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Algorithmic Issues in some Disjoint Clustering Problems in Combinatorial Circuits

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Algorithmic Issues in some Disjoint Clustering Problems in Combinatorial Circuits

Zola Nailah Donovan

Dissertation submitted
to the College of Engineering and Mineral Resources
at West Virginia University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in
Computer Science

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ABSTRACT

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Zola Nailah Donovan

As the modern integrated circuit continues to grow in complexity, the design of very large-scale integrated (VLSI) circuits involves massive teams employing state-of-the-art computer-aided design (CAD) tools. An old, yet significant CAD problem for VLSI circuits is physical design automation. In this problem, one needs to compute the best physical layout of millions to billions of circuit components on a tiny silicon surface. The process of mapping an electronic design to a chip involves several physical design stages, one of which is clustering. Even for combinatorial circuits, there exist several models for the clustering problem. In particular, we consider the problem of disjoint clustering in combinatorial circuits for delay minimization (CN). The problem of clustering with replication for delay minimization has been well-studied and known to be solvable in polynomial time. However, replication can become expensive when it is unbounded.

Consequently, CN is a problem worth investigating. In this dissertation, we establish the computational complexities of several variants of CN. We also present approximation and exact exponential algorithms for some variants of CN. In some cases, we even obtain an approximation factor of strictly less than two. Furthermore, our exact exponential algorithms beat brute force.

TO MY MOTHER, L. ELIZABETH BRYANT-RIVERA

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Chapter 1

Introduction

In this dissertation, we consider the problem of disjoint clustering in combinatorial circuits for delay minimization (CN). Combinatorial circuits implement Boolean functions and produce a unique output for every combination of input signals [Kos04]. The gates and their interconnections in the circuit represent implementations of one or more Boolean function(s). The Boolean functions are realized by the assignment of the gates to chips.

Due to manufacturing process requirements and capacity constraints, it is generally not possible to place all of the circuit elements in one chip. Consequently, the circuit is partitioned into clusters, where each cluster represents a chip in the overall circuit design. The circuit elements are assigned to clusters while satisfying specific design constraints (e.g., cluster capacity) [RW93]. However, such an assignment might cause some source to sink path(s) to cross multiple cluster boundaries. The resulting design may not be the most efficient implementation.

Gates and their interconnections usually have delays. The delays of the interconnections are determined by the way the circuit is clustered. Intra-cluster delays d are associated with the interconnections between gates in the same cluster. Inter-cluster delays D are associated with the interconnections between gates in different clusters. The

delay along a path from an input to an output is the sum of the delays of the gates and interconnections on the respective path. The delay of the overall circuit, with respect to a specific clustering, is the maximum delay among all paths that connect an input to any output in the clustered-circuit.

For example, consider the directed path P representing a simple combinatorial circuit in Figure 1.1(a). Let every vertex of P be unweighted (i.e., the weight of each vertex is one) with a zero delay. Let the cluster capacity equal two, the intra-cluster delay $d = 0$, and the inter-cluster delay D be any positive integer, where $D \gg d$. Suppose that we wish to find a disjoint clustering of P such that no cluster exceeds the cluster capacity, and the delay of P is minimized. Observe that the clustering of P in Figure 1.1(b), where every vertex of P belongs to a singleton cluster, is in an inefficient clustering with delay $4 \cdot D$. However, notice that the clustering of P shown in Figure 1.1(c) induces the minimum delay of $2 \cdot D$.

(a) Simple directed path P .

(b) An example of an inefficient clustering of P , where the delay is $4 \cdot D$.

(c) An example of an optimal clustering of P , where the delay is $2 \cdot D$.

Figure 1.1: Simple example of clusterings of a directed path.

Next, we consider another simple example of disjoint clustering of a combinatorial circuit represented by a directed acyclic graph (DAG) G in Figure 1.2(a). Again, let every vertex of G be unweighted with a zero delay. Let the cluster capacity equal two, the intra-cluster delay $d = 0$, and the inter-cluster delay D be any positive integer, where $D \gg d$. Suppose that we wish to find a disjoint clustering of G such that no cluster exceeds the cluster capacity, and the delay of G is minimized. Observe that the clustering of G in Figure 1.2(b), where every vertex of G belongs to a singleton cluster, is in an inefficient clustering with delay $3 \cdot D$. The delay is not improved in the clustering of the DAG in Figure 1.2(c). However, notice that the clustering of G shown in Figure 1.2(d) induces the minimum delay of $2 \cdot D$.

(a) An example of a DAG G . (b) Clustering with max delay $3 \cdot D$.

(c) Clustering with max delay $3 \cdot D$. (d) Clustering with max delay $2 \cdot D$.

Although we did not enumerate all clusterings of the graphs shown in Figures 1.1(a) and 1.2(a), it can be quickly done. Therefore, finding the overall delay of each graph is not challenging in such small instances. However, it is easy to see that as the number of vertices increases, we encounter a combinatorial explosion when attempting to enumerate all possibilities.

1.1 Clustering when replication is allowed

The problem of clustering combinatorial circuits for delay minimization when logic replication is allowed (CA) is well-studied [LLT69, RW93]. It frequently arises in VLSI design. In CA, the goal is to find a clustering of a circuit that minimizes the delay of the overall circuit. CA has been shown to be solvable in polynomial time [LLT69, RW93]. However, when replication is allowed, circuit elements may be assigned to more than one cluster.

Rajaraman and Wong [RW93] studied CA under the so-called *general delay model* in which the following holds:

1. each gate v of the network has a delay given by $\delta(v)$,
2. no delay is encountered on an interconnection linking two gates in the same cluster, and
3. a delay of D time units (D is a specified constant) is encountered on every interconnection linking two gates in different clusters.

In [RW93], they define a *clustering* as follows:

Definition 1.1.1. A *clustering* of a network $G = (V, E)$ is a triple (H, ϕ, Σ) , where

1. $H = (V', E')$ is a directed acyclic graph
2. ϕ is a function mapping V' to V such that
 - (a) for every edge $(u', v') \in E'$, $[\phi(u'), \phi(v')] \in E$,
 - (b) for every node $v' \in V'$ and edge $[u, \phi(v')] \in E$, there exists a unique $u' \in V'$ such that $\phi(u') = u$ and $(u', v') \in E'$, and
 - (c) for every PO node $v \in V$, there exists a unique $v' \in V'$ such that $\phi(v') = v$, and
3. Σ is a partition of V' .

Using Definition 1.1.1, we proved the following two lemmas.

Lemma 1.1.1. The pre-image of an output is also an output.

Proof. Let $G = (V, E)$ have a clustering (H, ϕ, Σ) , and by way of contradiction, suppose there exists a pre-image $v' \in V(H)$ of an output $v \in V(G)$, such that v' is not also an output. This means v' has a successor in H , say w' . By 2b of Definition 1.1.1, for every

node $w' \in V(H)$ and edge $[v, \phi(w')] \in E(G)$, there exists a unique $v' \in V(H)$ such that $\phi(v') = v$ and $(v', w') \in E(H)$. But edge $[v, \phi(w')] \notin E(G)$, since v is an output in G . So, there is exists no $v' \in V(H)$ such that $\phi(v') = v$ and $(v', w') \in E(H)$. Therefore, v 's pre-image $v' \in V(H)$ must also be an output. $\square$

Lemma 1.1.2. *Each path in a DAG G has a corresponding path of pre-images in a DAG H (resulting from the replication of vertices of G), such that the length of the path of pre-images in H is the same as the length of the path in G .*

Proof. Let $G = (V, E)$ have a clustering (H, ϕ, Σ) . Let $v_1, \dots, v_k$ represent the vertices of some path in G , with length k and endpoints v_1 and v_k , where v_k is an output in G . By 2c of Definition 1.1.1, v_k has a unique pre-image $v'_k \in V(H)$; and from the previous result, we know that this unique pre-image v'_k is also an output. By 2b of Definition 1.1.1, vertex $v'_k \in V(H)$ and edge $[v_{k-1}, \phi(v'_k)] \in E(G)$, implies there exists a unique $v'_{k-1} \in V(H)$ such that $\phi(v'_{k-1}) = v_{k-1}$ and $(v'_{k-1}, v'_k) \in E(H)$. Similarly, vertex $v'_{k-1} \in V(H)$ and edge $[v_{k-2}, \phi(v'_{k-1})] \in E(G)$, implies there exists a unique $v'_{k-2} \in V(H)$ such that $\phi(v'_{k-2}) = v_{k-2}$ and $(v'_{k-2}, v'_{k-1}) \in E(H)$. This argument continues until we reach vertex $v'_2 \in V(H)$ and edge $[v_1, \phi(v'_2)] \in E(G)$, which implies there exists a unique $v'_1 \in V(H)$ such that $\phi(v'_1) = v_1$ and $(v'_1, v'_2) \in E(H)$. Thus, there is a path of length k in H consisting of pre-images $v'_1, \dots, v'_k$. $\square$

1.1.1 Formulation of CA

The precise formulation of CA as stated in [RW93] is as follows:

Given a combinational network $G = (V, E)$ with weight function $w : V \rightarrow \mathbb{R}^+$, weight capacity M , and delay function $\delta : V \mapsto \mathbb{R}^+$:

1. *A clustering $\Gamma = (H, \phi, E)$ of G is feasible if for every cluster $C \in E, W(C)$ is at most M .*

2. The circuit clustering problem (or CA as we call it) is to compute a feasible clustering Γ of G such that the delay of Γ is minimum among all feasible clusterings of G .

1.1.2 An example of CA

In [LLT69], the authors considered the example of a DAG shown in Figure 1.2. They describe a polynomial time algorithm that returns a clustering solution with replicated vertices. In this case, the vertices are unweighted with a zero delay, the cluster capacity is five, the intra-cluster delay $d = 0$, and the inter-cluster delay $D = 1$. Their algorithm does not purport to return a solution with the least amount of replication.

Figure 1.2: An instance of the clustering problem when replication of vertices is allowed.

The clustering of the DAG with replication of vertices shown in Figure 1.3(a) is a solution of the algorithm described in [LLT69]. Observe that 11 extra vertices that are

introduced by the algorithm occur in the clustering solution (i.e., there are duplicates of some vertices in the original DAG).

(a) An optimal clustering with replicated vertices.

Now, notice that the clustering of the DAG with replication shown in Figure 1.3(b) is an optimal clustering solution with fewer replicated vertices than that shown in Figure 1.3(a).

(b) An optimal clustering with fewer replicated vertices.

Note that unbounded replication can be quite expensive. As systems become increasingly more complex, the need for clustering without logic replication is crucial. It follows that CN is an important problem in VLSI design.

1.2 Motivation and related work

Lawler, Levitt, and Turner [LLT69] were the first to present an exact polynomial time algorithm for CA. They also show that in the special case when the undirected underlying graph is a tree, CN is polynomial time solvable. We refer to the model under which the problems were studied in [LLT69] as the “unit delay model” [MBSV91], where the delay $\delta(v) = 0, \forall v \in V$ and the inter-cluster delay $D = 1$. A more general delay model is presented by Murgai et al. [MBSV91], where $\delta(v) \geq 0, \forall v \in V$ and $D \geq 0$. As per [RW93], this extension of the unit delay model is said to be more powerful

and realistic. The algorithm for the general delay model proposed in [MBSV91] does not always return an optimal solution. However, they specify the conditions under which their algorithm returns an optimum.

Cong and Ding [CD94] also consider CA under the “unit delay model”. However, instead of putting a constraint on the cluster capacity, they bound the number of incoming edges that are incident to a cluster (i.e., a “pin constrained variant of the CA problem). Such a variant of CA has applications in field-programmable gate array (FPGA) designs.

Rajaraman and Wong [RW93] consider CA under the more general delay model proposed in [MBSV91] and discuss a two-phase algorithm. The two phases include a labeling phase and a clustering phase, and the algorithm returns an optimal clustering in polynomial time.

Yang and Wong [YW97] expand on the work done in [RW93], and consider both area and pin constraints. Note that [MBSV91] and [RW93] consider area constraints only. In [YW97], they propose an efficient clustering algorithm that achieves an optimal solution under either the area constraint only or the pin constraint only. However, when both constraints are present, the algorithm does not always return the optimal solution. They specify the rare condition under which the algorithm fails to do so.

Cong and Romesis extend the study of CA to multi-level circuit clustering, with application to hierarchical field programmable gate array (FPGA) architecture, where clustering is applied recursively in two stages [CR01]. They show that the multi-level clustering problem for delay minimization with replication is **NP-hard**. They also propose an efficient heuristic for two-level clustering, providing a trade-off between area and delay by controlling the amount of replication.

Goldschmidt and Hochbaum [GH88] studied the multiway partition problem, where the goal is to find a partition of an edge-weighted graph G into some non-empty clusters, such that the total edge weight between the clusters is minimum. This problem remains

NP-hard, even when the input graph is unweighted, and there is no restriction on the cluster capacity. If the number of clusters is fixed (say r), then there is an algorithm that runs in time $O(n^{r^2})$ that solves this restriction exactly. Here n is the number of vertices of G . The case of the multiway partition problem, in which $r = 2$, is frequently encountered in the literature. This case is called the bipartition problem. It is **NP-hard** for k -regular graphs [BCLS87], where $k \geq 3$ is a fixed constant. On the positive side, there is a dynamical programming based algorithm for solving this problem in the case of trees [BJ89, GM88, Mac88].

Mak and Wong [MW96] examine the amount of replication needed for clusterings that reduce the cut size. They focus on the bipartition problem with the goal of finding a minimum-cut that minimizes the size of the replication set. They present an efficient network-flow based algorithm that finds a min-cut partitioning which requires the least amount of replication to separate the gates of the circuit into two subsets. They also show how their algorithm applies to the problem of area-constrained min-cut partitioning with replication.

Since unbounded logic replication can be quite expensive, CN has been a focus of interest. Kagaris [Kag03] considered CN under area constraints and pin constraints, separately. Both area-constrained and pin-constrained problems were shown to be **NP-hard**. Although not explicitly stated, the proof for the area-constrained problem established the **NP-completeness** of the decision version of a restriction of a variant of CN that we studied (cf. main theorem in [DGMS18]). They presented an efficient heuristic that makes use of the clustering algorithm described in [RW93]. Their experimental results showed that the delay is about 1.5 times the optimum (on average) for small inter-cluster delays, but increases with large inter-cluster delays and large cluster capacities. However, they did not establish provable bounds.

[SG11] and [SG14] explore the advantage of evolutionary algorithms aimed at reducing the delay and area in partitioning and floorplanning. In turn, this would reduce

the wirelength. A hybrid of the evolutionary algorithms is used to find optimal solutions to VLSI physical design problems.

In [MWK15], the authors present an algorithm for simultaneous multilayer interconnect spacing. While satisfying maximum delay constraints, their unique algorithm guarantees to minimize the total dynamic power dissipation caused by an interconnect.

In [SHY⁺10], adjustable delay buffers (ADBs) are used to minimize clock skew under different power modes. The ADBs have delays which can be tuned or adjusted. When the positions of some fixed number of ADBs are assumed to be predetermined, the authors propose a linear-time optimal algorithm. This algorithm assigns the values of the ADBs to minimize clock skew among all possible ADB assignments. In this case, there is a possibility of latency penalty. They also propose a modified algorithm to find an optimal solution with no latency penalty. Additionally, they give an efficient heuristic for finding good ADB positions.

Similar to [SHY⁺10], the author of [Kao15] studies the use of ADBs to minimize clock skew under different power modes. To generate zero clock skew in a given clock tree, they start by assigning ADB positions. If the number of ADBs assigned do not meet the constraints of the previous solution, they use a bottom-up approach for removing ADBs to minimize clock skew while satisfying all constraints.

[SSA10] examines the methods used to solve bi-criterion VLSI circuit partitioning problems. The authors present a hybrid genetic algorithm (GA) which employs the Taguchi method for local search. They test their hybrid algorithm with a variety of benchmarks circuits and found it superior in comparison to the standard GA and tabu search algorithms reported in the literature.

A routability-driven clustering technique for area and power reduction in clustered FPGAs is presented in [SPMS02]. This technique uses a cell connectivity metric to identify seeds for efficient clustering. Effective seed selection, coupled with an interconnect-resource aware clustering and placement, can have a remarkable impact

on circuit routability. It leads to better device utilization, reduction in power consumption and savings in area [SPMS02]. Additionally, the routing area is reduced by 35%. The authors also show that their clustering technique can reduce the overall device power usage by an average of 13%.

In [ML14], effective circuit partitioning techniques are employed by using clustering algorithms. The technique presented in [ML14] uses the circuit netlist to cluster the circuit in partitioning steps. It also minimizes the interconnection distance with the required iteration level. For the standard benchmark circuits the well-known clustering algorithms like *K*-Mean, *Y*-Mean, *K*-Medoid are performed. The results obtained in [ML14] show that the proposed techniques improve the delay. They also minimize the area by reducing the interconnection distance.

Aggarwal and Vitter [AV88] studied the complexity of sorting and other related problems under a two-level input/output (I/O) model (see Figure), where internal memory is limited, and there exists no bound (conceptually) on external storage. They obtained tight upper and lower bounds on the I/O communication required between internal memory and external storage for the problems.

