2008

Solutions for 2-dimensional stabilized Kuramoto -Sivashinsky system

Maomao Cai
West Virginia University

Follow this and additional works at: <https://researchrepository.wvu.edu/etd>

Recommended Citation

Cai, Maomao, "Solutions for 2-dimensional stabilized Kuramoto -Sivashinsky system" (2008). *Graduate Theses, Dissertations, and Problem Reports*. 4359.
<https://researchrepository.wvu.edu/etd/4359>

This Dissertation is protected by copyright and/or related rights. It has been brought to you by the The Research Repository @ WVU with permission from the rights-holder(s). You are free to use this Dissertation in any way that is permitted by the copyright and related rights legislation that applies to your use. For other uses you must obtain permission from the rights-holder(s) directly, unless additional rights are indicated by a Creative Commons license in the record and/ or on the work itself. This Dissertation has been accepted for inclusion in WVU Graduate Theses, Dissertations, and Problem Reports collection by an authorized administrator of The Research Repository @ WVU. For more information, please contact researchrepository@mail.wvu.edu.

Solutions for 2-Dimensional Stabilized Kuramoto-Sivashinsky System

Maomao Cai

Dissertation submitted to the
Eberly College of Arts and Sciences
at West Virginia University
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in
Mathematics

Dening Li, Ph.D., Chair
Harumi Hattori , Ph.D.
Harvey Diamond, Ph.D.
Weifu Fang, Ph.D.
Andrei Smirnov, Ph.D.

Department of Mathematics

Morgantown, West Virginia
2008

Keywords: Energy estimate, Continuation method, Linear iteration method,
Kuramoto-Sivashinsky-Kdv equation, Local solution, Global solution.

Copyright 2008 Maomao Cai

ABSTRACT

Solutions for 2-Dimensional Stabilized Kuramoto-Sivashinsky System

Maomao Cai

We study a stabilized Kuramoto-Sivashinsky system in two-dimensional space. A model consists of a mixed Kuramoto-Sivashinsky-Korteweg-de Vries equation, linearly coupled to an extra linear dissipative equation. The model is proposed to describe the surface waves on multi-layered liquid films. In this work, we investigate the stability of the solution to this system by establishing a priori energy estimate for the linearized problem of this non-linear system. We use linear iteration to prove the local existence of the solution to this system. Based on a weak global priori energy estimate, we further prove the global existence and uniqueness of classical solution for this system.

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Dr. Dening Li, for his valuable guidance, advice, and continual encouragement. Dr. Li was very patient and generous in advising my research work. It is a precious gift from the God to work under his supervision.

I would also like to thank Dr. Weifu Fang, Dr. Harvey Diamond, Dr. Harumi Hattori, and Dr. Andrei Smirnov at West Virginia University, for serving on my committee and for providing advice, discussion, and comments on my studies.

And finally, I would like to thank the Department of Mathematics and Eberly College of Arts and Sciences at West Virginia University for providing me with an excellent study environment and support during my study as a graduate student.

DEDICATION

To

my father Xueqin Cai , my mother Limin Ma

and

my husband Yong Zhang

Contents

1	Introduction	1
2	Linear Stability and High Order Estimate	5
2.1	Linearization	5
2.2	Zero order estimate	6
2.3	First order estimate	12
2.4	High order estimate	16
3	The Existence of Solution for the Linearized Problem	22
3.1	The linearized system with $u^0 \in C^{k+1}([0, T], R^2)$	22
3.2	The linearized system with $u^0 \in C([0, T], H_{ul}^{k+2}(R^2))$	29
4	The Existence of Local Solution for the Non-linear Problem	31
4.1	The local solution for the non-linear problem	31
4.2	The local classical solution for the non-linear problem	38

5	The Solution for a Linear Problem	39
5.1	An apriori estimate	40
5.2	The solution for a linear problem with $f \in L^1([0, T], L^2(R^2))$	42
6	The Global Weak Solution for the Non-linear Problem	45
6.1	Global estimate	45
6.2	The uniqueness of the global solution for the non-linear problem with $(u_0, v_0) \in L^2 \times L^2$	50
6.3	The local solution for the non-linear problem with $(u_0, v_0) \in L^2 \times L^2$	52
6.4	The global solution for the non-linear problem with $(u_0, v_0) \in L^2 \times L^2$. . .	57
7	The Global Classical Solution for the Non-linear Problem	60
7.1	The global solution for the linear problem with $u^0 \in C([0, T], L^2(R^2))$. . .	61
7.2	The global solution for the non-linear problem with $(u_0, v_0) \in H^2 \times H^1$. .	65
7.3	The regularity of the global solution for the non-linear problem with $(u_0, v_0) \in H^{k+2} \times H^{k+1}$	67

Chapter 1

Introduction

We study a two-dimensional stabilized Kuramoto-Sivashinsky-Korteweg-de Vries equation, linearly coupled to an extra linear dissipative equation. The model was first introduced in [15] to describe the surface waves on multi-layers liquid films, and the two dimensional model was proposed in [7].

The Kortweg-de Vries (KdV) equation is a third order nonlinear partial differential equation:

$$u_t + uu_x + u_{xxx} = 0, \tag{1.1}$$

which is first formulated as part of an analysis of shallow-water waves in canals. It has subsequently been found to be involved in a wide range of physics phenomena, especially those exhibiting travelling waves and solitons [9][12][25]. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior, and mass transport [12][16].

The Kuramoto-Sivashinsky (KS) equation is a well-known model of one-dimensional turbulence, which was derived in various physical contexts, including chemical-reaction waves, propagation of combustion fronts in gases, surface waves in a film of a viscous liquid

flowing along an inclined plane, patterns in thermal convection, rapid solidification, and others [23][21]. It has the form:

$$u_t + uu_x = -\alpha u_{xx} - \gamma u_{xxx}, \quad (1.2)$$

where $\alpha, \gamma > 0$ are coefficients accounting for the long-wave instability (gain) and short-wave dissipation respectively.

An one-dimensional generalized form of the KS equation that combines conservative and dissipative effects is a mixed Kuramoto-Sivashinsky -Korteweg-de Vries (KS-KdV) equation:

$$u_t + uu_x + u_{xxx} = -\alpha u_{xx} - \gamma u_{xxxx}, \quad (1.3)$$

which was first introduced by Benney [1] and is often called the Benney equation. This equation finds various applications in plasma physics, hydrodynamics and other fields [2][4][5]. In particular, a subject of considerable interest was the study of solitary-pulse (SP) solutions of the above equations [3][22]. However, solitary-wave (SW) cannot be stable in the Benney equation, since the zero solution, a background on top of which SWs are to be found, is unstable in this equation due to the presence of the linear gain, which is accounted for by the coefficient α .

A stabilized version of the Benney equation is proposed by Malomed [15]. It is based on the KS-Kdv equation for a real wave field $u(x, t)$, which is linearly coupled to an additional linear dissipative equation for an extra real wave field $v(x, t)$, that provides for the stabilization of the zero background. The model is as follows:

$$u_t + uu_x + u_{xxx} = -\alpha u_{xx} - \gamma u_{xxxx} + \epsilon_1 v_x \quad (1.4)$$

$$v_t + a_1 v_x = \Gamma v_{xx} + \epsilon_2 u_x. \quad (1.5)$$

The system describes, for instance, the propagation of surface waves in a two-layer liquid film in the case when one layer is dominated by viscosity. Here $\alpha, \gamma > 0$ and $\Gamma > 0$ are coefficients accounting for the gain and loss in the u subsystem and loss in the v subsystem respectively. a_1 is a group-velocity mismatch between the two wave fields. The coupling

parameters ϵ_1, ϵ_2 are positive. The linear coupling via the first derivatives is the same as that in known models of coupled internal waves propagating in multi-layered fluids [8]. The linear dissipative equation (1.5) implies that the substrate layer is essentially more viscous than the upper one [15]. In [15], the stability of SP solutions in the system of equations (1.4) and (1.5) is investigated by treating the gain and the dissipation constants α, γ, Γ as small parameters while the group-velocity mismatch a_1 need not be small.

In this work, we consider the following initial value problem:

$$u_t + uu_x + \Delta u_x = -\alpha u_{xx} - \gamma \Delta^2 u + \epsilon_1 v_x \quad (1.6)$$

$$v_t + a_1 v_x = \Gamma \Delta v + \epsilon_2 u_x \quad (1.7)$$

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y). \quad (1.8)$$

The above system is proposed in the study of cylindrical solitary pulses [7]. Notice that the two space variables (x, y) are not symmetric. This is because of the underlying non-symmetric physics [15][7]. The stability of steady-state soliton solutions is analyzed by perturbation theory and wave model [8][7]. In this work, we first establish the linear stability of the solution by deriving the energy estimate for a linearized system. Then, we obtain an energy estimate using an η -weighted norm for the non-linear problem and use this estimate to establish the existence of global solution for a large class of general initial data. In particular, our results show that the solution in [7] is not only linearly stable, but also globally existent with even modified initial data.

The thesis is arranged as follows. In chapter 2, we investigate the linear stability of the solution to the non-linear problem (1.6)-(1.8) by establishing the energy estimate for its linearized problem. In chapter 3, we use the obtained energy estimate and a continuation method to show the existence of the solution for the linearized problem. In chapter 4, we use a linear iteration method to obtain the existence of the local solution for the nonlinear problem. In chapter 5, we establish the existence of the solution for a linear problem with weak requirement on the right side term by treating the only nonlinear term uu_x as a known function. In chapter 6, we derive a global energy estimate. The existence and uniqueness of the global solution for the nonlinear problem are obtained under the assumption that $u_0 \in L^2(R^2)$ and $v_0 \in L^2(R^2)$. In chapter 7, we obtain the existence of

the global solution for the nonlinear problem under the assumption that $u_0 \in H^{k+2}(R^2)$ and $v_0 \in H^{k+1}(R^2)$, where k is a nonnegative integer.

Chapter 2

Linear Stability and High Order Estimate

2.1 Linearization

In this section, we linearize equations (1.6) and (1.7). Let $(\tilde{u}, \tilde{v})$ be a small perturbation of a bounded C^∞ solution (u^0, v^0) of (1.6) and (1.7), i.e.

$$\begin{aligned} u &\sim u^0 + \epsilon \tilde{u} \\ v &\sim v^0 + \epsilon \tilde{v}, \end{aligned} \tag{2.1}$$

with $\epsilon \ll 1$.

Substituting (u, v) into equations (1.6) and (1.7) and omitting the higher order terms of ϵ , we obtain a linearized system for $(\tilde{u}, \tilde{v})$:

$$\tilde{u}_t + u^0 \tilde{u}_x + u_x^0 \tilde{u} + \Delta \tilde{u}_x = -\alpha \tilde{u}_{xx} - \gamma \Delta^2 \tilde{u} + \epsilon_1 \tilde{v}_x \tag{2.2}$$

$$\tilde{v}_t + a_1 \tilde{v}_x = \Gamma \Delta \tilde{v} + \epsilon_2 \tilde{u}_x. \tag{2.3}$$

Notice that the linear stability of solution (u^0, v^0) is determined by the energy estimate for $(\tilde{u}, \tilde{v})$ under small initial perturbation. Hence, we discuss the following initial value

problem for the linearized system (2.2)-(2.3):

$$\tilde{u}_t + u^0 \tilde{u}_x + u_x^0 \tilde{u} + \Delta \tilde{u}_x = -\alpha \tilde{u}_{xx} - \gamma \Delta^2 \tilde{u} + \epsilon_1 \tilde{v}_x + \tilde{f} \quad (2.4)$$

$$\tilde{v}_t + a_1 \tilde{v}_x = \Gamma \Delta \tilde{v} + \epsilon_2 \tilde{u}_x + \tilde{g} \quad (2.5)$$

$$\tilde{u}(x, y, 0) = \tilde{u}_0(x, y), \tilde{v}(x, y, 0) = \tilde{v}_0(x, y), \quad (2.6)$$

where the coefficients $\alpha, \gamma, \Gamma, \epsilon_1, \epsilon_2$ are all positive constants with possible smooth small perturbations. Set $\alpha \equiv \alpha_0 + \alpha_1(x, y, t)$, $\gamma \equiv \gamma_0 + \gamma_1(x, y, t)$, $\Gamma \equiv \Gamma_0 + \Gamma_1(x, y, t)$. α_0, γ_0 and Γ_0 are positive constants. u^0 is a given bounded smooth function, and a_1 is a given bounded smooth function.

