar{\tau}_{\bar{\varepsilon}}), \dots, Y_N^{\bar{w}}(\bar{\tau}_{\bar{\varepsilon}}))] + E^{0, \zeta_n} [\int_0^{\bar{\tau}_{\bar{\varepsilon}}} e^{-\delta t} \frac{c^\gamma(X(t))}{\gamma} dt] \\ & \geq E^{0, \zeta_n} [e^{-\delta \bar{\tau}_{\bar{\varepsilon}}} f(X^{\bar{w}}(\bar{\tau}_{\bar{\varepsilon}}), Y_1^{\bar{w}}(\bar{\tau}_{\bar{\varepsilon}}), \dots, Y_N^{\bar{w}}(\bar{\tau}_{\bar{\varepsilon}}))] + E^{0, \zeta_n} [\int_0^{\bar{\tau}_{\bar{\varepsilon}}} e^{-\delta t} \frac{c^\gamma(X(t))}{\gamma} dt] \\ & = f(\zeta_n) + E^{0, \zeta_n} [\int_0^{\bar{\tau}_{\bar{\varepsilon}}} e^{-\delta t} L^c f(X^{\bar{w}}(t), Y_1^{\bar{w}}(t), \dots, Y_N^{\bar{w}}(t)) dt] \\ & \quad + E^{0, \zeta_n} [\int_0^{\bar{\tau}_{\bar{\varepsilon}}} e^{-\delta t} \frac{c^\gamma(X(t))}{\gamma} dt], \end{aligned} \quad (2.42)$$

where

$$\begin{aligned} & L^c f(x, y_1, y_2, \dots, y_N) \\ &= -\delta f + (rx - c) \frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} \end{aligned}$$

and for all n . From (2.42),

$$\begin{aligned} & E^{0, \zeta_n} [\int_0^{\bar{\tau}_{\bar{\varepsilon}}} e^{-\delta t} \{L^c f(X^{\bar{w}}(t), Y_1^{\bar{w}}(t), \dots, Y_N^{\bar{w}}(t)) + \frac{c^\gamma(X(t))}{\gamma}\} dt] \\ & \leq \Phi(\zeta_n) - f(\zeta_n) \end{aligned} \quad (2.43)$$

for all n . Taking the limit as $n \rightarrow \infty$, we obtain

$$E^{0, \zeta_0} [\int_0^{\bar{\tau}_{\bar{\varepsilon}}} H(t) dt] \leq 0, \quad (2.44)$$

where

$$H(t) = e^{-\delta t} (L^c f(X^{\bar{w}}(t), Y_1^{\bar{w}}(t), \dots, Y_N^{\bar{w}}(t)) + \frac{c^\gamma(X(t))}{\gamma}). \quad (2.45)$$

By dividing the left-hand side by $E^{0, \zeta_0}[\bar{\tau}_{\bar{\varepsilon}}]$ we get

$$\frac{E^{0, \zeta_0}[\int_0^{\bar{\tau}_{\bar{\varepsilon}}} H(t) dt]}{E^{0, \zeta_0}[\bar{\tau}_{\bar{\varepsilon}}]} = \frac{E^{0, \zeta_0}[\int_0^{\bar{\tau}_{\bar{\varepsilon}}} [H(t) - H(0)] dt] + H(0)E^{0, \zeta_0}[\bar{\tau}_{\bar{\varepsilon}}]}{E^{0, \zeta_0}[\bar{\tau}_{\bar{\varepsilon}}]} \rightarrow H(0) \quad (2.46)$$

as $\bar{\varepsilon} \rightarrow 0$, since $H(t)$ is continuous at $t = 0$. Thus, we have

$$H(0) = [-\delta f + (rx - c) \frac{\partial f}{\partial x} + \sum_{i=1}^N \alpha_i y_i \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^N \sigma_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j} + \frac{c^\gamma}{\gamma}](\zeta_0) \leq 0$$

for all $c \geq 0$ such that $\bar{w}_{\bar{\varepsilon}, c}$ is admissible for $\bar{\varepsilon}$ small enough [20].

The proofs for the cases where $x = 0$ & $y_{i_1} = 0$ & $y_{i_2} = 0$ & ... & $y_{i_k} = 0$ or $y_{i_1} = 0$ & $y_{i_2} = 0$ & ... & $y_{i_k} = 0$ are similar.

