2019

Infinite Matroids and Transfinite Sequences

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Infinite Matroids and Transfinite Sequences

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Dissertation submitted
to the Eberly College
at West Virginia University

in partial fulfillment of the requirements for the degree of
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2019

Keywords: Infinite Matroids, Matroid Intersection Conjecture, Patchwork Matroids, Cyclic Flats

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Abstract

Infinite Matroids and Transfinite Sequences

Martin Storm

A matroid is a pair $M = (E, \mathcal{I})$ where E is a set and $\mathcal{I}$ is a set of subsets of E that are called independent, echoing the notion of linear independence. One of the leading open problems in infinite matroid theory is the Matroid Intersection Conjecture by Nash-Williams which is a generalization of Hall's Theorem. In [31] Jerzy Wojciechowski introduced μ -admissibility for pairs of matroids on the same ground set and showed that it is a necessary condition for the existence of a matching. A pair of matroids (M, W) with common ground set E is μ -admissible if a subset of sequences in $E \times \{0, 1\}$ have a certain property. In order to determine if this property implies anything about the length of the sequence, we modify μ -admissibility to obtain μ' -admissibility, an equivalent property for pairs of matroids. We then use μ' -admissibility to show that for every successor ordinal of the form $\alpha + 2n$ there is a pair of partition matroids such that a shortest sequence in $E \times \{0, 1\}$ that fails to have the desired property has length $\alpha + 2n$. Furthermore, we introduce the class of patchwork matroids, which contains all finite matroids and all uniform matroids, and provide a method for their construction, prove a characterization theorem, show the class is closed under duality and taking minors, as well as several other properties. Lastly, a cyclic flat is a flat which is a union of circuits. We combine this notion with that of trees of matroids, which are introduced in [7] to show that the lattice of cyclic flats of a locally finite tree of finite matroids contains an atomic element.

Acknowledgments

Jerzy Wojciechowski for immeasurable patience, guidance. Nathan Bowler for inspiration and direction. Hong-Jian Lai for introducing me to matroids.

Dedication

To my father, Mr. Storm, and to my grandfather, Rev. Storm.

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1 Introduction

This dissertation introduces results concerning three topics in the field of infinite matroids; μ -admissibility for pairs of matroids, patchwork matroids and cyclic flats. For each of these topics, we devote a section to our results on it. The necessary preliminaries for each topic can be found in the background section and how they fit in the field of infinite matroids is discussed here in the Introduction.

One motivating example of finite matroids comes from vector spaces where given a finite subset E of a vector space, and the set $\mathcal{I}$ of all linearly independent subsets of E , the pair $(E, \mathcal{I})$ is a finite matroid. Finite graphs also provide another example of finite matroids. One way of attempting to extend this concept to infinite sets E is to require that an infinite set is independent only when all of its finite subsets are independent. This requires that all circuits are finite (leading to such matroids being called finitary) and much research has been done concerning them. However, the class of finitary matroids is not closed under duality, a property that plays an important role in finite matroid theory. So when Rado asked in [24] whether or not it is possible for a non-trivial theory to avoid requiring a finitary condition for independence, a natural property to desire is closure under duality. With this in mind, in [17] Higgs was motivated by the similarity between the duality in finite matroids and the duality in Fréchet V-spaces (shown by Sierpiński in [25]) to develop B-Matroids. B-Matroids provide a solution to Rado's Problem, and in [20] Oxley showed that the class of B-Matroids is the largest possible class of pre-independence spaces that is closed under duality and taking minors. Incidentally, the class of Tame Matroids (those matroids whose circuits and cocircuits have finite intersection) is also closed under duality and taking minors, and is thus a smaller class than B-matroids, and also serves as a solution to Rado's Problem. Oxley also rephrased the axiomatization of B-matroids in a more combinatorial sense, and directly pointed out the necessary condition that is needed to axiomatize infinite matroids. However, it was not widely known that this solution exists. Then, in [13] Bruhn *et al.* rediscovered infinite matroids, and showed their importance in infinite combinatorics with many examples and

important counterexamples. They also developed five axiomatizations of matroids in terms of independent sets, bases, circuits, closure and rank. This paper also gave more publicity to this solution of Rado's Problem, and has spurred more research regarding them.

One of the driving problems in the field of infinite matroids is the Matroid Intersection Conjecture due to Nash-Williams [1]. It originally was posed for finitary matroids, but can easily be viewed in the context of all matroids. Letting $cl_M(A)$ represent the closure of the set A in the matroid M , the conjecture is commonly written as:

Conjecture 1.1. *Let M and N be Infinite Matroids on a common ground set E . Then there exists a set I independent in both M and N which admits a partition J_M and J_N such that $cl_M(J_M) \cup cl_N(J_N) = E$.*

One reason for the importance of this conjecture is due to its connection with several proofs. For instance, in [3] it is shown that the matroid intersection conjecture implies the Aharoni-Berger theorem (which is the infinite generalization of Menger's theorem). In fact, the matroid intersection conjecture has been shown to be true in a number of separate cases: Bowler and Carmesin in [6] show that the result holds if the matroids M and N are derived from a graph G and Borel sets of the ends of G where G has a tree decomposition into parts of adhesion at most 2. Aharoni and Ziv showed in [1] that the conjecture holds if one matroid is finitary and the other is a direct sum of finite rank matroids. Furthermore, Bowler *et al.* show in [10] that it is sufficient for one of M and N to be patchwork.

In fact, Edmond's theorem [29] shows the conjecture holds for finite pairs of matroids, although it is more appropriate to look at the Matroid Intersection Conjecture as the infinite version of Edmond's theorem. It has been shown to be equivalent to the packing/covering conjecture by Bowler and Carmesin in [8]. Since Edmond's theorem is itself a generalization of Hall's Theorem, it can similarly be shown that the Matroid Intersection Conjecture would generalize Aharoni's generalization of König's Theorem proved in [2]. Investigating this notion, and inspired by the condition of q-admissibility formulated by Nash-Williams in [19], Jerzy Wojciechowski introduced

μ -admissibility for bipartite graphs in [30] and for pairs of matroids in [31] where he showed it is a necessary condition for matchings on infinite matroids.

In Section 3 we investigate μ -admissibility further. This requires working with strings, which are injective functions whose domain is an ordinal. The pair of matroids (M, W) with a common ground set E is μ -admissible when a subsets of the strings whose range is a subset of $E \times \{0, 1\}$ have a certain property. We define the μ -failure set of (M, W) as the set of ordinals α such that there is a string with domain α which causes (M, W) to not be μ -admissible. If the μ -failure set of (M, W) is nonempty, we call the least element of the μ -failure set the μ -barrier. In the case where it is empty, we use ∞ for the μ -barrier. In order to simplify finding the μ -barrier, we introduce μ' -admissibility, which considers all strings whose range is a subset of E , and show it is equivalent to μ -admissibility. We then go on to prove the following theorem which establishes the relationship between the μ -barrier and the μ' -barrier.

Lemma 1.2. *Let (M, W) be a pair of matroids on E which is not μ -admissible. If $\alpha + n$ is the μ' -barrier of (M, W) , then $\alpha + 2n$ is the μ -barrier of (M, W) .*

With this lemma, we then prove the central result of Section 3, which is as follows:

Theorem 1.3. *Let α be successor ordinal. There exists a pair of matroids (M, W) such that the μ' -barrier of (M, W) is α .*

The corresponding result for the μ -barrier is then obvious.

In Section 4 we are concerned with patchwork matroids. While there are patchwork matroids which exist in ZFC, in order to ensure the existence of the more interesting examples it is convenient assume extra axioms. For example, in [9] Bowler and Geschke use a fragment of Martin's Axiom to show the existence of an infinite self-dual uniform matroid, but it is still unknown whether or not such a matroid exists in ZFC alone. The majority of Section 4 is devoted to showing how to construct patchwork matroids. In particular, we define an equivalence relation $\simeq$ on the subsets of E such that $A \simeq B$ whenever $|A \triangle B|$ is finite. We denote the equivalence class of A by $[A]$ and partially order them by saying that $[A] \leq [B]$ whenever $|B \setminus A|$ is finite. A patch on $[A]$ is a

subset of $[A]$ which will be a subset of the bases of some matroid. We then call an antichain $\mathcal{A}$ in this partial order strong if for any pair of comparable equivalence classes b and c there is an $a \in \mathcal{A}$ which is comparable to both b and c . After describing several different kinds of patches, we prove the following theorem.

Theorem 1.4. *Let E be a set and $\mathcal{A}$ be a strong antichain of the lattice generated by $\simeq$. Let $\mathcal{B} = \bigcup_{[A] \in \mathcal{A}} \mathcal{B}_{[A]}$, where $\mathcal{B}_{[A]}$ is a patch on $[A]$. Then $\mathcal{B}$ is the set of bases for a matroid on E . We call these matroids Patchwork.*

This theorem demonstrates the large amount of freedom in constructing patchwork matroids. However, although this theorem tells us a fair bit about the construction of patchwork matroids, the characterization theorem which follows is much more important.

Theorem 1.5. *Let M be a matroid on E . M is patchwork if and only if for every $K \subseteq E$ one of the following holds:*

- (1) K is independent
- (2) K is spanning
- (3) There exists a base B with $|K \triangle B|$ finite

This theorem serves as a much more intuitive definition of patchwork matroids, and allows for many results to easily be shown using it. For instance it becomes clear that the class of patchwork matroids is closed under duality and taking minors. Furthermore, with Bowler *et al.*'s results in [10] it becomes immediately clear that the Matroid Intersection Conjecture holds whenever one of the matroids is patchwork. We finish Section 4 by showing the following two results

Proposition 1.6. *Let M be a patchwork matroid that is neither finitary nor cofinitary. Then M is wild and infinitely connected.*

Corollary 1.7. *Let E be an infinite set, P an infinite and coinfinite subset of E and $M = (E_M, \mathcal{I}_M)$ a finite matroid with $E_M \subset E$. There is a patch $\mathcal{B}$ on $[P]$ which can be constructed from M such that any patchwork matroid N constructed using $\mathcal{B}$ has M as a minor.*

The first result can be thought of as saying that patchwork matroids which are neither finitary nor cofinitary have all their circuits messily tangled together, like a hopelessly tangled ball of yarn. The second one is a corollary which results from extending the notion of a free product as presented in [15]. It essentially says that when we construct a patchwork matroid we have the option to specify any finite matroid as a minor.

The last topic we address in this dissertation relates to the lattice of cyclic flats of a Matroid. It is already known that every finite lattice is the lattice of cyclic flats of some finite matroid [4], and indeed it has been proven by Julie Sims [26] that any finite height lattice is the lattice of cyclic flats of some (possibly infinite) Matroid. Furthermore, it is easy to show that an arbitrary lattice must be bounded in order for it to be the lattice of cyclic flats of a matroid. This then raises the question of what other properties of a lattice must be required and what they imply about the matroid itself. One question revolved around whether or not such lattices need to have atomic elements. It turns out that they do not, and we use Ann-Kathrin Elm's formulation the construction found in [27] to clearly identify a countable matroid which does not possess an atomic element. We therefore say that a matroid is atomless if its lattice of cyclic flats has no atomic elements, and call the matroid weakly atomic if there is an atomic element in its lattice of cyclic flats. Section 5 focuses on results concerning when such atomic elements are present, and what happens if they aren't. The primary results concern matroids which are constructed using a tree of matroids, which were introduced in [7]. A tree of matroids is a pair consisting of a tree T and a function M which assigns to each vertex of T a matroid. Together with a set of branches (also called ends), it is shown in [7] that under certain conditions a matroid can be generated. We first show the following lemma

Lemma 1.8. *Let $\mathcal{T} = (T, M)$ be a locally finite tree of finite matroids and Ψ a set of branches of T such that $\mathcal{T}$ together with Ψ induces a matroid M . Let C be a circuit of M and let $N = M|_{\text{span}(C)}$. If N has no finite circuits then there is a locally finite tree of finite matroids $\mathcal{T}' = (T_N, N)$ such that N is induced by $\mathcal{T}'$ together with Ψ_N , where Ψ_N is the set of all branches of T_N .*

And then use it to show the primary result of Section 5

Theorem 1.9. *Let M be a matroid induced by a locally finite tree of finite matroids $\mathcal{T} = (T, M)$ and Ψ , a set of branches of T . Then M is weakly atomic.*

2 Background

2.1 Set Theory

Most of our definitions will be taken from [18] and [22]. We will start by introducing ordinals, cardinals and transfinite induction. To do this, we first define a partial order as follows:

Definition 2.1. A *partial order* is a binary relation $<$ on a set P such that

- 1) $p \not< p$ for all $p \in P$
- 2) if $p < q$ and $q < r$ then $p < r$

We then call $(P, <)$ a *partially ordered set*. In particular, $<$ is a *linear ordering* if

- (3) $p < q$ or $p = q$ or $q < p$ for all $p, q \in P$.

When we have a partial order $(P, <)$ we will use $\leq$ to mean that either $a < b$ or $a = b$. Given a partial order $(P, <)$ and a set $S \subseteq P$, we say that $e \in S$ is a *minimal element* of S if $x \not< e$ for all $x \in S$. If we have $e \leq x$ for all $x \in S$, we call e the *least element* of S . Similarly, we say that $e \in S$ is a *maximal element* of S if for all $x \in S$ we have $e \not< x$, and the *greatest element* of S if $x \leq e$ for all $x \in S$. If $e \in P$ and $x \leq e$ for all $x \in S$, then e is an *upper bound* of S , and similarly, if $e \leq x$ for all $x \in S$ then e is a *lower bound*. If e is the least upper bound of S , we call e the *supremum* of S , and write $\sup(S)$ and if e is the greatest lower bound of S , then we call e the *infimum* of S and write $\inf(S)$. We call a linearly ordered set $(P, <)$ *well-ordered* if every subset of P has a minimal element.

A set T is *transitive* if every element of T is a subset of T . That is, for all $t \in T$ we have $t \subset T$. We then say that a set γ is an *ordinal number* (or just an *ordinal*) if γ is transitive and well-ordered by $\in$. We denote the class of ordinals as Ord and for two ordinals α and β , $\alpha < \beta$ is taken to mean that $\alpha \in \beta$. We define $\alpha + 1 = \alpha \cup \{\alpha\}$ to be the *successor* of α . Given an

ordinal α , if there is an ordinal β such that $\alpha = \beta + 1$, then we call α a *successor ordinal*. If an ordinal α is not a successor ordinal, then we say it is a limit ordinal. Observe that limit ordinals can be written in the form $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$. In this context we define $0 = \sup(\emptyset) = \emptyset$, and then the natural numbers can be defined using the successor relationship. We define ω as the least non-empty limit ordinal. Furthermore the operations of $+$, $\cdot$ and exponentiation for ordinals are defined using transfinite recursion as follows:

Definition 2.2. For all ordinal numbers α

- (i) $\alpha + 0 = \alpha$,
- (ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β ,
- (iii) $\alpha + \beta = \lim_{\xi \rightarrow \beta} (\alpha + \xi)$ for all limit $\beta > 0$.

Definition 2.3. For all ordinal numbers α

- (i) $\alpha \cdot 0 = 0$,
- (ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β ,
- (iii) $\alpha \cdot \beta = \lim_{\xi \rightarrow \beta} (\alpha \cdot \xi)$ for all limit $\beta > 0$.

Definition 2.4. For all ordinal numbers α

- (i) $\alpha^0 = 1$,
- (ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β ,
- (iii) $\alpha^\beta = \lim_{\xi \rightarrow \beta} \alpha^\xi$ for all limit $\beta > 0$.

We state theorem 2.14 from [18]

Theorem 2.5. (*Transfinite Induction*) Let C be a class of ordinals and assume that:

- (i) $0 \in C$;
- (ii) if $\alpha \in C$ then $\alpha + 1 \in C$;
- (iii) if α is a nonzero ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$.

Then C is the class of all ordinals.

We will also use Cantor's Normal Form Theorem which can be found in [18].

Theorem 2.6. [Cantor] *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \dots + \omega^{\beta_n} \cdot k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > \dots > \beta_n$ and $k_1 \dots k_n$ are non-zero natural numbers.

Given two sets X, Y we say that they have the same cardinality (denoted $|X| = |Y|$) if there exists a bijection $f : X \rightarrow Y$. Since this defines an equivalence relation, we identify $|X|$ as the *cardinal number* associated with X . Because we can express the set of natural numbers $\mathbb{N}$ using ordinals as noted previously, we then say that a set X is finite (or has finite cardinality) if there is some $n \in \mathbb{N}$ such that $|X| = |n|$ (although we will typically write n instead of $|n|$). We call an ordinal α a *cardinal number* if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$.

One particular result of note is Zorn's Lemma.