Let $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product in R^2 and $H^k(R^2)$ be the usual Sobolev space defined by the norm:

$$\|f\|_k^2 = \int_{R^2} \sum_{|j| \leq k} |D^j f|^2 dx dy, \quad (2.7)$$

with $H^0(R^2) = L^2(R^2)$ and $\|f\| = \|f\|_0$.

Let $S(R^2)$ be the Schwartz rapidly decaying function space[19]. In the following sections, we discuss energy estimates for the above linearized problem (2.4)-(2.6).

2.2 Zero order estimate

Theorem 2.2.1 *Any solution $(\tilde{u}, \tilde{v}) \in C^\infty([0, T], S(R^2))$ of (2.4), (2.5) and (2.6) satisfies the estimate:*

$$\begin{aligned} & \partial_t (\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2) + \|\tilde{u}(t)\|_2^2 + \|\tilde{v}(t)\|_1^2 \\ & \leq C(\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2 + \|\tilde{f}(t)\|^2 + \|\tilde{g}(t)\|^2), \end{aligned} \quad (2.8)$$

$\forall t \in [0, T]$,

and

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2) + \int_0^T (\|\tilde{u}(s)\|_2^2 + \|\tilde{v}(s)\|_1^2) ds \\ & \leq C(T)(\|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2 + \int_0^T (\|\tilde{f}(s)\|^2 + \|\tilde{g}(s)\|^2) ds). \end{aligned} \quad (2.9)$$

$C(T)$ denotes a constant depending on T and the coefficients of the above linearized system. In particular, the constant $C(T)$ depends upon u^0 only in its $L^\infty([0, T], H^2(R^2))$ norm.

Proof. Take L^2 inner product of equation (2.4) with $\tilde{u}$ and equation (2.5) with $\tilde{v}$ over R^2 respectively. Notice that $\frac{1}{2}\partial_t w^2 = w\partial_t w$, we have:

$$\begin{aligned} & \frac{1}{2}\partial_t(\|\tilde{u}\|^2 + \|\tilde{v}\|^2) + \langle \gamma \Delta^2 \tilde{u}, \tilde{u} \rangle + \langle \alpha \tilde{u}_{xx}, \tilde{u} \rangle - \langle \Gamma \Delta \tilde{v}, \tilde{v} \rangle = \\ & -\langle u^0 \tilde{u}_x, \tilde{u} \rangle - \langle u_x^0 \tilde{u}, \tilde{u} \rangle - \langle a_1 \tilde{v}_x, \tilde{v} \rangle - \langle \Delta \tilde{u}_x, \tilde{u} \rangle + \langle \epsilon_1 \tilde{v}_x, \tilde{u} \rangle + \langle \tilde{f}, \tilde{u} \rangle + \langle \epsilon_2 \tilde{u}_x, \tilde{v} \rangle + \langle \tilde{g}, \tilde{v} \rangle. \end{aligned} \quad (2.10)$$

Let M_1, M_2, M ($M_1 + M_2 = M$) be constants such that:

$$\sup_{0 \leq t \leq T} (\|\Gamma_x\|_{L^\infty(R^2)}, \|\Gamma_y\|_{L^\infty(R^2)}, \|a_1\|_{L^\infty(R^2)}, \|\alpha\|_{L^\infty(R^2)}, \|\epsilon_1\|_{L^\infty(R^2)}, \|\epsilon_2\|_{L^\infty(R^2)}) < M_1,$$

$$\sup_{0 \leq t \leq T} (\|u^0\|_2^2, \|\gamma_x\|_{L^\infty(R^2)}, \|\gamma_y\|_{L^\infty(R^2)}, \|\gamma_{xx}\|_{L^\infty(R^2)}, \|\gamma_{xy}\|_{L^\infty(R^2)}, \|\gamma_{yy}\|_{L^\infty(R^2)}) < M_2.$$

Integrating by parts in (x, y) and applying Cauchy-Schwarz inequality [11]:

$$|\langle a, b \rangle| \leq \|a\| \|b\| \leq \varepsilon \|a\|^2 + \frac{1}{4\varepsilon} \|b\|^2, \quad (2.11)$$

where $\varepsilon < 1$ is any positive constant.

Notice that:

$$\|u\|_2 \sim \|u_{xx}\| + \|u_{yy}\| + \|u_{xy}\| + \|u\|. \quad (2.12)$$

We have:

$$\begin{aligned} & \langle \gamma \Delta^2 \tilde{u}, \tilde{u} \rangle \\ &= -\langle (\gamma \tilde{u})_x, \tilde{u}_{xxx} \rangle - 2\langle (\gamma \tilde{u})_y, \tilde{u}_{xxy} \rangle - \langle (\gamma \tilde{u})_y, \tilde{u}_{yyy} \rangle \\ &= (\langle \gamma \tilde{u}_{xx}, \tilde{u}_{xx} \rangle + 2\langle \gamma \tilde{u}_{xy}, \tilde{u}_{xy} \rangle + \langle \gamma \tilde{u}_{yy}, \tilde{u}_{yy} \rangle) + 2(\langle \tilde{u}_{xx}, \gamma_x \tilde{u}_x \rangle + \langle \tilde{u}_{yy}, \gamma_y \tilde{u}_y \rangle + \langle \tilde{u}_{xy}, \gamma_x \tilde{u}_y \rangle + \\ & \quad \langle \tilde{u}_{xy}, \gamma_y \tilde{u}_x \rangle) + \langle \tilde{u}_{xx}, \gamma_{xx} \tilde{u} \rangle + \langle \tilde{u}_{yy}, \gamma_{yy} \tilde{u} \rangle + 2\langle \tilde{u}_{xy}, \gamma_{xy} \tilde{u} \rangle, \end{aligned} \quad (2.13)$$

where

$$\langle \gamma \tilde{u}_{xx}, \tilde{u}_{xx} \rangle + 2\langle \gamma \tilde{u}_{xy}, \tilde{u}_{xy} \rangle + \langle \gamma \tilde{u}_{yy}, \tilde{u}_{yy} \rangle \geq \frac{\gamma_0}{2} (\|\tilde{u}_{xx}\|^2 + \|\tilde{u}_{yy}\|^2) + \gamma_0 \|\tilde{u}_{xy}\|^2. \quad (2.14)$$

$$\begin{aligned} -\langle \Gamma \Delta \tilde{v}, \tilde{v} \rangle &= \langle (\Gamma \tilde{v})_x, \tilde{v}_x \rangle + \langle (\Gamma \tilde{v})_y, \tilde{v}_y \rangle \\ &= \langle \Gamma \tilde{v}_x, \tilde{v}_x \rangle + \langle \Gamma \tilde{v}_y, \tilde{v}_y \rangle + \langle \Gamma_x \tilde{v}, \tilde{v}_x \rangle + \langle \Gamma_y \tilde{v}, \tilde{v}_y \rangle, \end{aligned} \quad (2.15)$$

where

$$\langle \Gamma \tilde{v}_x, \tilde{v}_x \rangle + \langle \Gamma \tilde{v}_y, \tilde{v}_y \rangle \geq \frac{\Gamma_0}{2} (\|\tilde{v}_x\|^2 + \|\tilde{v}_y\|^2). \quad (2.16)$$

$$\begin{aligned} \langle \Delta \tilde{u}_x, \tilde{u} \rangle &= -\langle \tilde{u}_{xx}, \tilde{u}_x \rangle - \langle \tilde{u}_{yy}, \tilde{u}_x \rangle \\ &\leq \beta_1 \|\tilde{u}_{xx}\|^2 + \frac{1}{4\beta_1} \|\tilde{u}_x\|^2 + \beta_2 \|\tilde{u}_{yy}\|^2 + \frac{1}{4\beta_2} \|\tilde{u}_x\|^2 \\ &\leq (\beta_1 + \beta_2) \|\tilde{u}\|_2^2 + \left(\frac{1}{4\beta_1} + \frac{1}{4\beta_2}\right) \|\tilde{u}_x\|^2 \\ &\leq (\beta_1 + \beta_2) \|\tilde{u}\|_2^2 + \beta \varepsilon_1 \|\tilde{u}\|_2^2 + \frac{\beta}{4\varepsilon_1} \|\tilde{u}\|^2, \end{aligned} \quad (2.17)$$

where β_1, β_2 are positive numbers, $\beta = \frac{\beta_1 + \beta_2}{4\beta_1\beta_2}$ and $\varepsilon_1 \ll \beta_1\beta_2$.

Here we estimate $\|\tilde{u}_x\|^2$ by means of the inequality:

$$\begin{aligned} \|\tilde{u}_x\|^2 &= |-\langle \tilde{u}_{xx}, \tilde{u} \rangle| \\ &\leq \|\tilde{u}\| \|\tilde{u}_{xx}\| \\ &\leq \varepsilon \|\tilde{u}_{xx}\|^2 + \frac{1}{4\varepsilon} \|\tilde{u}\|^2 \\ &\leq \varepsilon \|\tilde{u}\|_2^2 + \frac{1}{4\varepsilon} \|\tilde{u}\|^2. \end{aligned} \quad (2.18)$$

$$\begin{aligned}
\langle \alpha \tilde{u}_{xx}, \tilde{u} \rangle &\leq \varepsilon_2 \|\alpha \tilde{u}_{xx}\|^2 + \frac{1}{4\varepsilon_2} \|\tilde{u}\|^2 \\
&\leq M\varepsilon_2 \|\tilde{u}\|_2^2 + \frac{1}{4\varepsilon_2} \|\tilde{u}\|^2.
\end{aligned} \tag{2.19}$$

According to Sobolev imbedding theorem [19], $H^2(R^2)$ can be continuously imbedded into bounded $C(R^2)$, we have:

$$\begin{aligned}
\langle u^0 \tilde{u}_x, \tilde{u} \rangle &\leq \varepsilon_3 \|u^0 \tilde{u}_x\|^2 + \frac{1}{4\varepsilon_3} \|\tilde{u}\|^2 \\
&\leq \varepsilon_3 (\sup_{x,y} |u^0|)^2 \|\tilde{u}_x\|^2 + \frac{1}{4\varepsilon_3} \|\tilde{u}\|^2 \\
&\leq C\varepsilon_3 \|u^0\|_2^2 \|\tilde{u}\|_1^2 + \frac{1}{4\varepsilon_3} \|\tilde{u}\|^2 \\
&\leq CM\varepsilon_3 \|\tilde{u}\|_1^2 + \frac{1}{4\varepsilon_3} \|\tilde{u}\|^2.
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
\langle a_1 \tilde{v}_x, \tilde{v} \rangle &\leq M\varepsilon_4 \|\tilde{v}_x\|^2 + \frac{1}{4\varepsilon_4} \|\tilde{v}\|^2 \\
&\leq M\varepsilon_4 \|\tilde{v}\|_1^2 + \frac{1}{4\varepsilon_4} \|\tilde{v}\|^2.
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\langle \epsilon_1 \tilde{v}_x, \tilde{u} \rangle &\leq M\varepsilon_5 \|\tilde{v}_x\|^2 + \frac{1}{4\varepsilon_5} \|\tilde{u}\|^2 \\
&\leq M\varepsilon_5 \|\tilde{v}\|_1^2 + \frac{1}{4\varepsilon_5} \|\tilde{u}\|^2.
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
\langle \epsilon_2 \tilde{u}_x, \tilde{v} \rangle &\leq M\varepsilon_6 \|\tilde{u}_x\|^2 + \frac{1}{4\varepsilon_6} \|\tilde{v}\|^2 \\
&\leq M\varepsilon_6 \|\tilde{u}\|_1^2 + \frac{1}{4\varepsilon_6} \|\tilde{v}\|^2.
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
& 2(\langle \tilde{u}_{xx}, \gamma_x \tilde{u}_x \rangle + \langle \tilde{u}_{yy}, \gamma_y \tilde{u}_y \rangle + \langle \tilde{u}_{xy}, \gamma_x \tilde{u}_y \rangle + \langle \tilde{u}_{xy}, \gamma_y \tilde{u}_x \rangle) + \langle \tilde{u}_{xx}, \gamma_{xx} \tilde{u} \rangle + \\
& \langle \tilde{u}_{yy}, \gamma_{yy} \tilde{u} \rangle + 2\langle \tilde{u}_{xy}, \gamma_{xy} \tilde{u} \rangle \\
& \leq \beta_3 \|\tilde{u}\|_2^2 + \frac{M}{4\beta_3} (\beta_4 \|\tilde{u}\|_2^2 + \frac{1}{4\beta_4} \|\tilde{u}\|^2) + \beta_5 \|\tilde{u}\|_2^2 + \frac{M}{4\beta_5} \|\tilde{u}\|^2 \\
& \leq \varepsilon_7 \|\tilde{u}\|_2^2 + \frac{1}{\varepsilon_7} \|\tilde{u}\|^2,
\end{aligned} \tag{2.24}$$

where $\beta_3, \beta_4, \beta_5$ are positive numbers, and $\beta_4 \ll \beta_3$.