For applications that store and process

vast and dynamically changing datasets, such communication can dramatically hinder performance [Vit98, Arg01]. Therefore, a great deal of attention is given to algorithms that seek to minimize the number of disk accesses used to solve a given problem [VV96, Vit01, MZ02, Vit06, AB10].

Asahiro et al. [AFIM06], established the computational complexity for a variant of

Figure 1.3: The I/O model [AB10].

the clustering problem proposed by Diwan et al. [DRSS96]. They called the variant MINIMUM BLOCK TRANSFERS WITH B (MBT(B)) and considered the I/O model introduced in [AV88]. MBT(B) is similar to an unweighted variant of CN that we studied. We are given a DAG and a block size B (i.e., a cluster capacity), the goal is to find a mapping of the nodes into blocks of size at most B that minimizes the number of external arcs (i.e., inter-cluster edges) along any path from source to sink. They establish **NP-hardness** of MBT(B) and obtain some bounds on inapproximability and approximability [AFIM06, AMY18].

The rest of this dissertation is organized as follows: The disjoint clustering problems that we study are formally described in Chapter 2. We provide some computational complexity results in Chapter 3. We also show that our hardness results imply inapproximability below a certain threshold for several variants of CN. In Chapter 4 and 5, we propose some approximation and exact exponential algorithms, respectively. We conclude the dissertation with Chapter 6, by summarizing our main results and identifying avenues for future work.

Chapter 2

Statement of Problems

In this chapter, we formally describe the problems studied in this dissertation. We start with graph preliminaries. Next, we formulate the main problem using the language of combinatorial circuits. Finally, we represent such circuits as directed acyclic graphs and formulate the main problem using graph-theoretic terminology.

2.1 Graph preliminaries

In this section, we define the main graph-theoretic concepts that are used in this dissertation.

Graphs considered in this dissertation do not contain loops or parallel edges. The *degree* of a vertex v of an undirected graph G is the number of edges of G that are incident with v . The *maximum degree* of G is denoted by $\Delta(G)$ or simply Δ when G is known from the context.

A *directed path* (or, just a *path*) of a directed graph G is a sequence $Q = v_0 e_1 v_1 \dots e_l v_l$, where $v_0, v_1, \dots, v_l$ are vertices of G , $e_1, \dots, e_l$ are edges (also called *arcs*) of G , and $e_j = (v_{j-1}, v_j)$, $1 \leq j \leq l$. We call l the *length* of the path Q , and sometimes we say that Q is an *l -path* of G . If $v_0 = v_l$, then Q is called a *directed cycle*

(or, just *cycle*). G is said to be a *directed acyclic graph (DAG)*, if it contains no directed cycles. For further terminology on graphs and directed graphs, one may consult [Wes01, BJG10].

A *cluster* is an arbitrary subset of the vertices of a DAG, and it does not have to be strongly connected. If C is a cluster in a DAG G , then an edge is said to be a *cut-edge* if it connects a vertex of C to a vertex from $V(G) \setminus C$. The *degree* of C is the number of cut-edges incident with a vertex in C .

The *indegree* and *outdegree* of a vertex are the number of arcs that enter and

leave the vertex, respectively. A *source* (*sink*, resp.) is a vertex with indegree zero (outdegree zero, resp.). It is well-known that every DAG has a source and a sink [BJG10]. Let $\mathcal{S}$ and $\mathcal{O}$ be the set of sources and sinks of G , respectively. Notice that $\mathcal{S} = \{a, b\}$ and $\mathcal{O} = \{e, f\}$ in the DAG of Figure 2.1; $C_1 = \{a, c, g\}$ and $C_2 = \{b, e, f\}$ represent a pair of disjoint clusters.

Figure 2.1: A DAG representing a combinatorial network with two sources and two sinks.

2.2 Formulation of CN using combinatorial circuits

A combinatorial circuit can be represented as a DAG $G = (V, E)$. In G , each vertex $v \in V$ represents a gate, and each edge $(u, v) \in E$ represents an interconnection between gates u and v . In general, each gate in a circuit has an associated delay [MBSV91]. In the model that we consider in this dissertation, each interconnection has one of the following types of delays: (1) an intra-cluster delay, d , when there is an interconnection between two gates in the same cluster, or (2) an inter-cluster delay, D , when there is an

interconnection between two gates in different clusters.

The delay along a path from an input to an output is the sum of the delays of the gates and interconnections that lie on the path. The delay of the overall circuit is the maximum delay among all source to sink paths in the circuit.

A clustering partitions the circuit into disjoint subsets. A clustering algorithm tries to achieve one or both of the following goals, subject to one or more constraints:

- (1) The delay minimization through the circuit [RW93].
- (2) The minimization of the total number of cut-edges [HG95].

In this dissertation, we study CN under the delay model described as follows:

1. Associated with every gate v of the circuit, there is a delay $\delta(v)$ and a size $w(v)$.
2. The delay of an interconnection between two gates within a single cluster is d .
3. The delay of an interconnection between two gates in different clusters is D , where $D \gg d$.

The size of a cluster is the sum of the sizes of the gates in the cluster. The precise formulation of CN is as follows:

Given a combinatorial circuit, with each gate having a size and a delay, maximum degree Δ , intra- and inter-cluster delays d and D , respectively, and a positive integer M called cluster capacity, the goal is to partition the circuit into clusters such that

- 1. The size of each cluster is bounded by M ,*
- 2. The delay of the circuit is minimized.*

2.3 Graph-theoretic formulation of CN

In the rest of the dissertation, we focus on a graph-theoretic formulation of CN. Given a clustering of a combinatorial circuit represented as a DAG $G = (V, E)$, the delays on the interconnections between gates induce an *edge-delay function* $\delta : E \rightarrow \{d, D\}$ of G . The *weight* of a cluster is the sum of the weights of the vertices in the cluster. The *delay-length* of a directed path $P = v_0 e_1 v_1 \dots e_l v_l$ of G is $\sum_{i=0}^l \delta(v_i) + \sum_{i=1}^l \delta(e_i)$, where $\delta(e_i)$ is equal to d if v_{i-1} and v_i are inside the same cluster, or D , otherwise.

We also employ the following notations and concepts. The symbol X in our 3-tuple notation can be either W , which means that the vertices are weighted, or N , which means that the vertices are unweighted, the symbol M is the cluster capacity, and Δ is the maximum number of arcs entering or leaving any vertex of the DAG (i.e., the maximum degree Δ of the underlying undirected graph of the DAG).

$\text{CN}\langle X, M, \Delta \rangle$ is formulated (graph-theoretically) as follows: *Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \mathbb{N}$, delay function $\delta : V \rightarrow \mathbb{N}$, maximum degree Δ , constants d and D , and a cluster capacity M , the goal is to partition V into clusters such that*

1. *The weight of each cluster is bounded by M ,*
2. *The maximum delay-length of any path from a source to a sink of G is minimized.*

A clustering of G , such that the weight of each cluster is bounded by M , is called *feasible*. Given a feasible clustering of G , one can consider the corresponding edge-length function $\delta : E \rightarrow \{d, D\}$ of G . A clustering of G is *optimal* if the maximum delay-length of any path from a source to a sink is the minimum among all clusterings.

In Figure 2.2, we consider a simple example of disjoint clustering in a combinatorial circuit represented by a DAG. In this example, the delays and weights of all vertices

Figure 2.2: An example of a DAG and clustering of the DAG.

are equal to 0 and 1, respectively (i.e., $\delta(v) = 0$ and $w(v) = 1$ for all vertices v in the DAG), the cluster capacity is $M = 2$, the intra-cluster delay is $d = 0$; and, the inter-cluster delay is $D = 1$. It can be easily seen that the partition $\Sigma = \{\{s, a\}, \{b, e\}, \{c, t\}\}$ forms a feasible clustering such that the maximum delay-length of any path from s to t is 2. Moreover, we can quickly check to see that this clustering is optimal.

In this dissertation, we focus on a restriction of $CN\langle X, M, \Delta \rangle$, when $\delta(v) = 0$ for every vertex v of G . Figure 2.3 shows a lattice of some of the variants of CN that we study. Observe that each node of the lattice is a variant of $CN\langle X, M, \Delta \rangle$, and each child in the lattice is a restriction of some parent(s).

Figure 2.3: Some cases of the delay minimization problem that we investigate in this dissertation.

Chapter 3

Computational Complexity

In this chapter, we establish the computational complexities of several variants of CN.

3.1 Computational complexities of several variants of CN

In this section, we obtain the main results that deal with the computational complexities of several variants of CN. We prove theorems that establish the **NP-hardness** of some variants. Moreover, our reductions imply that some variants of CN are inapproximable within a certain factor.

For many of our results, we consider CN_{dec} , which is formulated as follows: *Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \mathbb{N}$, delay function $\delta : V \rightarrow \mathbb{N}$, maximum degree Δ , constants d and D , cluster capacity M , and a positive integer k , decide whether we can partition V into clusters such that*

1. *The weight of each cluster is bounded by M ,*
2. *The maximum delay-length of any path from a source to a sink of G is at most k .*

It is not hard to see that CN_{dec} is the decision version of $CN\langle W, M, \Delta \rangle$. We make this correspondence explicit by writing CN_{dec} as $CN_{dec}\langle W, M, \Delta \rangle$. We use the same notation for restrictions of $CN\langle W, M, \Delta \rangle$. If A is a subset of positive integers, then $CN_{dec}\langle A, M, \Delta \rangle$ denotes the restriction of $CN_{dec}\langle W, M, \Delta \rangle$, when the weights of vertices of the input DAG are from A .

Our first theorem establishes the **NP-completeness** of $CN_{dec}\langle W, M, \Delta \rangle$. Clearly, this means that $CN\langle W, M, \Delta \rangle$ is **NP-hard**.

Theorem 3.1.1. $CN_{dec}\langle W, M, \Delta \rangle$ is **NP-complete**.

Proof. We recall $CN_{dec}\langle W, M, \Delta \rangle$ as follows: Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \mathbb{N}$, $\delta(v) = 0 \forall v \in V$, maximum degree Δ , constants d and D , cluster capacity M , and a positive integer k , decide whether we can partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is at most k .

It is clear that $CN_{dec}\langle W, M, \Delta \rangle$ is in **NP** since it follows from the well-known fact that finding a maximum weighted path in an edge-weighted DAG is polynomial time solvable [CLRS09].

In order to establish **NP-hardness** of $CN_{dec}\langle W, M, \Delta \rangle$, we present a reduction from PARTITION. For that purpose, we recall PARTITION as follows: Given a set $A = \{a_1, a_2, \dots, a_n\}$, the goal is to check whether there is a set $A_1 \subset A$, such that $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A \setminus A_1} a_i$, where $i \in \{1, 2, \dots, n\}$. Without loss of generality, we assume that $\sum_{a_i \in A} a_i = B$ is even, otherwise the problem is trivial.

We now construct an instance I' of $CN_{dec}\langle W, M, \Delta \rangle$ as shown in Figure 3.1. There is a source s connected to a sink t through n vertices labeled u_1 through u_n . Let U denote the set of all vertices u_i ($1 \leq i \leq n$). The vertices in U are pairwise nonadjacent. Each vertex $u_i \in U$ has a weight a_i , and both s and t have weight $\frac{B}{2}$. We set $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity is set to B , and we set $k = D$. The description of I' is complete.

Figure 3.1: Reduction from PARTITION to $\text{CN}_{dec}\langle W, M, \Delta \rangle$.

Observe that I' can be constructed from an instance I of PARTITION in polynomial time. In order to complete the proof of the theorem, we show that I is a “yes” instance of PARTITION if and only if I' is a “yes” instance of $\text{CN}_{dec}\langle W, M, \Delta \rangle$.

Assume that I is a “yes” instance of PARTITION. This means that there exists a partition of A into A_1 and $A \setminus A_1$ such that $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A \setminus A_1} a_i = \frac{B}{2}$. Group the vertices corresponding to the elements in A_1 with s , and the remaining vertices with t . Observe that the cluster capacity constraint is met. Moreover, the maximum delay-length of any path from s to t is D . This means that I' is a “yes” instance of $\text{CN}_{dec}\langle W, M, \Delta \rangle$.

For the proof of the converse statement, assume that I' is a “yes” instance of $\text{CN}_{dec}\langle W, M, \Delta \rangle$. This means that there is a way of partitioning the vertices of G into clusters such that the maximum delay-length of any path from s to t is at most D . We observe that every vertex must be packed with either s or t , otherwise the maximum delay-length must equal $2 \cdot D > k$. Let U_s and U_t denote the subsets of the vertices in U that are packed with s and t , respectively. Let $w(s)$ and $w(t)$ be the weights of vertices s and t , respectively. Let $w(U_s)$ and $w(U_t)$ be the sums of the weights of the vertices in U that are packed with s and t , respectively. Clearly,

$$w(s) + w(U_s) + w(t) + w(U_t) = 2 \cdot B.$$

Since

$$w(s) + w(U_s) \leq B \text{ and } w(t) + w(U_t) \leq B,$$

we have

$$w(U_s) \leq \frac{B}{2} \text{ and } w(U_t) \leq \frac{B}{2}.$$

This implies that

$$w(U_s) = \frac{B}{2} \text{ and } w(U_t) = \frac{B}{2}.$$

Thus, we have obtained the desired partition of A . Hence, I is a “yes” instance of PARTITION.

□

The proof of Theorem 3.1.1 implies **NP-hardness** of $CN\langle W, M, \Delta \rangle$, even for planar networks and therefore, strengthens the result in [Kag03]. Moreover, we obtain the following inapproximability result.

Corollary 3.1.1. *$CN\langle W, M, \Delta \rangle$ does not admit a $(2 - \varepsilon)$ -approximation algorithm for each $\varepsilon > 0$, unless $\mathbf{P}=\mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(2 - \varepsilon)$ -approximation algorithm for $CN_{dec}\langle W, M, \Delta \rangle$. We construct a polynomial time algorithm for PARTITION as follows:

Algorithm 1: A polynomial time algorithm for PARTITION

input : An instance I of PARTITION.**output:** “Yes” or “No”.

- 1 Construct an instance I of PARTITION, we construct an instance G of $CN_{dec}\langle W, M, \Delta \rangle$ using the reduction described in the proof of Theorem 3.1.1.
 - 2 Run the $(2 - \varepsilon)$ -approximation algorithm for $CN_{dec}\langle W, M, \Delta \rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(2 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of PARTITION, then $OPT \leq D$. Moreover, the maximum delay of any clustering solution of G returned by the $(2 - \varepsilon)$ -approximation algorithm for $CN_{dec}\langle W, M, \Delta \rangle$ is at most $(2 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $2 \cdot D$. Thus, the $(2 - \varepsilon)$ -approximation algorithm solves the instance I of PARTITION exactly. □

The next theorem serves to strengthen Theorem 3.1.1.

Theorem 3.1.2. $CN_{dec}\langle W, M, 3 \rangle$ is **NP-complete**.

Figure 3.2: Reduction from PARTITION to $CN_{dec}\langle W, M, 3 \rangle$.

Proof. We recall $CN_{dec}\langle W, M, 3 \rangle$ as follows: Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \mathbb{N}$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 3$, constants d and D , cluster capacity M , and a positive integer k , decide whether we can partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is at most k .

It is clear that $CN_{dec}\langle W, M, 3 \rangle$ is in **NP** since it follows from the well-known fact that finding a maximum weighted path in an edge-weighted DAG is polynomial time solvable [CLRS09].

In order to establish **NP-hardness** of $CN_{dec}\langle W, M, 3 \rangle$, we present a reduction from PARTITION. For that purpose, we recall PARTITION as follows: Given a set $A = \{a_1, a_2, \dots, a_n\}$, the goal is to check whether there is a set $A_1 \subset A$, such that $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A \setminus A_1} a_i$, where $i \in \{1, 2, \dots, n\}$. Without loss of generality, we assume that $\sum_{a_i \in A} a_i = B$ is even, otherwise the problem is trivial.

We now construct an instance I' of $CN_{dec}\langle W, M, 3 \rangle$ as shown in Figure 3.2. Let U

denote the set of all vertices u_i ($1 \leq i \leq n$). The vertices in U are pairwise nonadjacent. Each vertex $u_i \in U$ belongs to a distinct path that connects the source s to the sink t . Let S denote the set of all vertices that are predecessors of the vertices in U . Let T denote the set of all vertices that are successors of the vertices in U . Note that in the underlying undirected graph, the subgraphs induced by S and T are isomorphic. Let m denote the size of S and T . Each vertex $u_i \in U$ has a weight of a_i . Every vertex in S and T has weight 1. We set $d = 0$ and let D be any positive integer. Every vertex is given a delay of 0. The cluster capacity M is set to $(\frac{B}{2} + m)$, and we set $k = D$. The description of I' is complete.

Observe that I' can be constructed from an instance I of PARTITION in polynomial time. In order to complete the proof of the theorem, we show that I is a “yes” instance of PARTITION if and only if I' is a “yes” instance of $\text{CN}_{dec}\langle W, M, 3 \rangle$.