Lemma 2.7. [Zorn] *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Given two sets X, Y we write X^Y to denote the family of all functions with domain Y and codomain X . We now turn our attention to introducing the notion of a lattice and some supporting definitions.

Definition 2.8. Let $(P, <)$ be a partial order with the property that for every $a, b \in P$ the infimum and supremum of the set $\{a, b\}$ both exist. For $a, b \in P$ define $a \wedge b = \inf(\{a, b\})$ and $a \vee b = \sup(\{a, b\})$. We call $\mathcal{L} = (P, \vee, \wedge)$ a lattice.

Since we will be talking about matroids, our lattices will be bounded.

Definition 2.9. A Lattice $\mathcal{L}$ is bounded if there exist elements $0, 1 \in \mathcal{L}$ such that $a \vee 0 = a$ and $a \vee 1 = 1$ for any choice of a . Equivalently, if $a \wedge 0 = 0$ and $a \wedge 1 = a$ for all a , then $\mathcal{L}$ is bounded.

Furthermore, since we will be concerned with the presence of atomic elements in lattices derived from matroids

Definition 2.10. Let $\mathcal{L}$ be a bounded lattice and $a \in L \setminus \{0\}$ such that $a \wedge b = a$ or 0 . Then a is an atomic element of $\mathcal{L}$. If for every $b \in L$ with $b > 0$ there is an atomic element a with $b \geq a$, then we say that $\mathcal{L}$ is *atomic*.

The last concept we introduce here is that of a strong antichain. Specifically, when we are dealing with a partially ordered set, it's nice to be able to talk about sets in which no two elements are comparable. As matroid theory deals with sets of sets, the partial order we will deal with is inclusion.

Definition 2.11. Let $(P, <)$ be a partially ordered set. A set $A \subset P$ is an *antichain* if for every $x, y \in A$, we have that $x \not\leq y$ and $x \not\geq y$. An antichain A is *strong* if for any $x, y \in P$ with $x \leq y$ there exists $c \in A$ such that one of the following is true

- 1) $c \leq x$
- 2) $x \leq c \leq y$
- 3) $y \leq c$

2.2 Graph Theory

Most of our definitions are taken from [16]. We define a graph $G = (V, E)$ as a pair of (possibly infinite) sets where V is the set of vertices and $E \subseteq [V]^2$ is the set of edges. An edge $e \in E$ is said to be *incident* to a vertex $v \in V$ if $v \in e$. Given a graph G and $u, v \in V$ we will frequently use (u, v) to denote $\{u, v\} \in E$. The *neighborhood* of $v \in V$ is $N(v) = \{u \in V : (u, v) \in E\}$. This concept can easily be extended to a set in the obvious manner: $N(S) = \bigcup_{s \in S} N(s)$. The degree of $v \in V$ is $d(v) = |\{e \in E : v \in e\}|$. We say that a vertex $v \in V$ is *isolated* if $d(v) = 1$ and that the graph G is *k-regular* if all vertices of G have degree k . When talking about multiple graphs, we will use $V(G)$ and $E(G)$ to talk about the vertices and edges of G . A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *complement* of a graph G is the graph $\bar{G}$ where $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{e \in [V]^2 : e \notin E(G)\}$.

We call a graph G *bipartite* if there is a partition of V into two disjoint sets A, B with $A \cup B = V$

such that every edge $e \in E$ contains a vertex in each part. In other words, $E \subseteq \{(a, b) : a \in A, b \in B\}$. Given a bipartite graph G we can define a second bipartite graph $\hat{G}$ with $V(\hat{G}) = V(G)$ and $E(\hat{G}) = \{(a, b) : a \in A, b \in B, (a, b) \notin E(G)\}$. We call $\hat{G}$ the *bipartite complement* of G .

A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \dots, x_k\} \quad E\{(x_0, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k)\}$$

such that each x_i is distinct. The vertices x_0 and x_k are *linked* by P and are called its *ends*, and we call P a path from x_0 to x_k (observe that order is not important. We can call P a path from x_k to x_0 just as easily). If P is a path from x_0 to x_k with $k \geq 2$, then the graph obtained from P by adding the edge $\{x_0, x_k\}$ to P is called a *cycle*. If a graph G has a subgraph P which is a path from u to v , then we say that G has a path from u to v (or sometimes that G has a *uv-path*). Similarly, if G has a subgraph which is a cycle, we say that G contains a cycle.

A graph G is *connected* if for each $u, v \in V$, G has a *uv-path*. A graph G is a *forest* if it does not contain any cycles, and a connected forest is called a *tree*. Given a tree T , a subgraph S of T is a *branch* if S is also a tree. A *rooted tree* is a pair (T, r) where T is a tree and $r \in V(T)$. A *ray* is a non-empty graph $R = (V, E)$ of the form

$$V = \{x_0, x_1, \dots\} \quad E\{(x_0, x_1), (x_1, x_2), \dots\}$$

We say that two rays R_1 and R_2 are equivalent if they cannot be separated from G by removing finitely many vertices.

Lastly a *directed graph* is a pair $D = (V, E)$ of disjoint sets together with two maps $\text{init} : E \rightarrow V$ and $\text{ter} : E \rightarrow V$ assigning to every edge an initial vertex and a terminal vertex. A *uv-path* P in a directed graph D , is a *uv-path* such that for each edge $(x_i, x_{i+1}) \in P$ the direction of (x_i, x_{i+1}) goes from x_i to x_{i+1} . In this case we say that $u = x_0$ is the start vertex and $v = x_k$ is the end vertex, and note that the order here does matter.

Given a graph G we say that M is a *matching* if there is no vertex of G which is incident to

Figure 1: Failure of Hall's Theorem in an Infinite Bipartite Graph

more than one edge of M . Alternatively, one can say that M induces a 1-regular subgraph of G . We say that a set of vertices $U \subseteq V(G)$ is *matched* if there is a matching M such that for each $v \in U$ there is an edge in M which is incident to v . A classical result concerning matchings is Hall's Theorem, which is restricted to finite graphs. It is stated as follows:

Theorem 2.12. (Hall) Let G be a finite bipartite graph with parts A and B . A is matched if and only if for each $H \subseteq A$ the set $N(H) = \{b \in B : \exists a \in H, (a, b) \in E\}$ satisfies $|H| \leq |N(H)|$.

That Hall's theorem fails to hold in the infinite case can be seen in figure 1.

2.3 Matroid Theory

We will make use of several of the axiomatizations for infinite matroids which can be found in [13], and in general we follow the notation of [21]. We start with the circuit definition of a matroid, where circuits are minimally dependent sets.

Definition 2.13. (Circuit Axioms) Let E be a set and $\mathcal{C}$ a family of subsets of E which satisfies

(C1) $\emptyset \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $(C_x | x \in X)$ is a family of elements of $\mathcal{C}$ such that $x \in C_y$ if and only if $x = y$ for all $x, y \in X$, then for every $z \in C \setminus (\bigcup_{x \in X} C_x)$ there exists an element $C' \in \mathcal{C}$

such that $z \in C' \subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X$.

(C4) The set $\mathcal{I} = \{I \subseteq E : C \not\subseteq I \forall C \in \mathcal{C}\}$ has the property that whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$ then the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

Then $M = (E, \mathcal{I})$ is a matroid, with $\mathcal{I}$ as defined above. $\mathcal{C}$ is the set of circuits of M and $\mathcal{I}$ is the set of independent sets of M . When working with multiple matroids, we use $\mathcal{C}_M$ to denote the set of circuits of M .

The second axiomatization we use is defined by the closure function.

Definition 2.14. (Closure Axioms) Let E be a set and $\text{span} : 2^E \rightarrow 2^E$ be a function. Suppose span satisfies the following:

(CL1) For all $X \subseteq E$ we have $X \subseteq \text{span}(X)$.

(CL2) For all $X \subseteq Y \subseteq E$ we have $\text{span}(X) \subseteq \text{span}(Y)$.

(CL3) For all $X \subseteq E$ we have $\text{span}(\text{span}(X)) \subseteq \text{span}(X)$.

(CL4) For all $Z \subseteq E$ and $x, y \in E$, if $y \in \text{span}(Z \cup \{x\}) - \text{span}(Z)$ then $x \in \text{span}(Z \cup \{y\})$.

(CLM) The set $\mathcal{I} = \{I \subseteq E : \text{for all } x \in I, x \notin \text{span}(I \setminus \{x\})\}$ has the property that whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$ then the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

Then $M = (E, \mathcal{I})$ is a matroid with $\mathcal{I}$ as defined above.

It should be noted that when we have the set of circuits $\mathcal{C}$ of a matroid M , then the closure operator for M is defined as

$$\text{span}_M(A) = A \cup \{x \in E : \exists C \in \mathcal{C} \text{ such that } x \in C, C \subseteq \{x\} \cup A\}$$

Given a matroid $M = (E, \mathcal{I})$, we say that $F \subseteq E$ is a *flat* if $\text{span}(F) = F$, and that F is a *cyclic flat* if F is flat which is a union of circuits.

A matroid can also be defined in terms of its bases, which are its maximally independent sets. The equivalency of all of these definitions can be found in [13].

Definition 2.15. Let E be a set and $\mathcal{B}$ be a collection of subsets of E satisfying:

(B1) $\mathcal{B} \neq \emptyset$

(B2) $\forall B_1, B_2 \in \mathcal{B}$ and all $x \in B_1 \setminus B_2$ there exists a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$

(BM) Let $\mathcal{I} = \{I \subseteq E : \exists B \in \mathcal{B} \text{ with } I \subseteq B\}$. Then for any $I \in \mathcal{I}$ and $X \subseteq E$ which satisfy $I \subseteq X$ the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

Then $M = (E, \mathcal{I})$ is a *matroid* with $\mathcal{B}$ as its set of bases and $\mathcal{I}$ its set of independent sets.

Before we introduce the next definition, we introduce the dual of a matroid.

Definition 2.16. The *dual* of M , denoted $M^* = (E, \mathcal{I}^*)$, is the matroid whose set of bases is $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$.

We use the prefix co- to refer to a property of the dual of the matroid. For example, a cocircuit of M is a circuit of M^* . Given this terminology, if our ground set E is countable, then from [7] we have the following orthogonality axioms

Definition 2.17. If E is countable, two sets $\mathcal{C}, \mathcal{D} \subseteq 2^E$ are the sets of circuits and cocircuits of a matroid on E if and only if they satisfy the following:

(C1) $\emptyset \notin \mathcal{C}$.

(C1*) $\emptyset \notin \mathcal{D}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C2*) No element of $\mathcal{D}$ is a subset of another.

(O1) $|C \cap D| \neq 1$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(O2) For all partitions $E = P \cup Q \cup \{e\}$ either $P \cup \{e\}$ includes an element of $\mathcal{C}$ containing e or $Q \cup \{e\}$ contains an element of $\mathcal{D}$ containing e .

(O3) For all $e \in C \in \mathcal{C}$ and $X \subseteq E$, among all sets $C' \in \mathcal{C}$ with $e \in C' \subseteq X \cup C$ there is one for which $C' \setminus X$ is minimal.

(O3*) For all $e \in D \in \mathcal{D}$ and $X \subseteq E$, among all sets $D' \in \mathcal{D}$ with $e \in D' \subseteq X \cup D$ there is one for which $D' \setminus X$ is minimal.

The following reversal of (B2) follows from the results of [13].

Proposition 2.18. *Let M be a matroid and $\mathcal{B}$ be its set of bases. Then for any $B_1, B_2 \in \mathcal{B}$ and any $y \in B_2 \setminus B_1$ there exists $x \in B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.*

Additionally we have the following proposition from [21]. It should be noted that although the proof is given in the context of finite matroids, no modifications are necessary for the infinite case.

Proposition 2.19. *Suppose that I is an independent set in a matroid M and e is an element of M such that $I \cup e$ is dependent. Then M has a unique circuit contained in $I \cup e$ and this circuit contains e .*

When B is a base of M , this circuit is called the *fundamental circuit of e with respect to B* .

There are also several kinds of matroids which are of interest to us. One such type of matroid are partition matroids. Given a set E , let $\{A_i\}$ be a partition of E . Then we can define a matroid M on E where I is independent if $|I \cap A_i| \leq 1$. With this in mind, we can see that from a bipartite graph G with parts A and B it is possible to define two partition matroids with ground set $E(G)$ by partitioning the edges of E according to which vertex of A (or, for the second matroid, B) they are incident to. The other type of matroids we will have some interest in are gammoids.

Definition 2.20. Let G be a finite directed graph with vertex set E , and let B_0 be a subset of E . There is a matroid M (called a *strict gammoid*) on E whose bases are all sets $B \subseteq E$ such that there is a set $\mathcal{P}$ of pairwise disjoint paths from B to B_0 such that every vertex of B is the start of some path $P \in \mathcal{P}$ and every vertex of B_0 is the end of some path $P \in \mathcal{P}$. A matroid N is a gammoid if it is isomorphic to a restriction of a strict gammoid.

We now introduce several methods of finding new matroids given one or more other matroids. We start off with the most commonly used operations of deletion, contraction. A more detailed discussion of them, and the dual can be found in [21, 13].

Definition 2.21. Let $M = (E, \mathcal{I})$ be a matroid, $\mathcal{B}$ its set of bases and $S \subset E$. Then each of the following is a matroid:

The *Deletion* of S from M , denoted $M \setminus S = (E \setminus S, \mathcal{I}')$, is the matroid whose set of bases is $\{B' \in \mathcal{I} : B' \cap S = \emptyset \text{ and } \forall I \in \mathcal{I} \text{ with } B' \subsetneq I, I \cap S \neq \emptyset\}$.

The *Contraction* of S in M , denoted $M/S = (E \setminus S, \mathcal{I}')$, is the matroid whose set of bases is the set of minimal elements of $\{I \in \mathcal{I} : \exists T \subseteq S \text{ with } I \cup T \in \mathcal{B}\}$.

If there exists sets $S, T \subset E$ such that $N = M/S \setminus T$ then we say that N is a *minor* of M .

There are also the concepts of a one-sum and a two-sum of matroids, which are described in more detail in [21].

Definition 2.22. Let $M = (E, \mathcal{I})$ and $N = (F, \mathcal{J})$ be two matroids with E and F disjoint. The one-sum of M and N is the matroid $M \oplus N = (E \cup F, \mathcal{I}')$ where $\mathcal{I}' = \{I \cup J : I \in \mathcal{I}, J \in \mathcal{J}\}$

Definition 2.23. Let $M = (E, \mathcal{I})$ and $N = (F, \mathcal{J})$ be two matroids such that $E \cap F = \{p\}$, $M \setminus \{p\} \neq M/\{p\}$ and $N \setminus \{p\} \neq N/\{p\}$. The two-sum of M and N , denoted $M \oplus_2 N$ is the matroid on $E \cup F \setminus \{p\}$ whose set of circuits is

$$\mathcal{C}_{M \oplus_2 N} = \{C_1 : p \notin C_1 \in \mathcal{C}_M\} \cup \{C_2 : p \notin C_2 \in \mathcal{C}_N\} \cup \{C_1 \cup C_2 \setminus \{p\} : p \in C_1 \in \mathcal{C}(M), p \in C_2 \in \mathcal{C}(N)\}$$

The notion of a two-sum easily leads to the idea of repeatedly performing it, and then a connection to trees can easily be formed. A focused discussion on this concept was done by Carmesin and Bowler, and can be found in [7]. We will be using rooted trees because later we will want an intuitive language for talking about the structure of the branches of our trees. Since, as previously noted, the choice of the root has no impact on the structure of the branches, the definitions and results will not depend on the vertex chosen to be the root.

Definition 2.24. We define a *tree of matroids* $\mathcal{T}$ to be a pair (T, M) where T is a rooted tree and M is a function assigning to each vertex t of T a matroid $M(t)$ on some set $E(t)$ such that for any two vertices t and t' of T the set $E(t) \cap E(t')$ consists of exactly one element if tt' is an edge of T and is empty otherwise. A *dummy edge* is any element in $E(t) \cap E(t')$ for some t and t' in T . We will use $e(tt')$ to denote the element of $E(t) \cap E(t')$, and will refer to tt' as the edge corresponding

to $e(tt')$. We define the dual tree of matroids $\mathcal{T}^*$ to be the pair (T, M^*) where $M^*(t)$ is the dual of $M(t)$ for every vertex t of T . Carmesin and Bowler also consider other trees of matroids and refer to the trees considered here as trees of matroids of overlap 1.