$$\langle \Gamma_x \tilde{v}, \tilde{v}_x \rangle + \langle \Gamma_y \tilde{v}, \tilde{v}_y \rangle \leq M\varepsilon_8 \|\tilde{v}\|_1^2 + \frac{1}{4\varepsilon_8} \|\tilde{v}\|^2. \tag{2.25}$$

$$\langle \tilde{f}, \tilde{u} \rangle \leq \varepsilon_9 \|\tilde{u}\|^2 + \frac{1}{4\varepsilon_9} \|\tilde{f}\|^2. \tag{2.26}$$

$$\langle \tilde{g}, \tilde{v} \rangle \leq \varepsilon_{10} \|\tilde{v}\|^2 + \frac{1}{4\varepsilon_{10}} \|\tilde{g}\|^2. \tag{2.27}$$

$$\begin{aligned}
\langle u_x^0 \tilde{u}, \tilde{u} \rangle & \leq \|u_x^0\| \|\tilde{u}\tilde{u}\| \\
& \leq \|u^0\|_1 \|\tilde{u}\| \sup_{x,y} |\tilde{u}| \\
& \leq C \|u^0\|_2 \|\tilde{u}\| \|\tilde{u}\|_2 \\
& \leq CM(\varepsilon_{11} \|\tilde{u}\|_2^2 + \frac{1}{4\varepsilon_{11}} \|\tilde{u}\|^2).
\end{aligned} \tag{2.28}$$

Combine from (2.14) to (2.28), we obtain:

$$\begin{aligned}
& \partial_t(\|\tilde{u}\|^2 + \|\tilde{v}\|^2) + \frac{\gamma_0}{2}(\|\tilde{u}_{xx}\|^2 + \|\tilde{u}_{yy}\|^2) + \gamma_0\|\tilde{u}_{xy}\|^2 + \frac{\Gamma_0}{2}(\|\tilde{v}_x\|^2 + \|\tilde{v}_y\|^2) \\
& \leq (\varepsilon_1 + \beta\varepsilon_8 + M\varepsilon_2 + \varepsilon_7 + CM\varepsilon_{11})\|\tilde{u}\|_2^2 + (CM\varepsilon_3 + M\varepsilon_6)\|\tilde{u}\|_1^2 \\
& + (\frac{\beta}{4\varepsilon_1} + \frac{1}{4\varepsilon_2} + \frac{1}{4\varepsilon_3} + \varepsilon_9 + \frac{1}{4\varepsilon_5} + \frac{1}{\varepsilon_7} + \frac{CM}{4\varepsilon_{11}})\|\tilde{u}\|^2 + (M\varepsilon_4 + M\varepsilon_5 + M\varepsilon_8)\|\tilde{v}\|_1^2 \\
& + (\frac{1}{4\varepsilon_6} + \frac{1}{4\varepsilon_4} + \varepsilon_{10} + \frac{1}{4\varepsilon_8})\|\tilde{v}\|^2 + \frac{1}{4\varepsilon_9}\|\tilde{f}\|^2 + \frac{1}{4\varepsilon_{10}}\|\tilde{g}\|^2.
\end{aligned} \tag{2.29}$$

Since ε_i ($i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$) are positive constants, we can choose C which is a constant depending only upon δ and ε_i . Adding $\|\tilde{u}\|^2$ and $\|\tilde{v}\|^2$ to both sides of (2.29), we have:

$$\partial_t(\|\tilde{u}\|^2 + \|\tilde{v}\|^2) + \|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2 \leq C(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 + \|\tilde{f}\|^2 + \|\tilde{g}\|^2). \tag{2.30}$$

Integrating the above inequality (2.30) on both sides, we have:

$$\begin{aligned}
& \|\tilde{u}\|^2 + \|\tilde{v}\|^2 \\
& \leq \|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2 - \int_0^t(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2)ds + C \int_0^t(\|\tilde{f}\|^2 + \|\tilde{g}\|^2)ds + C \int_0^t(\|\tilde{u}\|^2 + \|\tilde{v}\|^2)ds.
\end{aligned} \tag{2.31}$$

To obtain (2.9), we apply Gronwall's theorem to the inequality:

$$P(t) \leq C + k \int_0^t P(s)ds, \tag{2.32}$$

with $C = \|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2 - \int_0^t(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2)ds + C \int_0^t(\|\tilde{f}\|^2 + \|\tilde{g}\|^2)ds$ and $k = C$.

Therefore we have $P(t) \leq Ce^{kt}$, which yields (2.9) and the proof is complete.

The inequalities (2.8) and (2.9) in Theorem (2.2.1) are obtained for Cauchy initial data. Obviously, it's also valid for the initial-boundary value problems with periodic boundary conditions studied for the periodic waves of KS-KdV system in [6].

2.3 First order estimate

Theorem 2.3.1 *Any solution $(\tilde{u}, \tilde{v}) \in C^\infty([0, T], S(R^2))$ of (2.4), (2.5) and (2.6) satisfies the estimate:*

$$\begin{aligned} \partial_t(\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2) + \|\tilde{u}\|_3^2 + \|\tilde{v}\|_2^2 \leq \\ C(\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{g}\|^2), \end{aligned} \quad (2.33)$$

$$\forall t \in [0, T],$$

and

$$\begin{aligned} \sup_{0 \leq t \leq T}(\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2) + \int_0^T (\|\tilde{u}\|_3^2 + \|\tilde{v}\|_2^2) ds \leq \\ C(T)(\|\tilde{u}_0\|_1^2 + \|\tilde{v}_0\|_1^2 + \int_0^T (\|\tilde{f}\|^2 + \|\tilde{g}\|^2) ds). \end{aligned} \quad (2.34)$$

Proof. Taking L^2 inner product of (2.4) with $\Delta \tilde{u}$ and (2.5) with $\Delta \tilde{v}$ and integrating by parts, we have:

$$\begin{aligned} \frac{1}{2} \partial_t(\|\tilde{u}_x\|^2 + \|\tilde{u}_y\|^2 + \|\tilde{v}_x\|^2 + \|\tilde{v}_y\|^2) - \langle \gamma \Delta^2 \tilde{u}, \Delta \tilde{u} \rangle - \langle \Delta \tilde{u}_x, \Delta \tilde{u} \rangle \\ - \langle \alpha \tilde{u}_{xx}, \Delta \tilde{u} \rangle - \langle u^0 \tilde{u}_x, \Delta \tilde{u} \rangle - \langle u_x^0 \tilde{u}, \Delta \tilde{u} \rangle + \langle \Gamma \Delta \tilde{v}, \Delta \tilde{v} \rangle - \langle a_1 \tilde{v}_x, \Delta \tilde{v} \rangle \\ = - \langle \epsilon_1 \tilde{v}_x, \Delta \tilde{u} \rangle - \langle \tilde{f}, \Delta \tilde{u} \rangle - \langle \epsilon_2 \tilde{u}_x, \Delta \tilde{v} \rangle - \langle \tilde{g}, \Delta \tilde{v} \rangle. \end{aligned} \quad (2.35)$$

Here

$$\begin{aligned} \langle \tilde{u}_t, \Delta \tilde{u} \rangle &= - \langle \tilde{u}_{xt}, \tilde{u}_x \rangle - \langle \tilde{u}_{yt}, \tilde{u}_y \rangle \\ &= - \frac{1}{2} \partial_t(\|\tilde{u}_x\|^2 + \|\tilde{u}_y\|^2). \end{aligned} \quad (2.36)$$

Consider

$$- \langle \gamma \Delta^2 \tilde{u}, \Delta \tilde{u} \rangle = - \langle \gamma \tilde{u}_{xxxx} + 2\gamma \tilde{u}_{xyxy} + \gamma \tilde{u}_{yyyy}, \tilde{u}_{xx} + \tilde{u}_{yy} \rangle, \quad (2.37)$$

where

$$\begin{aligned} - \langle \gamma \tilde{u}_{xxxx}, \tilde{u}_{xx} \rangle &= \langle (\gamma \tilde{u}_{xx})_x, \tilde{u}_{xx} \rangle \\ &= \langle \gamma \tilde{u}_{xxx}, \tilde{u}_{xxx} \rangle + \langle \gamma_x \tilde{u}_{xx}, \tilde{u}_{xxx} \rangle. \end{aligned} \quad (2.38)$$

Here we have:

$$\langle \gamma \tilde{u}_{xxx}, \tilde{u}_{xxx} \rangle \geq \frac{\gamma_0}{2} \|\tilde{u}_{xxx}\|^2. \quad (2.39)$$

Let M be a fixed positive constant such that:

$$\langle \gamma_x \tilde{u}_{xx}, \tilde{u}_{xxx} \rangle \leq \varepsilon \|\tilde{u}\|_3^2 + \frac{M}{4\varepsilon} \|\tilde{u}\|_2^2. \quad (2.40)$$

Similarly we can get:

$$\begin{aligned} -\langle \gamma \tilde{u}_{xxx}, \tilde{u}_{yy} \rangle &= \langle (\gamma \tilde{u}_{yy})_x, \tilde{u}_{xxx} \rangle \\ &= \langle \gamma_x \tilde{u}_{yy}, \tilde{u}_{xxx} \rangle + \langle \gamma \tilde{u}_{xyy}, \tilde{u}_{xxx} \rangle \\ &= \langle \gamma_x \tilde{u}_{yy}, \tilde{u}_{xxx} \rangle - \langle (\gamma \tilde{u}_{xyy})_x, \tilde{u}_{xx} \rangle \\ &= \langle \gamma_x \tilde{u}_{yy}, \tilde{u}_{xxx} \rangle - \langle \gamma_x \tilde{u}_{xyy}, \tilde{u}_{xx} \rangle - \langle \gamma \tilde{u}_{xxyy}, \tilde{u}_{xx} \rangle \\ &= \langle \gamma_x \tilde{u}_{yy}, \tilde{u}_{xxx} \rangle - \langle \gamma_x \tilde{u}_{xyy}, \tilde{u}_{xx} \rangle + \langle \gamma_y \tilde{u}_{xx}, \tilde{u}_{xxy} \rangle + \langle \gamma \tilde{u}_{xxy}, \tilde{u}_{xxy} \rangle. \end{aligned} \quad (2.41)$$

$$\langle \Delta \tilde{u}_x, \Delta \tilde{u} \rangle \leq \varepsilon_1 \|\tilde{u}\|_3^2 + \frac{1}{4\varepsilon_1} \|\tilde{u}\|_2^2. \quad (2.42)$$

$$\begin{aligned} \langle \alpha \tilde{u}_{xx}, \Delta \tilde{u} \rangle &= -\langle \alpha \Delta \tilde{u}_x, \tilde{u}_x \rangle - \langle \alpha_x \Delta \tilde{u}, \tilde{u}_x \rangle \\ &\leq M\varepsilon_2 \|\tilde{u}\|_3^2 + \frac{1}{4\varepsilon_2} \|\tilde{u}\|_1^2 + M\varepsilon_3 \|\tilde{u}\|_2^2 + \frac{1}{4\varepsilon_3} \|\tilde{u}\|_1^2. \end{aligned} \quad (2.43)$$