Assume that I is a “yes” instance of PARTITION. This means that there exists a partition of A into A_1 and $A \setminus A_1$ such that $\sum_{a_i \in S_1} a_i = \sum_{a_i \in A \setminus A_1} a_i = \frac{B}{2}$. Group the vertices corresponding to the elements in A_1 with S , and the remaining vertices with T . Observe that the cluster capacity constraint is met. Moreover, the maximum delay-length of any path from the source s to the sink t is D . This means that I' is a “yes” instance of $\text{CN}_{dec}\langle W, M, 3 \rangle$.

Conversely, assume that I' is a “yes” instance of $\text{CN}_{dec}\langle W, M, 3 \rangle$. This means that there is a way of partitioning the vertices of the DAG in Figure 3.2 into clusters, such that the cluster capacity constraint is satisfied, and the maximum delay-length of any path from s to t is at most D . Since S and T have the same underlying structure and $|S| = |T|$, then without loss of generality, we may assume that every vertex in S is clustered together, and every vertex in T is clustered together. Furthermore, each vertex $u_i \in U$ must be clustered with the vertices in either S or T . Otherwise, the delay-length of the path from s to t is strictly greater than D . Let U_S and U_T denote the subsets of the vertices in U that are packed with S and T , respectively. Observe that $U_S \cup U_T = U$.

Notice that the delay-length of any path from s to a vertex in U_S is 0, and the delay-length of any path from a vertex in U_T to t is also 0. Let $w(S)$ and $w(T)$ denote the sum of the weights of all vertices in S and T , respectively. Notice that $w(S) = w(T) = m$. Let $w(U_S)$ and $w(U_T)$ denote the sum of the weights of all vertices in U_S and U_T , respectively. Clearly,

$$w(U_S) + w(U_T) + w(S) + w(T) = B + 2 \cdot m.$$

Since

$$w(U_S) + w(S) \leq \left(\frac{B}{2} + m\right) \text{ and } w(U_T) + w(T) \leq \left(\frac{B}{2} + m\right),$$

then

$$w(U_S) \leq \frac{B}{2} \text{ and } w(U_T) \leq \frac{B}{2}.$$

This implies that

$$w(U_S) = \frac{B}{2} \text{ and } w(U_T) = \frac{B}{2}.$$

Thus, we have obtained the desired partition of A . Hence, I is a “yes” instance of PARTITION.

□

The proof of Theorem 3.1.2 implies an inapproximability result for $\text{CN}\langle W, M, 3 \rangle$.

Corollary 3.1.2. *$\text{CN}\langle W, M, 3 \rangle$ does not admit a $(2 - \varepsilon)$ -approximation algorithm for each $\varepsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(2 - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle W, M, 3 \rangle$. We construct a polynomial time algorithm for PARTITION as follows:

Algorithm 2: A polynomial time algorithm for PARTITION**input** : An instance I of PARTITION.**output**: “Yes” or “No”.

- 1 Construct an instance I of PARTITION, we construct an instance G of $\text{CN}_{dec}\langle W, M, 3 \rangle$ using the reduction described in the proof of Theorem 3.1.2.
- 2 Run the $(2 - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle W, M, 3 \rangle$ to get a clustering Γ for G .
- 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(2 - \varepsilon) \cdot D$.

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of PARTITION, then $OPT \leq D$. Moreover, the maximum delay of any clustering solution of G returned by the $(2 - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle W, M, 3 \rangle$ is at most $(2 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $2 \cdot D$. Thus, the $(2 - \varepsilon)$ -approximation algorithm solves the instance I of PARTITION exactly.

□

The next theorem implies **NP-hardness** of $\text{CN}\langle [3], 3, 3 \rangle$. In the proof, we use a 3SAT reduction modeled after the one presented in [Kag03].

Theorem 3.1.3. $\text{CN}_{dec}\langle [3], 3, 3 \rangle$ is **NP-complete**.

Proof. We recall $\text{CN}_{dec}\langle [3], 3, 3 \rangle$ as follows: Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \{1, 2, 3\}$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 3$, constants d and D , cluster capacity $M = 3$, and a positive integer k , decide whether we can partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is at most k .

It is clear that $\text{CN}_{dec}\langle [3], 3, 3 \rangle$ is in **NP** since it follows from the well-known fact that finding a maximum weighted path in an edge-weighted DAG is polynomial time

solvable [CLRS09].

In order to establish **NP-hardness** of $\text{CN}_{dec}\langle[3], 3, 3\rangle$, we present a reduction from 3SAT. For that purpose, we recall 3SAT as follows: Given a 3-CNF formula ϕ with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$, the goal is to check whether ϕ has a satisfying assignment. Without loss of generality, for all $i \in \{1, \dots, n\}$ we assume that each variable x_i in ϕ appears at most three times and each literal at most twice. (Any 3SAT instance can be transformed to satisfy these properties in polynomial time [Pap94].)

Figure 3.3: Gadgets used to represent variables and clauses.

Let each variable x_i ($1 \leq i \leq n$) be represented by a variable gadget as shown in Figure 3.3(a). Let each clause C_j ($1 \leq j \leq m$) be represented by a clause gadget as shown in Figure 3.3(b). If a variable x_i or its complement $\bar{x}_i$ is the 1st, 2nd, or 3rd literal of a clause C_j , then the corresponding vertex labeled x_i (or $\bar{x}_i$) is connected to a sink labeled C_j through a pair of vertices labeled $\{y_{j1}, z_{j1}\}$, $\{y_{j2}, z_{j2}\}$, or $\{y_{j3}, z_{j3}\}$, respectively. A simple example of the construction of an instance I' is shown in Figure 3.4, where $\phi = C_1 \wedge C_2$, with $C_1 = (x_1, \bar{x}_2, x_3)$ and $C_2 = (x_2, x_3, x_4)$.

We now construct an instance I' of $\text{CN}_{dec}\langle[3], 3, 3\rangle$ as shown in Figure 3.5. The resulting DAG G represents a combinatorial circuit. Let U denote the set of all vertices labeled x_i or $\bar{x}_i$ ($1 \leq i \leq n$). There are n sources labeled T_i ($1 \leq i \leq n$) and m sinks labeled C_j ($1 \leq j \leq m$). They are connected through some vertices in U and $3 \cdot m$ pairs of vertices labeled $\{y_{jp}, z_{jp}\}$ ($1 \leq j \leq m, 1 \leq p \leq 3$). Each y_{jp} is connected to exactly

Figure 3.4: Simple example

one variable gadget. For every j , no two vertices in the set $\{y_{j1}, y_{j2}, y_{j3}\}$ are adjacent to both x_i and $\bar{x}_i$ of the same variable gadget. In other words, x_i and $\bar{x}_i$ cannot both be connected to the same clause gadget. Every T_i , z_{jp} , and C_j has a weight of 1, every $x_i, \bar{x}_i \in U$ has a weight of 2, and every y_{jp} has a weight of 3. We set $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity M is set to 3, and we set $k = 3 \cdot D$. The description of I' is complete.

Observe that I' can be constructed from I in polynomial time. In order to complete the proof of the theorem, we show that I is a “yes” instance of 3SAT if and only if I' is a “yes” instance of $\text{CN}_{dec}([3], 3, 3)$.

Suppose that I is a “yes” instance of 3SAT. This means that there exists an assignment of ϕ such that every clause has at least one **true** literal. If a literal x_i is set to **true**, then the corresponding vertex x_i (or $\bar{x}_i$) is clustered with T_i . However, if a literal x_i is set to **false**, then the corresponding vertex x_i is clustered alone. Since $M = 3$, every y_{jp} must be clustered alone. Since each clause C_j has at least one **true** literal, the vertex z_{jp} along the path of the vertex x_i (or $\bar{x}_i$) corresponding to that **true** literal is clustered alone. The resulting delay-length of the corresponding source to sink path is $3 \cdot D$. It

Figure 3.5: Reduction from the 3SAT problem to $\text{CN}_{dec}\langle[3], 3, 3\rangle$. The edges connecting variable gadgets to clause gadgets belong in the area with the shaded cloud.

is safe to cluster the remaining two z_{jp} vertices with C_j , even if they both belong to paths corresponding to **true** literals. In this case, the resulting paths have delay-length $2 \cdot D < 3 \cdot D = k$. However, if either one of these two z_{jp} vertices belongs to a path corresponding to a **false** literal, then it must be clustered with C_j to avoid exceeding the bound on the delay-length. Observe that the cluster capacity constraint is satisfied, and the maximum delay-length of any path from a source T_i to a sink C_j is $3 \cdot D$. This means that I' is a “yes” instance of $\text{CN}_{dec}\langle[3], 3, 3\rangle$.

Conversely, suppose that I' is a “yes” instance of $\text{CN}_{dec}\langle[3], 3, 3\rangle$. This means that there is a way of partitioning the vertices of G into clusters of capacity $M = 3$, such that the maximum delay-length of any path from a source to a sink is at most $3 \cdot D$.

Since $M = 3$, again notice that every y_{jp} must be clustered alone. Each sink C_j may be clustered with at most two vertices. This means that at least one z_{jp} is clustered alone. Consider the vertex x_i (or $\bar{x}_i$) along a path corresponding to a z_{jp} clustered alone. Since any source to sink path with a z_{jp} clustered alone has a delay-length of at least

$3 \cdot D$, then T_i must be clustered with vertex x_i (or $\bar{x}_i$). Otherwise, the delay-length of the path would be $4 \cdot D > 3 \cdot D = k$. Furthermore, since the cluster capacity is satisfied, either x_i or $\bar{x}_i$ (but not both) can be clustered with T_i . Take each literal that corresponds to a vertex x_i clustered with T_i , and set its value to **true**. Now, notice that any z_{jp} along a path in which x_i (or $\bar{x}_i$) is clustered alone, must be clustered with the sink C_j . Otherwise, the delay-length of the path would be $4 \cdot D > 3 \cdot D = k$. Take each literal that corresponds to a vertex x_i not clustered with T_i and set its value to **false**. Notice that at least one **true** literal appears in every clause. Thus, a satisfying clustering for G yields a satisfying assignment for ϕ . Hence, I is a “yes” instance of 3SAT.

□

The proof of Theorem 3.1.3 implies an inapproximability result for $\text{CN}\langle[3], 3, 3\rangle$.

Corollary 3.1.3. *$\text{CN}\langle[3], 3, 3\rangle$ does not admit a $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for any $\varepsilon > 0$, unless $\mathbf{P}=\mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[3], 3, 3\rangle$. We construct a polynomial time algorithm for 3SAT as follows:

Algorithm 3: A polynomial time algorithm for 3SAT

input : An instance I of 3SAT.

output: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}_{dec}\langle[3], 3, 3\rangle$ using the reduction described in the proof of Theorem 3.1.3.
 - 2 Run the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[3], 3, 3\rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(4 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of 3SAT, then $OPT \leq 3 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $CN_{dec}\langle[3], 3, 3\rangle$ is at most $(4 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $4 \cdot D$. Thus, the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm solves the instance I of 3SAT exactly.

□

The next theorem implies **NP-hardness** of $CN\langle[2], 2, 4\rangle$ – a restriction of $CN\langle W, 2, \Delta\rangle$.

Theorem 3.1.4. $CN_{dec}\langle[2], 2, 4\rangle$ is **NP-complete**.

Proof. We recall $CN_{dec}\langle[2], 2, 4\rangle$ as follows: Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \{1, 2\}$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 4$, constants d and D , cluster capacity $M = 2$, and a positive integer k , decide whether we can partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is at most k .

It is clear that $CN_{dec}\langle[2], 2, 4\rangle$ is in **NP** since it follows from the well-known fact that finding a maximum weighted path in an edge-weighted DAG is polynomial time solvable [CLRS09].

In order to establish **NP-hardness** of $CN_{dec}\langle[2], 2, 4\rangle$, we present a reduction from 3-BOUNDED POSITIVE 1-IN-3SAT (3-BP 1-IN-3SAT). For that purpose, we recall 3-BP 1-IN-3SAT as follows: Given a 3-CNF formula ϕ with n positive variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$, such that each variable appears in at most three clauses, the goal is to check whether ϕ has a satisfying assignment such that every clause of ϕ has exactly one **true** literal [DF09].

Let each variable x_i ($1 \leq i \leq n$) be represented by a variable gadget as shown in Figure 3.6(a). Let each clause C_j ($1 \leq j \leq m$) be represented by a clause gadget as

shown in Figure 3.6(b). If a variable x_i is the 1st, 2nd, or 3rd literal of a clause C_j , then the corresponding vertex labeled x_i is connected to a sink labeled C_j through a pair of vertices labeled $\{y_{j1}, z_{j1}\}$, $\{y_{j2}, z_{j2}\}$, or $\{y_{j3}, z_{j3}\}$, respectively.

Figure 3.6: Gadgets used to represent variables and clauses.

Figure 3.7: Reduction from 3-BP 1-IN-3SAT to $\text{CN}_{dec}([2], 2, 4)$. The edges connecting variable gadgets to clause gadgets belong in the area with the shaded cloud.

We now construct an instance I' of $\text{CN}_{dec}([2], 2, 4)$ as shown in Figure 3.7. The resulting DAG G represents a combinatorial circuit. Let U denote the set of all vertices labeled x_i or $\bar{x}_i$ ($1 \leq i \leq n$). There are n sources labeled F_i ($1 \leq i \leq n$) and m sinks labeled C_j ($1 \leq j \leq m$). They are connected through some vertices in U and $3 \cdot m$ pairs

of vertices labeled $\{y_{jp}, z_{jp}\}$ ($1 \leq j \leq m, 1 \leq p \leq 3$). Each y_{jp} is connected to exactly one variable gadget. Every $x_i, \bar{x}_i \in U$, every F_i, z_{jp} , and C_j has a weight of 1. Every y_{jp} has a weight of 2. We set $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity M is set to 2, and we set $k = 3 \cdot D$. The description of I' is complete.

Observe that I' can be constructed from I in polynomial time. In order to complete the proof of the theorem, we show that I is a “yes” instance of 3-BP 1-IN-3SAT if and only if I' is a “yes” instance of $\text{CN}_{dec}\langle[2], 2, 4\rangle$.

Suppose that I is a “yes” instance of 3-BP 1-IN-3SAT. This means that there exists an assignment of ϕ such that every clause has exactly one **true** literal. If a literal x_i is set to **true**, then the corresponding vertex x_i is clustered alone. However, if a literal x_i is set to **false**, then the corresponding vertex is clustered with F_i . Since $M = 2$, every y_{jp} must be clustered alone. Since each clause C_j has exactly one **true** literal, the vertex z_{jp} along the path of the vertex x_i corresponding to that **true** literal is clustered with C_j . The resulting delay-length of the corresponding source to sink path is $3 \cdot D$. The other two vertices belonging to the same clause gadget are clustered alone. Observe that the cluster capacity constraint is satisfied, and the maximum delay-length of any path from a source F_i to a sink C_j is $3 \cdot D$. This means that I' is a “yes” instance of $\text{CN}_{dec}\langle[2], 2, 4\rangle$.

Conversely, suppose that I' is a “yes” instance of $\text{CN}_{dec}\langle[2], 2, 4\rangle$. This means that there is a way of partitioning the vertices of G into clusters of capacity $M = 2$, and the maximum delay-length of any path from a source to a sink is at most $3 \cdot D$.

Since $M = 2$, again notice that every y_{jp} must be clustered alone. Each sink C_j is clustered with at most one vertex, so the remaining two z_{jp} vertices are clustered alone. Consider a vertex x_i along a path corresponding to a z_{jp} that is clustered alone. Since such a source to sink path has a delay-length of at least $3 \cdot D$, then the source F_i must be clustered with vertex x_i . Otherwise, the delay-length of the path would be $4 \cdot D > 3 \cdot D = k$. Furthermore, since the cluster capacity is satisfied, F_i can be clustered

with either x_i or $\bar{x}_i$ (but not both). Take each literal that corresponds to a vertex x_i clustered with F_i and set its value to **false**. Take each literal x_i that corresponds to a vertex x_i clustered alone and set its value to **true**. Notice that any z_{jp} along a path in which vertex x_i is clustered alone must be clustered with the sink C_j . Otherwise, the delay-length of the path would be $4 \cdot D > 3 \cdot D = k$. Observe that exactly one **true** literal appears in every clause. Thus, a satisfying clustering for G yields a satisfying assignment for ϕ . Hence, I is a “yes” instance of 3-BP 1-IN-3SAT.

□

The proof of Theorem 3.1.4 implies an inapproximability result for $\text{CN}\langle[2], 2, 4\rangle$.