Let $\mathcal{T} = (T, M)$ be a tree of matroids and (Ψ, Ψ^C) be a partition of the set of branches of T . We then define a Ψ -precircuit of $\mathcal{T}$ to be a pair (C, o) where C is a subtree of T with all branches of C in Ψ and o is a function assigning to each vertex t of C a circuit of $M(t)$ in such a way that $o(t)$ contains the dummy edge $e(tt')$ if and only if tt' is an edge of C . The underlying set of the Ψ -precircuit (C, o) is defined to be

$$C' = \bigcup_{t \in V(C)} o(t) \setminus \bigcup_{tt' \in E(C)} e(tt').$$

Let $\mathcal{C}$ be the set of all minimal nonempty underlying sets of Ψ -precircuits of $\mathcal{T}$ and $\mathcal{D}$ be the set of all minimal nonempty underlying sets of Ψ^C -precircuits of $\mathcal{T}^*$. We then say that $\mathcal{T}$ and Ψ induce a matroid $\mathcal{M}$ whose ground set is

$$E = \bigcup_{t \in V(T)} E(t) \setminus \bigcup_{tt' \in E(T)} e(tt').$$

if $\mathcal{C}(\mathcal{M}) = \mathcal{C}$ and $\mathcal{C}(\mathcal{M}^*) = \mathcal{D}$. It is actually possible for $\mathcal{C}$ to be the family of circuits of a matroid $\mathcal{M}$ on E and $\mathcal{C}(\mathcal{M}^*) \neq \mathcal{D}$. When this happens, we say that $\mathcal{T}$ and Ψ do not induce a matroid. In the particular case where Ψ is a Borel subset of the branches of $\mathcal{T}$, Carmesin and Bowler showed that $\mathcal{T}$ and Ψ will induce a matroid. For our concerns, however, it is sufficient to know that a set of branches is Borel if it is either all or none of the branches.

Remark 2.25. It follows from the definition that $\mathcal{M}^*$ is induced by the tree of matroids $\mathcal{T}^* = (T, M^*)$, and $\mathcal{M}|K$ is induced by $\mathcal{T}_K = (T, K)$ and Ψ where $K(t) = M(t)|K$.

We now define several properties of matroids:

Definition 2.26. Let $M = (E, \mathcal{I})$ be a matroid.

M is *paving* if every circuit C contains a set I for which there exists an $x \in E$ such that $I \cup \{x\}$ is a base.

M is *uniform* if every circuit C contains a base.

M is *finitary* (*cofinitary*) if every circuit (cocircuit) C of M is finite.

M is *tame* if for every circuit C and cocircuit D of M , $|C \cap D|$ is finite.

From [14] we borrow the notion of connectivity. It should be noted that while there are connections to connectivity for graphs, it's not quite as direct as one might expect.

Definition 2.27. Let $M = (E, \mathcal{I})$ be a matroid. Let (X, Y) be a partition of E and let B and B' be bases of $M \setminus X$ and $M \setminus Y$ respectively. We define $\kappa(X, Y) = \min\{|F| : F \subseteq B \cup B', (B \cup B') \setminus F \in \mathcal{I}\}$ and say that (X, Y) is a k -separation if $\kappa(X, Y) \leq k - 1$ and $|X|, |Y| \geq k$. M is k -connected if it has no k -separations.

We refer to $\{e\}$ as a loop if $\{e\} \in \mathcal{C}$, and observe that every cyclic flat of a matroid M contains all loops of M and none of its coloops. We say that a matroid M is *atomic* if the lattice of cyclic flats of M is atomic. However, if the lattice of cyclic flats of M contains an atomic element, then we say that M is *weakly atomic*.

A matroid M is *representable over a field k* if there exists a function $\phi : E \rightarrow V$ where V is a vector field over k such that a set $I \subseteq E$ is independent precisely when $\phi(I)$ contains no non-zero linear dependence. This notion can actually be extended in the following manner: A function $f : E \rightarrow k^A$ is a *thin family* if for every function $c : E \rightarrow k$ the sum $\sum_{e \in E} c(e)f(e)(a)$ is well defined for each $a \in A$. Then $M = (E, \mathcal{I})$ is a matroid where $I \in \mathcal{I}$ if and only if there is not a nonzero function $c : I \rightarrow k$ such that $\sum_{e \in E} c(e)f(e)(a) = 0$ for all $a \in A$. These matroids are called thin sum matroids and are discussed in more detail in [5].

We close out this section by presenting Ann-Kathrin Elm's formulation of the construction provided in [27] of a cofinitary matroid. The lattice of cyclic flats of this matroid does not contain an atomic element, but we will leave confirmation of this until after we have proved the necessary proposition. Our ground set will be a subset of a countably infinite dimensional vector space over

the real numbers constructed in the following manner: Let e_i be the infinite vector whose i^{th} entry is 1, and zero everywhere else. Denote $V_n = \text{span}(\{e_i : i \leq n\})$ in that vector space, and pick $f_n \in V_n$ such that f_n is not a linear combination of fewer than n elements of $\{e_1 \dots e_n, f_1 \dots f_{n-1}\}$. To see that such a f_n exists, first note that f_1 is any multiple of e_1 . Then presume that the result hold for all $k < n$. Now consider whether or not there exists a f_n^* with a 0 in the first coordinate which is not a linear combination of fewer than $n - 1$ elements of $\{e_2 \dots e_n, f_2^* \dots f_{n-1}^*\}$ where each $f_k^* = f_k$ except at the first coordinate where $f_k^* = 0$. The existence of such a f_n^* is assured by the assumption that the result holds for $k < n$. We can obtain a f_n by modifying the first coordinate f_n^* in the following manner: If $f_n^* = \sum_{i=2}^n a_i e_i + \sum_{i=2}^{n-1} b_i f_{n-1}^*$, with exactly $n - 1$ of the a_i 's and b_i 's being nonzero, then consider $s = \sum_{i=2}^{n-1} b_i r_i$ where r_i is the first coordinate of f_i . Notice that there can only be finitely many different possibilities for s , and if we select any real number which is not one of them as our new choice for the first coordinate of f_n then we ensure that f_n will have the desired property.

With this in mind, let $E = \{e_i : i \in \mathbb{N}\} \cup \{f_i : i \in \mathbb{N}\}$, and define $\mathcal{C} = \{C \subseteq E : C \text{ is a finite minimal linearly dependent set}\}$, so that $M = (E, \mathcal{C})$ is the resulting matroid. That M is finitary should be clear. We then claim that for any circuit C the span of C with respect to M is $V_k \cap E$ for some $k \in \mathbb{N}$. Since C is a circuit, it is finite, so let k be the smallest integer such that $C \subseteq V_k$. Then we know that $f_k \in C$ and therefore $|C| \geq k + 1$. But since $C \subseteq V_k$ it means that the largest independent set contained in C must have size at most k . So $|C| \leq k + 1$, and hence $|C| = k + 1$ and thus must span all of $V_k \cap E$. That C does not span any $v \notin V_k \cap E$ should be clear. Thus every cyclic flat of M is of the form $V_k \cap E$. We will refer to M^* as the ES Matroid.

2.4 Mu Admissibility

In order to talk about μ -admissibility properly, some initial definitions and terminology are needed, and our introduction loosely follows that of [31]. We define a *string* as an injective function f from an ordinal into some set S , and we say that such a string f is in S . For this Section (and

Section 3) if $\beta \leq \text{dom}(f)$ we denote f_β as the restriction of f to β . Now if we have a pair of matroids (M, W) on a common ground set E we can consider the set $\Gamma(M, W)$ of all strings f in $E \times \{0, 1\}$. In this context $(e, 0)$ and $(e, 1)$ refer to e viewed as an element of the matroid M and W respectively. We let

$$\text{rng}_M(f) = \{e \in E : (e, 0) \in \text{rng}(f)\}$$

and define $\text{rng}_W(f)$ similarly. Furthermore we say that a string f is *saturated* whenever $\text{rng}_M(f_\beta) \subseteq \text{rng}_W(f_\beta)$ for each $\beta \leq \text{dom}(f)$. Lastly, given a string f in $\Gamma(M, W)$, we say that e is *positive* in f if $e \notin \text{span}_W(\text{rng}_W(f))$ and *negative* in f if $e \notin \text{span}_M(E \setminus \text{rng}_M(f))$.

Let μ (called the μ -margin) be the function whose domain is $\Gamma(M, W)$ and whose range is the set $\mathbb{Z} \cup \{-\infty, \infty\}$ defined using transfinite induction as follows: $\mu(f) = 0$ if the domain of f is 0. If the domain is not zero, then

$$\mu(f_{\beta+1}) = \begin{cases} \mu(f_\beta) + 1 & \text{if } f(\beta) = (e, 1) \text{ and } e \text{ is positive in } f_\beta \\ \mu(f_\beta) - 1 & \text{if } f(\beta) = (e, 0) \text{ and } e \text{ is negative in } f_{\beta+1} \\ \mu(f_\beta) & \text{otherwise} \end{cases}$$

and $\mu(f_\beta) = \liminf_{\alpha \rightarrow \beta} \mu(f_\alpha)$ if β is a limit ordinal.

We say that a pair of matroids (M, W) is μ -admissible if for every saturated string f in $\Gamma(M, W)$ we have that $\mu(f) \geq 0$. Again, recall from [31], that if a pair of matroids (M, W) has a matching, then it is μ -admissible (that is, μ -admissibility is a necessary condition for matchability).

We now introduce a few definitions from [30]. First, we say that f is a *shuffle* of two strings h and g , if h and g are substrings of f such that $\text{rng}(h)$ and $\text{rng}(g)$ form a partition of $\text{rng}(f)$. Alternatively, it is possible to think of f as weaving two strings h and g together. Let h be a string in E and f be a string in $E \times \{0, 1\}$. We say that h is the *men's substring* of f if $h = \pi_1 \circ g$ where g is the substring of f satisfying $\text{rng}(g) = \text{rng}_M(f) \times \{0\}$. We define the *women's substring* of f similarly.

Also from [30] if we have a string f , and $\alpha \leq \text{dom}(h')$ where h' is the women's substring, then the f -lift of α is the unique ordinal α' such that $f(\alpha') = h'(\alpha) \times \{1\}$, or $\alpha' = \text{dom}(f)$ if

$\alpha = \text{dom}(h')$. Now suppose we have two strings f and g in $E \times \{0, 1\}$ with the same women's substring and for each α we let α' and α'' be the f -lift and g -lift of α respectively. We say that g is a *deferment* of f if for each α we have that $\text{rng}_M(g_{\alpha''}) \subseteq \text{rng}_M(f_{\alpha'})$. Observe then that if g is a deferment of f , this means that every element of $E \times \{0\}$ which appears in f either does not appear in g or does so at a later position. We also recall the following corollary from [30]

Proposition 2.28. *Let f and g be strings in $\Gamma(M, W)$ with the same women's substring. If g is a deferment of f , then $\mu(f) \leq \mu(g)$.*

Let f be a string in E , and let $\Psi(f)$ be the set of strings g in $E \times \{0, 1\}$ such that f is men's substring and the women's substring of g . Then, let $\psi(f) \in \Psi(f)$ be the string in $E \times \{0, 1\}$ such that for every $e \in \text{rng}(f)$ if $\psi(f)(\alpha) = (e, 1)$ then $\psi(f)(\alpha + 1) = (e, 0)$. That is, whenever, $(e, 1)$ shows up in $\psi(f)$, that $(e, 0)$ immediately follows it. It is then obvious that $\psi(f)$ is saturated, and that every saturated string in $\Psi(f)$ is a deferment of $\psi(f)$.

3 Mu Admissibility

In this section we introduce μ' -admissibility, show that it is equivalent to μ -admissibility, introduce the notion of the μ -failure set and the μ -barrier. We also fully categorize which ordinals can be the μ -barrier of a pair of matroids.

3.1 μ' -admissibility

Essentially, we would much rather work with a string in E rather than a string in $E \times \{0, 1\}$. To that end, let $\Gamma'(M, W)$ be the set of strings in E , and given $f \in \Gamma'(M, W)$, we say that e is positive in f when $e \notin \text{span}_W(\text{rng}(f))$ and e is *negative* in f when $e \notin \text{span}_M(E \setminus \text{rng}(f))$. It is worthwhile to note that given $f \in \Gamma'(M, W)$ and viewing $\psi(f)$ as a string in $\Gamma(M, W)$, that e is positive in f exactly when it is positive in $\psi(f)$.

We now define the μ' -margin of f as a function from $\Gamma'(M, W)$ whose range is the set $\mathbb{Z} \cup$

$\{-\infty, \infty\}$ as follows: let $f \in \Gamma'(M, W)$, and let $\mu'(f) = 0$ if $\text{dom}(f) = \emptyset$. Then use transfinite induction as follows.

$$\mu'(f_{\beta+1}) = \begin{cases} \mu'(f_\beta) + 1 & \text{if } f(\beta) \text{ is positive in } f_\beta \text{ and not negative in } f_{\beta+1} \\ \mu'(f_\beta) - 1 & \text{if } f(\beta) \text{ is negative in } f_{\beta+1} \text{ and not positive in } f_\beta \\ \mu'(f_\beta) & \text{otherwise} \end{cases}$$

with $\mu'(f_\alpha) = \liminf_{\beta \rightarrow \alpha} \mu'(f_\beta)$ for limit ordinals.

Similarly, we say a pair of matroids (M, W) is μ' -admissible if for any string f in $\Gamma'(M, W)$ we have $\mu'(f) \geq 0$. We now prove several results tying μ' -admissibility and μ -admissibility together.

Proposition 3.1. *Let M and W be two matroids on a common ground set E , and let f be a string in E . Then $\mu'(f) = \mu(\psi(f))$.*

Proof. Our proof will be by induction on the length of f . Let $g = \psi(f)$, and observe that if $\text{dom}(f) = 0$, then the result clearly holds. So assume that if $\alpha < \beta$ and α' is the g -lift of α that $\mu'(f_\alpha) = \mu(g_{\alpha'})$. Now if β is a limit ordinal and β' is the g -lift of β , due to the definitions of μ and μ' , along with the induction hypothesis, we can say that

$$\mu'(f_\beta) = \lim_{\alpha \rightarrow \beta} \mu'(f_\alpha) = \lim_{\alpha' \rightarrow \beta'} \mu(g_{\alpha'}) = \mu(g_{\beta'})$$

So our only concern is if $\beta = \delta + 1$, a successor ordinal. By letting δ' be the g -lift of δ , observe then, that in this case there is an $e \in E$ such that $\text{rng}(f_\delta) \setminus \text{rng}(f_\beta) = \{e\}$ and $\text{rng}(g_{\delta'}) \setminus \text{rng}(g_{\beta'}) = \{(e, 1), (e, 0)\}$. Thus we can consider three cases: □

Case 1. $\mu'(f_\beta) = \mu'(f_\delta) + 1$. In this case, we have that e is positive and not negative in f_β .

Then, e is positive and not negative in $g_{\beta'}$. So consequently $\mu(g_{\beta'}) = \mu(g_{\delta'}) + 1$.

Case 2. $\mu'(f_\beta) = \mu'(f_\delta) - 1$. In this case, we have that e is negative and not positive in f_β .

Then, e is negative and not positive in $g_{\beta'}$. So consequently $\mu(g_{\beta'}) = \mu(g_{\delta'}) - 1$.

Case 3. $\mu'(f_\beta) = \mu'(f_\delta)$. We actually have two subcases here, but they both handle similarly. In the first e is neither positive nor negative in f_β , and so it's neither positive nor negative in $g_{\beta'}$. And so no increases or decreases happen at all, giving $\mu(g_{\beta'}) = \mu(g_{\delta'})$. In the other case, e is both positive and negative in f_β and so it will be both positive and negative in $g_{\beta'}$. But since g is saturated, the increase due to e being positive in $g_{\beta'}$ occurs first (thus preventing any differing cases of μ or μ' -admissibility from cropping up). In either case, there is a decrease to offset it, and so $\mu(g_{\beta'}) = \mu(g_{\delta'})$ as desired, completing the proof.

With this proposition in hand, we now have the following corollary

Corollary 3.2. *The pair of matroids (M, W) on E is μ -admissible if and only if it is μ' -admissible.*

Proof. Assume that (M, W) is μ -admissible. Let $f \in \Gamma'(M, W)$, and observe that $\mu(\psi(f)) \geq 0$ since (M, W) is μ -admissible. Therefore $\mu'(f) \geq 0$ and so (M, W) is μ' -admissible. So then assume that (M, W) is μ' -admissible and let $f \in \Gamma(M, W)$ be a saturated string. Let h be the women's substring of f , and observe that f is a deferment of $\psi(h)$. Thus from Propositions 2.28 and 3.1 we have that

$$0 \leq \mu'(h) = \mu(\psi(h)) \leq \mu(f)$$

completing the proof. □

One issue of working with μ -admissibility and μ' -admissibility is that of dealing with the variations of what can happen at the limit step. This is best exemplified by the following definition.

Definition 3.3. Let g be a string and $S \neq \emptyset$ a set with $S \subseteq \text{rng}(g)$. If $g^{-1}(S)$ has a maximal element e , then we say that S is *locked in* g and $g(e)$ is the *key* of S in g . If no such maximal element exists, then we say that S is *unlocked in* g .