$$\begin{aligned}
\langle u^0 \tilde{u}_x, \Delta \tilde{u} \rangle &\leq \varepsilon_4 \|\Delta \tilde{u}\|^2 + \frac{1}{4\varepsilon_4} \|u^0 \tilde{u}_x\|^2 \\
&\leq \varepsilon_4 \|\Delta \tilde{u}\|^2 + \frac{(\sup_{x,y} |u^0|)^2}{4\varepsilon_4} \|\tilde{u}_x\|^2 \\
&\leq \varepsilon_4 \|\tilde{u}\|_2^2 + \frac{C \|u^0\|_2^2}{4\varepsilon_4} \|\tilde{u}\|_1^2 \\
&\leq \varepsilon_4 \|\tilde{u}\|_2^2 + \frac{CM}{4\varepsilon_4} \|\tilde{u}\|_1^2.
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
\langle \Gamma \Delta \tilde{v}, \Delta \tilde{v} \rangle &= \Gamma_0 \|\Delta \tilde{v}\|^2 + \langle \Gamma_1 \Delta \tilde{v}, \Delta \tilde{v} \rangle \\
&\geq \frac{\Gamma_0}{2} \|\Delta \tilde{v}\|^2.
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
\langle a_1 \tilde{v}_x, \Delta \tilde{v} \rangle &\leq M \varepsilon_5 \|\Delta \tilde{v}\|^2 + \frac{1}{4\varepsilon_5} \|\tilde{v}_x\|^2 \\
&\leq M \varepsilon_5 \|\tilde{v}\|_2^2 + \frac{1}{4\varepsilon_5} \|\tilde{v}\|_1^2.
\end{aligned} \tag{2.46}$$

$$\begin{aligned}
-\langle \epsilon_1 \tilde{v}_x, \Delta \tilde{u} \rangle &\leq M \varepsilon_6 \|\Delta \tilde{u}\|^2 + \frac{1}{4\varepsilon_6} \|\tilde{v}_x\|^2 \\
&\leq M \varepsilon_6 \|\tilde{u}\|_2^2 + \frac{1}{4\varepsilon_6} \|\tilde{v}\|_1^2.
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
-\langle \tilde{f}, \Delta \tilde{u} \rangle &\leq \varepsilon_7 \|\Delta \tilde{u}\|^2 + \frac{1}{4\varepsilon_7} \|\tilde{f}\|^2 \\
&\leq \varepsilon_7 \|\tilde{u}\|_2^2 + \frac{1}{4\varepsilon_7} \|\tilde{f}\|^2.
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
-\langle \epsilon_2 \tilde{u}_x, \Delta \tilde{v} \rangle &\leq M \varepsilon_8 \|\Delta \tilde{v}\|^2 + \frac{1}{4\varepsilon_8} \|\tilde{u}_x\|^2 \\
&\leq M \varepsilon_8 \|\tilde{v}\|_2^2 + \frac{1}{4\varepsilon_8} \|\tilde{u}\|_1^2.
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
-\langle \tilde{g}, \Delta \tilde{v} \rangle &\leq \varepsilon_9 \|\Delta \tilde{v}\|^2 + \frac{1}{4\varepsilon_9} \|\tilde{g}\|^2 \\
&\leq \varepsilon_9 \|\tilde{v}\|_2^2 + \frac{1}{4\varepsilon_9} \|\tilde{g}\|^2.
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
\langle u_x^0 \tilde{u}, \Delta \tilde{u} \rangle &\leq \|u_x^0\| \|\tilde{u} \Delta \tilde{u}\| \\
&\leq \|u^0\|_1 \|\Delta \tilde{u}\| (\sup_{x,y} |\tilde{u}|) \\
&\leq C \|u^0\|_2 \|\tilde{u}\|_2 \|\tilde{u}\|_2 \\
&\leq CM \|\tilde{u}\|_2^2.
\end{aligned} \tag{2.51}$$

Since ε_i ($i=1,2,3,4,5,6,7,8,9$) are positive constants, we can choose a constant C depending upon the coefficients of (2.4)-(2.5) and ε_i . In particular, the constant C depends upon u^0 only in its $L^\infty([0, T], H^2(R^2))$ norm. Combining with (2.8), we obtain:

$$\begin{aligned}
&\partial_t (\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2) + \|\tilde{u}\|_3^2 + \|\tilde{v}\|_2^2 \\
&\leq C (\|\tilde{u}\|_1^2 + \|\tilde{v}\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{g}\|^2),
\end{aligned} \tag{2.52}$$

$\forall t \in [0, T]$.

By applying Gronwall's inequality to (2.52), we obtain (2.34) and the proof is completed.

2.4 High order estimate

Theorem 2.4.1 *Any solution $(\tilde{u}, \tilde{v}) \in C^\infty([0, T], S(R^2))$ of (2.4), (2.5) and (2.6) satisfies the estimate:*

$$\begin{aligned} & \partial_t(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2) + \|\tilde{u}\|_4^2 + \|\tilde{v}\|_2^2 \\ & \leq C(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2 + \|\tilde{f}\|^2 + \|\tilde{g}\|^2), \end{aligned} \quad (2.53)$$

$\forall t \in [0, T]$,

and

$$\begin{aligned} & \sup_{0 \leq t \leq T}(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_1^2) + \int_0^T (\|\tilde{u}\|_4^2 + \|\tilde{v}\|_2^2) ds \\ & \leq C(T)(\|\tilde{u}_0\|_2^2 + \|\tilde{v}_0\|_1^2 + \int_0^T (\|\tilde{f}\|^2 + \|\tilde{g}\|^2) ds). \end{aligned} \quad (2.54)$$

Proof. Taking L^2 inner product of (2.4) with $\Delta^2 \tilde{u}$ and (2.5) with $\Delta \tilde{v}$ and integrating by parts, we obtain:

$$\begin{aligned} & \frac{1}{2} \partial_t (\|\tilde{u}_{xx}\|^2 + \|\tilde{u}_{yy}\|^2 + 2\|\tilde{u}_{xy}\|^2 + \|\tilde{v}_x\|^2 + \|\tilde{v}_y\|^2) + \langle \gamma \Delta^2 \tilde{u}, \Delta^2 \tilde{u} \rangle + \langle \Delta \tilde{u}_x, \Delta^2 \tilde{u} \rangle \\ & + \langle \alpha \tilde{u}_{xx}, \Delta^2 \tilde{u} \rangle + \langle u^0 \tilde{u}_x, \Delta^2 \tilde{u} \rangle + \langle u_x^0 \tilde{u}, \Delta^2 \tilde{u} \rangle + \langle \Gamma \Delta \tilde{v}, \Delta \tilde{v} \rangle - \langle a_1 \tilde{v}_x, \Delta \tilde{v} \rangle \\ & = \langle \epsilon_1 \tilde{v}_x, \Delta^2 \tilde{u} \rangle + \langle \tilde{f}, \Delta^2 \tilde{u} \rangle - \langle \epsilon_2 \tilde{u}_x, \Delta \tilde{v} \rangle - \langle \tilde{g}, \Delta \tilde{v} \rangle. \end{aligned} \quad (2.55)$$

Here

$$\begin{aligned} \langle \tilde{u}_t, \Delta^2 \tilde{u} \rangle &= -\langle \tilde{u}_{xt}, \tilde{u}_{xxx} \rangle - 2\langle \tilde{u}_{yt}, \tilde{u}_{xxy} \rangle - \langle \tilde{u}_{yt}, \tilde{u}_{yyy} \rangle \\ &= \langle \tilde{u}_{xxt}, \tilde{u}_{xx} \rangle + 2\langle \tilde{u}_{xyt}, \tilde{u}_{xy} \rangle - \langle \tilde{u}_{yyt}, \tilde{u}_{yy} \rangle \\ &= \frac{1}{2} \partial_t (\|\tilde{u}_{xx}\|^2 + \|\tilde{u}_{yy}\|^2) + \partial_t \|\tilde{u}_{xy}\|^2. \end{aligned} \quad (2.56)$$

Consider

$$\begin{aligned}
\langle \gamma \Delta^2 \tilde{u}, \Delta^2 \tilde{u} \rangle &= \gamma_0 \langle \Delta^2 \tilde{u}, \Delta^2 \tilde{u} \rangle + \langle \gamma_1 \Delta^2 \tilde{u}, \Delta^2 \tilde{u} \rangle \\
&= \gamma_0 \|\Delta^2 \tilde{u}\|^2 + \langle \gamma_1 \Delta^2 \tilde{u}, \Delta^2 \tilde{u} \rangle \\
&\geq \frac{\gamma_0}{2} \|\Delta^2 \tilde{u}\|^2.
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
\langle \Delta \tilde{u}_x, \Delta^2 \tilde{u} \rangle &\leq \varepsilon_1 \|\Delta^2 \tilde{u}\|^2 + \frac{1}{4\varepsilon_1} \|\Delta \tilde{u}_x\|^2 \\
&\leq \varepsilon_1 \|\tilde{u}\|_4^2 + \frac{\varepsilon}{4\varepsilon_1} \|\tilde{u}\|_4^2 + \frac{1}{16\varepsilon_1 \varepsilon} \|\tilde{u}\|_2^2,
\end{aligned} \tag{2.58}$$

where we estimate $\|\Delta \tilde{u}_x\|^2$ by means of the inequality:

$$\begin{aligned}
\|\Delta \tilde{u}_x\|^2 &= |\langle \tilde{u}_{xxx}, \tilde{u}_{xxx} \rangle + 2\langle \tilde{u}_{xyy}, \tilde{u}_{xxx} \rangle + \langle \tilde{u}_{xyy}, \tilde{u}_{xyy} \rangle| \\
&= |-\langle \tilde{u}_{xx}, \tilde{u}_{xxx} \rangle - 2\langle \tilde{u}_{xx}, \tilde{u}_{xxy} \rangle - \langle \tilde{u}_{yy}, \tilde{u}_{xxy} \rangle| \\
&\leq \varepsilon \|\tilde{u}\|_4^2 + \frac{1}{4\varepsilon} \|\tilde{u}\|_2^2,
\end{aligned} \tag{2.59}$$

where $\varepsilon \ll \varepsilon_1 \ll 1$.

$$\begin{aligned}
\langle \alpha \tilde{u}_{xx}, \Delta^2 \tilde{u} \rangle &\leq \varepsilon_2 \|\Delta^2 \tilde{u}\|^2 + \frac{1}{4\varepsilon_2} \|\alpha \tilde{u}_{xx}\|^2 \\
&\leq \varepsilon_2 \|\tilde{u}\|_4^2 + \frac{M}{4\varepsilon_2} \|\tilde{u}\|_2^2.
\end{aligned} \tag{2.60}$$

$$\begin{aligned}
\langle u^0 \tilde{u}_x, \Delta^2 \tilde{u} \rangle &\leq \varepsilon_3 \|\Delta^2 \tilde{u}\|^2 + \frac{(\sup_{x,y} |u^0|)^2}{4\varepsilon_3} \|\tilde{u}_x\|^2 \\
&\leq \varepsilon_3 \|\tilde{u}\|_4^2 + \frac{C \|u^0\|_2^2}{4\varepsilon_3} \|\tilde{u}\|_1^2 \\
&\leq \varepsilon_3 \|\tilde{u}\|_4^2 + \frac{CM}{4\varepsilon_3} \|\tilde{u}\|_1^2.
\end{aligned} \tag{2.61}$$

$$\begin{aligned}
\langle \Gamma \Delta \tilde{v}, \Delta \tilde{v} \rangle &= \Gamma_0 \|\Delta \tilde{v}\|^2 + \langle \Gamma_1 \Delta \tilde{v}, \Delta \tilde{v} \rangle \\
&\geq \frac{\Gamma_0}{2} \|\Delta \tilde{v}\|^2.
\end{aligned} \tag{2.62}$$

$$\begin{aligned}
-\langle a_1 \tilde{v}_x, \Delta \tilde{v} \rangle &\leq \varepsilon_4 \|\Delta \tilde{v}\|^2 + \frac{1}{4\varepsilon_4} \|a_1 \tilde{v}_x\|^2 \\
&\leq \varepsilon_4 \|\tilde{v}\|_2^2 + \frac{M}{4\varepsilon_4} \|\tilde{v}\|_1^2.
\end{aligned} \tag{2.63}$$

$$\begin{aligned}
\langle \epsilon_1 \tilde{v}_x, \Delta^2 \tilde{u} \rangle &\leq \varepsilon_5 \|\Delta^2 \tilde{u}\|^2 + \frac{M}{4\varepsilon_5} \|\tilde{v}_x\|^2 \\
&\leq \varepsilon_5 \|\tilde{u}\|_4^2 + \frac{M}{4\varepsilon_5} \|\tilde{v}\|_1^2.
\end{aligned} \tag{2.64}$$

$$\begin{aligned}
\langle \tilde{f}, \Delta^2 \tilde{u} \rangle &\leq \varepsilon_6 \|\Delta^2 \tilde{u}\|^2 + \frac{1}{4\varepsilon_6} \|\tilde{f}\|^2 \\
&\leq \varepsilon_6 \|\tilde{u}\|_4^2 + \frac{1}{4\varepsilon_6} \|\tilde{f}\|^2.
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
-\langle \epsilon_2 \tilde{u}_x, \Delta \tilde{v} \rangle &\leq \varepsilon_7 \|\Delta \tilde{v}\|^2 + \frac{M}{4\varepsilon_7} \|\tilde{u}_x\|^2 \\
&\leq \varepsilon_7 \|\tilde{v}\|_2^2 + \frac{M}{4\varepsilon_7} \|\tilde{u}\|_1^2.
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
-\langle \tilde{g}, \Delta \tilde{v} \rangle &\leq \varepsilon_8 \|\Delta \tilde{v}\|^2 + \frac{1}{4\varepsilon_8} \|\tilde{g}\|^2 \\
&\leq \varepsilon_8 \|\tilde{v}\|_2^2 + \frac{1}{4\varepsilon_8} \|\tilde{g}\|^2.
\end{aligned} \tag{2.67}$$

$$\begin{aligned}
\langle u_x^0 \tilde{u}, \Delta^2 \tilde{u} \rangle &\leq \|u_x^0\| \|\tilde{u} \Delta^2 \tilde{u}\| \\
&\leq \|u^0\|_1 \|\Delta^2 \tilde{u}\| (\sup_{x,y} |\tilde{u}|) \\
&\leq C \|u^0\|_2 \|\tilde{u}\|_4 \|\tilde{u}\|_2 \\
&\leq CM(\varepsilon_9 \|\tilde{u}\|_4^2 + \frac{1}{4\varepsilon_9} \|\tilde{u}\|_2).
\end{aligned} \tag{2.68}$$

Since ε_i ($i=1,2,3,4,5,6,7,8,9$) are positive constants, we can choose a constant C depending upon the coefficients of (2.4)-(2.5) and ε_i . Combining with (2.33), we obtain (2.53).