Corollary 3.1.4. *$\text{CN}\langle[2], 2, 4\rangle$ does not admit a $(\frac{4}{3} - \epsilon)$ -approximation algorithm for each $\epsilon > 0$, unless $\mathbf{P}=\mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(\frac{4}{3} - \epsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[2], 2, 4\rangle$. We construct a polynomial time algorithm for 3-BP 1-IN-3SAT as follows:

Algorithm 4: A polynomial time algorithm for 3-BP 1-IN-3SAT

input : An instance I of 3-BP 1-IN-3SAT.

output: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}_{dec}\langle[2], 2, 4\rangle$ using the reduction described in the proof of Theorem 3.1.4.
 - 2 Run the $(\frac{4}{3} - \epsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[2], 2, 4\rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(4 - \epsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of 3-BP 1-IN-3SAT, then $OPT \leq 3 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{4}{3} - \epsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[2], 2, 4\rangle$ is at

most $(4 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $4 \cdot D$. Thus, the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm solves the instance I of 3-BP 1-IN-3SAT exactly.

□

The next theorem implies **NP-hardness** of $\text{CN}\langle[2], 2, 4\rangle$ – a restriction of $\text{CN}\langle W, 2, 3\rangle$.

Theorem 3.1.5. $\text{CN}_{dec}\langle[2], 2, 3\rangle$ is **NP-complete**.

Proof. It is clear that $\text{CN}_{dec}\langle[2], 2, 3\rangle$ is in **NP**. This follows from the well-known fact that a maximum weighted path in an edge-weighted DAG can be found in polynomial time [CLRS09].

In order to establish **NP-hardness** of $\text{CN}_{dec}\langle[2], 2, 3\rangle$, we reduce from 1-IN-3SAT and recall the problem as follows:

1-IN-3SAT: Given a 3-CNF formula ϕ with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$, the goal is to check whether ϕ has a satisfying assignment such that every clause of ϕ has exactly one *true* literal. Without loss of generality, for all $i \in \{1, \dots, n\}$ we assume that each variable x_i in ϕ appears at most 3 times and each literal at most twice. (Any 3SAT instance can be transformed to satisfy these properties in polynomial time [Pap94].)

Let each variable x_i ($1 \leq i \leq n$), be represented by a variable gadget as shown in Figure 3.8(a). Let each clause C_j ($1 \leq j \leq m$), be represented by a clause gadget as shown in Figure 3.8(b). If a variable x_i or its complement $\bar{x}_i$ is the 1st, 2nd, or 3rd literal of a clause C_j , then the corresponding vertex labeled x_i (or $\bar{x}_i$) is connected to a sink labeled C_j by traversing the edge (y_{j1}, z_{j1}) , (y_{j2}, z_{j2}) , or (y_{j3}, z_{j3}) , respectively.

We now construct an instance I' of $\text{CN}_{dec}\langle[2], 2, 3\rangle$ as shown in Figure 3.9. The resulting DAG G represents a combinatorial circuit. Let U be the set of all vertices labeled x_i or $\bar{x}_i$ ($1 \leq i \leq n$). There are n sources S_i connected to m sinks C_j ($1 \leq j \leq m$)

Figure 3.8: Gadgets used to represent variables and clauses.

through some vertices in U and $3 \cdot m$ pairs of vertices labeled y_{jp} and z_{jp} ($1 \leq j \leq m, 1 \leq p \leq 3$). Notice that every y_{jp} is connected to exactly one vertex gadget. For each i , we have that x_i and $\bar{x}_i$ cannot both be connected to the same clause gadget. We let every vertex in U , and each S_i , z_{jp} , and C_j have a weight of 1. We let each y_{jp} have a weight of 2. We set $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity M is set to 2, and we let $k = 3 \cdot D$. The description of I' is complete.

Figure 3.9: Reduction from 1-IN-3SAT to $\text{CN}_{\text{dec}}([2], 2, 3)$. The edges connecting variable gadgets to clause gadgets belong in the shaded area.

Observe that I' can be constructed from I in polynomial time. In order to complete

the proof of the theorem, we show that I is a “yes” instance of 1-IN-3SAT, if and only if I' is a “yes” instance of $\text{CN}_{dec}\langle[2], 2, 3\rangle$.

Suppose that I is a “yes” instance of 1-IN-3SAT. This means that there exists an assignment of ϕ such that every clause has exactly one *true* literal. If a literal is set to *true*, then the corresponding vertex (x_i or $\bar{x}_i$) should be clustered alone. But, if a literal is set to *false*, then the corresponding vertex should be clustered with S_i . Since $M = 2$, every y_{jp} must be clustered alone. Each clause C_j has exactly one *true* literal, so the vertex z_{jp} along the path of the vertex (x_i or $\bar{x}_i$) corresponding to that literal should be clustered with the sink C_j . Every z_{jp} not clustered with C_j must be clustered alone. Observe that the cluster capacity constraint is met, and the maximum delay-length of any path from a source S_i to a sink C_j is $3 \cdot D$. This means that I' is a “yes” instance of $\text{CN}_{dec}\langle[2], 2, 3\rangle$.

Conversely, suppose that I' is a “yes” instance of $\text{CN}_{dec}\langle[2], 2, 3\rangle$. This means that there is a way of packing the vertices of G into clusters of capacity $M = 2$, such that the delay-length of any path from a source to a sink is at most $3 \cdot D$.

Since $M = 2$, again notice that every y_{jp} must be clustered alone. Each sink C_j is clustered with at most one z_{jp} . So, the remaining two z_{jp} vertices are clustered alone. Consider a vertex x_i (or $\bar{x}_i$) along a path corresponding to a z_{jp} that is clustered alone. Since such a source to sink path has delay-length at least $3 \cdot D$, then the source S_i must be clustered with vertex x_i (or $\bar{x}_i$). Otherwise, the delay-length of the path would be $4 \cdot D > 3 \cdot D = k$. Furthermore, since the cluster capacity constraint is satisfied, S_i can be clustered with either x_i or $\bar{x}_i$ (but not both). Take each literal that corresponds to a vertex x_i (or $\bar{x}_i$) that is clustered with S_i and set its value to *false*. Take each literal that corresponds to a vertex x_i (or $\bar{x}_i$) that is clustered alone and set its value to *true*. Finally, notice that any z_{jp} along a path in which vertex x_i (or $\bar{x}_i$) is clustered alone must be clustered with the sink C_j . Otherwise, the delay-length of the path would be $4 \cdot D > 3 \cdot D = k$. Observe that exactly one *true* literal appears in every clause. Thus,

a satisfying clustering for G yields a satisfying assignment for ϕ . Hence, I is a “yes” instance of 1-IN-3SAT.

□

The proof of Theorem 3.1.5 implies an inapproximability result for $\text{CN}\langle[2], 2, 3\rangle$.

Corollary 3.1.5. $\text{CN}\langle[2], 2, 3\rangle$ does not admit a $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for each $\varepsilon > 0$, unless $\mathbf{P}=\mathbf{NP}$.

Proof. By way of contradiction, suppose there exists a $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[2], 2, 3\rangle$. We construct a polynomial time algorithm for 1-IN-3SAT as follows:

Algorithm 5: A polynomial time algorithm for 1-IN-3SAT

input : An instance I of 1-IN-3SAT.

output: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}_{dec}\langle[2], 2, 3\rangle$ using the reduction described in the proof of Theorem 3.1.5.
 - 2 Run the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[2], 2, 3\rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(4 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of 1-IN-3SAT, then $OPT \leq 3 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}_{dec}\langle[2], 2, 3\rangle$ is at most $(4 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $4 \cdot D$. Thus, the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm solves the instance I of 1-IN-3SAT exactly.

□

In [Kag03], CN is considered under area constraints and pin constraints, separately. The decision version of the area-constrained problem is formulated as follows: Given a directed acyclic graph $G(V, E)$ representing a combinatorial circuit, a delay $\delta(v)$ and area $\alpha(v)$ for each $v \in V$, an inter-cluster delay constant $D \geq 0$, a cluster area bound M , and a maximum delay bound B , determine whether there exists a clustering with no replication so that in each cluster C , $\sum_{v \in C} \alpha(v) \leq M$, and for any path $P = (p_1, p_2, \dots, p_n)$ from a primary input to a primary output, $\sum_{i=1}^n \delta(p_i) + k \cdot D \leq B$, where $k = |\{(p_i, p_{i+1}) : (p_i, p_{i+1}) \in P \wedge (p_i, p_{i+1}) \text{ appear in different clusters}\}|$. The decision version of the pin-constrained problem has an analogous formulation. However, the area of each cluster C is not restricted, while the total number of I/O pins of each cluster must not exceed a given constant Q .

The next results establish **NP-hardness** and inapproximability of $\text{CN}\langle[4], 5, \Delta\rangle$.

Theorem 3.1.6. *$\text{CN}\langle[4], 5, \Delta\rangle$ is **NP-hard**.*

Proof. We recall $\text{CN}\langle[4], 5, \Delta\rangle$ as follows: Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \{1, 2, 3, 4\}$, $\delta(v) = 0 \forall v \in V$, maximum degree Δ , constants d and D , and cluster capacity $M = 5$, the goal is to partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is minimized.

To show that $\text{CN}\langle[4], 5, \Delta\rangle$ is **NP-hard**, we reduce from 3SAT (cf. Theorem 2.1 in [Kag03]). For that purpose, we recall 3SAT as follows: Given a 3-CNF formula ϕ with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$, the goal is to check whether ϕ has a satisfying assignment.

Let each variable x_i ($1 \leq i \leq n$) be represented by a variable gadget as shown in Figure 3.10(a). Let each clause C_j ($1 \leq j \leq m$) be represented by a clause gadget as shown in Figure 3.10(b). If a variable x_i or its complement $\bar{x}_i$ is the p th literal of a clause C_j , where $p \in \{1, 2, 3\}$, then we add edges (x_i, z_{jp}) or $(\bar{x}_i, z_{jp})$, respectively. The resulting DAG G represents a combinatorial circuit. Let U denote the set of all

Figure 3.10: Gadgets used to represent variables and clauses.

vertices labeled x_i or $\bar{x}_i$ ($1 \leq i \leq n$). There are n sources labeled T_i ($1 \leq i \leq n$) and m sinks labeled C_j ($1 \leq j \leq m$). They are connected through some vertices in U and $3 \cdot m$ vertices labeled z_{jp} ($1 \leq j \leq m, 1 \leq p \leq 3$). Each z_{jp} is connected to exactly one variable gadget. For every j , no two vertices in the set $\{z_{j1}, z_{j2}, z_{j3}\}$ are adjacent to both x_i and $\bar{x}_i$ of the same variable gadget. In other words, x_i and $\bar{x}_i$ cannot both be connected to the same clause gadget. Every T_i and C_j has a weight of 1, every $x_i, \bar{x}_i \in U$ has a weight of 4, and every z_{jp} has a weight of 2. Let $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity M is set to 5, and set $k = 2 \cdot D$. It is shown that an instance I of 3SAT is a “yes” instance if and only if an instance I' of $\text{CN}\langle[4], 5, \Delta\rangle$ is a “yes” instance. $\square$

Corollary 3.1.6. $\text{CN}\langle[4], 5, \Delta\rangle$ does not admit a $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for any $\varepsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$.

Proof. By way of contradiction, suppose there exists a $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle[4], 5, \Delta\rangle$. We construct a polynomial time algorithm for 3SAT as follows:

Algorithm 6: A polynomial time algorithm for 3SAT

input : An instance I of 3SAT.**output**: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}\langle[4], 5, \Delta\rangle$ using the reduction described in the proof of Theorem 3.1.6.
 - 2 Run the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle[4], 5, \Delta\rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(3 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of 3SAT, then $OPT \leq 2 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle[4], 5, \Delta\rangle$ is at most $(3 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $3 \cdot D$. Thus, the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm solves the instance I of 3SAT exactly.

□

We further observe that if we reduce from the variant of 3SAT such that each variable occurs at most three times and each literal occurs at most twice in the Boolean formula, then we obtain the next result.

Theorem 3.1.7. $\text{CN}\langle[4], 5, 3\rangle$ is **NP-hard**.

Proof. We recall $\text{CN}\langle[4], 5, 3\rangle$ as follows: Given a DAG $G = (V, E)$, with vertex-weight function $w : V \rightarrow \{1, 2, 3, 4\}$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 3$, constants d and D , and cluster capacity $M = 5$, the goal is to partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is minimized.

To show that $\text{CN}\langle[4], 5, 3\rangle$ is **NP-hard**, we reduce (in polynomial time) from a variant of 3SAT (cf. Theorem 2.1 in [Kag03]). For that purpose, we recall the variant of

3SAT as follows:

$3\text{SAT}_{\leq 3, \leq 2}$: Given a 3-CNF formula ϕ with n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$, such that each variable occurs at most three times and each literal occurs at most twice, the goal is to check whether ϕ has a satisfying assignment. Note that the requirement that each clause has exactly three literals is relaxed in this restriction of 3SAT. Any variable, say x_i , with q occurrences (for some $q > 3$) can be replaced with q new variables $w_1, \dots, w_q$. The clauses $(\bar{w}_1 \vee w_2) \wedge (\bar{w}_2 \vee w_3) \wedge (\bar{w}_q \vee w_1)$ can then be added to ϕ to ensure that the q new variables retain the truth assignment of the original variable x_i [Pap94].

Figure 3.11: Gadgets used to represent variables and clauses.

Given an instance I of $3\text{SAT}_{\leq 3, \leq 2}$, we construct an instance I' of $\text{CN}([4], 5, 3)$. Let each variable x_i ($1 \leq i \leq n$) be represented by a variable gadget as shown in Figure 3.11(a). Let each clause C_j ($1 \leq j \leq m$) be represented by a clause gadget as shown in Figure 3.11(b). If a variable x_i or its complement $\bar{x}_i$ is the p th literal of a clause C_j , where $p \in \{1, 2, 3\}$, then we add edges (x_i, z_{jp}) or $(\bar{x}_i, z_{jp})$, respectively. The resulting DAG G represents a combinatorial circuit. Let U denote the set of all vertices labeled x_i or $\bar{x}_i$ ($1 \leq i \leq n$). There are n sources labeled T_i ($1 \leq i \leq n$) and m sinks labeled C_j ($1 \leq j \leq m$). They are connected through some vertices in U and $3 \cdot m$ vertices labeled z_{jp} ($1 \leq j \leq m, 1 \leq p \leq 3$). Each z_{jp} is connected to exactly one variable gadget. For every j , no two vertices in the set $\{z_{j1}, z_{j2}, z_{j3}\}$ are adjacent to both x_i and $\bar{x}_i$ of the same variable gadget. In other words, x_i and $\bar{x}_i$ cannot both be connected to the same clause gadget. Every T_i and C_j has a weight of 1, every $x_i, \bar{x}_i \in U$ has a weight of 4, and

every z_{jp} has a weight of 2. We set $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity M is set to 5, and we set $k = 2 \cdot D$. The description of I' is complete.

Observe that an instance I' of $\text{CN}\langle[4], 5, 3\rangle$, constructed from any instance I of $3\text{SAT}_{\leq 3, \leq 2}$, has maximum degree three. We can show (as in [Kag03]), that I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$ if and only if I' is a “yes” instance of $\text{CN}\langle[4], 5, 3\rangle$, as desired. $\square$

Corollary 3.1.7. *$\text{CN}\langle[4], 5, 3\rangle$ does not admit a $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for any $\varepsilon > 0$, unless $\mathbf{P}=\mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle[4], 5, 3\rangle$. We construct a polynomial time algorithm for $3\text{SAT}_{\leq 3, \leq 2}$ as follows:

Algorithm 7: A polynomial time algorithm for $3\text{SAT}_{\leq 3, \leq 2}$

input : An instance I of $3\text{SAT}_{\leq 3, \leq 2}$.

output: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}\langle[4], 5, 3\rangle$ using the reduction described in the proof of Theorem 3.1.7.
 - 2 Run the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle[4], 5, 3\rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(3 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$, then $OPT \leq 2 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle[4], 5, 3\rangle$ is at most $(3 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $3 \cdot D$. Thus, the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm solves the instance I of $3\text{SAT}_{\leq 3, \leq 2}$ exactly.

□

In [AFIM06], $\text{MBT}(B)$ is shown to be **NP-hard**. They formulate $\text{MBT}(B)$ as follows: Given a DAG $G = (V, A)$ and a block size B , the goal is to find a packing $\mathcal{P}$ that minimizes the block transfer $bt_{\mathcal{P}}(G)$ such that the size of every block $P_i \in \mathcal{P}$ is bounded by B . Furthermore, they obtain a $(\frac{3}{2} - \varepsilon)$ -inapproximability bound and a $(2 - \varepsilon)$ -approximability bound for $\text{MBT}(2)$, where $\varepsilon = \frac{2}{h}$ when height $h \geq 4$ is even and $\varepsilon = \frac{2}{h+1}$ when $h \geq 3$ is odd.

The next results establish **NP-hardness** and inapproximability of $\text{CN}\langle N, 2, 5 \rangle$.

Theorem 3.1.8. *$\text{CN}\langle N, 2, 5 \rangle$ is **NP-hard**.*

Proof. We recall $\text{CN}\langle N, 2, 5 \rangle$ as follows: Given a DAG $G = (V, E)$, with $w(v) = 1 \forall v \in V$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 5$, constants d and D , and cluster capacity $M = 2$, the goal is to partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is minimized.