The idea behind sets being locked or unlocked can correspond to whether or not there is a 'last element' and thus whether or not an element is likely to be negative or not (thus causing a decrease when calculating the μ' margin). This notion comes into play in the following lemma.

Lemma 3.4. *Let G be a bipartite graph with parts A and B and let (M_G, W_G) be the pair of partition matroids generated from G . Let α be the limit ordinal such that $\bar{\mu}'(M_G, W_G) = \alpha + n$, and let H be obtained from G by adding a single vertex u to B and a single edge $e = (u, v)$ where $v \in A$ is not an isolated vertex of G . Then every string $h \in \Gamma'(M_H, W_H)$ with $\text{dom}(h) \leq \alpha$ and $e \in \text{rng}(h)$ has $\mu'(h) \geq 1$.*

Proof. Let S be the set of all edges incident to v in G . Let $h \in \Gamma'(M_H, W_H)$ with $\text{dom}(h) < \alpha$, $e \in \text{rng}(h)$ and let g be the substring of h such that $\text{rng}(g) = \text{rng}(h) \setminus \{e\}$. By way of contradiction, assume that $\mu'(h) = 0$. If $S \cup \{e\}$ is not locked in h , then we clearly have a contradiction, so $S \cup \{e\}$ is locked in h . Also notice that if S is locked in g , then again we will have a contradiction, so S must be unlocked in g and therefore e must be the key of $S \cup \{e\}$ in h . Pick some $f \in S$ (recall that v is not isolated, so such an f does exist.) and let h' be the string obtained from h by swapping e and f . Since M_H and W_H are partition matroids, it is easy to verify that $\mu'(h) = \mu'(h')$. But $S \cup \{e\}$ is locked in h' and e is not the key of $S \cup \{e\}$ in h' , so we know that $\mu'(h') > 0$ which gives the desired contradiction. $\square$

Recall, though, that the aim of this chapter is to prove things regarding the shortest length of a saturated string $f \in \Gamma(M, W)$ for which $\mu(f) < 0$. To simplify our language, we define the μ -failure set of (M, W) to be

$$\Omega(M, W) = \{\alpha : \text{There exists a saturated string } f \in \Gamma(M, W) \text{ with } \mu(f) < 0 \text{ and } \text{dom}(f) = \alpha\}$$

Then we define the μ -barrier of (M, W) to be

$$\bar{\mu}(M, W) = \begin{cases} \min(\Omega(M, W)) & \text{if } \Omega(M, W) \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

Similarly we can define $\Omega'(M, W)$ and $\bar{\mu}'(M, W)$ as the μ' -failure set and μ' -barrier of (M, W) respectively. Now we introduce the following theorem which will allow us to work solely with the

μ' -barrier.

Theorem 3.5. *Let (M, W) be a pair of matroids on E which is not μ -admissible. If $\alpha + n$ is the μ' -barrier of (M, W) , then $\alpha + 2n$ is the μ -barrier of (M, W) .*

Proof. The proof is by contradiction. Assume that $\alpha + n$ is μ' -barrier of (M, W) but that $\beta < \alpha + 2n$ is instead the μ -barrier of (M, W) . Let g be a saturated string in $\Gamma(M, W)$ with $\mu(g) = -1$ and $\text{dom}(g) = \beta$ (that is, g witnesses that the μ -barrier is β). Let h be the women's substring of g , and observe that due to propositions 2.28 and 3.1 we know that $\mu'(h) = \mu(\psi(h)) \leq \mu(g) = -1$. Now since $\alpha + n$ is the μ' -barrier of (M, W) we have that $\alpha + n \leq \text{dom}(h)$ and so $\alpha + 2n \leq \text{dom}(\psi(h))$ follows from the definition of $\psi(h)$. At the same time, $\text{dom}(h) \leq \beta$, so we have that $\alpha + n \leq \beta < \alpha + 2n$. So the finite tail of g must contain fewer than $2n$ elements, and at the same time, at least n of those elements must come from $E \times \{1\}$ due to h . Thus, since g is saturated, we can conclude that there must be some $e \in E$ such that $e \in \text{rng}_M(\psi(h)) \setminus \text{rng}_M(g)$, and that this e must come from the finite portion of the tail of h . But then we could use the substring f of g which removes $(e, 1)$, and have $\mu(f) \leq \mu(g)$ and at the same time $\mu'(h') \leq \mu'(h) < 0$ where h' is the women's substring of f . And since $\text{dom}(h') < \text{dom}(h)$ we would get a contradiction with the μ' -barrier of (M, W) being $\alpha + n$. $\square$

3.2 μ -Barrier

We will now start proving statements about the μ' -barrier, and then leave it up to the reader to draw the obvious corollaries in light of Theorem 3.5. We start off with the easiest result:

Proposition 3.6. *There does not exist a pair of matroids (M, W) such that the μ' -barrier of (M, W) is a limit ordinal.*

Proof. Suppose there was such a pair of matroids (M, W) such that $\bar{\mu}'(M, W) = \alpha$ a limit ordinal. Then let $f \in \Gamma'(M, W)$ be such that $\text{dom}(f) = \alpha$ and $\mu'(f) = -1$, in other words, f witnesses that $\bar{\mu}'(M, W) = \alpha$. Since α is a limit ordinal, we know that $\mu(f) = \liminf_{\beta \rightarrow \alpha} \mu(f_\beta)$. But since $\mu'(f) < 0$ this implies that there exists at least one β with $\mu'(f_\beta) < 0$, which is a contradiction. $\square$

Unfortunately, the main goal of this section is not so easy to prove. Our proof will be constructive, and use transfinite induction, in addition to using Cantor's normal form theorem to recognize that any ordinal can be written as $\alpha = \omega \cdot \gamma + n$.

We are now prepared to show the main result.

Theorem 3.7. *Let α be successor ordinal. There exists a pair of matroids (M, W) such that the μ' -barrier of (M, W) is α .*

Proof. If α is finite, then let $M_{0,\alpha}$ be the matroid on α elements with no circuits, and let $W_{0,\alpha}$ be the matroid on α elements with a single circuit consisting of the entire ground set. It is easy to verify that (M_0, W_0) is not μ' -admissible, and that any string $f \in \Gamma'(M_{0,\alpha}, W_{0,\alpha})$ with $\mu'(f) < 0$ must have $\text{dom}(f) = \alpha$.

If α is not finite, by theorem 2.6 we can write $\alpha = \omega \cdot \gamma + n$. We will now perform induction on γ by constructing a bipartite graph $G_{\omega\gamma+n}$ with parts A and B , and then letting $M_{\omega\gamma+n}$ and $W_{\omega\gamma+n}$ be the partition matroids arising from $G_{\omega\gamma+n}$ by using A and B respectively. For $\gamma = 1$, we construct $G_{\omega+n}$ in the following manner: Let $A = \{0\} \times (\omega + n)$, $B = \{1\} \times (\omega + n - 1)$ and then $V(G_{\omega+n}) = A \cup B$. We define the edge set of $G_{\omega+n}$ as

$$E(G_{\omega+n}) = \{((0, k), (1, k)) : k \in \omega\} \cup \{((0, \omega + k), v) : 0 \leq k < n, v \in B\}$$

For a visual representation of $G_{\omega+1}$, see figure 2. Let $\bar{V} = \{(0, \omega + k) : 0 \leq k < n\}$ be the set of all vertices of G with infinite degree.

We now define f with $\text{dom}(f) = \omega + n$ and let $f(0) = ((0, 0), (1, 0))$. Then on $(0, n(n-1)]$ we define f to be all edges incident to any vertex of the form $(1, \omega + k)$, And on $[n(n-1) + 1, n^2 + 1]$ we continue with all edges incident to $(1, 1)$, using first the edge not incident to any vertex in $\bar{V}$, and then following with the edges incident to vertices in $\bar{V}$. We then repeat this for all vertices of the form $(1, k)$ where $k \geq 2$ is an integer. After the limit step, we then use all edges incident to $(1, 0)$ that have not already been used.

It is easily verified that $\mu'(f) = -1$, and so $\bar{\mu}'(M_{\omega+n}, W_{\omega+n}) \leq \omega + n$. Furthermore, by

Figure 2: $G_{\omega+1}$

proposition 3.6 we know that $\bar{\mu}'(M_{\omega+n}, W_{\omega+n}) \neq \omega$.

Now we show that if g is a string with $\text{dom}(g) < \omega + n$ then $\mu'(g) \geq 0$. Define $L(g)$ where g is a string in $\Gamma'(M_{\omega+n}, W_{\omega+n})$, as the set of all vertices $v \in B$ such that there is an edge incident to v in $\text{rng}(g)$ and every edge incident to v in $\text{rng}(g)$ is also incident to a vertex in $\bar{V}$. Similarly, we define $R(g)$ as the set of all $v \in \bar{V} \subset A$ such that the set S of all edges incident to v is locked in g . Then by induction we can show that $\mu'(g) \geq |L(g)| - |R(g)|$. It now suffices to show that $|L(g)| - |R(g)| \geq 0$ whenever $\text{dom}(g) < \omega + n$. If $|R(g)| > 0$ then each vertex of the form $(1, \omega + k)$ belongs to $L(g)$, so $|L(g)| \geq n - 1$. But, since $\text{dom}(g) < \omega + n$ we have that $|R(g)| < n$ and so $|L(g)| - |R(g)| \geq 0$ as desired. Thus we have that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) = \omega\gamma + n$.

Now assume that for $\beta < \gamma$ we have that $\bar{\mu}'(M_{\omega\beta+n}, W_{\omega\beta+n}) = \omega\beta + n$ and furthermore when $n = 1$, there is a vertex $v \in A$ in $G_{\omega\beta+1}$ such that the following two properties hold:

1. There exists a string $g \in \Gamma'(M_{\omega\beta+1}, W_{\omega\beta+1})$ using all edges incident to v with $\text{dom}(g) = \omega\beta$, and for any $\tau \leq \text{dom}(g)$ we have $\mu'(g \upharpoonright \tau) = 0$.

2. If a vertex u is added to B and connected to v to form the bipartite graph H_β , the pair of partition matroids $(M_{H_\beta}, W_{H_\beta})$ arising from H_β is μ' -admissible.

For $G_{\omega+1}$ we identify v as the only element in $\bar{V}$. That property (2) is satisfied is clear. For property (1) we use the string g defined by

$$g(\alpha) = \begin{cases} ((1, \frac{\alpha}{2}), (0, \frac{\alpha}{2})) & \alpha \text{ is even} \\ ((1, \frac{\alpha-1}{2}), (0, \omega)) & \alpha \text{ is odd} \end{cases}$$

Induction Step: Assume that $\gamma = \beta + 1$, a successor ordinal. We construct $G_{\omega\gamma+n}$ in the following manner:

Figure 3: $G_{\omega\gamma+2}$

Let H_{β} be the graph obtained from $G_{\omega\beta+1}$ by adding a vertex u to B and attaching it to the vertex v identified for properties (1) and (2). Then let H_i be a copy of H_{β} for each $i \in \omega$ and let A_i, B_i be the corresponding vertex partitions for H_i and let u_i and v_i be the corresponding vertices for u and v . We let $\bar{V} = \{v_{\omega+i} : 0 \leq i < n\}$ and $\bar{U} = \{u_{\omega+i} : 0 \leq i < n-1\}$ be two distinct sets of additional vertices. We then let $A = \bar{V} \cup (\bigcup_{i \in \omega} A_i)$ and $B = \bar{U} \cup (\bigcup_{i \in \omega} B_i)$. We define $V(G_{\omega\gamma+n}) = A \cup B$, let $U = \{u_i : i \in \omega\}$, and $T = \{(u, v) : u \in U \cup \bar{U}, v \in \bar{V}\}$. The edge set of $G_{\omega\gamma+n}$ is then defined to be $E(G_{\omega\gamma+n}) = T \cup \bigcup_{i \in \omega} E(H_i)$. See figure 2 for a sketch of $G_{\omega\gamma+2}$.

We now show that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) = \omega\gamma + n$. First, we use that $G_{\omega\beta+1}$ satisfies property (1) to identify ω different strings h_i in $G_{\omega\gamma+n}$ with $\text{dom}(h_i) = \omega\beta$ for which $\mu'(h_i \upharpoonright \tau) = 0$ for all $\tau \leq \text{dom}(h_i)$. Using a diagonalization argument, we can combine these strings into a single string h with $\text{dom}(h) = \omega\beta$ and $\mu'(h \upharpoonright \tau) = 0$ for all $\tau \leq \omega\beta$. We then extend h to a string f with $\text{dom}(f) = \omega\gamma + n$ by defining $f(\tau) = h(\tau)$ for $\tau < \omega\beta$ and $f(\omega\beta)$ is the edge incident to u_0 which is not incident to any vertex in $\bar{V}$. Then on $(\omega\beta, \omega\beta + n(n-1)]$ we define f to be all edges incident to any vertex in $\bar{U}$. On $[\omega\beta + n(n-1) + 1, \omega\beta + n^2 + 1]$ we continue with all edges incident to u_1 using first any edge not incident to any vertex in $\bar{V}$ and then following with all edges incident to vertices in $\bar{V}$. We repeat this for all vertices u_i with $i \geq 2$. On $[\omega\gamma, \omega\gamma + n)$ we use the remaining n edges incident to u_0 . Once again, it is easy to verify that $\mu'(f) = -1$ which shows that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) \leq \omega\gamma + n$.

It now suffices to show that every string $g \in \Gamma'(M_{\omega\gamma+n}, W_{\omega\gamma+n})$ with $\text{dom}(g) \leq \omega\gamma + n$ has $\mu'(g) \geq 0$. So fix $g \in \Gamma'(M_{\omega\gamma+n}, W_{\omega\gamma+n})$ and let $L(g)$ be the set of all vertices $u \in U \cup \bar{U}$ such that $\text{rng}(g)$ contains an edge incident to u , and every edge incident to u in $\text{rng}(g)$ is incident to a vertex in $\bar{V}$. For each $v \in \bar{V}$ we let S_v be the set of all edges incident to v . We then let $R(g) = \{e_v : e_v \text{ is the key of } S_v \text{ in } g\}$. Lastly, we let h_i be the substring of g with $\text{rng}(h_i) = \text{rng}(g) \cap E(H_i)$. Observe that by property (2) of the induction hypothesis that $\mu'(h_i) \geq 0$. A straightforward induction then shows that

$$\mu'(g) \geq \sum_{i=0}^{\omega} \mu'(h_i) + |L(g)| - |R(g)|$$

Now in order for $\sum_{n=0}^{\omega} \mu'(h_i) + |L(g)| - |R(g)| < 0$ we must have that $R(g) \neq \emptyset$. So let $\tau = \min(g^{-1}(R(g)))$. We will show that $|R(g)| = n$ and $\tau \geq \omega\gamma$, which will force $\text{dom}(g) \geq \omega\gamma + n$. Now since $R(g \upharpoonright \tau + 1) \neq \emptyset$, we know that there is some $v \in \bar{V}$ for which all edges incident to v are contained in $\text{rng}(g \upharpoonright \tau + 1)$. This forces $\bar{U} \subseteq L(g \upharpoonright \tau + 1)$ and since $|\bar{U}| = n - 1$, we must have that $\bar{V} = R(g)$, showing that $|R(g)| = n$. Furthermore, to avoid $L(g \upharpoonright \tau + 1)$ being infinite, we must have that for all but finitely many $i \in \omega$ the edge $(u_i, v_i) \in \text{rng}(g \upharpoonright \tau)$. Now observe that $G_{\omega\cdot\beta+1}$ and H_i satisfy the conditions of lemma 3.4 and so in order to prevent $\mu'(h_i \upharpoonright \tau + 1) \geq 1$ being true for infinitely many $i \in \omega$, we must have that for all but finitely many $i \in \omega$ that $\text{dom}(h_i \upharpoonright \tau + 1) > \omega\beta$. Thus we have that $\tau \geq \omega\beta + \omega = \omega\gamma$.

Thus we have $\bar{\mu}'(M_{\omega\cdot\gamma+n}, W_{\omega\cdot\gamma+n}) = \omega \cdot \gamma + n$. To show that $G_{\omega\gamma+1}$ satisfies properties (1) and (2), we identify v as the only element of $\bar{V}$. With this identification, that $G_{\omega\gamma+1}$ satisfies property (2) should be obvious.