(3) (1) and (2) imply that Φ is a viscosity solution of (2.30) $\square$

2.5 Numerical results

2.5.1 Two risky assets

Consider $N = 2$, this is the case when there are two risky assets. For the numerical calculation, we choose $\lambda = 0.1$, $k = 0.03$ (if $k \neq 0$), $\alpha_1 = 0.11$, $\alpha_2 = 0.13$, $\sigma_{11} = 0.3$, $\sigma_{12} = 0.2$, $\sigma_{21} = 0.2$, $\sigma_{22} = 0.3$, $r = 0.07$, $\delta = 0.1$, $\gamma = 0.3$. Then, m_1 , m_2 , and m_{12} are given by $m_1 = 0.96$, $m_2 = 0.89$, $m_{12} = 0.51$. As mentioned before, we discuss the cases where no borrowing or short-selling is allowed. Depending on the values of k and λ , we discuss two cases of the problem (2.25) numerically and analytically in Subsections 2.5.1.1 to 2.5.1.2.

2.5.1.1 The case when $k = 0$.

This is the case where there are no fixed transaction costs but there are proportional transaction costs. We first explain various notations in Figure 2.1. It is easy to see that the first octant has been divided into an outer space and an inner space by four curved planes. The outer space are the transaction regions which will be discussed in Figure 2.2 while the inner space is the No-Transaction region. In Figure 2.1, the curves C_1 , C_2 , C_3 , and C_4 are the sets of points $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum values while trading two stocks. The optimal portfolio curve C_1 is the set of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by buying two stocks; curve C_2 is the set of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by buying the first stock but selling the second stock; curve C_3 is the set of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by selling the first stock but buying the second stock; curve C_4 is the set of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains

its maximum by selling two stocks. The curves B_1s , B_2s , B_3s and B_4s are the sets of points $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum values while trading only one stock out of the two stocks. The optimal portfolio curves B_1s are the sets of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by buying the first stock but do not trade the second stock; curves B_2s are the sets of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by selling the first stock but do not trade the second stock; curves B_3s are the sets of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by do not trade the first stock but buying the second stock; curves B_4s are the sets of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum by do not trade the first stock but selling the second stock.

In Figure 2.2, regions R_1 , R_2 , R_3 and R_4 are the transaction regions of trading two stocks. For instance, if the values of our portfolio (x_1, y_1, z_1) are in R_1 , then we need to buy both of the two stocks. The optimal portfolio will be $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$, which is one of the points on C_1 . The transaction size will be determined by the distance between the optimal portfolio and original portfolio which is $(\bar{x}_1 - x_1, \bar{y}_1 - y_1, \bar{z}_1 - z_1)$. Regions R_5 , R_6 , R_7 and R_8 are the transaction regions of trading only one stock. For example, if our portfolio with coordinates which is in region R_5 , we need to buy the first stock but do not trade the second stock. The optimal portfolio will be one of the points on B_1s . The transaction size will be calculated by the distance between the optimal portfolio and the original portfolio. Region R_1 is the transaction region of buying two stocks; region R_2 is the transaction region of buying the first stock and selling the second stock; region R_3 is the transaction region of selling the first stock but buying the second stock; region R_4 is the transaction region of selling two stocks; region R_5 is the transaction region of buying the first stock but do not trade the second stock; region R_6 is the transaction region of selling the first stock but do not trade the second stock; region R_7 is the transaction region of do not trade the first stock but buying the second stock; region R_8 is the transaction region of do not trade the first stock but selling the second stock.

2.5.1.2 The case when $k > 0$.

This case consists of both fixed and proportional transaction costs. Similar to Figure 2.1, Figure 2.3 shows the first octant has been divided into two parts by four curved planes. The outer part are the transaction regions which will be discussed in Figure 2.4 while the inner part is the No-Transaction region. In Figure 2.3, the curves C_1 , C_2 , C_3 , and C_4 are the sets of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum values while trading two stocks and the curves B_1s , B_2s , B_3s and B_4s are the sets of $(\bar{x}, \bar{y}, \bar{z})$ at which the value function attains its maximum values while trading only one stock. In Figure 2.4, regions R_1 , R_2 , R_3 and R_4 are the transaction regions of trading two stocks. Regions R_5 , R_6 , R_7 and R_8 are the transaction regions of trading only one stock.

It is obvious to see that every two connected curved planes have intersect curves which are C_1 , C_2 , C_3 and C_4 shown in Figure 2.1. While in Figure 2.3, except for the parts of the curved planes bounded by the four curved planes,

Figure 2.1: Boundaries of transaction regions and NT region when $k = 0$. Figure 2.2: Transaction regions when $k = 0$.

Figure 2.3: Boundaries of transaction regions and NT region when $k > 0$. Figure 2.4: Transaction regions when $k > 0$.