To show that $\text{CN}\langle N, 2, 5 \rangle$ is **NP-hard**, we construct an instance I' of $\text{CN}\langle N, 2, 5 \rangle$ from an instance I of 3SAT as described in the proof of Theorem 5 in [AFIM06]. The theorem follows from the fact that I' is a “yes” instance if and only if I is a “yes” instance.

□

Corollary 3.1.8. *$\text{CN}\langle N, 2, 5 \rangle$ does not admit a $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for each $\varepsilon > 0$, unless $\mathbf{P}=\mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 5 \rangle$. We construct a polynomial time algorithm for 3SAT as follows:

Algorithm 8: A polynomial time algorithm for 3SAT

input : An instance I of 3SAT.**output**: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}\langle N, 2, 5 \rangle$ using the reduction described in the proof of Theorem 5 in [AFIM06].
 - 2 Run the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 5 \rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(3 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of 3SAT, then $OPT \leq 2 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 5 \rangle$ is at most $(3 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $3 \cdot D$. Thus, the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm solves the instance I of 3SAT exactly.

□

We further observe that if we reduce from the variant of 3SAT such that each variable occurs at most three times and each literal occurs at most twice, then we obtain the next result.

Theorem 3.1.9. $\text{CN}\langle N, 2, 4 \rangle$ is **NP-hard**.

Proof. We recall $\text{CN}\langle N, 2, 4 \rangle$ as follows: Given a DAG $G = (V, E)$, with $w(v) = 1 \forall v \in V$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 4$, constants d and D , and cluster capacity $M = 2$, the goal is to partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is minimized.

Consider the proof of Theorem 5 in [AFIM06]. To show that $\text{CN}\langle N, 2, 4 \rangle$ is **NP-hard**, we reduce (in polynomial time) from a variant of 3SAT, instead. For that purpose,

we recall the variant of 3SAT as follows:

$3\text{SAT}_{\leq 3, \leq 2}$: Given a 3-CNF formula ϕ with n variables $u_1, \dots, u_n$ and m clauses $C_1, \dots, C_m$, such that each variable occurs at most three times and each literal occurs at most twice, the goal is to check whether ϕ has a satisfying assignment. Note that the requirement that each clause has exactly three literals is relaxed in this restriction of 3SAT. Any variable, say u_i , with q occurrences (for some $q > 3$) can be replaced with q new variables $w_1, \dots, w_q$. The clauses $(\bar{w}_1 \vee w_2) \wedge (\bar{w}_2 \vee w_3) \wedge (\bar{w}_q \vee w_1)$ can then be added to ϕ to ensure that the q new variables retain the truth assignment of the original variable u_i [Pap94].

Given an instance I of $3\text{SAT}_{\leq 3, \leq 2}$, we construct an instance I' of $\text{CN}\langle N, 2, 4 \rangle$ using the reduction described in the proof of Theorem 5 in [AFIM06]. Notice that every instance I' of $\text{CN}\langle N, 2, 4 \rangle$ has maximum degree four. It is easy to see based on the proof in [AFIM06], that I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$ if and only if I' is a “yes” instance of $\text{CN}\langle N, 2, 4 \rangle$, as desired. $\square$

Corollary 3.1.9. *$\text{CN}\langle N, 2, 4 \rangle$ does not admit a $(\frac{3}{2} - \epsilon)$ -approximation algorithm for each $\epsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$.*

Proof. By way of contradiction, suppose there exists a $(\frac{3}{2} - \epsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 4 \rangle$. We construct a polynomial time algorithm for $3\text{SAT}_{\leq 3, \leq 2}$ as follows:

Algorithm 9: A polynomial time algorithm for $3\text{SAT}_{\leq 3, \leq 2}$

input : An instance I of $3\text{SAT}_{\leq 3, \leq 2}$.

output: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}\langle N, 2, 4 \rangle$ using the reduction described in the proof of Theorem 5 in [AFIM06].
 - 2 Run the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 4 \rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(3 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$, then $OPT \leq 2 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 4 \rangle$ is at most $(3 - \varepsilon) \cdot D$. Otherwise, the maximum delay of any clustering solution of G must be at least $3 \cdot D$. Thus, the $(\frac{3}{2} - \varepsilon)$ -approximation algorithm solves the instance I of $3\text{SAT}_{\leq 3, \leq 2}$ exactly.

□

For the purpose of our next reduction, we state the following problems:

CUBIC MONOTONE 1-IN-3SAT (CM 1-IN-3SAT): Given a 3-CNF formula ϕ with n positive variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$, such that each variable appears in exactly three clauses, the goal is to check whether ϕ has a satisfying assignment such that every clause of ϕ has exactly one *true* literal. Note that even when restricted to planar graphs, CM 1-IN-3SAT is **NP-hard** [MR01].

$\text{CN}_{dec}\langle N, 2, 4 \rangle$: Given a DAG $G = (V, E)$, with $w(v) = 1 \ \forall v \in V$, $\delta(v) = 0 \ \forall v \in V$, maximum degree $\Delta = 4$, constants d and D , cluster capacity $M = 2$, and a positive integer k , decide whether we can partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is at most k .

By further constraining the clustering of every sink, we consider the following modification of $CN_{dec}\langle N, 2, 4 \rangle$:

$CN'_{dec}\langle N, 2, 4 \rangle$: Given a DAG $G = (V, E)$, with $w(v) = 1 \forall v \in V$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 4$, constants d and D , cluster capacity $M = 2$, and a positive integer k , decide whether we can partition V into clusters such that the weight of each cluster is bounded by M , every sink is clustered with one immediate predecessor, their other immediate predecessors are clustered alone, and the maximum delay-length of any path from a source to a sink of G is at most k .

Theorem 3.1.10. $CN'_{dec}\langle N, 2, 4 \rangle$ is **NP-complete**.

Proof. It is clear that $CN'_{dec}\langle N, 2, 4 \rangle$ is in **NP**. This follows from the well-known fact that a maximum weighted path in an edge-weighted DAG can be found in polynomial time [CLRS09]. Moreover, we can easily verify that the other constraints are met.

In order to establish **NP-hardness** of $CN'_{dec}\langle N, 2, 4 \rangle$, we present a reduction from CM 1-IN-3SAT. Let each variable x_i ($1 \leq i \leq n$) be represented by a variable gadget as shown in Figure 3.12(a). Let each clause C_j ($1 \leq j \leq m$) be represented by a clause gadget as shown in Figure 3.12(b). Consider the clause gadget's underlying undirected graph, and let Y_{j1}, Y_{j2}, Y_{j3} ($1 \leq j \leq m$) denote the three 4-cycles, respectively. If a variable x_i is the 1st, 2nd, or 3rd literal of a clause C_j , then the corresponding vertex labeled x_i is connected to a sink labeled C_j through some vertices, including the pair $\{y_{j1}, z_{j1}\}$, $\{y_{j2}, z_{j2}\}$, or $\{y_{j3}, z_{j3}\}$, respectively.

We now construct an instance I' of $CN'_{dec}\langle N, 2, 4 \rangle$ as shown in Figure 3.13. The resulting DAG G represents a combinatorial circuit. Let U denote the set of all vertices labeled x_i ($1 \leq i \leq n$). There are n sources labeled S_i and m sinks labeled C_j ($1 \leq j \leq m$). They are connected through some vertices in U and $3 \cdot m$ sets of vertices including $\{y_{jp}, w_{jp}, z_{jp}\}$ ($1 \leq j \leq m, 1 \leq p \leq 3$). Each y_{jp} is connected to exactly one variable gadget. Every vertex has a weight of 1. We set $d = 0$ and let $D = 1$. All vertices are

Figure 3.12: Gadgets used to represent variables and clauses.

given a delay of 0. The cluster capacity M is set to 2, and we set $k = 4$. The description of I' is complete.

Observe that I' can be constructed from I in polynomial time. In order to complete the proof of the theorem, we show that I is a “yes” instance of CM 1-IN-3SAT, if and only if I' is a “yes” instance of $CN'_{dec}\langle N, 2, 4 \rangle$.

Suppose that I is a “yes” instance of CM 1-IN-3SAT. This means that there exists an assignment of ϕ such that every clause has exactly one *true* literal. If a literal x_i is set to *true*, then the corresponding vertex x_i should be clustered alone. However, if a literal x_i is set to *false*, then the corresponding vertex x_i is clustered with S_i . Cluster the vertices of each Y_{jp} so that the intra-cluster edges correspond to a perfect matching. Since each clause C_j has exactly one *true* literal, the vertex z_{jp} along the path corresponding to that *true* literal is clustered with the sink C_j . The resulting delay-length of the corresponding source to sink path is $4 \cdot D$. Each remaining z_{jp} not clustered with C_j is clustered alone. Observe that the cluster capacity constraint is satisfied, every sink is clustered with one immediate predecessor, their other immediate predecessors are clustered alone, and the delay-length of every path from a source S_i to a sink C_j is 4. This means that I' is a “yes” instance of $CN'_{dec}\langle N, 2, 4 \rangle$.

Conversely, suppose that I' is a “yes” instance of $CN'_{dec}\langle N, 2, 4 \rangle$. This means that

Figure 3.13: Reduction from CM 1-IN-3SAT to $CN'_{dec}(N, 2, 4)$. The edges connecting variable gadgets to clause gadgets belong in the area with the shaded cloud.

there is a way of partitioning the vertices of G into clusters of capacity $M = 2$, such every sink is clustered with one immediate predecessor, their other immediate predecessors are clustered alone, and the delay-length of every path from a source S_i to a sink C_j is at most 4.

Each sink C_j is clustered with one vertex, so the remaining two z_{jp} vertices are clustered alone. Observe that any source to sink path in which a z_{jp} is clustered alone, has a delay-length of at least 4. Since the delay-length constraint is satisfied, each S_i must be clustered with the vertex x_i corresponding to a z_{jp} that is clustered alone. Also, the vertices of the set Y_{jp} must be clustered in such a way that the intra-cluster edges form a perfect matching. Take each literal that corresponds to a vertex x_i clustered with a source and set its value to *false*. Now, notice that each z_{jp} along a path in which vertex x_i is clustered alone, must be clustered with the sink C_j . Otherwise, the delay-length of the path would be strictly greater than $4 = k$. Take each literal x_i that corresponds to a vertex x_i clustered alone and set its value to *true*. Notice that exactly one *true*

literal appears in every clause. Thus, a satisfying clustering for G yields a satisfying assignment for ϕ . Hence, I is a “yes” instance of CM 1-IN-3SAT.

□

Theorem 3.1.11. $CN\langle N, 2, 3 \rangle$ is **NP-hard**.

Proof. We recall $CN\langle N, 2, 3 \rangle$ as follows: Given a DAG $G = (V, E)$, with $w(v) = 1 \forall v \in V$, $\delta(v) = 0 \forall v \in V$, maximum degree $\Delta = 3$, constants d and D , cluster capacity $M = 2$, and a positive integer k , the goal is to partition V into clusters such that the weight of each cluster is bounded by M , and the maximum delay-length of any path from a source to a sink of G is minimized.

In order to establish **NP-hardness** of $CN\langle N, 2, 3 \rangle$, we present a reduction from $3SAT_{\leq 3, \leq 2}$. For that purpose, we recall $3SAT_{\leq 3, \leq 2}$ as follows: Given a 3-CNF formula ϕ with n variables $u_1, \dots, u_n$ and m clauses $C_1, \dots, C_m$, such that each variable occurs at most three times and each literal occurs at most twice, the goal is to check whether ϕ has a satisfying assignment. Note that the requirement that each clause has exactly three literals is relaxed in this restriction of 3SAT. Any variable, say u_i , with q occurrences (for some $q > 3$) can be replaced with q new variables $w_1, \dots, w_q$. The clauses $(\bar{w}_1 \vee w_2) \wedge (\bar{w}_2 \vee w_3) \wedge (\bar{w}_q \vee w_1)$ can then be added to ϕ to ensure that the q new variables retain the truth assignment of the original variable u_i [Pap94].

Figure 3.14: Gadgets used to represent variables and clauses.

Given an instance I of $3\text{SAT}_{\leq 3, \leq 2}$, we construct an instance I' of $\text{CN}\langle N, 2, 3 \rangle$. Let each variable u_i ($1 \leq i \leq n$) be represented by a variable gadget as shown in Figure 3.14(a), where the dashed arrows indicate possible successors. Note that since each variable u_i occurs at most three times, then the size of the neighborhood of $\{u_i, \bar{u}_i\}$ is at most three. Let each clause C_j ($1 \leq j \leq m$) be represented by a clause gadget as shown in Figure 3.14(b). A set of edges also connects clause gadgets to variable gadgets. For example, if the p -th literal of clause C_j is the α -th occurrence of some literal u_a , where $p \in \{1, 2, 3\}$, $\alpha \in \{1, 2\}$ and $a \in \{1, \dots, n\}$, then we add edge $(x_{jp}, u'_{a\alpha})$. Every vertex has a weight of 1. We set $d = 0$ and let D be any positive integer. All vertices are given a delay of 0. The cluster capacity M is set to 2, and we set $k = 3 \cdot D$. The description of I' is complete.

Observe that I' can be constructed from I in polynomial time. To complete the proof of the theorem, we show that I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$ if and only if I' is a “yes” instance of $\text{CN}\langle N, 2, 3 \rangle$.

Suppose that I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$. This means that there exists an assignment of ϕ such that every clause has at least one *true* literal. If literal u_i (or $\bar{u}_i$) is set to *true*, then we cluster the vertices as follows:

1. t_i is clustered with v_i and f_i is clustered with $\bar{v}_i$ (or t_i is clustered with $\bar{v}_i$ and f_i is clustered with v_i).
2. For each $r \in \{1, 2\}$, $\bar{u}_{ir}$ is clustered with $\bar{u}'_{ir}$ (or u_{ir} is clustered with u'_{ir}).
3. If the r -th occurrence of literal u_i (or $\bar{u}_i$) is the p -th literal of clause C_j , then u'_{ir} (or $\bar{u}'_{ir}$) is clustered with clause gadget vertex x_{jp} , where $p \in \{1, 2, 3\}$.
4. For any p -th literal of clause C_j that is set to *true*, then x_{jp} is clustered with its successor.
5. The successors of the variable gadget vertex f_i , say V_{f_i} , are clustered in such a

way that the edges of the underlying undirected graph of $G[V_{f_i}]$ form a perfect matching.

6. The clause gadget vertex c_{j5} and its predecessors, say $V_{c_{j5}}$, are clustered in such a way that the edges of the underlying undirected graph of $G[V_{c_{j5}} \cup c_{j5}]$ form a perfect matching.
7. All other vertices are clustered alone.

Observe that the cluster capacity constraint is satisfied, and the maximum delay-length of any path from a source to a sink is $3 \cdot D$. This means that I' is a “yes” instance of $\text{CN}\langle N, 2, 3 \rangle$.

Conversely, suppose that I' is a “yes” instance of $\text{CN}\langle N, 2, 3 \rangle$. This means that there is a way of partitioning the vertices of G into clusters of capacity $M = 2$, such that the delay-length of any path from a source to a sink is at most $3 \cdot D$. Observe that under any partitioning, the delay-length of any path from a source to a sink is at least $3 \cdot D$. In any partitioning with delay-length equal to $3 \cdot D$, we have that either t_i is clustered with v_i or t_i is clustered with $\bar{v}_i$, for every $i \in \{1, \dots, n\}$. Furthermore, in any partitioning with delay-length equal to $3 \cdot D$, there is at least one x_{jp} that must be clustered with its successor. If t_i is clustered with v_i , then for each $r \in \{1, 2\}$, $\bar{u}_{ir}$ must be clustered with $\bar{u}'_{ir}$. Set literal u_i to *true* and consider each u'_{ir} *free*. Otherwise, if t_i is clustered with $\bar{v}_i$, then for each $r \in \{1, 2\}$, u_{ir} must be clustered with u'_{ir} . Set literal $\bar{v}_i$ to *true* and consider each $\bar{u}'_{ir}$ *free*. Since at least one x_{jp} is clustered with its successor, namely some *free* vertex. This means that at least one *true* literal appears in every clause. Thus, a satisfying clustering for G yields a satisfying assignment for ϕ . Hence, I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$. $\square$

Corollary 3.1.10. $\text{CN}\langle N, 2, 3 \rangle$ does not admit a $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for any $\varepsilon > 0$, unless $\mathbf{P} = \mathbf{NP}$.

Proof. By way of contradiction, suppose there exists a $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 3 \rangle$. We construct a polynomial time algorithm for $3\text{SAT}_{\leq 3, \leq 2}$ as follows:

Algorithm 10: A polynomial time algorithm for $3\text{SAT}_{\leq 3, \leq 2}$

input : An instance I of $3\text{SAT}_{\leq 3, \leq 2}$.

output: “Yes” or “No”.