For property (1) we construct the desired string g in the following manner: We use the fact that $G_{\omega\cdot\beta+1}$ satisfies property (1) to identify ω different strings h_i in $G_{\omega\cdot\gamma+1}$ with $\text{dom}(h_i) = \omega\beta$ for which $\mu'(h_i \upharpoonright \tau) = 0$ for all $\tau \leq \text{dom}(h_i)$. Using a diagonalization argument, we can combine these strings into a single string h with $\text{dom}(h) = \omega\beta$ and $\mu'(h \upharpoonright \tau) = 0$ for all $\tau \leq \omega\beta$. Then we extend h to a string g with $\text{dom}(g) = \omega\gamma$ by

Figure 4: $G_{\omega\gamma+2}$

$$g(\tau) = \begin{cases} h(\tau) & \tau < \omega\beta \\ ((u_k, v_k)) & \tau = \omega\beta + 2k \\ ((u_k, v_\omega)) & \tau = \omega\beta + 2k + 1 \end{cases}$$

That $\mu'(g \upharpoonright \tau) = 0$ for all $\tau \leq \omega\gamma$ should be clear.

Limit Step: We now consider the case when γ is a limit ordinal, and we construct $G_{\omega\gamma+n}$ in the following manner:

For each $0 < \beta < \gamma$, we take the graph $G_{\omega\beta+1}$ and add a vertex u to B and attach it to the vertex v identified by the induction hypothesis to form the graph H_β . We then take a copy of H_β for each $0 < \beta < \gamma$ and identify the vertex partitions of H_β as A_β and B_β , in addition to the vertices u and v as u_β and v_β respectively. We let $\bar{V} = \{v_{\gamma+i} : 0 \leq i < n\}$ and $\bar{U} = \{u_{\gamma+i} : 0 \leq i < n-1\}$ be two distinct sets of additional vertices. Then let $A = \bar{V} \cup \bigcup_{\tau < \gamma} A_\tau$, $B = \bar{U} \cup \bigcup_{\tau < \gamma} B_\tau$ and identify $U = \{u_i : i < \gamma\}$ with $T = \{(u, v) : u \in U \cup \bar{U}, v \in \bar{V}\}$. The vertex set of $G_{\omega\gamma+n}$ is $A \cup B$, and we let the edge set $E(G_{\omega\gamma+n}) = T \cup \bigcup_{0 < \beta < \gamma} E(H_\beta)$.

To show that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) \leq \omega\gamma + n$, we use property (1) of the induction hypothesis on $G_{\omega\beta+1}$ for each $0 < \beta < \gamma$. This yields γ different strings h_β with the property that $\text{dom}(h_\beta) = \omega\beta$, and for any $\tau \leq \text{dom}(h_\beta)$ we have $\mu'(h_\beta \upharpoonright \tau) = 0$. Now for $1 < \beta < \gamma$ we extend h_β to $\bar{h}_\beta$ by adding first the edge (u_β, v_β) and then all remaining edges incident to u_β at the end of h_β . For h_1

we only add the edge (u_1, v_1) to the end of h_1 to obtain $\bar{h}_1$. Observe that for each $0 < \beta < \gamma$ we still have that $\bar{h}_\beta$ still has the property that for any $\tau \leq \text{dom}(\bar{h}_\beta)$ we have $\mu'(\bar{h}_\beta \upharpoonright \tau) = 0$. Observe that we can now combine all of the strings $\bar{h}_\beta$ into a single string h with $\text{dom}(h) = \omega\gamma$ and for any $\tau \leq \text{dom}(h)$ we have $\mu'(h \upharpoonright \tau) = 0$. We then form a string g using all edges incident any vertex in $\bar{U}$. We then let f be the string obtained by extending g by h . Lastly, we extend f to a string $\bar{f}$ with $\text{dom}(\bar{f}) = \omega\gamma + n$ by adding the unused edges incident to u_1 onto the end of f . It should be clear that $\mu'(\bar{f}) = -1$, thus showing that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) \leq \omega\gamma + n$.

To show that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) \geq \omega\gamma + n$, fix a string $g \in \Gamma'(M_{\omega\gamma+n}, W_{\omega\gamma+n})$ and let $L(g)$ be the set of all vertices $u \in U \cup \bar{U}$ such that $\text{rng}(g)$ contains an edge incident to u , and every edge incident to u in $\text{rng}(g)$ is incident to a vertex in $\bar{V}$. For each $v \in \bar{V}$ we let S_v be the set of all edges incident to v . We then let $R(g) = \{e_v : e_v \text{ is the key of } S_v \text{ in } g\}$. Lastly, for each $0 < \beta < \gamma$ we let h_β be the substring of g with $\text{rng}(h_\beta) = \text{rng}(g) \cap E(H_\beta)$. Once again, by property (2) of the induction hypothesis $\mu'(h_\beta) \geq 0$. A straightforward induction then shows that

$$\mu'(g) \geq \sum_{0 < \beta < \gamma} \mu'(h_\beta) + |L(g)| - |R(g)|$$

Once again, we want $\sum_{0 < \beta < \gamma} \mu'(h_i) + |L(g)| - |R(g)| < 0$, and by property (2) of the induction hypothesis we know that $\mu'(h_\beta) \geq 0$ for each $0 < \beta < \gamma$. Thus we must have that $R(g) \neq \emptyset$ and so we let $\tau = \min(g^{-1}(R(g)))$. We will show that we actually need $|R(g)| = n$ and $\tau \geq \omega\gamma$. Since $R(g \upharpoonright \tau) \neq \emptyset$ in order for $|L(g)|$ to be finite we must have that all but finitely many edges (u_β, v_β) must belong to $\text{rng}(g \upharpoonright \tau)$. But since $G_{\beta\omega+1}$ with v_β satisfy the conditions of lemma 3.4, in order to prevent $\sum_{0 < \beta < \gamma} \mu'(h_i \upharpoonright \tau)$ from being infinite, we must have that all but finitely many h_β have $\text{dom}(h_\beta \upharpoonright \tau) > \omega\beta$ thus forcing $\tau \geq \omega\gamma$. On the other hand, we know that if $R(g) \neq \emptyset$ then $\bar{U} \subseteq L(g)$ and so $|L(g)| \geq n - 1$. Since $R(g) \subseteq \bar{V}$, we know that $|R(g)| \leq n$, so in order for $\sum_{0 < \beta < \gamma} \mu'(h_i) + |L(g)| - |R(g)| < 0$ to be true we must have $|R(g)| = n$. This then shows that $\bar{\mu}'(M_{\omega\gamma+n}, W_{\omega\gamma+n}) \geq \omega\gamma + n$.

For properties (1) and (2), we identify the single element of $\bar{V}$ as the required v for $G_{\omega\gamma+1}$.

That property (2) is satisfied is obvious. For property (1) we construct the desired string in the following manner:

We use property (1) of the induction hypothesis on $G_{\omega\beta+1}$ for each $0 < \beta < \gamma$. This yields γ different strings h_β with the property that $\text{dom}(h_\beta) = \omega\beta$, and for any $\tau \leq \text{dom}(h_\beta)$ we have $\mu'(h_\beta \upharpoonright \tau) = 0$. Now for $0 < \beta < \gamma$ we extend h_β to $\bar{h}_\beta$ by adding first the edge $(u_\beta v_\beta)$ and then all of the remaining edges incident to u_β at the end of h_β . Observe that for each $0 < \beta < \gamma$ we still have that $\bar{h}_\beta$ still has the property that for any $\tau \leq \text{dom}(\bar{h}_\beta)$ we have $\mu'(\bar{h}_\beta \upharpoonright \tau) = 0$. Observe that we can now combine all of the strings $\bar{h}_\beta$ into a single string h with $\text{dom}(h) = \omega\gamma$ and for any $\tau \leq \text{dom}(h)$ we have $\mu'(h \upharpoonright \tau) = 0$, showing property (1) and completing the proof. $\square$

4 Patchwork Matroids

In this Section we introduce patchwork matroids and show several results concerning them. We show how to construct patchwork matroids using patches, introduce several kinds of patches and prove the patchwork characterization theorem. We then conclude this section by showing several properties of patchwork matroids.

4.1 Patches

In order to define a patch, we must first introduce an equivalence relation. Let E be a set and $A, B \subseteq E$. We say that $A \simeq B$ if and only if $|A \triangle B|$ is finite. We will denote the $\simeq$ -equivalence class of A by $[A]$, and partially order the set of equivalence classes by saying that $[A] \leq [B]$ when $|A \setminus B|$ is finite. We then define a patch in the following manner.

Definition 4.1. Let E be a set and A a subset of E . We then say that $\mathcal{B} \subset [A]$ is a *patch* on $[A]$ if $\mathcal{B}$ satisfies

(P1) $\mathcal{B} \neq \emptyset$

(P2) $\forall B_1, B_2 \in \mathcal{B}$ and each $x \in B_1 \setminus B_2$ there exists a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$

(P3) If $[C] < [A]$ then for all $C \in [C]$ there exists $B \in \mathcal{B}$ such that $C \subset B$

(P4) If $[A] < [C]$ then for all $C \in [C]$ there exists $B \in \mathcal{B}$ such that $B \subset C$.

The easiest example of patches are uniform patches. A uniform patch on $[A]$ can be constructed by picking an arbitrary $B \in [A]$ and then letting $\mathcal{B} = \{B' \in [A] : |B' \setminus B| = |B \setminus B'|\}$. Note that calling this a uniform patch is appropriate, as elements of $\mathcal{B}$ are basically swapping out finitely many elements from B for an equal number of arbitrarily chosen elements from $E \setminus B$, mimicking the behavior of bases of uniform matroids. However, the definition of a patch leaves room for plenty of flexibility, and we can then consider other kinds of patches, such as those that arise from locally finite bipartite graphs (which we call bipartite patches).

Thus we have:

Proposition 4.2. *Let G be a locally finite bipartite graph with vertex partitions A_1 and A_2 . Let $\mathcal{A}$ be the family of all subgraphs of the bipartite complement of G which are induced by a finite matching. Then the set*

$$\mathcal{B} = \{(A_1 \setminus X) \cup Y : \exists H \in \mathcal{A} \text{ such that } X = V(H) \cap A_1, Y = V(H) \cap A_2\}$$

is a patch on $[A_1]$. Patches constructed in such a fashion are bipartite patches.

Proof. Clearly (P1) is satisfied. To show (P2), let $B_1 = (A_1 \setminus X_1) \cup Y_1$ and $B_2 = (A_1 \setminus X_2) \cup Y_2$, with $x \in B_1 \setminus B_2$. Thus $x \in ((A_1 \setminus X_1) \cup Y_1) \setminus ((A_1 \setminus X_2) \cup Y_2)$, or more simply $x \in (Y_1 \setminus Y_2) \cup (X_2 \setminus X_1)$. We want to show that there exists a $y \in (Y_2 \setminus Y_1) \cup (X_1 \setminus X_2)$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. Let H_1 and H_2 be the graphs associated with B_1 and B_2 respectively. Let H be the graph with $E(H) = E(H_1) \cup E(H_2)$ and $V(H) = V(H_1) \cup V(H_2)$, and observe that x has degree 1 in H , and that the edge incident to x is not in both H_1 and H_2 . Thus we know there exists a path P from x to some vertex y of degree 1 in H , and that the edges of P alternate as to whether or not they belong to H_1 or H_2 . Let $M = (E(H_1) \setminus E(P)) \cup (E(H_2) \cap E(P))$ and let T be the graph induced by the set of edges M . We first claim that $T \in \mathcal{A}$, since the edge in H incident to x cannot belong to both H_1 or H_2 , and so the edges in P must alternate being in H_1 and H_2 (two consecutive edges from the same graph would imply that the graph has a vertex of degree 2). Thus $T \in \mathcal{A}$, since

for each edge added to H_1 from H_2 we have an edge incident to the same vertex being removed (if such an edge exists). Finally, we claim that $(A_1 \setminus V(T)) \cup (V(T) \cap A_2) = (B_1 \setminus \{x\}) \cup \{y\}$ for some $y \in (Y_2 \setminus Y_1) \cup (X_1 \setminus X_2)$. To realize this, consider the following: if $x \in Y_1$ then there was an edge incident to x in H_1 and in P , and so that edge isn't present in T . On the other hand, if $x \in X_2$ then there wasn't an edge incident to x in H_1 and so the edge which is incident to it in P is also present in T . A similar conclusion follows for y (at the other end of the path P).

To show (P3) suppose $[F] < [A_1]$. Then $|F \setminus A_1|$ is finite and $|A_1 \setminus F|$ is infinite. Since G is locally finite, we can clearly find a 1-regular graph H with $V(H) \cap A_2 = F \setminus A_1$ and $V(H) \cap A_1 \subset A_1 \setminus F$ such that H is a subgraph of the complement of G . Since this $H \in \mathcal{A}$, clearly $(A_1 \setminus V(H)) \cup (F \setminus A_1) \in \mathcal{B}$.

The proof of (P4) is similar, except in this case $|A_1 \setminus F|$ is infinite and $|F \setminus A_1|$ is finite. But this doesn't prevent us from finding H , and so the desired result will hold. $\square$

It should be noted that if we let G be a graph with no edges, then the associated bipartite patch will be a uniform patch.

Proposition 4.3. *Let G be a directed graph, $E \subseteq V(G)$ and $B \subset E$. Let $\mathcal{B}$ be the collection of all subsets C of E such that there is a finite set of disjoint directed paths from $E \setminus B$ to B with $C \setminus B$ as the set of start vertices and $B \setminus C$ as the set of end vertices. If B and G satisfy*

(1) *For every $x \in E \setminus B$ and $B' \subseteq B$ there are infinitely many disjoint paths from x to B' which avoid $E \setminus B$.*

(2) *For every $x \in B$ and $B' \subset E \setminus B$ there are infinitely many disjoint paths from x to B' which avoid B .*

Then $\mathcal{B}$ is a patch on $[B]$, and we call such patches Gammoid patches.

Proof. (P1) follows easily, since clearly $B \in \mathcal{B}$. (P2) follows from finite matroid theory in the following manner: Let $B_1, B_2 \in \mathcal{B}$, and let $\mathcal{P}_1, \mathcal{P}_2$ be sets of disjoint paths which witnessing that B_1 and B_2 belong to $\mathcal{B}$ respectively. Let P_1 be the set of all vertices not in B or B_1 but in some path in $\mathcal{P}_1$ and define P_2 similarly. Let $H = (B_1 \cup B_2) \setminus B$ and let $K = B \setminus (B_1 \cap B_2)$. Observe that $R = H \cup K \cup P_1 \cup P_2$ is finite. Let D be the directed graph induced by R . Then we can build

a finite gammoid using D and K , where $B_1 \cap R$ and $B_2 \cap R$ are maximally independent. Base exchange in the finite gammoid D then yields the desired result.

For (P3), assume that $[C] < [B]$. Since $[C] < [B]$ we know that $|B \setminus C|$ is infinite and $|C \setminus B|$ is finite. Then let $B' = B \setminus C$. That (1) extends to finite sets $X \subset E \setminus B$ is easily shown by induction given that there are infinitely many disjoint paths and each of the paths is finite in length. (P4) can be shown similarly, using (2) instead. $\square$

The definition of a Gammoid patch makes it clear that we can actually define a second patch simply by reversing the direction of the edges and using $E \setminus B$ in place of B . This naturally gives rise to what we'd expect to call the dual of the gammoid patch. That is, if we have $P \subseteq E$ and $\mathcal{B}$ a Gammoid patch on $[P]$, then $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ will be a Gammoid patch on $[E \setminus P]$. We can quickly extend this to all patches as follows:

Lemma 4.4. *Let $P \subseteq E$ and let $\mathcal{B}$ be a patch on $[P]$. Then $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ is a patch on $[E \setminus P]$ and we say that $\mathcal{B}^*$ is the dual of $\mathcal{B}$*

Proof. Let $\mathcal{B}$ be a patch on $[P]$. We wish to show that $\mathcal{B}' = \{E \setminus B : B \in \mathcal{B}\}$ is a patch on $[E \setminus P]$. Clearly (P1) will be satisfied. To show (P2), let $B'_1, B'_2 \in \mathcal{B}'$ with $B'_1 = E \setminus B_1$, $B'_2 = E \setminus B_2$, and let $x \in B'_1 \setminus B'_2$. Then $x \in (E \setminus B_1) \setminus (E \setminus B_2) = B_2 \setminus B_1$. So since $\mathcal{B}$ is subset of the bases of a matroid we know by Proposition 2.18 that there exists $y \in B_1 \setminus B_2 = B'_2 \setminus B'_1$ with $B_1 \setminus \{y\} \cup \{x\} \in \mathcal{B}$. Notice then that $E \setminus (B_1 \setminus \{y\} \cup \{x\}) = (E \setminus B_1) \setminus x \cup \{y\} = B'_1 \setminus \{x\} \cup \{y\}$.