Ticker Name	AXP	BA	CAT	CSCO	CVX
TRT1M	0.00743	0.009417	0.015416	0.00261	0.013074
Ticker Name	DD	DIS	HD	HPQ	IBM
TRT1M	0.005802	0.007068	0.004486	0.006387	0.010469

Table 2.1: TRT1M

there are some extra parts of the planes exceed the intersect curves. The reason makes this happen is because of the fixed transaction cost k . And comparing the No-Transaction regions between the cases where $k = 0$ and $k > 0$, we see the No-Transaction region where $k = 0$ is smaller than the No-Transaction region where $k > 0$. This is easy to see from the hollow regions in Figure 2.2 and Figure 2.4

2.5.2 N risky assets

In section 2.3, the explicit solution to equation (2.11) is given by (2.18). A simplified explicit solution which also satisfies (2.11) is given by (2.26). We use (2.26) to be our value function in this subsection. Our aim is to figure out how to trade N risky assets. We study the data from ten Dow Jones Industrial Average companies. The average TRT1M (Monthly Total Return) for these ten stocks from January 2005 to December 2011 are given in Table 2.1. The symmetric variance and covariance matrix for these stocks are given in Table 2.2.

2.5.2.1 Algorithm I

The algorithm to specify all of the combinations for trading N stocks is: If only one stock is traded out of N stocks, there will be C_N^1 choices, and if exactly two stocks are traded, there will be C_N^2 choices. Similarly, exactly i ($i = 1, 2, \dots, N$) stocks are traded will lead to C_N^i choices. Thus, there will be $C_N^1 + C_N^2 + \dots + C_N^N = 2^N - 1$ combinations in total.

For the numerical calculation, we choose $\lambda = 0.002$, $k = 7$, $r = 0.001$, $\delta = 0.002$, $\gamma = 0.3$, $x = 2000$, $(y_1, y_2, \dots, y_{10}) = (0, 0, \dots, 0)$ and by using the data from Table 2.1 and Table 2.2, we get the optimal ξ values (how much we should put in each stocks) after enumerate all the $2^{10} - 1$ choices in Table 2.3. The ten stocks' monthly closed prices in January 2012 are shown in Table 2.4. Table 2.3 and Table 2.4 provide us the information on how to trade the ten stocks in order to make the investment to be optimal. If we have \$2000 in our bank account and no money in the stocks accounts at the beginning, theoretically we have to buy stock AXP for \$463.4 (9.24 shares), buy stock CSCO for \$807.8 (41.12 shares), buy stock DD for \$195.7 (3.85 shares), buy stock HD for \$290.5 (6.54 shares), and buy stock HPQ for \$76.9 (2.75 shares). We will not trade stocks BA, CAT, CVX, DIS and IBM, since their ξ values are either less than the fixed transaction cost which is \$7 or less than the corresponding stock prices. In the real life, we can only trade integer shares of stocks. Hence we buy AXP for 9 shares; buy

	AXP	BA	CAT	CSCO	CVX	DD	DIS	HD	HPQ	IBM
AXP	0.016047	0.005455	0.008433	0.005006	0.002142	0.007219	0.005313	0.003829	0.004364	0.002422
BA		0.00649	0.004119	0.003245	0.002147	0.00406	0.003549	0.001814	0.003008	0.001448
CAT			0.011752	0.004992	0.003855	0.006242	0.004752	0.003452	0.005172	0.00254
CSCO				0.006567	0.002017	0.003805	0.003117	0.002895	0.004189	0.002493
CVX					0.003844	0.002251	0.002005	0.000866	0.002374	0.00108
DD						0.006717	0.004248	0.00298	0.003374	0.002537
DIS							0.004655	0.002011	0.003367	0.001856
HD								0.004977	0.001884	0.001581
HPQ									0.006058	0.002321
IBM										0.00331

Table 2.2: Variance and Covariance matrix

Ticker Name	AXP	BA	CAT	CSCO	CVX
ξ (dollars)	463.4	13.14	7.0	807.8	4.3
Ticker Name	DD	DIS	HD	HPQ	IBM
ξ (dollars)	195.7	28.8	290.5	76.9	6.0

Table 2.3: ξ values

Ticker Name	AXP	BA	CAT	CSCO	CVX
Stock prices (dollars)	50.14	74.18	109.12	19.65	103.12
Ticker Name	DD	DIS	HD	HPQ	IBM
Stock prices (dollars)	50.89	38.9	44.39	27.98	192.6

Table 2.4: Stock prices

CSCO for 41 shares; buy DD for 3 shares; buy HD for 6 shares; and buy HPQ for 2 shares; and there are $\$(2000 - 7 \times 5 - (1 + 0.002) \times 50.14 \times 9 - (1 + 0.002) \times 19.65 \times 41 - (1 + 0.002) \times 50.89 \times 3 - (1 + 0.002) \times 44.39 \times 6 - (1 + 0.002) \times 27.98 \times 2) = \229.66 left in the bank account. The optimal value for the value function is 10179.