- 1 Construct an instance G of $\text{CN}\langle N, 2, 3 \rangle$ using the reduction described in the proof of Theorem 3.1.11.
 - 2 Run the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 3 \rangle$ to get a clustering Γ for G .
 - 3 **return** “Yes” if and only if the maximum delay induced by Γ is less than or equal to $(4 - \varepsilon) \cdot D$.
-

Let OPT denote the delay of the optimal clustering of G . If I is a “yes” instance of $3\text{SAT}_{\leq 3, \leq 2}$, then $OPT \leq 3 \cdot D$. Moreover, the maximum delay of any clustering solution of G returned by the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm for $\text{CN}\langle N, 2, 3 \rangle$ is at most $(4 - \varepsilon) \cdot D$. Otherwise, the maximum delay-length of any clustering solution of G must be at least $4 \cdot D$. Thus, the $(\frac{4}{3} - \varepsilon)$ -approximation algorithm solves the instance I of $3\text{SAT}_{\leq 3, \leq 2}$ exactly.

□

Fixed-parameter tractability of $\text{CN}\langle W, M, \Delta \rangle$

We have shown that $\text{CN}\langle N, 2, 3 \rangle$ is **NP-hard**. This result has some direct consequences about the non-fixed parameter tractability of the clustering problem when one is minimizing the delay.

Corollary 3.1.11. *The clustering problem $(\text{CN}\langle W, M, \Delta \rangle)$ is not fixed-parameter tractable (FPT) with respect to $M, \Delta, d, D, \max_{z \in V} w(z)$ and $w(V) = \sum_{z \in V} w(z)$ unless $\mathbf{P} = \mathbf{NP}$.*

Proof. We prove only for M . Others can be done similarly. Assume that the problem is FPT with respect to M . This means that the problem is polynomial time solvable when $M = 2$. Since by hardness result the $M = 2$ case is **NP**-hard, we have that **P=NP**. The proof is complete. $\square$

Chapter 4

Approximation algorithms

In this chapter, we propose approximation algorithms for several **NP-hard** variants of $\text{CN}\langle X, M, \Delta \rangle$ and analyze them.

4.1 An integer program for $\text{CN}\langle W, M, \Delta \rangle$

Let w_j be the weight of vertex j . Define x_{ij} to be an integer variable that is set to 1, if vertices i and j are in the same cluster, and 0 otherwise. We present the following integer program (IP):

Packing constraints

$$x_{ii} = 1, \quad \forall i \in V \quad (4.1)$$

$$\sum_{j=1}^n w_j \cdot x_{ij} \leq M, \quad \forall i \in V \quad (4.2)$$

Consistency constraints

$$x_{ij} = x_{ji}, \quad \forall i, j \in V \quad (4.3)$$

$$x_{ik} \geq x_{ij} + x_{jk} - 1, \quad \forall i, j, k \in V \quad (4.4)$$

Condition (4.1) ensures that every vertex is clustered. Condition (4.2) ensures that every cluster has weight at most M . Condition (4.3) ensures that either i and j are in the same cluster or they are in different clusters. Likewise, condition (4.4) ensures that if i and j are in one cluster, and j and k are in one cluster, then i and k must be in the same cluster and all clusters are disjoint.

We now come to the objective function. For any vertex j , let δ_j be the delay at j in a clustering. This delay is completely dependent upon its predecessors. We can write

$$\delta_j = \max_{i:(i,j) \in E} \{ \delta_i + d \cdot x_{ij} + D \cdot (1 - x_{ij}) \}. \quad (4.5)$$

Hence the function to be minimized is δ_t where t is the sink of the circuit.

Observation 1. *The transitivity constraint (i.e., Condition (4.4)) in the IP formulation is necessary in the case even when $M = 2$.*

Consider the feasible clustering in the small example shown in Figure 4.1 where $M = 2$. Notice that in this example, the cluster capacity constraint is satisfied (i.e., $\{s\}, \{c\}, \{d\}, \{a, b\}, \{b, t\}$ are all clusters of capacity ≤ 2).

Also, the symmetric constraint is satisfied (i.e., $x_{ij} = x_{ji}$ for each pair of vertices).

However, observe that vertex a is clustered with vertex b , and vertex b is clustered with vertex t , but vertex a is not clustered with vertex t . Now, with this 3rd constraint, the cluster capacity would be violated, and this would not be a feasible clustering. We need the constraint to return a feasible clustering containing a set of disjoint clusters.

Figure 4.1: Small example to show that the transitivity constraint for the IP is necessary.

IP reduction

Let $\text{IP}_{\text{CN}\langle W, M, \Delta \rangle}$ denote the IP formulation for $\text{CN}\langle W, M, \Delta \rangle$. To prove that $\text{IP}_{\text{CN}\langle W, M, \Delta \rangle}$ is equivalent to $\text{CN}\langle W, M, \Delta \rangle$, we consider their corresponding decision versions. For the decision version $\text{IP}_{\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle}$, the goal is to decide whether or not there exists a feasible solution $\mathbf{x} \in \mathbb{B}^n \times \mathbb{B}^n$ (i.e., some $\mathbf{x} \in \mathbb{B}^n \times \mathbb{B}^n$ that satisfies every constraint). For the decision version $\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle$, the goal is to decide if there exists a feasible clustering Γ (i.e., a disjoint clustering Γ of G in which no cluster $C \in \Gamma$ exceeds the cluster capacity M).

Proposition 4.1.1: $\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle \leq_P \text{IP}_{\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle}$.

Proof. Given any instance $G = (V, E)$ of $\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle$, we construct an instance I' of $\text{IP}_{\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle}$. Without loss of generality, we assume that G has a single sink. Let $|V| = n$ and $|E| = m$. Let each edge (i, j) of the underlying undirected graph of G be represented by variable x_{ij} . Let the weight of each vertex $j \in V$ be represented by a parameter w_j . For every $i \in V$, we construct the inequality $w_1 \cdot x_{i1} + \dots + w_i \cdot x_{ii} + \dots + w_n \cdot x_{in} \leq M$. For every pair $i, j \in V$, we construct the equality $x_{ij} = x_{ji}$. For every triple $i, j, k \in V$, we construct the inequality $x_{ik} \geq x_{ij} + x_{jk} - 1$. For every $i \in V$, we construct the equality $x_{ii} = 1$. The intra- and inter-cluster delays correspond to constants d and D , respectively. We let variable $x_{ij} = 1$ if vertex i is clustered with vertex j , and 0 otherwise. Observe that instance I' of $\text{IP}_{\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle}$ can be constructed in polynomial time. $\square$

Proposition 4.1.2: Let $\mathcal{C}$ be the set of all feasible clusterings of G such that G is an instance of $\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle$. Let $\mathcal{F}$ be the set of all feasible solutions of I such that I is an instance of $\text{IP}_{\text{CN}_{\text{dec}}\langle W, M, \Delta \rangle}$. If there exists a solution $\mathbf{x} \in \mathcal{F}$, then there exists a clustering $\Gamma \in \mathcal{C}$.

Proof. Suppose there exists a solution $\mathbf{x}$ of I such that $\mathbf{x} \in \mathcal{F}$. This means that $\mathbf{x} \in \mathbb{B}^n \times \mathbb{B}^n$, and the constraints (4.2), (4.3), and (4.4) are all satisfied. We find a clustering

Γ of G as follows: For each x_{ij} , such that $x_{ij} = 1$ and $i \neq j$, cluster vertices i and j together. Put the remaining vertices into singleton clusters. Since all constraints of the IP are satisfied, then we have a partition Γ of the vertex set such that for all clusters $C \in \Gamma$, we have that $w(C) \leq M$. This means that Γ is a feasible clustering of G . Hence, $\Gamma \in \mathcal{C}$. $\square$

Proposition 4.1.3: Let $\mathcal{C}$ be the set of all feasible clusterings of G such that G is an instance of $CN_{dec}(W, M, \Delta)$. Let $\mathcal{F}$ be the set of all feasible solutions of I such that I is an instance of $IP_{CN_{dec}(W, M, \Delta)}$. If there exists a clustering $\Gamma \in \mathcal{C}$, then there exists a solution $\mathbf{x} \in \mathcal{F}$.

Proof. Suppose there exists a partition Γ of G such that $\Gamma \in \mathcal{C}$. This means that $w(C) \leq M$ for every cluster $C \in \Gamma$. Since $x_{ij} = 1$ for all vertices j clustered with vertex i , and every vertex is clustered, then $x_{ii} = 1$ for all i . Hence, constraint (4.1) is satisfied. Moreover, in order to meet the cluster capacity constraint, the sum of the weights of each vertex j clustered with i , including the weight of vertex i , must not exceed M . This means that for each i , we satisfy the inequality $w_1 \cdot x_{i1} + \dots + w_i \cdot x_{ii} + \dots + w_n \cdot x_{in} \leq M$. Hence, constraint (4.2) of $IP_{CN_{dec}(W, M, \Delta)}$ is satisfied.

Next, observe the relationship between the clustering of each pair $i, j \in V$. If vertex i is clustered with vertex j , then $x_{ij} = 1 = x_{ji}$. Otherwise, it is clear that $x_{ij} = 0 = x_{ji}$. Hence, constraint (4.3) of $IP_{CN_{dec}(W, M, \Delta)}$ is satisfied.

Finally, notice the relationship between the clustering of any three vertices $i, j, k \in V$. Consider the following cases:

Case 1: If vertex i is clustered with vertex j , and vertex j is clustered with vertex k , then vertex i is also clustered with vertex k . In this case, we have $x_{ij} = 1$, $x_{jk} = 1$, and

$x_{ik} = 1$. Observe that

$$\begin{aligned}
 x_{ik} &\geq x_{ij} + x_{jk} - 1 \\
 \implies x_{ik} &\geq 1 + 1 - 1 \\
 \implies x_{ik} &\geq 1 \\
 1 &\geq 1
 \end{aligned}$$

Case 2: If vertex i is not clustered with vertex j , and vertex j is clustered with vertex k , then vertex i cannot be clustered with vertex k . In this case, we have $x_{ij} = 0$, $x_{jk} = 1$, and $x_{ik} = 0$. Observe that

$$\begin{aligned}
 x_{ik} &\geq x_{ij} + x_{jk} - 1 \\
 \implies x_{ik} &\geq 0 + 1 - 1 \\
 \implies x_{ik} &\geq 0 \\
 0 &\geq 0
 \end{aligned}$$

Case 3: If vertex i is clustered with vertex j , and vertex j is not clustered with vertex k , then vertex i cannot be clustered with vertex k . In this case, we have $x_{ij} = 1$, $x_{jk} = 0$, and $x_{ik} = 0$. Observe that

$$\begin{aligned}
 x_{ik} &\geq x_{ij} + x_{jk} - 1 \\
 \implies x_{ik} &\geq 1 + 0 - 1 \\
 \implies x_{ik} &\geq 0 \\
 0 &\geq 0
 \end{aligned}$$

Case 4: If vertex i is not clustered with vertex j , and vertex j is not clustered with vertex k , then vertex i can either be clustered with vertex k or not. In this case, we have $x_{ij} = 0$,

$x_{jk} = 0$, and $(x_{ik} = 1 \text{ or } x_{ik} = 0)$. Observe that

$$\begin{aligned}
 x_{ik} &\geq x_{ij} + x_{jk} - 1 \\
 \implies x_{ik} &\geq 0 + 0 - 1 \\
 \implies x_{ik} &\geq -1 \\
 0 &\geq -1 \text{ and } 1 \geq -1
 \end{aligned}$$

Hence, constraint (4.4) of $\text{IP}_{\text{CN}_{dec}\langle W, M, \Delta \rangle}$ is satisfied.

Since every constraint of $\text{IP}_{\text{CN}_{dec}\langle W, M, \Delta \rangle}$ is satisfied, then there exists a solution $\mathbf{x} \in \mathcal{F}$.

□

Corollary 4.1.1. *There exists a solution $\mathbf{x} \in \mathcal{F}$ if and only if there exists a clustering $\Gamma \in \mathcal{C}$. In other words, there exists a bijective function $f : \mathcal{F} \rightarrow \mathcal{C}$ such that $f(\mathbf{x}) = \Gamma$.*

4.2 An LP-rounding algorithm for $\text{CN}\langle N, 2, \Delta \rangle$

In this section, we present an LP-rounding algorithm for $\text{CN}\langle N, 2, \Delta \rangle$. We make use of the linear programming relaxation of the IP formulated in Section 4.1.

Let $\text{LP}_{\text{CN}\langle N, 2, \Delta \rangle}$ be the linear programming relaxation obtained from $\text{IP}_{\text{CN}\langle W, M, \Delta \rangle}$ when vertices are unweighted and $M = 2$ (i.e., the problem restricted to $\text{CN}\langle N, 2, \Delta \rangle$) by replacing its 0-1 integrality constraints for x_{ij} with $x_{ij} \in [0, 1]$.

Algorithm 11: An LP rounding algorithm for $CN\langle N, 2, \Delta \rangle$ **input** : A DAG $G = (V, E)$, where $|V| = n$ and $|E| = m$.**output**: A clustering Γ of G .

- 1 Solve $LP_{CN\langle N, 2, \Delta \rangle}$. Let each $\hat{x}_{ij}$ denote the delay on the edge connecting vertices i and j and let $\hat{\delta}_j$ denote the delay at vertex j .
- 2 Let $G' = G$.
- 3 **while** $G' \neq \emptyset$ **do**
 - 4 Consider a source s of G' such that the delay-length of the path from s to the sink t is maximum.
 - 5 Let vertex $v \in N^+(s)$ be such that $\hat{\delta}_v = \min_{j \in N^+(s)} \{\hat{\delta}_j\}$.
 - 6 We round some $\hat{x}_{ij}$ to 0-1 values $\bar{x}_{ij}$ as follows: set $\bar{x}_{sv} = 1$, set $\bar{x}_{sj} = 0 \forall j \in N^+(s) \setminus v$, and set $\bar{x}_{vj} = 0 \forall j \in (N^-(v) \cup N^+(v)) \setminus s$.
 - 7 Let $G' = G'[V \setminus \{s, v\}]$
- 8 Cluster together all vertices i and j such that $\bar{x}_{ij} = 1$, where $i \neq j$. Put the remaining vertices into singleton clusters.
- 9 **return** Γ .

Corollary 4.2.1. Algorithm 11 runs in time $O(T_{LP}(n, m))$, where $T_{LP}(n, m)$ is the running time of the fastest linear programming algorithm for solving $LP_{CN\langle N, 2, \Delta \rangle}$.

Theorem 4.2.1. Algorithm 11 is a 2-approximation algorithm.

Proof. Let Q be a path of G from a source to the sink t with maximum delay-length. Let OPT be the delay of an optimal clustering of G . This means that OPT is the sum of the fractional intra- and inter-cluster delays of the edges along Q . Let ALG be the delay of the clustering of G returned by Algorithm 11. Since the algorithm returns a solution with an integral delay, notice that for each intra-cluster edge $(i, j) \in Q$, the delay is increased by $d \cdot (1 - \hat{x}_{ij})$. Moreover, for each inter-cluster edge $(i, j) \in Q$, the

delay is decreased by $D \cdot (1 - \hat{x}_{ij})$. Hence,

$$\begin{aligned}
\frac{ALG}{OPT} &\leq \frac{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij} + D \cdot (1 - \hat{x}_{ij}) + \sum_{(i,j) \in Q} d \cdot (1 - \hat{x}_{ij}) - D \cdot (1 - \hat{x}_{ij})}{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij} + D \cdot (1 - \hat{x}_{ij})} \\
&= 1 + \frac{\sum_{(i,j) \in Q} d \cdot (1 - \hat{x}_{ij}) - D \cdot (1 - \hat{x}_{ij})}{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij} + D \cdot (1 - \hat{x}_{ij})} \\
&\leq 1 + \frac{\sum_{(i,j) \in Q} d \cdot (1 - \hat{x}_{ij})}{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij} + D \cdot (1 - \hat{x}_{ij})} \\
&\leq 1 + \frac{\sum_{(i,j) \in Q} D \cdot (1 - \hat{x}_{ij})}{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij} + D \cdot (1 - \hat{x}_{ij})} \\
&= 2 - \frac{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij}}{\sum_{(i,j) \in Q} d \cdot \hat{x}_{ij} + D \cdot (1 - \hat{x}_{ij})}
\end{aligned}$$

□

Corollary 4.2.2. *Algorithm 11 has approximation factor strictly less than 2 when the intra-cluster delay $d > 0$.*

4.3 A 3-approximation algorithm for $CN\langle N, 2, \Delta \rangle$

In this section, we present a combinatorial 3-approximation algorithm for $CN\langle N, 2, \Delta \rangle$. Our algorithm makes use of the fact that there is a polynomial-time algorithm for finding a path with a maximum number of edges in DAGs. In each iteration, the algorithm picks a path P with a maximum number of edges. Then it considers the central edge $e = (u, v)$ of P , and puts u and v in the same cluster. After that u and v are removed from G . The algorithm iterates until all edges of the input DAG are exhausted.

Theorem 4.3.1. *Algorithm 12 is a 3-approximation algorithm for $CN\langle N, 2, \Delta \rangle$.*

Proof. For a path P , let $l(P)$ be the length of P (i.e., the number of edges of P). More-

Algorithm 12: A 3-approximation algorithm for $CN\langle N, 2, \Delta \rangle$.