To show (P3) let $[F] < [E \setminus P]$ with $F \in [F]$. Then $[P] < [E \setminus F]$, and by (P3) on $\mathcal{B}$ we know that there exists $B \subset E \setminus F$ with $B \in \mathcal{B}$. This then gives that $F \subset E \setminus B$ and since $E \setminus B \in \mathcal{B}'$ we are done. (P4) follows similarly. $\square$

One of our goals is to find a general way of constructing paving matroids. To do this we take inspiration from Proposition 2.1.21 of [21]. As such we will need to worry about the relative size of various sets, so we introduce the following equivalence relation: $A \sim B$ if $|A \setminus B| = |B \setminus A|$ is finite. We let $\langle A \rangle$ be the equivalence class of A under $\sim$. Then we can introduce the general notion of a paving partition as follows.

Definition: Let E be a set and $P \subseteq E$ and $A \in [P]$. A *paving partition* of $[P]$ is a set $\mathcal{S} \subset [P]$ with $E \notin \mathcal{S}$ such that for each $A' \in \langle A \rangle$ there is a unique $S \in \mathcal{S}$ such that $A' \subseteq S$. We will call $S \in \mathcal{S}$ a *stone*, $A' \in \langle A \rangle$ a *pebble*, $\langle A \rangle$ the *ground layer*, and $\langle B \rangle$, where $|B \setminus A| = 1$, is the base layer.

Once we have paving partitions, we can use them to define paving patches in the following manner.

Theorem 4.5. *Let $\mathcal{S}$ be a paving partition of $[P]$ with $\langle T \rangle$ the ground layer and $\langle B \rangle$ the base layer. Let $\mathcal{B} = \{B' \in \langle B \rangle : \exists S \in \mathcal{S}, B \subseteq S\}$. Then $\mathcal{B}$ is a patch. Patches constructed like this are called paving patches.*

Proof. Since $E \notin \mathcal{S}$, we know there exists an $S \in \mathcal{S}$ and a $y \in E \setminus S$. Let $T \subseteq S$ be a pebble, observe that $T \cup \{y\} \in \mathcal{B}$ and so (P1) is satisfied. For (P2), select any $B_1, B_2 \in \mathcal{B}$, let $x \in B_1 \setminus B_2$, and let $B_1 \setminus \{x\} \subseteq S_1 \in \mathcal{S}$. Since $B_2 \not\subseteq S_1$, there exists $y \in B_2 \setminus S_1$ and so $B \setminus \{x\} \cup \{y\} \not\subseteq S_1$. Since $\mathcal{S}$ is a paving partition, S_1 is the only element of $\mathcal{S}$ for which $B_1 \setminus \{x\}$ is a subset. Thus $B \setminus \{x\} \cup \{y\} \in \mathcal{B}$ since it cannot be a subset of any element of $\mathcal{S}$ and clearly belongs to $\langle B \rangle$.

For (P3) assume that $[F] < [P]$ and $A \in [F]$. By definition of $[F] < [P]$ we know there exists some $C \in [P]$ such that $A \subset C$. Let T be any stone, and observe that $C \triangle T$ is finite so $|C \setminus T|$ is finite hence $|A \setminus T| = k$ is finite, while $|T \setminus A|$ is infinite. Pick any $R \subset T \setminus A$ with $|R| = k$ and observe that $(T \cup A) \setminus R$ is a pebble with A as a subset. Since every pebble is contained in a base (P3) then clearly follows.

To show (P4), let $[P] < [F]$ and $A \in [F]$. First observe that A contains a pebble, because for any pebble T , we know there exists $A_T \in [F]$, and $|A_T \setminus A|$ is finite. So adjust T by removing $T \setminus A$ and adding sufficient elements from $A \setminus T$ (Which is infinite because $A \not\subseteq T$). Next notice that since $\mathcal{S}$ is a paving partition we have that for any $S \in \mathcal{S}$, $S \in [P]$ and so $|A \setminus S| > 0$. Thus if $T \subseteq S_T \in \mathcal{S}$, we know there exists $t \in A \setminus S$ and so $T \cup \{t\} \in \mathcal{B}$. $\square$

The following example illustrates how easy it is to construct paving patches: let $B \subset E$ be infinite and coinfinite. Define a graph J with $\langle B \rangle$ as the set of vertices, connect the vertices A

and B with an edge if $|A \triangle B| = 2$. To generate a paving partition, let $\mathcal{N}$ be a set of independent vertices in J , and our paving partition then is $\mathcal{N}$ plus any element of $\langle T \rangle$ which isn't contained in an element of $\mathcal{N}$. Clearly, this generates a paving partition, and the corresponding patch is simply $\langle B \rangle \setminus \mathcal{N}$. We refer to these paving patches as *sparse paving patches*.

Proposition 4.6. *There is a paving patch which is not a gammoid patch, and a gammoid patch which is not a paving patch.*

Proof. For our paving patch let $E = \mathbb{Z} \setminus \{0\}$ and $\langle \mathbb{Z}^+ \rangle$ be the base layer. Let $\mathcal{T} = \{\mathbb{Z}^+ \setminus \{n\} \cup \{-n\} : n \in \mathbb{N}\} \cup \{\mathbb{Z}^+ \setminus \{1, 2, 3\} \cup \{-1, -2, -3\}\}$ and let our paving partition $\mathcal{S}$ contain $\mathcal{T}$ and any pebbles which are not a subset of $\mathcal{T}$. In order to show that the paving patch $\mathcal{B}$ derived from our paving partition $\mathcal{S}$ is not a gammoid patch, we will show that it is impossible to build a digraph which generates $\mathcal{B}$. Assume that such a digraph D exists. We know that each $n \in \mathbb{Z}^-$ does not have a path to $-n$ since $\{\mathbb{Z}^+ \setminus \{n\} \cup \{-n\} : n \in \mathbb{N}\} \subset \mathcal{N}$. At the same time, we also know that there does not exist a set of three disjoint paths from $\{-1, -2, -3\}$ to $\{1, 2, 3\}$ since $\mathbb{Z}^+ \setminus \{1, 2, 3\} \cup \{-1, -2, -3\} \in \mathcal{N}$. This means there are at most two vertices $\{x, y\}$ such that any path from $\{-1, -2, -3\}$ to $\{1, 2, 3\}$ must pass through either x or y . Furthermore, every vertex of $\{1, 2, 3\}$ is reachable via some path from $\{x, y\}$, since every vertex of $\{1, 2, 3\}$ is reachable via some path from $\{-1, -2, -3\}$. So without loss of generality there exists two paths starting at x and ending at different vertices of $\{1, 2, 3\}$. However, notice that at most one of $\{-1, -2, -3\}$ can then have a path to x . But this then forces the other two vertices to have paths to y generating the same contradiction we sought to avoid by not using x .

For our bipartite patch, let G be a bipartite graph with parts A_1 and A_2 both infinite and $x, y \in A_1, z \in A_2$ and exactly two edges: (x, z) and (y, z) . Observe that $A_1 \setminus \{x, y\} \cup \{z\}$ must be a pebble if the patch is also paving. However, $A_1 \setminus \{x, y\} \cup \{z\}$ is dependent, so the patch clearly can't be paving. $\square$

We close out this section by remarking that we can define a fundamental circuit with respect to patches. Specifically given a patch $\mathcal{B}$ on $[P]$, some $B \in \mathcal{B}$ and $e \in E \setminus B$, we call C a *fundamental*

circuit of e with respect to B if $C \not\subseteq B'$ for all $B' \in \mathcal{B}$, and $|C|$ is minimal. Clearly if such a C exists, then $C \in [P]$. That these circuits exist and are unique will be a consequence of using patches to form matroids.

4.2 Properties of Patchwork Matroids

Our first objective in this section is to show how patchwork matroids are generated from patches. We then show the characterization in Theorem 1.5 describes exactly the same class of matroids as those produced by our construction.

Theorem 4.7. *Let E be a set and $\mathcal{A}$ be a strong antichain of the lattice generated by $\simeq$. Let $\mathcal{B} = \bigcup_{[A] \in \mathcal{A}} \mathcal{B}_{[A]}$, where $\mathcal{B}_{[A]}$ is a patch on $[A]$. Then $\mathcal{B}$ is the set of bases for a matroid on E . We call these matroids Patchwork.*

Proof. That $\mathcal{B}$ satisfies (B1) is obvious. To show (B2), let B_1, B_2 be arbitrary bases, and let $x \in B_1 \setminus B_2$. Let $A = \{y \in E : B \setminus \{x\} \cup \{y\} \notin \mathcal{B}\}$. Observe that for all $B' \in \mathcal{B}_{[B_1]}$ we have that $B' \not\subseteq A$, since $\mathcal{B}_{[B_1]}$ satisfies (B2). Thus we know that $[B_1] \not\leq [A]$. However, since $B_1 \setminus \{x\} \subseteq A$ we have that $[B_1] \leq [A]$, and so $[B_1] = [A]$. Thus $|A \setminus B_1|$ is finite and so if $|B_2 \setminus B_1|$ is infinite, then clearly there exists $y \in B_2 \setminus B_1$ such that $y \notin A$. In this case $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$. On the other hand, if $|B_2 \setminus B_1|$ is finite, then $[B_2] = [B_1]$ and (B2) already holds.

To show (BM), let $I \subseteq B \in \mathcal{B}$ and $I \subseteq X \subseteq E$. Since $\mathcal{A}$ is a strong antichain, we know that there exists an $[A] \in \mathcal{A}$ such that $[I] > [A]$ or $[I] \leq [A]$ and either $[A] \leq [X]$ or $[X] \leq [A]$. However, we can immediately disregard the case where $[I] > [A]$, since that would imply that there is a $B \in \mathcal{B}$ with $B \subset I$. Furthermore in the case where $[I] < [A]$, since $\mathcal{B}_{[A]}$ is a patch, we know that there exists $A_B \in [A]$ with $I \subseteq A_B \in \mathcal{B}$. Without loss of generality, we can assume that $A_B = B$ and so $[A] = [B]$.

Notice that if $[I] = [B]$, then $|B' \setminus I|$ is finite for any $B' \in [B]$, hence $|B' \setminus X|$ is finite, so take a B' such that $|B' \setminus X|$ is minimal to complete the proof. Otherwise consider the following:

If $[X] < [B]$ there exists a $B_A \in \mathcal{B}_{[B]}$ with $X \subset B_A$ from the definition of a patch. Thus

$\{B \cap X | (I \subset B) \wedge (B \in \mathcal{B})\}$ clearly has a maximal element. If $[B] = [X]$ then $|B' \setminus X|$ is finite for all $B' \in [B]$. Thus we can pick a $B' \in \mathcal{B}_{[A]}$ which minimizes $|B' \setminus X|$ and contains I . Lastly, suppose that $[B] < [X]$. Then since $\mathcal{B}_{[B]}$ is a patch, there exists a $B' \in \mathcal{B}_{[B]}$ with $B' \subset X$. Thus $|B \setminus X|$ is finite, and so by using (B2) on B with B' (possibly repeatedly) within the patch $\mathcal{B}_{[B]}$ we can find a $B'' \in \mathcal{B}_{[B]}$ for which $I \subset B'' \subset X$, completing the proof. $\square$

Remark 4.8. The fundamental circuits of patches are precisely the fundamental circuits of a matroid constructed using those patches. This ensures the existence and uniqueness of the fundamental circuits of patches.

Theorem 4.9. *Let M be a matroid on E . M is patchwork if and only if for every $K \subseteq E$ one of the following holds:*

- (1) K is independent
- (2) There exists a base B with $B \subseteq K$
- (3) There exists a base B with $|K \triangle B|$ finite

Proof. Let $\mathcal{A} = \{[B] : B \in \mathcal{B}\}$. We must show for each $[A] \in \mathcal{A}$ that $[A] \cap \mathcal{B}$ is a patch and that $\mathcal{A}$ is a strong antichain. This then implies that if M satisfies the statement then M is patchwork.

To show that $\mathcal{A}$ is a strong antichain, take any two equivalence classes with $[C] \leq [D]$, let $C \in [C]$ and observe that we can find $D \in [D]$ with $C \subset D$. Now notice that if D is independent, then we can find $B \in \mathcal{B}$ with $D \subset B$ and so $[D] \leq [B]$. Similarly, if C contains a base B then $[B] \leq [C]$. In fact if case (3) holds for either C or D , then we get that $[C] = [B]$ or $[D] = [B]$. So the only case left is when C is independent and D contains a base. But by (IM) we know that we can extend C to a maximally independent set B with $B \subset D$, and since D contains a base, we know that $B \in \mathcal{B}$ also holds. Thus we get $[C] \leq [B] \leq [D]$ and have shown that $\mathcal{A}$ is a strong antichain.

Showing that $[A] \cap \mathcal{B}$ is a patch is even easier. We get (P1) by the definition of $\mathcal{A}$ and we get (P2) by virtue of the fact that $\mathcal{B}$ is the set of bases. Then (P3) and (P4) easily follow from the condition implied by the theorem.

On the other hand, if M is patchwork, let $A \in E$ be an arbitrary set. If $[A]$ is an element of the strong antichain $\mathcal{A}$ for M (since M is patchwork), then A satisfies (3). Otherwise, since $\mathcal{A}$ is strong, we know that there is some element $[C] \in \mathcal{A}$ with a patch on it, such that $[C] < [A]$ in which case A contains a base, or $[A] < [C]$, in which case we know A is contained in a base. Thus either (2) or (1) is satisfied. $\square$

It follows that patchwork Matroids are closed under duality and taking minors. However, using theorem 1.5 to prove these facts doesn't show us the effect of duality and taking minors has on patches in a matroid. We start by considering duality.

Theorem 4.10. *The class of patchwork matroids is closed under the operations of duality and taking minors.*

Proof. That the class of patchwork matroids is closed under duality follows immediately from Theorem 1.5. Therefore it suffices to show it is closed under deletion (and contraction will follow). Let M be a patchwork matroid on E , and partition $E = X \cup R$ where R is the set to be removed. Let $\mathcal{A}' = \{[A] \in \mathcal{A} : [A] \leq [X]\}$ be our new antichain. To show that it is strong, let $[C] \leq [D] \leq [X]$ and notice that since $\mathcal{A}$ is a strong antichain we have that there exists some $[A] \in \mathcal{A}$ with one of the following three conditions holding

(1) $[X] \leq [A]$. In this case, the entire proof is done. Since M is patchwork, we know that $X \subset B$ a base of M and so $M \setminus R$ is a trivial matroid.

(2) $[D] \leq [A] \leq [X]$. Notice that $[D] \leq [A]$ and $[A] \in \mathcal{A}'$.

(3) $[A] \leq [D] \leq [X]$. If this case holds, then notice that no $[A'] \in \mathcal{A}' \subset \mathcal{A}$ can satisfy $[D] \leq [A']$. So there exists some $[A'] \in \mathcal{A}$ with either $[C] \leq [A'] \leq [D]$ or $[A'] \leq [C]$, and in both cases such an $[A'] \in \mathcal{A}'$.

Thus $\mathcal{A}'$ is a strong antichain of the $\simeq$ lattice on X .

So let $\mathcal{B}$ be a patch on $[P]$ (with $P \cap R = \emptyset$) viewing $[P]$ as an element of the $\simeq$ lattice on E . Once we show that $\mathcal{B}' = \{B \in \mathcal{B} : |B \setminus R| \text{ is minimal}\}$ is a patch on $[P]$ viewed as an element of the $\simeq$ lattice on X we will be done.

Since $\mathcal{B}$ is a patch on $[P]$ and $P \cap R = \emptyset$, then we know that $|B \setminus R|$ is finite for every $B \in \mathcal{B}$, so there is a B for which $|B \setminus R|$ is minimal. Thus $\mathcal{B}'$ satisfies (P1). (P2) is inherited from $\mathcal{B}$ as well, since it will be impossible to violate the minimality condition using basis exchange. For (P3) let $F \in [F] < [P]$ with $F \cap R = \emptyset$. Since $\mathcal{B}$ is a patch, there is some $B \in \mathcal{B}$ with $F \subset B$. Let $B' \in \mathcal{B}'$. Since $|B \setminus R|$ is finite, we can use (P2) on $\mathcal{B}$ to swap elements from $B \cap R$ with elements from $B' \setminus R$ to find a B'' with $F \subset B''$ and $B' \in \mathcal{B}'$. For (P4), let $F \cap R = \emptyset$ and $[P] < [F]$. Since $\mathcal{B}$ is a patch on $[P]$ we already have that there some $B \in \mathcal{B}$ with $B \subset F$, completing the proof. $\square$

We can now turn our attention to other results, concerning which properties a patchwork Matroid can have.