Note that there are five stocks that we will not trade. Next we consider trading the rest five stocks only. We use the same parameters as we do for $N = 10$, but in this case $N = 5$. Let p and y start from $p = (0, 0, 0, 0, 0)$ and $(y_1, y_2, \dots, y_5) = (0, 0, \dots, 0)$. The optimal $\xi = (485.3, 852.4, 202.5, 310.9, 76.4)$ after we enumerate all $2^5 - 1$ choices. Together with the information given in Table 2.4, the conclusion is: buy AXP, CSCO, DD, HD and HPQ for 9 shares, 43 shares, 3 shares, 7 shares and 2 shares respectively. The amount of money left in the bank account will be \$145.8. The optimal value for the value function is 10562. We know the optimal value for the value function when trade the ten stocks is 10179, while the optimal value for the value function when trade the five stocks is 10562. The actual difference between the two optimal values is $10562 - 10179 = 383$, and the percentage difference is $\frac{10562 - 10179}{10562} = 3.63\%$.

2.5.2.2 Algorithm II

The numerical results shown in subsection 2.5.2.1 are the results specifying all of the $2^{10} - 1$ cases. For some large power N , explicit enumeration is normally impossible due to the exponentially increasing number of potential solutions. Branch and Bound method is the mostly widely used tool for solving large scale combinatorial optimization problems. The briefly idea for iteration has three main components: selection of the node to process, bound calculation, and branching. A tentative algorithm to solve our problem by using similar idea from the B&B method is given below. Figure 2.5 illustrates the algorithm.

- Let p be a $1 \times N$ vector with each entries are either 0 or 1, which means we will not trade the i th stock if the entry on the i th position of p is 0 or we will trade the i th stock if the entry on the i th position of p is 1.

- Set the initial $p = (0, 0, \dots, 0)$ if we start from have not traded any one of the N stocks (or set the initial $p = (1, 1, \dots, 1)$ if we start from trading all of the N stocks). Denote this to be node level 0. Calculate the value for the intervention operator

$$\begin{aligned}
Mv &= ax'^\gamma + \sum_{i=1}^N b_i y_i'^{m_i} + \sum_{i,j=1, i \neq j}^N b_{ij} (y_i' y_j')^{m_{ij}} \\
&= a(x - k\tilde{p} - \sum_{i=1}^N \xi_i - \lambda \sum_{i=1}^N |\xi_i|)^\gamma + \sum_{i=1}^N b_i (y_i + \xi_i)^{m_i} \\
&\quad + \sum_{i,j=1, i \neq j}^N b_{ij} [(y_i + \xi_i)(y_j + \xi_j)]^{m_{ij}}, \tag{2.47}
\end{aligned}$$

where $\tilde{p}$ is the sum of all 1s in p . Store the current Mv value to be the incumbent value.

- Branch into two branches $p = (0, 0, 0, \dots, 0)$ and $p = (1, 0, 0, \dots, 0)$, and this indicates to trade the 1st stock or not to trade it. Both of the two nodes are assumed in node level 1 (If use the initial $p = (1, 1, \dots, 1)$, branch into $p = (1, 1, \dots, 1)$ and $p = (0, 1, \dots, 1)$). Calculate the two Mv values. If any Mv value is greater than the incumbent value, keep a record for the new Mv and also store the corresponding p . For example, at node $p = (1, 0, 0, \dots, 0)$ we have a greater Mv value. We will branch this node into two branches with nodes $p = (1, 0, 0, \dots, 0)$ and $p = (1, 1, 0, \dots, 0)$.
- Repeat the procedures provided in the above steps, we may get the optimal strategy on trading N stocks.

If we use the algorithm illustrated above, the way to bound the calculations at each nodes will be studied in the near future. A short cut to solve the problem is: set initial $p = (1, 1, \dots, 1)$; get $(\xi_1, \xi_2, \dots, \xi_N)$ by using the function 'fmincon' in Matlab; determine the maximum of $|\xi_i|$ happens on the i th position of $(\xi_1, \xi_2, \dots, \xi_N)$; make the i th position of p be 0 and recalculate the problem; continue doing this until we get $p = (0, 0, \dots, 0)$; among the N times of calculations the optimal result has been stored. Identical result shows up if we use the same example in subsection 2.5.2.1.

Figure 2.5: Branch and Bound graph

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