By applying Gronwall's inequality to (2.53), we obtain (2.54) and the proof is complete.

It is easy to see that the constant $C(T)$ in (2.54) depends upon u^0 only in its $L^\infty([0, T], H^2(R^2))$ norm.

Theorem 2.4.2 *Any solution $(\tilde{u}, \tilde{v}) \in C^\infty([0, T], S(R^2))$ of (2.4), (2.5) and (2.6) satisfies the estimate, for any integer $k \geq 0$,*

$$\begin{aligned}
&\partial_t (\|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2) + \|\tilde{u}\|_{k+4}^2 + \|\tilde{v}\|_{k+2}^2 \\
&\leq C_k (\|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2 + \|\tilde{f}\|_k^2 + \|\tilde{g}\|_k^2),
\end{aligned} \tag{2.69}$$

$\forall t \in [0, T]$,

and

$$\begin{aligned}
&\sup_{0 \leq t \leq T} (\|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2) + \int_0^T (\|\tilde{u}\|_{k+4}^2 + \|\tilde{v}\|_{k+2}^2) ds \\
&\leq C_k(T) (\|\tilde{u}_0\|_{k+2}^2 + \|\tilde{v}_0\|_{k+1}^2 + \int_0^T (\|\tilde{f}\|_k^2 + \|\tilde{g}\|_k^2) ds).
\end{aligned} \tag{2.70}$$

Here $C_k(T)$ is a constant depending on T and the coefficients of (2.4) and (2.5). In particular, the constant $C_k(T)$ depends upon u^0 only in its $L^\infty([0, T], H^{k+2}(R^2))$ norm.

Proof. We will prove this theorem by induction.

For $k = 0$, we proved it in Theorem (2.4.1). Therefore this theorem is true for $k = 0$.

Next, suppose that this theorem is true for a positive integer k , $k \geq 0$, then we have:

$$\begin{aligned} & \partial_t(\|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2) + \|\tilde{u}\|_{k+4}^2 + \|\tilde{v}\|_{k+2}^2 \\ & \leq C_k(\|\tilde{u}\|_{k+2}^2 + \|\tilde{v}\|_{k+1}^2 + \|\tilde{f}\|_k^2 + \|\tilde{g}\|_k^2). \end{aligned} \quad (2.71)$$

We show in the following that this theorem is also true for $k + 1$, that is:

$$\begin{aligned} & \partial_t(\|\tilde{u}\|_{k+3}^2 + \|\tilde{v}\|_{k+2}^2) + \|\tilde{u}\|_{k+5}^2 + \|\tilde{v}\|_{k+3}^2 \\ & \leq C_{k+1}(\|\tilde{u}\|_{k+3}^2 + \|\tilde{v}\|_{k+2}^2 + \|\tilde{f}\|_{k+1}^2 + \|\tilde{g}\|_{k+1}^2). \end{aligned} \quad (2.72)$$

Taking $(k+1)$ -th partial derivative of equations (2.4)-(2.5) with respect to x and taking L^2 inner product of (2.4) with $\Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}$ and (2.5) with $\Delta \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{v}$, we have:

$$\langle \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_t, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u} \rangle = \frac{1}{2} \partial_t(\|\frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_{xx}\|^2 + \|\frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_{yy}\|^2) + \partial_t \|\frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_{xy}\|^2. \quad (2.73)$$

$$\langle \gamma \frac{\partial^{k+1}}{\partial x^{k+1}} \Delta^2 \tilde{u}, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u} \rangle \geq \frac{\gamma_0}{2} \|\Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}\|^2. \quad (2.74)$$

$$\begin{aligned} \langle \frac{\partial^{k+1}}{\partial x^{k+1}} \Delta \tilde{u}_x, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u} \rangle & \leq \varepsilon_1 \|\tilde{u}\|_{k+5}^2 + \frac{1}{4\varepsilon_1} \|\tilde{u}\|_{k+4}^2 \\ & \leq \varepsilon_1 \|\tilde{u}\|_{k+5}^2 + \frac{\varepsilon}{4\varepsilon_1} \|\tilde{u}\|_{k+5}^2 + \frac{1}{16\varepsilon_1\varepsilon} \|\tilde{u}\|_{k+3}^2, \end{aligned} \quad (2.75)$$

where $\varepsilon \ll \varepsilon_1 \ll 1$.

In particular, we have:

$$\begin{aligned} \langle \frac{\partial^{k+1}}{\partial x^{k+1}} u_x^0 \tilde{u}, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u} \rangle & \leq \|\frac{\partial^{k+1}}{\partial x^{k+1}} u_x^0 \tilde{u}\| \|\Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}\| \\ & \leq C \|u^0\|_{k+2} \|\tilde{u}\|_{k+3} \|\tilde{u}\|_{k+5}. \end{aligned} \quad (2.76)$$

We have the following inequality:

$$\begin{aligned} & \frac{1}{2} \partial_t (\| \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_{xx} \|^2 + \| \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_{yy} \|^2) + \partial_t \| \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u}_{xy} \|^2 + \frac{\gamma_0}{2} \| \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{u} \|^2 + \frac{\Gamma_0}{2} \| \Delta \frac{\partial^{k+1}}{\partial x^{k+1}} \tilde{v} \|^2 \\ & \leq C_{k+1} (\| \tilde{u} \|_{k+3}^2 + \| \tilde{v} \|_{k+2}^2 + \| \tilde{f} \|_{k+1}^2 + \| \tilde{g} \|_{k+1}^2). \end{aligned} \quad (2.77)$$

Similarly, for any non-negative integers α, β , with $\alpha + \beta = k + 1$, we have the following inequalities:

$$\begin{aligned} & \frac{1}{2} \partial_t (\| \partial_x^\alpha \partial_y^\beta \tilde{u}_{xx} \|^2 + \| \partial_x^\alpha \partial_y^\beta \tilde{u}_{yy} \|^2) + \partial_t \| \partial_x^\alpha \partial_y^\beta \tilde{u}_{xy} \|^2 + \frac{\gamma_0}{2} \| \Delta^2 \partial_x^\alpha \partial_y^\beta \tilde{u} \|^2 + \frac{\Gamma_0}{2} \| \Delta \partial_x^\alpha \partial_y^\beta \tilde{v} \|^2 \\ & \leq C_{k+1} (\| \tilde{u} \|_{k+3}^2 + \| \tilde{v} \|_{k+2}^2 + \| \tilde{f} \|_{k+1}^2 + \| \tilde{g} \|_{k+1}^2). \end{aligned} \quad (2.78)$$

Taking summation for all $\alpha + \beta = k + 1$ and combining (2.78) with (2.71), we get the estimate (2.72).

Remark 2.4.1 For convenience, we use the notation Π_T^k in the following to denote the product space of (u, v) : $(u, v) \in \Pi_T^k$ if

$$\begin{aligned} u & \in C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+4}(R^2)) \\ v & \in C([0, T], H^{k+1}(R^2)) \cap L^2([0, T], H^{k+2}(R^2)). \end{aligned}$$

Π_T^k is a Banach space equipped with the norm:

$$\sup_{0 \leq t \leq T} (\|u(t)\|_{k+2}^2 + \|v(t)\|_{k+1}^2) + \int_0^T (\|u(s)\|_{k+4}^2 + \|v(s)\|_{k+2}^2) ds.$$

Chapter 3

The Existence of Solution for the Linearized Problem

3.1 The linearized system with $u^0 \in C^{k+1}([0, T], R^2)$

Several methods are available to establish the existence of solution to the linear problem (2.4)-(2.6). In this chapter, we use the continuation method to prove the following existence and uniqueness of the solution to (2.4)-(2.6).

Theorem 3.1.1 *In the initial value problem (2.4)-(2.6), let $k \geq 0$ be any integer, and assume that*

- $u_0 \in H^{k+2}(R^2)$ and $v_0 \in H^{k+1}(R^2)$ and $f, g \in L^2([0, T], H^k(R^2))$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (2.4)-(2.5) are k -th order continuously differentiable and are constants outside a bounded domain, u^0 is $(k+1)$ -th order continuously differentiable and is a constant outside a bounded domain.

Then (2.4)-(2.6) has a unique solution (u, v) in the space Π_T^k satisfying the estimate (2.70).

Remark 3.1.1 By Sobolev imbedding theorem, for $k \geq 2$, the solution (u, v) is continuously differentiable in (x, y) up to fourth and second order. If $k \geq 4$, and $f, g \in H^1([0, T], H^k(R^2))$, then by equation (2.4)-(2.6), we can derive that (u_t, v_t) are also continuous. Therefore the solution (u, v) in Theorem (3.1.1) is a classical solution.

First of all, we rewrite (2.4)-(2.6) briefly as follows:

$$\begin{cases} u_t + \mathcal{L}_1(u, v) = f \\ v_t + \mathcal{L}_2(u, v) = g \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y), \end{cases} \quad (3.1)$$

where

$$\begin{cases} \mathcal{L}_1(u, v) = u^0 u_x + u_x^0 u + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x \\ \mathcal{L}_2(u, v) = a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x. \end{cases} \quad (3.2)$$

Consider the following one-parameter family of initial value problems, denoted as Λ_λ ($\lambda \in [0, 1]$):

$$\begin{cases} u_t + \lambda \mathcal{L}_1(u, v) + (1 - \lambda) \Delta^2 u = f \\ v_t + \lambda \mathcal{L}_2(u, v) - (1 - \lambda) \Delta v = g \\ u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y). \end{cases} \quad (3.3)$$

Obviously, for $\lambda = 1$, the problem Λ_1 in (3.3) is the same as the initial value problem (2.4)-(2.6). It is readily checked that the energy estimate (2.70) is valid uniformly for the solution (u, v) of (2.4)-(2.6) with the constant $C_k(T)$ in (2.70) independent of the parameter $\lambda \in [0, 1]$.

We show in the following that the conclusion on the existence of solution (u, v) in Theorem (3.1.1) is also true for (3.3) for all $\lambda \in [0, 1]$. Then the conclusion in Theorem (3.1.1) is a particular case for $\lambda = 1$.

To achieve this, we denote Υ as the subset of $[0, 1]$ such that for parameter $\lambda \in \Upsilon$, Theorem (3.1.1) is true for (3.3), i.e., the initial value problem (3.3) has a unique solution in Π_T^k satisfying (2.70). We need to show that $\Upsilon = [0, 1]$, i.e., we need to prove that the subset Υ is not empty, and it is both closed and open.

1. Υ is not empty.