Theorem 4.11. *Let M be a patchwork matroid which is neither finitary nor cofinitary. Then M is infinitely connected and wild.*

Proof. We start by showing that M is wild. Since the class of patchwork Matroids is closed under duality, M^* is also a patchwork matroid. Let $\mathcal{A}$ and $\mathcal{A}^*$ be the strong antichains associated with M and M^* respectively. Let B be a base of M and observe there is some $[B'] \in \mathcal{A}^*$ such that $[B] \leq [B']$ or $[B'] \leq [B]$. Let C be a fundamental circuit of e with respect to B and D be a fundamental cocircuit of f with respect to B' . Without loss of generality, assume that $[B] \leq [B']$. Thus $C \setminus D$ is finite. Since M is neither finitary nor cofinitary, by Theorem 1.5 C must be infinite, so $C \cap D$ is infinite and M is wild.

To show that M is infinitely connected suppose that (X, Y) is a k -separation with k finite. Then from the previous theorem, we know that X must either be independent, contain a base, or have finite symmetric difference with a base. If X contains a base, then Y must have rank k . So Y clearly can't contain a base, or have finite symmetric difference with one. Therefore Y is independent and so $|Y| = k$. But then (X, Y) is not a k -separation. Suppose then that X has finite symmetric difference from a base. Then Y must have finite rank, hence independent, and thus finite. But, X contains finitely many elements not in a base. So thus M is cofinitary, which is a contradiction. So the only way for (X, Y) to be a finite separation, is for both X and Y to be

independent. But then, once again, M becomes cofinitary which is a contradiction. So M cannot have any finite separations. $\square$

At this point we observe that in [14] the existence of an infinitely connected matroid was posed as an open problem, and although a solution was presented in [9], its existence required the use of Martin's Axiom. As the matroid constructed in [9] is uniform, it is patchwork, and so is covered by the above theorem. However, the above theorem doesn't show that such matroids exist in ZFC alone, so the problem is still open.

Next, recall from earlier that one of the most important problems in Infinite Matroid Theory is the Matroid Intersection Conjecture which is as follows:

Conjecture 4.12. *Any two matroids M and N on a common ground set E have a common independent set I admitting a partition $I = J_M \cup J_N$ such that $cl_M(J_M) \cup cl_N(J_N) = E$.*

Where cl is the closure function. It can then be easily seen that if B is a base of M , then $cl(B) = E$. Combined with Theorem 1.5 the following result seems natural.

Theorem 4.13. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be patches on $[P]$ with P infinite. Then there exists B a base in one patch and independent in the other.*

Proof. Let $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ be selected such that $|B_1 \triangle B_2| = n$ is minimal. If $n < 2$ then we are done. If $n > 2$ then let $x \in B_1 - B_2$ and $y \in B_2 - B_1$ (If one doesn't exist, done). Then let $C_y^{B_1}$ and $C_x^{B_2}$ be the respective cocircuits and notice both are elements of $[P]$. Therefore there exists $z \in C_y^{B_1} \cap C_x^{B_2}$, and so $B_1 \setminus \{z\} \cup \{y\} \in \mathcal{B}_1$ and $B_2 \setminus \{z\} \cup \{x\} \in \mathcal{B}_2$ with smaller symmetric difference, generating a contradiction. $\square$

This theorem together with the example of a non-gammoid paving patch leads to the interesting question of whether or not finite matroids can be embedded in a patch of arbitrary size. With the free product as defined by Crapo and Schmitt in [15] we can obtain the following two results.

Lemma 4.14. *Let E_1 and E_2 be disjoint sets, $\mathcal{B}_1$ a patch on $[\emptyset]$ with ground set E_1 , and $\mathcal{B}_2$ a patch on $[E_2]$ with ground set E_2 . Define*

$$\mathcal{B} = \{B_1 \cup B_2 \cup M \setminus N : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, \\ N \subseteq B_2, M \subseteq E_1 \setminus B_1, |N| = |M| \text{ finite}\}$$

. Then $\mathcal{B}$ is a patch on $[E_2]$ with ground set $E_1 \cup E_2$. We will denote this as $\mathcal{B}_1 \square \mathcal{B}_2 = \mathcal{B}$.

Proof. (P1) is satisfied by taking $M = N = \emptyset$. For (P2) let $D_1 = ((B_1 \cup B_2) \setminus N_1) \cup M_1$ and $D_2 = ((B_3 \cup B_4) \setminus N_2) \cup M_2$ be elements of $\mathcal{B}$. Let $x \in D_1 \setminus D_2$ and we wish to find a $y \in D_2 \setminus D_1$ such that $D_1 \setminus \{x\} \cup \{y\}$ is an element of $\mathcal{B}$. So suppose $x \in E_2$. If there exists $y \in (B_3 \cup M_2) \setminus (B_1 \cup M_1)$, then we are done, since $((B_1 \cup B_2) \setminus (N_1 \cup \{x\})) \cup (M_1 \cup \{y\}) = D_1 \setminus \{x\} \cup \{y\}$. If, however, $(B_3 \cup M_2) \setminus (B_1 \cup M_1) = \emptyset$, then we know that $|N_2| \leq |N_1|$ since $\mathcal{B}_1$ consists of finite sets. Thus $|(B_2 \setminus N_1) \setminus (B_4 \setminus N_2)| \leq |(B_4 \setminus N_2) \setminus (B_2 \setminus N_1)|$. At the same time, let $\mathcal{M}$ be the matroid on E_2 with $\mathcal{B}_2$ as its set of bases. (This happens because the antichain consisting only of $[E_2]$ is strong). Let $T = (B_2 \setminus N_1) \cap (B_4 \setminus N_2)$ and consider $\mathcal{M}/T$. Since elements of $\mathcal{B}_2$ are cofinite in E_2 we know that $\mathcal{M}/T$ is a finitary matroid and therefore there exists a $y \in (B_4 \setminus N_2)$ for which $(B_2 \setminus N_1) \cup \{y\} \setminus \{x\}$ is independent in $\mathcal{M}/T$. This y is the desired element. If $x \in E_1$ and there is a base $\bar{B}$ of $\mathcal{B}_1$ with $x \in \bar{B} \subseteq B_1 \cup M_1$ use (P2) in $\mathcal{B}_1$ with $\bar{B}$ and B_3 to find y . Otherwise notice that $x \in M_1 \setminus M_2$ is true regardless of the base chosen for B_1 . Now if $(B_3 \cup M_2) \setminus (B_1 \cup M_1)$ is nonempty, any element of it will work for y . Otherwise $(B_3 \cup M_2) \subseteq (B_1 \cup M_1)$ so $|M_2| < |M_1|$, and so $|N_2| < |N_1|$. Thus any element of $(B_4 \setminus N_2) \setminus (B_2 \setminus N_1)$ will work, and there must be one since $|B_4 \setminus B_2| = |B_2 \setminus B_4|$.

For (P3), suppose $[F] < [E_2]$ with ground set $E_1 \cup E_2$. If E_2 is finite, then this case is irrelevant because $[E_2]$ is then the finite equivalence class. So we assume that E_2 is infinite. Let $F_1 = F \cap E_1$ and $F_2 = F \cap E_2$. Since F_1 must be finite, we know that $[F_2] < [E_2]$. Therefore, there is some $B_2 \in \mathcal{B}_2$ with $F_2 \subset B_2$. Since F_1 is finite, we can find a $B_1 \in \mathcal{B}_1$ such that $F_1 \cap B_1$ is maximal (recall $\mathcal{B}_1$ is a finite patch). If $F_1 \subseteq B_1$ we are done, as $B_1 \cup B_2 \in \mathcal{B}$. If $B_1 \subset F_1$, we let $M = F_1 \setminus B_1$ (and therefore clearly finite) and N any subset of $B_2 \setminus F_2$ with $|N| = |M|$. We know such a N

exists since $B_2 \setminus F_2$ is infinite due to $[F_2] < [E_2]$.

For (P4), we have that $[E_2] < [F]$. Once again, let $F_1 = F \cap E_1$ and $F_2 = F \cap E_2$. Since $\mathcal{B}_2$ is a patch on $[E_2]$, each $B \in \mathcal{B}_2$ is cofinite in E_2 . Thus $[E_2] = [F_2]$. Now if E_1 is finite, then we have that $[\emptyset] = [F_1]$ and $[E_2] < [F]$ is impossible (and so trivially satisfied). Otherwise $[\emptyset] < [F_1]$ and so we can find a $B_1 \in \mathcal{B}_1$ such that $B_1 \subset F_1$, and more importantly, $|F_1 \setminus B_1|$ is infinite. Since $[E_2] = [F_2]$ we can select any $B_2 \in \mathcal{B}_2$ and let $N = B_2 \setminus F_2$, and notice that $|N|$ is finite. Since $|F_1 \setminus B_1|$ is infinite, we then let M be any subset of $F_1 \setminus B_1$ with $|M| = |N|$. With this, we then have that $(B_1 \cup M) \cup (B_2 \setminus N) \subset F$, completing the proof. $\square$

Corollary 4.15. *Let E be an infinite set, P an infinite and coinfinite subset of E and $M = (E_M, \mathcal{I}_M)$ a finite matroid with $E_M \subset E$. There is a patch $\mathcal{B}$ on $[P]$ which can be constructed from M such that any patchwork matroid N constructed using $\mathcal{B}$ has M as a minor.*

Proof. Partition $E \setminus E_M$ into two infinite sets E_F and E_T . Let $\mathcal{F} = \{E_F\}$ be the patch on $[E_F]$ with ground set E_F and $\mathcal{T} = \{\emptyset\}$ be the patch on $[\emptyset]$ with ground set E_T . Observe that the bases of M form a patch on E_M . Then $\mathcal{B} = T \square (M \square F)$ will function as the desired patch. $\square$

5 Cyclic Flats

For this chapter, we assume that all matroids possess no loops or coloops. We do this because they have no impact of the lattice of cyclic flats (either being present in all or none of the cyclic flats.).

5.1 Introductory Results

We start with some general results concerning atomic cyclic flats. The primary result of this section shows that if M is a matroid generated from a tree of finite matroids that M has an atomic cyclic flat. In the course of doing this, we show that any restriction of M to the span of a circuit can also be generated from a tree of matroids using all or none of the ends of its tree.

Theorem 5.1. *Let M be a matroid and $\mathcal{L}$ be its lattice of cyclic flats. $\mathcal{L}$ has an atomic element if and only if M has a cyclic flat F such that $M|F$ is a uniform matroid.*

Proof. Assume that $\mathcal{L}$ has an atomic element F . Therefore F is a cyclic flat, and furthermore, for every circuit $C \subseteq F$ we have that $\text{span}(C) = F$. Thus every circuit C in $M|F$ contains a base of $M|F$. Hence $M|F$ is uniform. Suppose that F is a cyclic flat such that $M|F$ is uniform. Then every circuit $C \subseteq F$ contains a base of $M|F$ and hence $\text{span}(C) = F$. If F isn't an atomic element of $\mathcal{L}$ then there exists a cyclic flat $F' \subsetneq F$ such that F' is not empty. Thus F' has at least one circuit $C' \in F'$. Since F' is a flat, $\text{span}(C') \subseteq F'$. But $C' \subseteq F$, and so $\text{span}(C') = F \subseteq F'$, which is a contradiction. $\square$

Then considering the situation where a matroid has no atomic cyclic flat, we have the following result.

Proposition 5.2. *If M is a matroid such that its lattice of cyclic flats contains no atomic elements, then for every circuit C_0 there is an infinite sequence of circuits $(C_i : i \in \mathbb{N})$ with C_i a circuit of M such that $\text{span}(C_i) \subset \text{span}(C_j)$ whenever $i > j$.*

Proof. We construct such an infinite sequence by induction. Observe that since $\text{span}(C_0)$ is a non-empty cyclic flat, and the lattice of cyclic flats of M does not have an atomic element, we know that there is a non-empty cyclic flat $F \subsetneq \text{span}(C_0)$. Let C_1 be a circuit contained in F , and observe that $\text{span}(C_1) \subsetneq \text{span}(C_0)$. By iterating this process infinitely many times we then generate an infinite sequence of circuits satisfying the statement of the theorem. $\square$

We can also easily obtain the following result.

Lemma 5.3. *Let M be a matroid. If M has a finite circuit, then M is weakly atomic.*

Proof. Suppose M is a matroid with a finite circuit. Then let C be a circuit of M such that $|C|$ is minimal and therefore finite. Let $F = \text{span}(C)$ and consider any circuit $D \subseteq F$. Since $M|F$ has rank equal to $|C| - 1$, we know that $|D| \leq |C|$. But since C was chosen so that $|C|$ is minimal,

we have $|D| = |C|$. Since $|C| = |D|$ is finite, we know that $|D|$ contains a base of $M|F$. Therefore $M|F$ is uniform, and hence by, Theorem 5.1, M is weakly atomic. $\square$

This lemma along with Proposition 5.2 give us a fairly interesting characterization: The span of any circuit of an atomless matroid must contain infinitely many infinite circuits with strictly smaller span.

Proposition 5.4. *Let M be a countable matroid on E and let F be a cyclic flat of M . Then $E \setminus F$ is a cyclic flat of M^* .*

Proof. We borrow the orthogonality axiom (O2) from [7] which states that for any element $e \in E$ and a disjoint partition of $E \cup \{e\}$ into $P \cup Q$ we have that there is either a circuit $C \subseteq P \cup \{e\}$ with $e \in C$ or a cocircuit $D \subseteq Q \cup \{e\}$ with $e \in Q$. Note that this axiom holds when M is countable, which is why we require that M be countable in the first place. We will also be using the fact that for any circuit C and any cocircuit D that $|C \cap D| \neq 1$.

So assume that F is a cyclic flat of M . To show that $E \setminus F$ is a flat of M^* , observe that for each $f \in F$, if were in the cospan of $E \setminus F$, then there would be a circuit $C_f \subseteq F$ and a cocircuit $D_f \subseteq (E \setminus F) \cup \{f\}$, which is impossible, since then $C_f \cap D_f = \{f\}$. Therefore the cospan of $E \setminus F$ contains no elements of F , and so $E \setminus F$ is a flat of M^* . To show that $E \setminus F$ is a cyclic flat of M^* , let $e \in E \setminus F$ and apply (O2) with $Q = E \setminus (F \cup \{e\})$ and $P = F$. Since $F \cup \{e\}$ contains no circuit through e , clearly $E \setminus F$ contains a cocircuit D with $e \in D$. Since our choice of e was general, this completes the proof. $\square$

With this result in hand, we can now easily see that the ES Matroid is atomless. Let the ES Matroid be M^* , and recall that the cyclic flats of M were all of the form $V_k \cap E$ and formed an ascending chain. By applying the previous proposition (Since the ES Matroid is countable), we can see that we get a descending chain of cyclic flats, and so M^* is atomless. Now since the ES Matroid requires an infinite field, the next question is whether or not such a result can be reproduced for matroids which are representable over a finite field. For this, we offer a partial

solution: in this case of cofinitary matroids the answer is no; such a matroid will have an atomic cyclic flat.

Theorem 5.5. *Let M be a cofinitary matroid on E which is representable over a finite field k . Then M has an atomic cyclic flat.*

Proof. Since M is representable, let $\phi : E \rightarrow k^A$ be a thin family representing M . Pick $e_0 \in E$ and let $\mathcal{D}$ be a maximal set of cocircuits of M subject to

- 1) No element of $\mathcal{D}$ contains e_0
- 2) For any $D \in \mathcal{C}(M^*)$ and any $e \in D$ if $D - e \subseteq \bigcup \mathcal{D}$ then $D \in \mathcal{D}$

That such a $\mathcal{D}$ exists can be seen by applying Zorn's Lemma to the set $\mathcal{S}$ of all sets of cocircuits satisfying both 1 and 2 ordered by inclusion. Then, by replacing M with $M \setminus (\bigcup \mathcal{D})$ if necessary, we may assume without loss of generality that $\bigcup \mathcal{D}$ is empty. This then implies that e_0 lies in the cospan of every cocircuit of M , and thus every circuit of M through e_0 meets every cocircuit of M and so is spanning. Now let C be any circuit of M with $e_0 \notin C$. We now claim that $|C| \leq |k|$.

Proof of the claim: Let $e_1 \in C$ and let B be a base of M extending $C \setminus \{e_1\}$ and not containing e_0 . Since $C_{e_0}^B$ is spanning, it must be $B \cup \{e_0\}$; so $C \setminus \{e_1\} \subseteq C_{e_0}^B$. Now let c_0 be a ϕ -thin dependence with support $C_{e_0}^B$ and C_1 a ϕ -thin dependence with support C . Suppose for a contradiction that $|C| > |k|$. Then $|C \setminus \{e_1\}| > |k^*|$ so by the pigeonhole principle there are two elements of $C \setminus \{e_1\}$ for which $\frac{c_0(e)}{c_1(e)}$ takes the same value λ . Let C' be a circuit of M with $e_0 \in C' \subseteq \text{support}(c_0 - \lambda c_1)$. Then $C' \setminus \{e_0\}$ is a base of M and $|(C' \setminus \{e_0\}) \setminus B| = |\{e_1\}| = 1 < 2 \leq |B \setminus (C' \setminus \{e_0\})|$ generating the desired contradiction.