- 1: Input: a DAG G ;
 - 2: Output: a clustering of vertices of G ;
 - 3: Take a longest path P in the DAG G ;
 - 4: Declare the central edge $e_{\lceil \frac{l}{2} \rceil}$ of P as a d -edge, where l denotes the length of P , and $e_1, \dots, e_l$ are the edges of P .
 - 5: The edges adjacent to $e_{\lceil \frac{l}{2} \rceil}$ should be declared as D -edges.
 - 6: Remove edge $e_{\lceil \frac{l}{2} \rceil}$ together with its adjacent edges.
 - 7: Continue this process until all edges of G are exhausted.
-

over, let

$$l = \max_P l(P).$$

So, l denotes the length of a longest path of G .

The following shows a lower bound for OPT , where OPT is the delay of the optimal clustering of G when $M = 2$.

$$OPT \geq \lceil \frac{l(P)}{2} \rceil \cdot d + \lfloor \frac{l(P)}{2} \rfloor \cdot D.$$

Since P represents any path, then the above inequality must also be true for the longest path. Thus,

$$OPT \geq \lceil \frac{l}{2} \rceil \cdot d + \lfloor \frac{l}{2} \rfloor \cdot D.$$

Now, let us estimate ALG , where ALG is the delay of the clustering found by the algorithm. We will consider 3 cases.

Case 1: $l = 1$. Then it can be easily seen that $ALG = OPT$.

Case 2: l is even. Then

$$\begin{aligned}
 ALG &\leq l \cdot D \\
 &\leq 2 \cdot \left(\left\lceil \frac{l}{2} \right\rceil \cdot d + \left\lfloor \frac{l}{2} \right\rfloor \cdot D \right) \\
 &\leq 2 \cdot OPT \\
 &< 3 \cdot OPT.
 \end{aligned}$$

Case 3: l is odd and $l \geq 3$. Then

$$\begin{aligned}
 ALG &\leq l \cdot D \\
 &\leq 3 \cdot \frac{l-1}{2} \cdot D \\
 &= 3 \cdot \left\lfloor \frac{l}{2} \right\rfloor \cdot D \\
 &\leq 3 \cdot \left(\left\lceil \frac{l}{2} \right\rceil \cdot d + \left\lfloor \frac{l}{2} \right\rfloor \cdot D \right) \\
 &\leq 3 \cdot OPT.
 \end{aligned}$$

The proof of the theorem follows. □

Figure 4.2 shows an example of a DAG for which the algorithm achieves an approximation factor of 3.

Figure 4.2: A DAG which obtains a factor 3 approximation.

Observe that in this example, $OPT = 2 \cdot d + D$ (see Figure 4.3(a)) and $ALG = 3 \cdot D$ (see Figure 4.3(b)). Hence, if $d = 0$, we have $ALG = 3 \cdot OPT$.

(a) An optimal clustering

(b) A worst-case clustering by the algorithm

Figure 4.3: An optimal clustering and a worst-case clustering

4.4 An improved 2-approximation algorithm for $CN\langle N, 2, \Delta \rangle$

In this section, we present a 2-approximation algorithm for $CN\langle N, 2, \Delta \rangle$. Our algorithm makes use of the fact that there is a polynomial time algorithm for finding a path with maximum edge-weight in DAGs. The algorithm tries to construct a so-called dominating matching of the input DAG G . We prove that this algorithm has a performance

ratio of 2.

We start with the following:

Definition 4.4.1. A matching in a DAG G is a collection of edges that do not share a vertex. A matching of a DAG G is perfect if any vertex of G is incident to an edge from the matching.

Let $G = (V, A)$ be a DAG. Clearly, any perfect matching of G contains exactly $\frac{|V|}{2}$ edges. Let l be the length of a longest path in G .

Definition 4.4.2. A matching I in G is dominating, if every longest path of G contains more than $\frac{l}{2}$ edges of I .

It is easy to see that if G contains a dominating matching, then l has to be odd.

Lemma 4.4.1. Let $G = (V, A)$ be a DAG and let l be the length of a longest path in G . There is a polynomial algorithm which decides whether G has a dominating matching and finds one if it exists.

Proof. We may assume that G is connected, i.e., its undirected underlying graph is connected. If l is even, then there is no dominating matching. So we may assume that l is odd.

Let $I = \emptyset$. Construct a longest path P in G and add odd edges to I . Construct a set S initially consisting of the edges of P . Note that I is a perfect matching in $G[S]$, the subgraph of G induced by S . This property of I in $G[S]$ is maintained.

Consider an edge a in $A \setminus S$ such that only one end-vertex of a is in S . If there is not such an edge a , remove the edges of $A \setminus S$ from G . Also, remove the resulting isolated vertices of G . If a does not belong to a longest path of G , delete it from G and remove the resulting isolated vertices from G . Otherwise, let Q be a longest path of G passing through a . Add all odd edges of Q to I . If an edge of Q assigned to I is incident to an edge of I in $G[S]$, then we have that G has no dominating matching. Otherwise, add all

edges of Q to S and observe that I is a perfect matching in $G[S]$. Continue for as long as $A \neq S$.

Now we have that $A = S$ and I is a perfect matching in $G[S]$. Since every longest path of G must start from an edge in I and end with an edge of I , the only possibility for a longest path to contain at most $\frac{l}{2}$ edges of I is if it contains two consecutive edges that are not from I . Thus, we do the following. Consider every pair of edges not from I forming a directed path of length 2 and check whether the pair is on any longest path of G . If so, G has no dominating matching. Otherwise, I is a dominating matching.

The above proof is an algorithm which runs in polynomial time. The proof of the lemma is complete. $\square$

Using Lemma 4.4.1, we obtain a 2-approximation algorithm for $CN\langle N, 2, \Delta \rangle$.

Theorem 4.4.1. *The problem $CN\langle N, 2, \Delta \rangle$ admits a 2-approximation algorithm.*

Algorithm 13: A 2-approximation algorithm for $CN\langle N, 2, \Delta \rangle$.

- 1: Input: a DAG G ;
 - 2: Output: a clustering of vertices of G ;
 - 3: Check whether G has a dominating matching.
 - 4: If it does not, return an arbitrary feasible clustering of G (for example, put each vertex in a separate cluster).
 - 5: If G contains a dominating matching I , then for each edge $e = uv \in I$, put u and v in the same cluster, and put the remaining vertices in a separate cluster.
 - 6: Output the resulting clustering of G .
-

Proof. Consider Algorithm 13, which is a generic algorithm for the problem. For a path P , let $l(P)$ be the length of P (i.e., the number of edges of P). Moreover, let l denote the length of a longest path of G .

The following shows a lower bound for OPT , where OPT is the delay of the optimal clustering of G when $M = 2$.

$$OPT \geq \left\lceil \frac{l(P)}{2} \right\rceil \cdot d + \left\lfloor \frac{l(P)}{2} \right\rfloor \cdot D.$$

Since P represents any path, then the above inequality must also be true for the longest path. Thus,

$$OPT \geq \left\lceil \frac{l}{2} \right\rceil \cdot d + \left\lfloor \frac{l}{2} \right\rfloor \cdot D.$$

Now, let us estimate ALG , where ALG is the delay of the clustering found by the algorithm. We consider 2 cases.

Case 1: G has no dominating matching. Then if l is even, we have

$$\begin{aligned} ALG &\leq l \cdot D \\ &\leq 2 \cdot \left(\left\lceil \frac{l}{2} \right\rceil \cdot d + \left\lfloor \frac{l}{2} \right\rfloor \cdot D \right) \\ &\leq 2 \cdot OPT. \end{aligned}$$

On the other hand, if l is odd, then since G has no dominating matching, we have

$$OPT \geq \left\lfloor \frac{l}{2} \right\rfloor \cdot d + \left\lceil \frac{l}{2} \right\rceil \cdot D.$$

Hence

$$\begin{aligned} ALG &\leq l \cdot D \\ &\leq 2 \cdot \left(\left\lfloor \frac{l}{2} \right\rfloor \cdot d + \left\lceil \frac{l}{2} \right\rceil \cdot D \right) \\ &\leq 2 \cdot OPT. \end{aligned}$$

Case 2: G has a dominating matching I . Then since any path of length l has an edge

from I , we have

$$\begin{aligned}
 ALG &\leq d + (l-1) \cdot D \\
 &\leq 2 \cdot \left(\left\lceil \frac{l}{2} \right\rceil \cdot d + \left\lfloor \frac{l}{2} \right\rfloor \cdot D \right) \\
 &\leq 2 \cdot OPT.
 \end{aligned}$$

The proof of the theorem is complete. $\square$

4.5 A simple $(2 + \frac{2}{l-1})$ -approximation algorithm

We present a simple approximation algorithm for $CN\langle N, 2, \Delta \rangle$ as follows:

Algorithm 14: A $(2 + \frac{2}{l-1})$ -approximation algorithm for $CN\langle N, 2, \Delta \rangle$

input : A DAG $G = (V, E)$ with longest source to sink path of length $l \geq 3$,

where $|V| = n$ and $|E| = m$.

output: A clustering Γ of G .

- 1 Find a maximum matching F of G .
 - 2 Cluster together all vertices u and v such that $(u, v) \in F$.
 - 3 Put the remaining vertices in singleton clusters.
 - 4 **return** Γ .
-

Corollary 4.5.1. Algorithm 14 runs in time $O(\sqrt{n} \cdot m)$.

Theorem 4.5.1. Algorithm 14 is a $(2 + \frac{2}{l-1})$ -approximation algorithm.

Proof. Let OPT be the delay of an optimal clustering of G , let ALG be the delay of the clustering found by Algorithm 14, and let l denote the length of a longest source to sink path in G . Observe that when the cluster capacity $M = 2$, then $OPT \geq \lceil l/2 \rceil \cdot d + \lfloor l/2 \rfloor \cdot D$, where d and D are the intra- and inter-cluster delays, respectively. Moreover, $OPT \geq \lfloor l/2 \rfloor \cdot D$ when $d = 0$. Since there are at most l inter-cluster edges in a clustering

of G , then we have

$$\begin{aligned} \frac{ALG}{OPT} &\leq \frac{l \cdot D}{\lfloor l/2 \rfloor \cdot D} \\ &\leq \begin{cases} 2, & \text{if } l \text{ is even} \\ 2 + \frac{2}{l-1}, & \text{if } l \text{ is odd} \end{cases} \end{aligned}$$

□

Corollary 4.5.2. *Let l denote the length of a longest source to sink path in G . If $l \geq 3$ and $0 < d \ll D < (l+1) \cdot d$, where d and D are the intra- and inter-cluster edges, respectively, then Algorithm 14 returns a clustering with a maximum delay-length that is strictly less than twice the optimum.*

Proof. Suppose $l \geq 2$ and $0 < d \ll D < (l+1) \cdot d$. Let OPT be the delay of an optimal clustering of G and let ALG be the delay of the clustering found by Algorithm 14. Observe that when the cluster capacity $M = 2$, then $OPT \geq \lceil l/2 \rceil \cdot d + \lfloor l/2 \rfloor \cdot D$. Since there are at most l inter-cluster edges in a clustering of G , then we have

$$\begin{aligned} \frac{ALG}{OPT} &\leq \frac{l \cdot D}{\lceil l/2 \rceil \cdot d + \lfloor l/2 \rfloor \cdot D} \\ &= \begin{cases} \frac{2 \cdot D}{d + D}, & \text{if } l \text{ is even} \\ \frac{2 \cdot l \cdot D}{(l+1) \cdot d + (l-1) \cdot D}, & \text{if } l \text{ is odd} \end{cases} \\ &= \begin{cases} 2 - 2 \cdot \frac{d}{d + D}, & \text{if } l \text{ is even} \\ 2 - 2 \cdot \frac{(l+1) \cdot d - D}{(l+1) \cdot d + (l-1) \cdot D}, & \text{if } l \text{ is odd} \end{cases} \\ &\leq 2 - 2 \cdot \min \left\{ \frac{d}{d + D}, \frac{(l+1) \cdot d - D}{(l+1) \cdot d + (l-1) \cdot D} \right\} \\ &< 2 \end{aligned}$$

□

4.6 A parameterized approximation algorithm for $\text{CN}\langle W, M, \Delta \rangle$

Here, we present a parameterized approximation algorithm for $\text{CN}\langle W, M, \Delta \rangle$.

Algorithm 15: A parameterized approximation algorithm for $\text{CN}\langle W, M, \Delta \rangle$

input : A DAG $G = (V, E)$, with a weight $w(v)$ for each vertex $v \in V$, a cluster capacity M , and maximum degree Δ .

output: A clustering Γ of G .

- 1 For $q = \Delta \cdot M$, determine if there exists a feasible $(w(C), p, q)$ -partition of the underlying undirected graph of G , where the function $w(C)$ denotes the weight of a cluster C and $p = M$.
 - 2 If a feasible partition exists, let Γ equal the partition. Otherwise, let Γ be a clustering of singleton clusters.
 - 3 **return** Γ .
-

Theorem 4.6.1. $(w(C), p, q)$ -PARTITION can be solved in time $2^{O(q)} \cdot |V|^{O(1)}$.
[CFK⁺15]

Corollary 4.6.1. Algorithm 15 can be solved in time $2^{O(\Delta \cdot M)} \cdot |V|^{O(1)}$.

Theorem 4.6.2. Algorithm 15 is an $(M + M^2)$ -factor parameterized approximation algorithm.

Proof. Let OPT be the delay of an optimal clustering of G and let ALG be the delay of the clustering found by Algorithm 15. Let l denote the length of a longest source to

sink path in G . Observe that

$$OPT \geq \begin{cases} [(l+1)/M] \cdot (M-1) \cdot d + [(l+1)/M-1] \cdot D, & \text{if } M \mid (l+1) \\ \lceil (l+1)/M \rceil \cdot (M-1) \cdot d + \lfloor (l+1)/M \rfloor \cdot D, & \text{if } M \nmid (l+1) \end{cases}$$

where d and D are the intra- and inter-cluster delays, respectively.

Notice that if $l+1 \leq M$, then all vertices on source to sink paths can be clustered together, and we are done. Therefore, we may assume that $(l+1) > M$. Next, we let $d = 0$ and obtain the following:

$$OPT \geq \begin{cases} [(l+1)/M-1] \cdot D, & \text{if } M \mid (l+1) \\ \lfloor (l+1)/M \rfloor \cdot D, & \text{if } M \nmid (l+1) \end{cases}$$

Since there are at most l inter-cluster edges in any clustering of G , then $ALG \leq l \cdot D$.

We now consider the following two cases:

Case 5: If $M \mid (l+1)$, then

$$\frac{ALG}{OPT} \leq \frac{l \cdot D}{[(l+1)/M-1] \cdot D} \quad (4.6)$$

$$= M + M \cdot \left(\frac{M-1}{l+1-M} \right) \quad (4.7)$$

$$\leq M + M \cdot (M-1) \quad (4.8)$$

$$\leq M + M^2 \quad (4.9)$$

The inequality in 4.8 follows from the assumption that $(l+1) > M$, which implies that $l+1-M \geq 1$.

Case 6: If $M \nmid (l+1)$, then $l+1 = M \cdot b + r$ for some positive integer b and an integer

$r \in \{1, \dots, M-1\}$. This means that

$$\begin{aligned}
 \frac{ALG}{OPT} &\leq \frac{l \cdot D}{\lfloor (l+1)/M \rfloor \cdot D} \\
 &= \frac{M \cdot l}{l+1-r} \\
 &= M + M \cdot \left(\frac{r-1}{l+1-r} \right) \\
 &\leq M + M \cdot \left(\frac{M-1}{l+1-M} \right) \\
 &\leq M + M \cdot (M-1) \\
 &\leq M + M^2
 \end{aligned}$$

□

Chapter 5

Exact Exponential Algorithms

In this chapter, we propose exact exponential algorithms for several **NP-hard** variants of $\text{CN}\langle X, M, \Delta \rangle$ and analyze them. We assume that $n = |V|$ and $m = |E|$. We also use a modified big-O notation, as defined in [FK10], that suppresses all polynomially bounded factors. For functions f and g we write $f(n) = O(g(n))$ if $f(n) = O(g(n)\text{poly}(n))$, where $\text{poly}(n)$ is a polynomial.

5.1 Exact exponential algorithms for $\text{CN}\langle W, M, \Delta \rangle$

Since $\text{CN}\langle W, M, \Delta \rangle$ is a partition problem (see [FK10] for details), we can solve this problem by simply generating all possible partition of the vertices of G . As it is stated in [FK10], the number of these partitions is most n^n , which is roughly $2^{O(n \cdot \log n)}$. Thus, we have:

Observation 2. $\text{CN}\langle W, M, \Delta \rangle$ can be solved in time $O^*(2^{n \cdot \log n})$.

When the vertices of G are partitioned, each edge e of G can have a delay d or D . Thus, we have at most 2^m possible assignments of delays to edges of G . If we generate all these 2^m assignments and for each them check in polynomial time whether it corresponds to a feasible clustering and if it is, we compute the optimal delay of G ,

we can find an optimal clustering in time $O^*(2^m)$. Thus, we have:

Observation 3. $CN\langle W, M, \Delta \rangle$ can be solved in time $O^*(2^m)$.