Actually, $0 \in \Upsilon$. The problem Λ_0 is the following two separate initial value problems for u and v :

$$\begin{cases} u_t + \Delta^2 u = f \\ u(x, y, 0) = u_0(x, y) \end{cases} \quad (3.4)$$

$$\begin{cases} v_t - \Delta v = g \\ v(x, y, 0) = v_0(x, y). \end{cases} \quad (3.5)$$

The existence of the solutions (u, v) for the problem (3.4) and (3.5) is the standard result for the Cauchy problem of general parabolic equations, see e.g. [13][19].

2. Υ is closed in $[0, 1]$.

Let $\lambda_j \in \Upsilon$, and $\lambda_j \rightarrow \lambda_0$. We need to show $\lambda_0 \in \Upsilon$.

Since $\lambda_j \in \Upsilon$, let (u_j, v_j) be the solution of the following initial value problem:

$$\begin{cases} u_{jt} + \lambda_j \mathcal{L}_1(u_j, v_j) + (1 - \lambda_j) \Delta^2 u_j = f \\ v_{jt} + \lambda_j \mathcal{L}_2(u_j, v_j) - (1 - \lambda_j) \Delta v_j = g \\ u_j(x, y, 0) = u_0(x, y), \quad v_j(x, y, 0) = v_0(x, y). \end{cases} \quad (3.6)$$

By (2.70), (u_j, v_j) is uniformly bounded in Π_T^k . In particular:

$$\begin{aligned} & \int_0^T (\|u_j(s)\|_{k+4}^2 + \|v_j(s)\|_{k+2}^2) ds \\ & \leq C_k(T) (\|u_0\|_{k+2}^2 + \|v_0\|_{k+1}^2 + \int_0^T (\|f\|_k^2 + \|g\|_k^2) ds) \\ & \leq K. \end{aligned} \quad (3.7)$$

(u_{j-1}, v_{j-1}) satisfies:

$$\begin{cases} u_{(j-1)t} + \lambda_{j-1} \mathcal{L}_1(u_{j-1}, v_{j-1}) + (1 - \lambda_{j-1}) \Delta^2 u_{j-1} = f \\ v_{(j-1)t} + \lambda_{j-1} \mathcal{L}_2(u_{j-1}, v_{j-1}) - (1 - \lambda_{j-1}) \Delta v_{j-1} = g \\ u_{j-1}(x, y, 0) = u_0(x, y), \quad v_{j-1}(x, y, 0) = v_0(x, y). \end{cases} \quad (3.8)$$

After adding $\lambda_j \mathcal{L}_1(u_{j-1}, v_{j-1}) - \lambda_j \mathcal{L}_1(u_j, v_j)$ and $(1 - \lambda_j) \Delta^2 u_{j-1} - (1 - \lambda_j) \Delta^2 u_j$ to the left side of the first equation in (3.6) and subtracting the first equation in (3.8), we get the first equation in (3.9).

Let $(\tilde{u}_j, \tilde{v}_j) \equiv (u_j - u_{j-1}, v_j - v_{j-1})$ ($j = 2, 3, \dots$) which satisfies:

$$\left\{ \begin{array}{l} \tilde{u}_{jt} + \lambda_j \mathcal{L}_1(\tilde{u}_j, \tilde{v}_j) + (1 - \lambda_j) \Delta^2 \tilde{u}_j \\ = -(\lambda_j - \lambda_{j-1}) [\mathcal{L}_1(u_{j-1}, v_{j-1}) - \Delta^2 u_{j-1}] \\ \tilde{v}_{jt} + \lambda_j \mathcal{L}_2(\tilde{u}_j, \tilde{v}_j) - (1 - \lambda_j) \Delta \tilde{v}_j \\ = -(\lambda_j - \lambda_{j-1}) (\mathcal{L}_2(u_{j-1}, v_{j-1}) - \Delta v_{j-1}) \\ \tilde{u}_j(x, y, 0) = 0, \quad \tilde{v}_j(x, y, 0) = 0. \end{array} \right. \quad (3.9)$$

Applying (2.70) to the solution $(\tilde{u}_j, \tilde{v}_j)$ of (3.9), we have:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\tilde{u}_j(t)\|_{k+2}^2 + \|\tilde{v}_j(t)\|_{k+1}^2) + \int_0^T (\|\tilde{u}_j(s)\|_{k+4}^2 + \|\tilde{v}_j(s)\|_{k+2}^2) ds \\ & \leq C_k(T) |(\lambda_j - \lambda_{j-1})|^2 \int_0^T (\|u_{j-1}(s)\|_{k+4}^2 + \|v_{j-1}(s)\|_{k+2}^2) ds \\ & \leq C_k(T) |(\lambda_j - \lambda_{j-1})|^2 K. \end{aligned} \quad (3.10)$$

Here we have:

$$\|u^0 u_{(j-1)x}\|_k^2 \leq C \|u^0\|_{C^k}^2 \|u_{j-1}\|_{k+1}^2 \quad (3.11)$$

$$\|u_x^0 u_{(j-1)}\|_k^2 \leq C \|u^0\|_{C^{k+1}}^2 \|u_{j-1}\|_k^2. \quad (3.12)$$

Since $\lambda_j \rightarrow \lambda_0$, it follows that (u_j, v_j) is a Cauchy sequence in Π_T^k and its limit (u, v) is obviously the solution of (3.3) for λ_0 . This shows Υ is closed in $[0, 1]$.

3. Υ is open in $[0, 1]$.

Let $\lambda_0 \in \Upsilon$, $\lambda \in [0, 1]$ and $|\lambda - \lambda_0| \leq \varepsilon$. We need to show that for $\varepsilon \ll 1$, $\lambda \in \Upsilon$.

Since $\lambda_0 \in \Upsilon$, we can construct a sequence of approximate solutions (u_n, v_n) ($n = 2, 3, \dots$) as the solutions of the following problems:

$$\left\{ \begin{array}{l} u_{nt} + \lambda_0 \mathcal{L}_1(u_n, v_n) + (1 - \lambda_0) \Delta^2 u_n \\ = f + (\lambda_0 - \lambda)(\mathcal{L}_1(u_{n-1}, v_{n-1}) - \Delta^2 u_{n-1}) \\ v_{nt} + \lambda_0 \mathcal{L}_2(u_n, v_n) - (1 - \lambda_0) \Delta v_n \\ = g + (\lambda_0 - \lambda)(\mathcal{L}_2(u_{n-1}, v_{n-1}) - \Delta v_{n-1}) \\ u_n(x, y, 0) = u_0(x, y), \quad v_n(x, y, 0) = v_0(x, y). \end{array} \right. \quad (3.13)$$

For $n = 1$, we simply take (u_1, v_1) to be the solution of the following problem:

$$\left\{ \begin{array}{l} u_{1t} + \lambda_0 \mathcal{L}_1(u_1, v_1) + (1 - \lambda_0) \Delta^2 u_1 = f \\ v_{1t} + \lambda_0 \mathcal{L}_2(u_1, v_1) - (1 - \lambda_0) \Delta v_1 = g \\ u_1(x, y, 0) = u_0(x, y), \quad v_1(x, y, 0) = v_0(x, y). \end{array} \right. \quad (3.14)$$

(u_{n-1}, v_{n-1}) is the solution of the following problem:

$$\left\{ \begin{array}{l} u_{(n-1)t} + \lambda_0 \mathcal{L}_1(u_{n-1}, v_{n-1}) + (1 - \lambda_0) \Delta^2 u_{n-1} \\ = f + (\lambda_0 - \lambda)(\mathcal{L}_1(u_{n-2}, v_{n-2}) - \Delta^2 u_{n-2}) \\ v_{(n-1)t} + \lambda_0 \mathcal{L}_2(u_{n-1}, v_{n-1}) - (1 - \lambda_0) \Delta v_{n-1} \\ = g + (\lambda_0 - \lambda)(\mathcal{L}_2(u_{n-2}, v_{n-2}) - \Delta v_{n-2}) \\ u_{n-1}(x, y, 0) = u_0(x, y), \quad v_{n-1}(x, y, 0) = v_0(x, y). \end{array} \right. \quad (3.15)$$

Let $(\tilde{u}_n, \tilde{v}_n) \equiv (u_n - u_{n-1}, v_n - v_{n-1})$ ($n = 2, 3, \dots$). $(\tilde{u}_n, \tilde{v}_n)$ is the solution of the following problem:

$$\left\{ \begin{array}{l} \tilde{u}_{nt} + \lambda_0 \mathcal{L}_1(\tilde{u}_n, \tilde{v}_n) + (1 - \lambda_0) \Delta^2 \tilde{u}_n \\ = (\lambda_0 - \lambda)(\mathcal{L}_1(\tilde{u}_{n-1}, \tilde{v}_{n-1}) - \Delta^2 \tilde{u}_{n-1}) \\ \tilde{v}_{nt} + \lambda_0 \mathcal{L}_2(\tilde{u}_n, \tilde{v}_n) - (1 - \lambda_0) \Delta \tilde{v}_n \\ = (\lambda_0 - \lambda)(\mathcal{L}_2(\tilde{u}_{n-1}, \tilde{v}_{n-1}) - \Delta \tilde{v}_{n-1}) \\ \tilde{u}_n(x, y, 0) = 0, \quad \tilde{v}_n(x, y, 0) = 0. \end{array} \right. \quad (3.16)$$

Since

$$\begin{aligned} & \|\mathcal{L}_1(\tilde{u}_{n-1}, \tilde{v}_{n-1}) - \Delta^2 \tilde{u}_{n-1}\|_k^2 + \|\mathcal{L}_2(\tilde{u}_{n-1}, \tilde{v}_{n-1}) - \Delta \tilde{v}_{n-1}\|_k^2 \\ & \leq M(\|\tilde{u}_{n-1}\|_{k+4}^2 + \|\tilde{v}_{n-1}\|_{k+2}^2), \end{aligned} \quad (3.17)$$

by (2.70), $(\tilde{u}_n, \tilde{v}_n)$ satisfies:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\tilde{u}_n\|_{k+2}^2 + \|\tilde{v}_n\|_{k+1}^2) + \int_0^T (\|\tilde{u}_n\|_{k+4}^2 + \|\tilde{v}_n\|_{k+2}^2) ds \\ & \leq MC_k(T)(\lambda_0 - \lambda)^2 \int_0^T (\|\tilde{u}_{n-1}\|_{k+4}^2 + \|\tilde{v}_{n-1}\|_{k+2}^2) ds \\ & \leq MC_k(T)(\lambda_0 - \lambda)^2 (\sup_{0 \leq t \leq T} (\|\tilde{u}_{n-1}\|_{k+2}^2 + \|\tilde{v}_{n-1}\|_{k+1}^2) + \int_0^T (\|\tilde{u}_{n-1}\|_{k+4}^2 + \|\tilde{v}_{n-1}\|_{k+2}^2) ds). \end{aligned} \quad (3.18)$$

Choose $\varepsilon \ll 1$ such that $MC_k(T)\varepsilon^2 < 1$. Hence (u_n, v_n) is a Cauchy sequence in Π_T^k and its limit (u, v) is the solution of (3.3) for λ . Therefore Υ is open in $[0, 1]$. This concludes the proof of Theorem (3.1.1).

3.2 The linearized system with $u^0 \in C([0, T], H_{ul}^{k+2}(R^2))$.

To prepare the study of the nonlinear system (1.6)-(1.7) in next chapter, we introduce the uniformly local Sobolev space $H_{ul}^s(R^2)$, see also [10].

Definition 3.2.1 *The uniformly local Sobolev space $H_{ul}^s(R^n)$ is defined by $f \in H_{ul}^s(R^n)$ if and only if $\forall \phi \in C_0^\infty(R^n)$ and $\phi_{X_0}(X) \equiv \phi(X - X_0)$,*

$$\sup_{X_0 \in R^n} \|f \phi_{X_0}\|_{H^s(R^n)} \leq \infty, \quad (3.19)$$

A corresponding uniformly local Sobolev norm can be defined as

$$\|f\|_{H^s(R^2), ul} = \sup_{X_0 \in R^2} \|f \phi_{X_0}\|_{H^s(R^2)} \quad (3.20)$$

for a fixed given $\phi \in C_0^\infty(R^2)$ with $\phi(X) = 1$ in $|X| \leq 1$.

It is readily verified that thus defined H_{ul}^s is a Hilbert space.

Then we have the following improved version of Theorem (3.1.1).