Thus, if there is such a circuit, it is finite, giving an atomic cyclic flat. Otherwise M must consist of a single circuit, and so it is uniform, again giving the existence of an atomic cyclic flat. $\square$

5.2 Trees of Matroids

One avenue of thought is whether or not an atomless matroid can be constructed from atomic matroids. While that might be possible, the following sections show that it cannot be done using a two-sum, or even by a tree of atomic matroids.

Proposition 5.6. *Let M and N be atomic matroids. Then $M \oplus_2 N$ is an atomic matroid.*

Proof. Let $e \in E(M) \cap E(N)$. Without loss of generality if e is a loop of M then we can rewrite $M \oplus_2 N$ as $(M \setminus \{e\}) \oplus (M/e)$ and it becomes obvious that $M \oplus_2 N$ is atomic. So let F be a cyclic flat of $M \oplus_2 N$ such that F is not the set of loops of $M \oplus_2 N$. We will show that F contains an atomic cyclic flat of $M \oplus_2 N$. Observe that $F = R \cup S$ where $R = F \cap E(M)$ and $S = F \cap E(N)$. If R contains an atomic cyclic flat of M , then we are done. Otherwise, observe that $\text{cl}_M(R) = R \cup e$ is a cyclic flat of M . Without loss of generality, we can assume that $R \cup e$ is an atomic cyclic flat of M . Similarly we can assume that $S \cup e$ is also an atomic cyclic flat of N . If F is not itself an atomic cyclic flat, then there is a smaller cyclic flat $G = R_G \cup S_G$ where $R_G = G \cap E(M)$ and $S_G = G \cap E(N)$ such that G is not the set of loops of $M \oplus_2 N$. Observe then that $R_G \cup e \subseteq R \cup e$. Since $R \cup e$ is an atomic cyclic flat, and $R_G \cup e$ is also cyclic flat then $R_G \cup e = R \cup e$. Otherwise $R_G \cup e$ is the set of loops of M , which contradicts that e is not a loop of M . Similarly $S_G \cup e = S \cup e$. Thus we have that $F = G$, which is a contradiction to the fact that G was smaller than F . Thus F is atomic, completing the proof. $\square$

Since we are dealing with infinite matroids, and the above process only works for finite 2-sums, the next step is to try to use a tree of matroids to do so. It turns out that such an attempt fails, but we need to prove several results to reach that conclusion. Due to the fact that cyclic flats have a fundamental relation with the closure operator we need to consider how the dummy edges interact with the closure operator. To do that we introduce the following definitions

Definition 5.7. Let $\mathcal{T} = (T, M)$ be a locally finite tree of finite matroids and Ψ be a set of branches of T such that $\mathcal{T}$ together with Ψ induces a matroid M . Let r, s be adjacent vertices

Figure 5: Picture for Fat/Thin

of T . Let R be the largest subtree of T which includes r but does not include s and define S similarly. Define Ψ_R as the subset of Ψ which includes only ends of R , and define Ψ_S similarly. Let M_S and M_R be the corresponding induced matroids. Observe that $e(rs)$ is an element of both M_S and M_R . We say that $e(rs)$ is *thin away from* $M(r)$ or *thin towards* $M(s)$ if M_S contains a circuit C with $e(rs)$ in it and $|C| \leq 2$. If $e(rs)$ is a loop we say it is *meager away from* $M(r)$. We say that $e(rs)$ is *fat away from* $M(r)$ (*fat towards* $M(s)$) if there exists a circuit C of M_S with $e(rs)$ in it, and every such circuit contains at least three elements. If $e(rs)$ is neither fat or thin in a given direction, then we say that $e(rs)$ is *ghostly*.

There are now a fair number of easy observations to make with regards to thin/fat/ghostly edges and the span operator. Firstly, a dummy edge $e(rs)$ need not be fat or thin in a specific direction. For instance if $e(rs)$ is a coloop of $M(r)$ then it cannot be thin or fat towards $M(r)$ because there is no Ψ_R -precircuit whose underlying set contains $e(rs)$. Also notice that a circuit of M “crossing” $e(rs)$ cannot exist if $e(rs)$ isn’t fat or thin in both directions. So if $e(rs)$ is thin in one direction, but neither in the other, then no circuit of M “crosses” $e(rs)$. The implication here then is that if we have a circuit C of M and an element $f \in E(M)$ the only way $f \in \text{span}(C)$ can happen is if in the tree T there is a path from a vertex used by C to the vertex of f such that all the edges used correspond to non-ghostly dummy edges.

5.3 The Fundamental Lemma

Let M be induced by a locally finite tree of finite matroids $\mathcal{T} = (T, M)$ and a set of branches Ψ of T . If Ψ is empty, then clearly all the circuits of M are finite, and so M has an atomic element by Lemma 5.3. Unfortunately this takes care of exactly one case out of potentially infinitely many. However, if we let C be a circuit of M and consider $N = M|_{\text{span}(C)}$ then it is easy to observe that any atomic element of N will be an atomic element of M . Then one possible approach would be to show that N can be induced from a locally finite tree of finite matroids say $\mathcal{T}_N = (T_N, N)$ and a set of branches Ψ_N that is either all or none of the branches of T_N . If we can do it using none of the branches, we've found a finite circuit and are done. Otherwise, if we use all of the branches, we don't have to worry about whether or not a particular end is in Ψ , which simplifies the problem.

It turns out that in general this is not true. The obvious approach would be to try and use Remark 2.25, but that fails because Remark 2.25 does not alter Ψ . The next attempt would be to take the vertices of T where $M(t) \cap \text{span}(C) \neq \emptyset$ and use those to induce T_N and thus a locally finite tree of matroids in the obvious fashion. However, the following example shows that this strategy fails as well.

Take a two way infinite path, and fix a vertex r as the root. To each vertex to the left of r , connect a leaf, and call the resulting rooted tree T . To each leaf that we added we assign a copy of uniform matroid $U_{1,2}$. To each non-leaf we assign a copy the uniform matroid $U_{1,3}$, and to r itself we assign the matroid $U_{2,4}$. Since all of the matroids involved are uniform, it doesn't matter how the dummy edges are assigned. We then let Ψ consist solely of the right branch of T , and let M be induced by $\mathcal{T}$ and Ψ .

Notice that if we let C be the two element circuit of M whose elements come from $M(r)$, then $M|_{\text{span}(C)} = M$, and that Ψ still consists of only one of the two branches of T . Hence this attempt fails to reduce T to a subtree T_N with all branches in Ψ , and thus we aren't ensured an atomic element.

However, in this particular case, the circuit C is finite and this case is already taken care of

Figure 6: The matroid M

due to Theorem 5.3. In fact, it turns out that if there are no finite circuits then we obtain the desired result.

Lemma 5.8. *Let $\mathcal{T} = (T, M)$ be a locally finite tree of finite matroids and Ψ a set of branches of T such that $\mathcal{T}$ together with Ψ induces a matroid M . Let C be a circuit of M and let $N = M|_{\text{span}(C)}$. If N has no finite circuits then there is a locally finite tree of finite matroids $\mathcal{T}' = (T_N, N)$ such that N is induced by $\mathcal{T}'$ together with Ψ_N , where Ψ_N is the set of all branches of T_N .*

Proof. Let M be generated from a tree of matroids $\mathcal{T} = (T, M)$ and a set of branches Ψ . Let C be a circuit of M and let $N = M|_{\text{span}(C)}$. In order to find a candidate for T_N , we assign to each circuit C_i of N a fixed Ψ -precircuit $\bar{C}_i = (T_i, o_i)$ such that the underlying set of $\bar{C}_i$ is precisely C_i . We then let T_A be the smallest subtree of T containing all vertices of every T_i . In order to form a tree of matroids, we let $A(t)$ be the restriction of $M(t)$ to $\text{span}(C)$ together with the dummy edges corresponding to the edges of T_A . The set of branches Ψ_A needed to induce N is the union of the family of sets of all branches of each T_i . Note that $\mathcal{T}_A$ is completely determined by the choice of the fixed Ψ -precircuits $\bar{C}_i$, and different choices of $\bar{C}_i$ may determine different $\mathcal{T}_A$. Furthermore, Ψ_A may not be all the branches of T_A as seen the previous example. We will show that if there are no finite circuits then there is a choice of the fixed Ψ -precircuits $\bar{C}_i$ which will determine a subtree T_N such that every branch of T_N is also a branch of some T_i (and thus Ψ_N is the set of all branches of T_N).

Assume then, that N has no finite circuits, and arbitrarily choose a set of fixed Ψ -precircuits $\bar{C}_i$. We now introduce the process we will call detachment. Let $\bar{P}_C = (T_C, o_C)$ be the fixed Ψ -

precircuit for C . Observe that the subgraph of T_A induced by $V(T_A) \setminus V(T_C)$ is a forest. Let H be one of the trees in that forest, and suppose that (T_i, o_i) and (T_j, o_j) are two (not necessarily fixed) Ψ -precircuits for which T_i and T_j contain vertices of T_C and H , but $o_i \upharpoonright V(H)$ and $o_j \upharpoonright V(H)$ are different. Then there exists a vertex t of H with $t \in V(T_i) \cap V(T_j)$ such that $o_i(t) \neq o_j(t)$. By using the circuit axiom (C3) we can then find a circuit D in $A(t)$ which does not use the dummy edge needed for any path in T_A from t to any vertex in T_C , but contains any one other edge in $o_i(t) \cup o_j(t)$ that we desire. We can then use D together with the obvious choices from (T_i, o_i) and (T_j, o_j) to form a non-empty Ψ -precircuit which is contained entirely in H .

We will now adjust our choice of the fixed Ψ -precircuits by considering each possible tree H .

If the matroids assigned to the vertices of H contribute no elements to N , then do the following: Pick a Ψ -precircuit (T_i, o_i) for which T_i contains a vertex of H . For every other Ψ -precircuit (T_j, o_j) , replace the part of T_j which lies in H with the part of T_i which lies in H , and let $o'_j(t) = o_i(t)$ for each $t \in V(T_i) \cap V(H)$. It is easy to verify that the result is another Ψ -precircuit, and after doing this, H can be reduced to just that part of T_i . Any branches of H remaining after this, must belong to Ψ as they are branches of T_i .

If the matroids of H contribute exactly 1 element to N , then observe that every Ψ -precircuit which passes into H must contain that element. Otherwise use detachment to construct a Ψ -precircuit whose underlying set is a singleton, thus identifying a loop and generating a contradiction. Thus we can repeat the previous procedure to remove the branches not in Ψ .

If the matroids of H contribute more than 1 element to N , then using detachment we can again identify a finite circuit (with either one or two elements). Since we have no finite circuits, this generates a contradiction, and so this case cannot happen.

Thus we let T_N be the tree that results from T_A by performing the above reductions on each tree H . Then $N(t)$ is the restriction of $A(t)$ to the vertices of T_N . Notice that each branch of T_N is either a branch of T_C (the tree associated with the Ψ -precircuit $\bar{P}_C$) or is a branch in one of the reduced subtrees H . In each case, the branch in question must be included in Ψ_N by virtue of being used in some precircuit so that $\mathcal{T}' = (T_N, N(t))$ and Ψ_N together induce N . $\square$

Once we have this lemma in hand, we can prove the desired result concerning locally finite trees of finite matroids.

5.4 Main Results

Theorem 5.9. *Let M be a matroid induced by a locally finite tree of finite matroids $\mathcal{T} = (T, M)$ and Ψ , a set of branches of T . Then M is weakly atomic.*

Proof. Assume that M is induced by $\mathcal{T} = (T, M)$ a locally finite tree of finite matroids with a set of ends Ψ . If M has any finite circuits which are not loops, we are done. Let C be a circuit of M , and let $N = M|_{\text{Span}(C)}$, and use Lemma 5.8 to express N as being induced by $\mathcal{T}_N = (T_N, N)$ with a subtree T_N of T and $N(t)$ a restriction of $M(t)$, and Ψ_N , the set of all ends of T_N .

Observe that all dummy edges $e(rs)$ of T_N are either fat or thin, due to the construction of N via Lemma 5.8. For each matroid $N(r)$ let L be the set of edges of $N(r)$ which are meager away from $N(r)$. We will then denote $N'(r)$ to be $N(r)/L$. Additionally, for any element e of N there is a circuit containing e . Then consider the following construction.

Pick an element e and consider the matroid $N(r)$ such that $e \in E(N(r))$. Select a circuit D_r of $N(r)$ such that $e \in D_r$, D_r uses as few edges of $N(r)$ which are fat away from $N(r)$. If there are multiple choices, pick one for which $\text{span}(D_r)$ is minimized in $N'(r)$. If we still have multiple choices, then choose one so that D_r uses as many edges which are meager away from $N(r)$. There might still be multiple options, but any of them will be sufficient. Note that since $N(r)$ is a finite matroid and T_N is finitely branching, all of these decisions are possible. Now, for each fat edge $e(rs)$, find a circuit D_s of $N(s)$ selected in the same manner as D_r was where $e(rs)$ is the required element of D_s . For each edge $e(rs)$ which is thin or meager, select the circuit D_s of $N(s)$ which causes $e(rs)$ to be thin or meager. Repeat this process for all dummy edges to obtain a precircuit (T_D, o_D) of N .

At this point it is useful to point out a few properties of the circuits D_s that we selected. First, suppose that D' is a circuit contained in $\text{span}(D_s)$ in $N'(s)$. D' must contain at least one edge fat away from $N(s)$, otherwise D' would be a finite circuit, causing a contradiction (M has no finite

circuits). Furthermore, if the fat edges contained in D are not also contained in D_s , then D' can't be used to show that the underlying set of the precircuit (T_D, o_D) is not a circuit, because any precircuit using D' must have elements not in the underlying set of (T_D, o_D) . Likewise, D' can't interfere with the span of the underlying set of (T_D, o_D) for the same reasons. Now if all the fat edges contained in D' are contained in D_s , then observe that every fat edge contained in D_s is also contained in D' , and similarly the edge e required to be in D_s must also be in D' . Otherwise the circuit axiom (C3) allows us to contradict the fact that D_s contained a minimal number of edges fat away from $N(s)$. There are now just two properties left to notice: First, that since D_s was chosen with minimal span in $N'(s)$ and D' is contained in that span, $\text{span}(D') = \text{span}(D_s)$. Secondly, since D_s contains a maximal number of meager edges, that D' must contained more, and so if we replaced D_s with D' in (T_D, o_D) it would be impossible to get the underlying set of the resulting precircuit to be a subset of the underlying set of (T_D, o_D) . The essential result of these properties is that D_s cannot be replaced by anything in $\text{span}(D_s)$ without either using more elements of N or getting exactly the same span in $N'(s)$.

The above then yields two necessary results. First that the underlying set of (T_D, o_D) is indeed a circuit. Because if it were not then there is another precircuit (T_R, o_R) which does generate a circuit, and the underlying set of (T_R, o_R) is strictly a subset of the underlying set of (T_D, o_D) . This implies that there is some vertex t of both T_D and T_R such that $o_D(t) \neq o_R(t)$, and the properties of $o_D(t)$ then cause the desired contradiction. So we can let D be the underlying set of (T_D, o_D) . The second result, which states that $\text{span}(D)$ is an atomic cyclic flat of N , finishes the proof.

If $\text{span}(D)$ is not an atomic cyclic flat of N , then there exists a circuit A in $\text{span}(D)$ with $\text{span}(A) \subsetneq \text{span}(D)$, and $\text{span}(A) \neq \text{span}(\emptyset)$. Let (T_A, o_A) be a Ψ -precircuit whose underlying set is A . Notice that if T_A shares no vertex with T_D , then A is a loop (this follows from the proof of Lemma 5.8) and this is a contradiction to our choice of A . So then let t be the vertex that T_A shares with T_D . Now the properties of $o_D(t)$ force $o_A(t)$ to use exactly the same fat edges, and have exactly the same span. And then these properties force the same behavior throughout all of

the vertices of T_A , and thus $\text{span}(A) = \text{span}(D)$ becomes forced, generating a contradiction. Thus $\text{span}(D)$ is an atomic cyclic flat of N , and the proof is complete. $\square$

With this theorem in hand, we might hope that all matroids arising from such trees are atomic. However, the following counterexample shows that this is not true. Let N be the ES matroid and take any locally finite tree T , and to the vertex t , and assign the direct sum $M(t) = N \oplus U_{k-1,k}$ where $k = \deg(t)$. This makes $M(t)$ a copy of N with a circuit of length k added to it. We use the added circuit to provide the dummy edges, and thus generate a matroid M for which any cyclic flat contains or is contained in some copy of N . Thus it cannot contain an atomic cyclic flat, and so M is atomless.

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