5.2 Exact exponential algorithms for $CN\langle N, 2, \Delta \rangle$

Now, we are going to discuss some ideas that will lead to exact algorithms for some restrictions of $CN\langle W, M, \Delta \rangle$. The running-time of these algorithms will be smaller than those presented by trivial algorithms in Observations 2 and 3. We focus on the case $M = 2$ which we know to be **NP-hard**. For this case, we will say that two edges of G are independent if they are not incident to the same vertex. A matching of G is a set of pairwise independent edges of G . A matching is maximal if it is not a subset of a larger matching.

Proposition 5.2.1: For any instance of $CN\langle W, M, \Delta \rangle$ there exists an optimal clustering, such that the edges of G with delay d form a maximal matching of G .

Proof. Consider an optimal clustering of G . Since $M = 2$, we have that any two edges with delay d are independent. Thus, they form a matching I . Now, if I is not maximal, then there is an edge e , such that $I \cup \{e\}$ is a matching. Put the end-vertices of e to the same cluster. Observe that the resulting clustering is feasible, moreover, its delay does not exceed the delay of the original clustering. Thus, the resulting clustering is again optimal. By continuing this process, we will end-up with a clustering such that the edges of delay d form a maximal matching. The proof is complete. $\square$

In order to describe our first algorithm, for each edge e let $N[e]$ be the closed neighborhood of e , that is, the set of edges adjacent to e together with e .

Proposition 5.2.2: Let e be any edge of the graph G . Then there is an optimal clustering of G , such that the matching of edges with delay d has non-empty intersection with $N[e]$.

Proof. Take any optimal clustering such that the edges with delay d do not intersect $N[e]$. Then we can put the end-vertices of e in one cluster, and e will become a d -edge. Observe that the resulting clustering is feasible, moreover, its delay does not exceed that of the original clustering. Thus, the resulting clustering is optimal. The proof is complete. $\square$

Proposition 5.2.2 prompts the following algorithm for $\text{CN}\langle N, 2, \Delta \rangle$. We can take any maximal matching $I = \{e_1, \dots, e_t\}$, and we can try all possible t -tuples of edges $N[e_1] \times \dots \times N[e_t]$. Since $t \leq \frac{n}{2}$ and for each $j = 1, \dots, t$ $|N[e_j]| \leq 2 \cdot \Delta - 1$, we have that the running-time of this algorithm is bounded by

$$O^*(N[e_1] \times \dots \times N[e_t]) \leq O^*((2 \cdot \Delta - 1)^{\frac{n}{2}}) = O^*((\sqrt{2 \cdot \Delta - 1})^n).$$

The formal description of the algorithm is (see Algorithm 16).

Algorithm 16: An exact algorithm for $\text{CN}\langle N, 2, \Delta \rangle$.

- 1: Take any maximal matching $I = \{e_1, \dots, e_t\}$.
 - 2: Try all t -tuples of edges $N[e_1] \times \dots \times N[e_t]$ as the set of d -edges of the clustering.
 - 3: Output the one for which the delay is smallest.
-

First observe that the maximal matching in step 1 of Algorithm 16 can be constructed in linear time. Also, note that when any vertex of G is of degree 3, the trivial algorithm in Observation 3 runs in time $O^*(2^{\frac{3n}{2}}) = O^*((\sqrt{8})^n)$, while Algorithm 16 runs in time $O^*((\sqrt{2 \cdot 3 - 1})^n) = O^*((\sqrt{5})^n)$.

In order to describe our next algorithm, let us give additional definitions. Two vertices of G are independent, if there is no edge of G , whose end-vertices are these two vertices. An independent set is maximal if it is not a subset of a larger independent set. Our next algorithm makes use of the following result:

Theorem 5.2.1. (page 11, [FK10]) Any graph G contains at most $3^{\frac{n}{3}}$ maximal independent set of vertices. Moreover, these independent sets can be enumerated in time

$O^*(3^{\frac{n}{3}})$.

For each graph G consider its line graph $L(G)$. In $L(G)$ the set of vertices are the edges of G , and two vertices of $L(G)$ are adjacent if they share a vertex in G . Let us take an arbitrary graph G and construct its line graph $L(G)$. By Theorem 5.2.1, it has at most $3^{\frac{|E|}{3}} \leq 3^{\frac{n\Delta}{6}}$ maximal independent sets. Moreover, these maximal independent sets can be enumerated in time $O^*(3^{\frac{n\Delta}{6}})$. Observe that any maximal independent set of $L(G)$ is just a maximal matching of G . Thus, G has at most $3^{\frac{n\Delta}{6}}$ maximal matchings. We try all these matchings as a candidate for the optimal clustering. Thanks to Proposition 5.2.1, at least one of them should be an optimal clustering. Thus, we can solve the clustering problem in time $O^*(3^{\frac{n\Delta}{6}})$.

The formal description of the algorithm is (see Algorithm 17).

Algorithm 17: An exact algorithm for $\text{CN}\langle N, 2, \Delta \rangle$.

- 1: For the graph G consider its line graph $L(G)$.
 - 2: Generate all maximal independent sets of $L(G)$ and try each of them as a clustering in G .
 - 3: Output the one for which the delay is smallest.
-

Note that when any vertex of G is of degree 3, Algorithm 16 runs in time $O^*((\sqrt{2 \cdot 3 - 1})^n) = O^*(\sqrt{5}^n)$, while Algorithm 17 runs in time $O^*(3^{\frac{n\Delta}{6}}) = O^*(\sqrt{3}^n)$.

5.3 An exact exponential algorithm for $\text{CN}\langle W, M, 3 \rangle$

We also considered the following alternative approach: Given a DAG $G = (V, E)$. Let $G' = (V', E')$, where $V' = V$ and $E' = \emptyset$. Start with a clustering Γ of singleton clusters of G' . Add an edge $e \in E$ to E' , such that e belongs to set of edges incident from a source in G . Re-cluster G' to minimize the maximum delay. Set Γ equal to the

re-clustering. While $E' \neq E$, continue to add an edge $e \in E$ to E' , then re-cluster to minimize the maximum delay of G' . Return Γ .

From this approach, we made the following observation.

Observation 4. *After adding the last edge, we may not have an optimal clustering. In the worst case, we may have to re-cluster the original graph, even after exhausting all intermediate re-clusterings. Hence, this approach runs in time $O^*(2^m)$.*

Consider the example of a simple DAG below.

(a) A simple example of a DAG G

(b) A clustering of singleton clusters

Observe that if we first add edge (x, v) , it makes sense to put vertices x and v into a cluster.

(c)

Next, suppose that we add edge (y, w) . Then it makes sense to put vertices y and w into a cluster.

(d)

Now, suppose that we add edge (s, u) . Then it makes sense to put vertices s and u into a cluster.

(e)

Suppose that we add edge (w, t) next. It does not make sense to re-cluster any other vertices, because at least one vertex on the 2-path must be clustered alone. Thus, keeping vertex t in a singleton cluster will suffice at this point.

(f)

Next, if we add edge (u, v) , it does not make sense to re-cluster any other vertices. No matter the clustering at this stage, the current maximum delay-length is minimum.

(g)

Finally, we add edge (v, w) . Observe that the delay-length of the final clustering of G is $3 \cdot D$. But this is not optimal.

(h)

The clustering of G shown below is an example of an optimal clustering.

(i) An optimal clustering of G with max delay-length $2 \cdot D$

The previous approach gave us some insights which lead to the next result.

Corollary 5.3.1. $CN\langle W, M, 3 \rangle$ can be solved exactly in time $O(4^n)$.

Proof. Given an instance G of $CN\langle W, M, 3 \rangle$, we partition V into levels $V_0, \dots, V_l$, where l is the “height” (as described in [AMY18]) (i.e., the longest length of any path from a source to a sink in G). Layer V_0 contains all sources of G , and vertex $v \in V \setminus V_0$ belongs to the layer whose index k represents the length of the longest path from a source to v . Moreover, there exists an edge (u, v) for every vertex $v \in V_k$ ($1 \leq k \leq l$) such that $u \in V_{k-1}$. Observe that V_l is the set of sinks in G such that the length from a source to a sink $t \in V_l$ is l . Note that such a partition can be found in $O(|V| + |E|)$ time.

Let $\delta_{G[V \setminus V_0]}$ be the maximum delay-length that is minimum over all possible $2^{|E(G[V \setminus V_0])|}$ clusterings of $G[V \setminus V_0]$. Then, we have

$$\delta_G = \begin{cases} \delta_{G[V \setminus V_0]} + d & \text{if } \forall u \in V_0, \text{ s.t. } u \text{ belongs to a path of length } l, \\ & \exists! v \in V_1 \text{ s.t. } ((u, v) \in E) \wedge (|N^-(v)| = 1) \wedge \\ & (v \text{ belongs to a cluster } C \text{ s.t. } w(u) + w(C) \leq M) \\ \delta_{G[V \setminus V_0]} + D & \text{otherwise} \end{cases}$$

Observe that the running time is $O(2^{|E(G[V \setminus V_0])|}) \leq O(2^{2 \cdot |V \setminus V_0|}) = O(4^{|V \setminus V_0|}) \leq O(4^n)$, where the first inequality is due to the fact that the outdegree of every vertex in $V \setminus V_0$ is at most two.

□

Chapter 6

Conclusion

In this chapter, we summarize our main results and identify avenues for future work

6.1 Summary of contributions

In this dissertation, we studied the problems of disjoint clustering in combinatorial circuits for delay minimization (CN). We obtained the computational complexities of several variants of CN. We also showed that for some variants, our results imply hardness of approximation within a certain factor. Furthermore, we proposed some approximation and exact exponential algorithms and analyzed them. For some cases, we obtained an approximation factor of strictly less than two. Moreover, for some variants, our exact exponential algorithms beat brute force. Our inapproximability and approximability results for variants of CN that we have studied are tabulated in Table 6.1. In Figure 6.1, we present results for the original lattice of problems (cf. Figure 2.3).

Table 6.1: Approximability and inapproximability results

Problem	Approximability	Inapproximability	Running time
$\text{CN}\langle W, M, \Delta \rangle$	$(M^2 + M)$ -approximable	$2 - \varepsilon$	$2^{O(\Delta \cdot M)} \cdot V ^{O(1)}$
$\text{CN}\langle W, M, 3 \rangle$	$(M^2 + M)$ -approximable	$2 - \varepsilon$	$2^{O(3 \cdot M)} \cdot V ^{O(1)}$
$\text{CN}\langle W, 2, 3 \rangle$	6-approximable	$\frac{4}{3} - \varepsilon$	$2^6 \cdot V ^{O(1)}$
$\text{CN}\langle W, 2, \Delta \rangle$	6-approximable	$\frac{4}{3} - \varepsilon$	$2^{O(\Delta \cdot 2)} \cdot V ^{O(1)}$
$\text{CN}\langle N, M, \Delta \rangle$	$(M^2 + M)$ -approximable	$\frac{3}{2} - \varepsilon$	$2^{O(\Delta \cdot M)} \cdot V ^{O(1)}$
$\text{CN}\langle N, 2, \Delta \rangle$	$(2 + \frac{2}{l-1})$ -approximable	$\frac{3}{2} - \varepsilon$	$O(\sqrt{n} \cdot m)$
$\text{CN}\langle N, M, 3 \rangle$	$(M^2 + M)$ -approximable	$\frac{4}{3} - \varepsilon$	$2^{O(3 \cdot M)} \cdot V ^{O(1)}$
$\text{CN}\langle N, 2, 3 \rangle$	2-approximable	$\frac{4}{3} - \varepsilon$	$O(T_{LP}(n, m))$

Figure 6.1: Results for some variants of the disjoint clustering problem that we investigated.

6.2 Future work

We are interested in further investigating the gap between the approximability and inapproximability of several variants of the problems that we have studied. To do so, we would like to prove inapproximability using different assumptions. In particular, we would work to obtain inapproximability results by assuming the truth of the UNIQUE GAMES CONJECTURE. In this effort, we hope to gain additional insights into the problems that we have studied. In turn, this may lead to some interesting algorithmic results.

6.2.1 Implementation and experimentation

We have designed several algorithms for which we have also provided formal mathematical analyses to measure the quality of the solutions obtained, as well as to measure performance. We want to further this work by implementing our algorithms and obtaining experimental results. For instance, it would be interesting to see, after implementing our algorithms and carrying out several experiments on some benchmark circuits, how our experimental results compare to other algorithms and heuristics (e.g., the DJCL heuristic described in [Kag03]).

6.2.2 Unique Games Conjecture

Throughout our research, we based every inapproximability result on the common assumption that $\mathbf{P} \neq \mathbf{NP}$. However, there are other less traditional assumptions that we could have made. For example, we could have assumed the UNIQUE GAMES CONJECTURE to be true. In future work, we would like to prove inapproximability of several variants of some of the problems that we studied based on other assumptions. For that purpose, we begin this chapter by defining the 2-PROVER-1-ROUND (2P1R) GAME, a constraint satisfaction problem which motivated Khot's study of the UNIQUE GAMES (UG) PROBLEM and lead to the UNIQUE GAMES CONJECTURE (UGC) [Kho02]. The goal of the 2P1R GAME is to find an (approximately) optimal labeling to a given instance

$$\mathcal{U}_{2p1r}(G(V, W, E), [m], [n], \{\pi_e \mid e \in E\}),$$

where $G(V, W, E)$ is a bipartite graph, each vertex $v \in V$ is assigned a label from the set $[m]$, each vertex $w \in W$ is assigned a label from the set $[n]$, and a constraint associated with each edge $e = (v, w) \in E$, is represented by a surjective function $\pi_e : [m] \rightarrow [n]$ with $m \geq n$ [Kho10].

A 2P1R GAME can be thought of as a game between two provers, P_1 and P_2 , and

a probabilistic polynomial time verifier. The verifier randomly chooses an edge $e = (v, w)$, from which vertices v and w are sent for labeling from the sets $[m]$ and $[n]$ (resp.) by provers P_1 and P_2 , respectively. The verifier accepts if and only if the provers' answers satisfy $\pi_e(i) = j$, where i and j correspond to the labels given by P_1 and P_2 , respectively. The probability of acceptance by the verifier is the same value as that of the provers' strategy [Kho10, KS13]. UG (as defined below) arises as a special case of 2P1R, when $m = n$, and the function $\pi_e : [m] \rightarrow [n]$ is bijective [Kho10].

Definition 6.2.1. A UNIQUE GAME $\mathcal{U}(G(V, E), [n], \{\pi_e \mid e \in E\})$ is a constraint satisfaction problem defined as follows: $G(V, E)$ is a directed graph whose vertices represent variables and edges represent constraints. The goal is to assign to each vertex a label from the set $[n]$. The constraint on an edge $e = (v, w) \in E$ is described by a bijection $\pi_e : [n] \mapsto [n]$. A labeling $L : V \mapsto [n]$ satisfies the constraint on edge $e = (v, w)$ if and only if $\pi_e(L(v)) = L(w)$. Let $OPT(\mathcal{U})$ denote the maximum fraction of constraints that can be satisfied by any labeling:

$$OPT(\mathcal{U}) := \max_{L: V \mapsto [n]} \frac{1}{|E|} \cdot |\{e \in E \mid L \text{ satisfies } e\}|$$

For the purposes of illustration, consider the binary constraint satisfaction problem given by the maximum cut problem (MAX CUT). We recall MAX CUT as follows: Given an undirected graph $G = (V, E)$ and nonnegative weights $w_{ij} \geq 0$ for all edges $(i, j) \in E$, the goal is to find a partition of the vertices S and $\bar{S}$ that maximizes the sum of the weights of the edges in the cut $(S, \bar{S})$. MAX CUT corresponds to the weighted constraint satisfaction problem in which $[n] = \{0, 1\}$, there is a variable x_i for each vertex $i \in V$, and there is a constraint $f(x_i, x_j) = x_i \oplus x_j$ for each edge (i, j) with weight w_{ij} , where $\oplus$ is the exclusive-or function (which is 1 precisely when $x_i \neq x_j$). This problem happens to be a UG on a universe of two elements, because for any constraint on an edge (i, j) , when given the value of $x_i \in \{0, 1\}$, there is only one value of x_j for

which the constraint is satisfied (likewise for a given value of x_j) [WS11].

The UNIQUE GAMES CONJECTURE asserts that it is **NP-hard** to distinguish between instances of the unique games problem in which almost every constraint is satisfied and instances in which almost no constraint is satisfied. We now give a formal definition of the UNIQUE GAMES CONJECTURE [Kho02].

UNIQUE GAMES CONJECTURE 1: For arbitrarily small constants $\zeta, \delta > 0$, there exists a constant $k = k(\zeta, \delta)$ such that it is **NP-hard** to determine whether a unique 2-prover game with answers from a domain of size k has value at least $1 - \zeta$ or at most δ .

Although still an open question, assuming that UGC is true was shown to imply optimal hardness results for many optimization problems [KKMO07, KR08, Rag08], and has motivated research leading to interesting algorithmic results [CMM06, Tre08, ABS15, Kol11]. Recent results are providing strong evidence towards correctness of UGC [DKK⁺18b, DKK⁺18a, KMS18].

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