Theorem 3.2.1 *In the initial value problem (2.4)-(2.6), let $k \geq 0$ be any integer. Under the same assumptions as in Theorem (3.1.1), except the requirement that $u^0 \in C^{k+1}([0, T], R^2)$ and u^0 be a constant outside a bounded domain. Instead, we assume*

$$u^0 \in C([0, T], H_{ul}^{k+2}(R^2)).$$

Then we have the same conclusion as in Theorem (3.1.1), i.e., (2.4)-(2.6) has a unique solution (u, v) in the space Π_T^k satisfying the estimate (2.70).

To prove Theorem (3.2.1), we notice that the estimate (2.70) can be obtained assuming that $u^0 \in C([0, T], H_{ul}^{k+2}(R^2))$. Since $H_{ul}^{k+2}(R^2)$ is a Banach algebra, in step 2 (to show Υ is closed) in the proof of Theorem (3.1.1), we have:

$$\begin{aligned} \|u^0 u_{(j-1)x}\|_k^2 &\leq \|u^0 u_{j-1}\|_{k+2}^2 \\ &\leq C \|u^0\|_{k+2}^2 \|u_{j-1}\|_{k+2}^2 \end{aligned} \quad (3.21)$$

$$\begin{aligned}
 \|u_x^0 u_{(j-1)}\|_k^2 &\leq \|u^0 u_{j-1}\|_{k+2}^2 \\
 &\leq C \|u^0\|_{k+2}^2 \|u_{j-1}\|_{k+2}^2.
 \end{aligned}
 \tag{3.22}$$

Hence (3.10) follows from (3.9). Same argument also applies to the estimate of (3.16) in step 3 (to show Υ is open) in the proof of Theorem (3.1.1) and we can obtain (3.18) correspondingly.

Chapter 4

The Existence of Local Solution for the Non-linear Problem

4.1 The local solution for the non-linear problem

In this section, we prove the existence of local solution for non-linear system (1.6)-(1.7). Consider the following initial value problem:

$$u_t + uu_x + \Delta u_x = -\alpha u_{xx} - \gamma \Delta^2 u + \epsilon_1 v_x \quad (4.1)$$

$$v_t + a_1 v_x = \Gamma \Delta v + \epsilon_2 u_x \quad (4.2)$$

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y). \quad (4.3)$$

As before, the coefficients $\gamma, \Gamma, \alpha, a_1, \epsilon_1, \epsilon_2$ are assumed to be positive constants or positive constants outside a bounded domain with possible smooth perturbations in bounded domain. In particular, γ, Γ are uniformly positive: $\gamma \geq \delta > 0, \Gamma \geq \delta > 0$. Actually, uu_x is the only non-linear term in (4.1)-(4.3). With minor modifications, our proof in this section can also apply to the case where the coefficients $\gamma, \Gamma, \alpha, a_1, \epsilon_1, \epsilon_2$ may depend upon (u, v) or their derivatives.

Also, we consider the case that the initial values (u_0, v_0) have the form:

$$(u_0, v_0) = (U_0 + \tilde{u}_0, V_0 + \tilde{v}_0) \quad (4.4)$$

with (U_0, V_0) being constants. For the cases that they are periodic functions, our proof can be easily adjusted by considering periodic domain and the conclusion obtained here remains true. The main conclusion of this section is the following theorem.

Theorem 4.1.1 *In the initial value problem (4.1)-(4.3), let k be a non-negative integer and assume that*

- *All the coefficients $\gamma, \Gamma, \alpha, a_1, \epsilon_1, \epsilon_2$ are positive constants with possible smooth perturbations in a bounded domain;*
- *$(u_0, v_0) = (U_0 + \tilde{u}_0, V_0 + \tilde{v}_0)$ with (U_0, V_0) being constants and $(\tilde{u}_0, \tilde{v}_0) \in H^{k+2}(R^2) \times H^{k+1}(R^2)$;*

Then there is a $T > 0$ such that (4.1)-(4.3) has a unique solution $(u, v) = (U_0 + \tilde{u}, V_0 + \tilde{v}) \in C([0, T], H_{ul}^{k+2}(R^2) \times H_{ul}^{k+1}(R^2))$ satisfying

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\tilde{u}(t)\|_{k+2}^2 + \|\tilde{v}(t)\|_{k+1}^2) + \int_0^T (\|\tilde{u}(s)\|_{k+4}^2 + \|\tilde{v}(s)\|_{k+2}^2) ds \\ & \leq C_k(T) (\|\tilde{u}_0\|_{k+2}^2 + \|\tilde{v}_0\|_{k+1}^2). \end{aligned} \quad (4.5)$$

Proof. We use linear iteration to prove Theorem (4.1.1). First of all, we construct an approximate solution (u_a, v_a) of the form:

$$(u_a, v_a) = (U_0 + \tilde{u}_a, V_0 + \tilde{v}_a), \quad (4.6)$$

where $(\tilde{u}_a, \tilde{v}_a)$ is the solution for the initial value problem of the linear system:

$$\tilde{u}_{at} + (U_0 + \tilde{u}_0)\tilde{u}_{ax} + \Delta\tilde{u}_{ax} + \alpha\tilde{u}_{axx} + \gamma\Delta^2\tilde{u}_a - \epsilon_1\tilde{v}_{ax} = 0 \quad (4.7)$$

$$\tilde{v}_{at} + a_1 \tilde{v}_{ax} - \Gamma \Delta \tilde{v}_a - \epsilon_2 \tilde{u}_{ax} = 0 \quad (4.8)$$

$$\tilde{u}_a(x, y, 0) = \tilde{u}_0(x, y), \quad \tilde{v}_a(x, y, 0) = \tilde{v}_0(x, y). \quad (4.9)$$

According to Theorem (3.1.1) and Theorem (3.2.1), the solution $(\tilde{u}_a, \tilde{v}_a) \in \Pi_T^k$ of (4.7)-(4.9) exists and satisfies the estimate (2.70). Also, from equations (4.7) and (4.8), we have:

$$\tilde{u}_{at} = -((U_0 + \tilde{u}_0)\tilde{u}_{ax} + \Delta \tilde{u}_{ax} + \alpha \tilde{u}_{axx} + \gamma \Delta^2 \tilde{u}_a - \epsilon_1 \tilde{v}_{ax}) \quad (4.10)$$

$$\tilde{v}_{at} = -(a_1 \tilde{v}_{ax} - \Gamma \Delta \tilde{v}_a - \epsilon_2 \tilde{u}_{ax}). \quad (4.11)$$

So $(\tilde{u}_a, \tilde{v}_a)$ satisfies:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\tilde{u}_a(t)\|_{k+2}^2 + \|\tilde{v}_a(t)\|_{k+1}^2) \\ & + \int_0^T (\|\tilde{u}_a(s)\|_{k+4}^2 + \|\tilde{v}_a(s)\|_{k+2}^2 + \|\tilde{u}_{at}(s)\|_k^2 + \|\tilde{v}_{at}(s)\|_k^2) ds \\ & \leq C_k(T) (\|\tilde{u}_0\|_{k+2}^2 + \|\tilde{v}_0\|_{k+1}^2). \end{aligned} \quad (4.12)$$

Now we look for the solution of (4.1)-(4.3) in the form of

$$(u, v) = (u_a + \dot{u}, v_a + \dot{v}). \quad (4.13)$$

Obviously (u, v) is the solution of (4.1)-(4.3) if and only if $(\dot{u}, \dot{v})$ is the solution of the following problem:

$$\dot{u}_t + (u_a + \dot{u})\dot{u}_x + \Delta \dot{u}_x + \alpha \dot{u}_{xx} + \gamma \Delta^2 \dot{u} - \epsilon_1 \dot{v}_x + \tilde{u}_{ax}\dot{u} = \tilde{f} \quad (4.14)$$

$$\dot{v}_t + a_1 \dot{v}_x - \Gamma \Delta \dot{v} - \epsilon_2 \dot{u}_x = \tilde{g} \quad (4.15)$$

$$\dot{u}(x, y, 0) = 0, \quad \dot{v}(x, y, 0) = 0 \quad (4.16)$$

with

$$\tilde{f} = -(\tilde{u}_{at} + u_a \tilde{u}_{ax} + \Delta \tilde{u}_{ax} + \alpha \tilde{u}_{axx} + \gamma \Delta^2 \tilde{u}_a - \epsilon_1 \tilde{v}_{ax}) \quad (4.17)$$

$$\tilde{g} = -(\tilde{v}_{at} + a_1 \tilde{v}_{ax} - \Gamma \Delta \tilde{v}_a - \epsilon_2 \tilde{u}_{ax}). \quad (4.18)$$

Since $u_a \in C([0, T], H_{ul}^{k+2}(R^2))$ and $H_{ul}^{k+2}(R^2)$ is a Banach algebra, we have:

$$\begin{aligned} \|u_a(t)\tilde{u}_{ax}(t)\|_k &\leq \|u_a(t)\tilde{u}_a(t)\|_{k+2} \\ &\leq C\|u_a(t)\|_{k+2}\|\tilde{u}_a(t)\|_{k+2}. \end{aligned} \quad (4.19)$$

Since

$$\begin{aligned} &\|\tilde{u}_{at} + \Delta\tilde{u}_{ax} + \alpha\tilde{u}_{axx} + \gamma\Delta^2\tilde{u}_a - \epsilon_1\tilde{v}_{ax}\|_k \\ &\leq \|\tilde{u}_{at}\|_k + \|\Delta\tilde{u}_{ax}\|_k + \|\alpha\tilde{u}_{axx}\|_k + \|\gamma\Delta^2\tilde{u}_a\|_k + \|\epsilon_1\tilde{v}_{ax}\|_k \\ &\leq C(\|\tilde{u}_a(t)\|_{k+4} + \|\tilde{v}_a(t)\|_{k+2} + \|\tilde{u}_{at}(t)\|_k), \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} &\|\tilde{v}_{at} + a_1\tilde{v}_{ax} - \Gamma\Delta\tilde{v}_a - \epsilon_2\tilde{u}_{ax}\|_k \\ &\leq C(\|\tilde{u}_a(t)\|_{k+4} + \|\tilde{v}_a(t)\|_{k+2} + \|\tilde{u}_{at}(t)\|_k). \end{aligned} \quad (4.21)$$

From (4.12) and (4.17)-(4.21), we know that $(\tilde{f}, \tilde{g}) \in L^2([0, T], H^k(R^2))$ and their norms $\int_0^T (\|\tilde{f}(s)\|_k^2 + \|\tilde{g}(s)\|_k^2) ds$ are controlled by $(\|\tilde{u}_0\|_{k+2}^2 + \|\tilde{v}_0\|_{k+1}^2)$, that is:

$$\int_0^T (\|\tilde{f}(s)\|_k^2 + \|\tilde{g}(s)\|_k^2) ds \leq C_k(T)(\|\tilde{u}_0\|_{k+2}^2 + \|\tilde{v}_0\|_{k+1}^2). \quad (4.22)$$

Denote the non-linear differential operators on the left side of the equations in (4.14)-(4.16) as $\mathcal{M}(\dot{u})$, we can rewrite (4.14)-(4.16) briefly as:

$$\begin{cases} \mathcal{M}(\dot{u})(\dot{u}, \dot{v}) = (\tilde{f}, \tilde{g}) \\ \dot{u}(x, y, 0) = 0, \dot{v}(x, y, 0) = 0. \end{cases} \quad (4.23)$$

Take $\dot{u}_0(x, y, t) = \dot{v}_0(x, y, t) = 0$. By Theorem (3.2.1), one can obtain a sequence of solutions $(\dot{u}_j, \dot{v}_j)$, for $j = 1, 2, 3, \dots$, in the product space Π_T^k by solving the linear problem:

$$\begin{cases} \mathcal{M}(\dot{u}_{j-1})(\dot{u}_j, \dot{v}_j) = (\tilde{f}, \tilde{g}) \\ \dot{u}_j(x, y, 0) = \dot{v}_j(x, y, 0) = 0. \end{cases} \quad (4.24)$$

According to (2.70) of Theorem (2.4.2), $(\dot{u}_j, \dot{v}_j)$ satisfies:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\dot{u}_j(t)\|_{k+2}^2 + \|\dot{v}_j(t)\|_{k+1}^2) + \int_0^T (\|\dot{u}_j(s)\|_{k+4}^2 + \|\dot{v}_j(s)\|_{k+2}^2) ds \\ & \leq C_{k,j}(T) \int_0^T (\|\tilde{f}(s)\|_k^2 + \|\tilde{g}(s)\|_k^2) ds. \end{aligned} \quad (4.25)$$

In particular, we claim that the constant $C_{k,j}(T)$ in (4.25) depends upon $\dot{u}_{j-1}$ only in its $L^\infty([0, T], H^{k+2}(R^2))$ norm, and it is uniform in T for small T (say, $T \leq 1$).

Since $\dot{u}_0(x, y, t) = \dot{v}_0(x, y, t) = 0$, this conclusion is true for the case $j = 0$.

Assume that there is a constant K such that

$$\sup_{0 \leq t \leq T} \|\dot{u}_{j-1}(t)\|_{k+2}^2 \leq K, \quad (4.26)$$

then we need to show (4.26) is also true for $\dot{u}_j$.

Notice that the constant $C_{k,j}(T)$ in (4.25) depends on K and is independent of specific $\dot{u}_{j-1}$, take $T_0 \ll 1$ such that:

$$C_k(T_0) \int_0^{T_0} (\|\tilde{f}(s)\|_k^2 + \|\tilde{g}(s)\|_k^2) ds \leq K, \quad (4.27)$$

then (4.25) implies that:

$$\sup_{0 \leq t \leq T_0} \|\dot{u}_j(t)\|_{k+2}^2 \leq K. \quad (4.28)$$

Therefore the constant $C_{k,j}(T) = C_k(T)$ in (4.25) is uniform in j .

We conclude that for such T_0 , the sequence $(\dot{u}_j, \dot{v}_j)$ is uniformly bounded in the product space $\Pi_{T_0}^k$.

Let $U_j = (\dot{u}_j, \dot{v}_j)$, we have:

$$\mathcal{M}(\dot{u}_j)U_j = \begin{cases} (\dot{u}_j)_t + (u_a + \dot{u}_j)(\dot{u}_j)_x + \Delta(\dot{u}_j)_x \\ + \alpha(\dot{u}_j)_{xx} + \gamma\Delta^2\dot{u}_j - \epsilon_1(\dot{v}_j)_x + \tilde{u}_{ax}\dot{u}_j \\ (\dot{v}_j)_t + (a_1)(\dot{v}_j)_x - \Gamma\Delta(\dot{v}_j) - \epsilon_2(\dot{u}_j)_x \end{cases}$$

$$\mathcal{M}(\dot{u}_{j-1})U_j = \begin{cases} (\dot{u}_j)_t + (u_a + \dot{u}_{j-1})(\dot{u}_j)_x + \Delta(\dot{u}_j)_x \\ + \alpha(\dot{u}_j)_{xx} + \gamma\Delta^2\dot{u}_{j-} - \epsilon_1(\dot{v}_j)_x + \tilde{u}_{ax}\dot{u}_j \\ (\dot{v}_j)_t + (a_1)(\dot{v}_j)_x - \Gamma\Delta(\dot{v}_j) - \epsilon_2(\dot{u}_j)_x. \end{cases}$$

Since $\mathcal{M}(\dot{u}_{j-1})U_j = \mathcal{M}(\dot{u}_j)U_{j+1} = (\tilde{f}, \tilde{g})$, we get $\mathcal{M}(\dot{u}_j)(U_{j+1} - U_j) = (\mathcal{M}(\dot{u}_{j-1}) - \mathcal{M}(\dot{u}_j))U_j = F_j$ where

$$F_j = \begin{cases} -(\dot{u}_j - \dot{u}_{j-1})(\dot{u}_j)_x = f_j \\ 0 = g_j. \end{cases}$$

Let $(w_j, z_j) = (\dot{u}_{j+1} - \dot{u}_j, \dot{v}_{j+1} - \dot{v}_j)$, (w_j, z_j) satisfies:

$$\begin{cases} \mathcal{M}(\dot{u}_j)(w_j, z_j) = (\mathcal{M}(\dot{u}_{j-1}) - \mathcal{M}(\dot{u}_j))(\dot{u}_j, \dot{v}_j) = -(w_{j-1}\partial_x\dot{u}_j, 0) \\ w_j(x, y, 0) = z_j(x, y, 0) = 0. \end{cases} \quad (4.29)$$

Again from (2.70), we have:

$$\begin{aligned}
 & \sup_{0 \leq t \leq T_0} (\|w_j(t)\|_{k+2}^2 + \|z_j(t)\|_{k+1}^2) \\
 & \quad + \int_0^{T_0} (\|w_j(s)\|_{k+4}^2 + \|z_j(s)\|_{k+2}^2) ds \\
 & \leq C_k(T_0) \int_0^{T_0} \|w_{j-1}(s) \dot{u}_{jx}(s)\|_k^2 ds \\
 & \leq C_k(T_0) \int_0^{T_0} \|w_{j-1}(s) \dot{u}_j(s)\|_{k+2}^2 ds \\
 & \leq C_k(T_0) \sup_{0 \leq t \leq T_0} \|\dot{u}_j\|_{k+2}^2 \int_0^{T_0} \|w_{j-1}(s)\|_{k+2}^2 ds \\
 & \leq C_k(T_0) K T_0 \sup_{0 \leq t \leq T_0} \|w_{j-1}(t)\|_{k+2}^2.
 \end{aligned} \tag{4.30}$$

Choose $T_0 \ll 1$ such that $C_k(T_0) K T_0 < 1$. Then the sequence $(\dot{u}_j, \dot{v}_j)$ converges to $(\dot{u}, \dot{v})$ which is the solution of (4.14)-(4.16). One can readily recover the solution (u, v) of (4.1)-(4.3) by (4.13).

The inequality (4.5) is obtained from (4.12), (4.22) and (4.25). This completes the proof of Theorem (4.1.1).

Remark 4.1.1 The conclusion in Theorem (4.1.1) is also true if the coefficients and the initial data (u_0, v_0) are all periodic functions. One need only to replace R^2 by a periodic domain T^2 . The proof proceeds the same way as that in Theorem (4.1.1). The details are omitted here.

4.2 The local classical solution for the non-linear problem

By Sobolev imbedding theorem, for $k \geq 4$, the solution (u, v) is continuously differentiable in (x, y) up to fourth and second order. Also, we can derive that (u_t, v_t) are continuous by equations (4.1)-(4.2). That is:

$$u_t = -(uu_x + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x) \quad (4.31)$$

$$v_t = -(a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x). \quad (4.32)$$

Therefore the solution (u, v) in Theorem (4.1.1) is a classical solution.

Chapter 5

The Solution for a Linear Problem

In this chapter, we study the following linear problem:

$$u_t + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x = f \quad (5.1)$$

$$v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x = 0 \quad (5.2)$$

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y). \quad (5.3)$$

The problem (5.1)-(5.3) is very similar to the linear problem (2.4)-(2.6) studied in chapter 3, except for the dropping of linearized terms $u^0 u_x$, $u_x^0 u$. Actually, the result obtained in this chapter remains valid if $u^0 u_x$, $u_x^0 u$ are included.

The main differences between the study in this chapter and that in chapter 3 are the “weaker” assumption on the initial data (u_0, v_0) , the right-hand side function f , and consequently a “weaker” form of solution (u, v) . Here, we assume that $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$ and $f \in L^1([0, T], L^2(R^2))$ instead of $(u_0, v_0) \in H^{k+2}(R^2) \times H^{k+1}(R^2)$ and $f \in L^2([0, T], H^k(R^2))$ as in Theorem (3.1.1). This relaxation on the requirements of the initial data and f turns out to be important in our following study of the global solution.

With the initial data $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$, the corresponding solution (u, v) has also reduced regularity. We look for solution (u, v) to the linear problem (5.1)-(5.3) in

the spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ respectively.

Since there are terms such as Δu_x , $\Delta^2 u$ and Δv in (5.1)-(5.3), the solution (u, v) in the spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ satisfies the problem (5.1)-(5.3) in the sense of distribution [19].

5.1 An apriori estimate

In this section, we derive the energy estimate for the linear problem (5.1)-(5.3).

Theorem 5.1.1 *Any solution (u, v) in $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ of (5.1)-(5.3) satisfies the estimate:*

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u(t)\|^2 + \|v(t)\|^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\ & \leq C(T) (\|u_0\|^2 + \|v_0\|^2 + (\int_0^T \|f(s)\| ds)^2). \end{aligned} \quad (5.4)$$

Proof. For $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$, we have $\Delta^2 u \in L^2([0, T], H^{-2}(R^2))$. Since $L^2([0, T], H^{-2}(R^2))$ is the dual space of $L^2([0, T], H^2(R^2))$, it follows that the dual of $\Delta^2 u$ with u is well-defined. Here we use the notation $\langle u, v \rangle_{-2}^*$ to denote the dual of functions u and v in the dual spaces $H^{-2}(R^2)$ and $H^2(R^2)$. According to the definition of the derivative of a distribution [19], we have:

$$\langle \Delta^2 u, u \rangle_{-2}^* = \langle \Delta u, \Delta u \rangle, \quad (5.5)$$

where the left-hand side term is the dual of $\Delta^2 u$ with u in the dual spaces $H^{-2}(R^2)$ and $H^2(R^2)$, and the right-hand side term is the inner product in the space $L^2(R^2)$.

Similarly we have $\Delta u_x \in L^2([0, T], H^{-1}(R^2))$. Notice that $L^2([0, T], H^{-1}(R^2)) \subseteq L^2([0, T], H^{-2}(R^2))$, we have $\Delta u_x \in L^2([0, T], H^{-2}(R^2))$ and $\langle \Delta u_x, u \rangle_{-2}^* = -\langle \Delta u, u_x \rangle$.

Notice that $L^\infty([0, T], L^2(R^2))$ is the dual space of $L^1([0, T], L^2(R^2))$, and $C([0, T], L^2(R^2)) \subseteq L^\infty([0, T], L^2(R^2))$. Since $f \in L^1([0, T], L^2(R^2))$, the dual of f with u is well defined.

For $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$, we have $\Delta v \in L^2([0, T], H^{-1}(R^2))$. Since $L^2([0, T], H^{-1}(R^2))$ is the dual space of $L^2([0, T], H^1(R^2))$, the dual of Δv with v is well defined and

$$\langle \Delta v, v \rangle_{-1}^* = -\langle v_x, v_x \rangle - \langle v_y, v_y \rangle. \quad (5.6)$$

Here we use the notation $\langle u, v \rangle_{-1}^*$ to denote the dual of functions u and v in the dual spaces $H^{-1}(R^2)$ and $H^1(R^2)$.

From the equation (5.1), we have $u_t = f - (\Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x)$. Notice that the dual of every term on the right-hand side of previous equation with u is well defined. So the dual of u_t with u is well-defined. Since $C_0^\infty([0, T], R^2)$ is dense in the space $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$, there exists a sequence of functions u_n in $C_0^\infty([0, T], R^2)$ which converges to u . We have:

$$\langle u_{nt}, u_n \rangle \rightarrow \langle u_t, u \rangle_{-2}^*, \quad (5.7)$$

and

$$\langle u_{nt}, u_n \rangle = \frac{1}{2} \partial_t \langle u_n, u_n \rangle \rightarrow \frac{1}{2} \partial_t \langle u, u \rangle. \quad (5.8)$$

Hence $\langle u_t, u \rangle^* = \frac{1}{2} \partial_t \langle u, u \rangle$.

Similarly $v_t \in L^2([0, T], H^{-1}(R^2))$ and $\langle v_t, v \rangle_{-1}^* = \frac{1}{2} \partial_t \langle v, v \rangle$.

The derivatives of other terms in equations (5.1)-(5.2) are in the space of $L^2(R^2)$. Here we use the Gronwall's theorem to the inequality:

$$\partial_t P(t) \leq CP(t) + M(t), \quad (5.9)$$

with $P(t) = \|u(t)\|^2 + \|v(t)\|^2 + \int_0^t (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds$ and $M(t) = \|u(t)\| \|f(t)\|$.

Therefore we have:

$$e^{-Ct} P(t) - P(0) \leq \int_0^t e^{-Cs} M(s) ds. \quad (5.10)$$

and hence

$$P(t) \leq C_1(t)(P(0) + \int_0^t M(s) ds). \quad (5.11)$$

Notice that:

$$\begin{aligned} \int_0^T \|f\| \|u\| ds &\leq \sup_{0 \leq t \leq T} \|u\| \int_0^T \|f\| ds \\ &\leq \varepsilon (\sup_{0 \leq t \leq T} \|u\|)^2 + \frac{1}{4\varepsilon} (\int_0^T \|f\| ds)^2, \end{aligned} \quad (5.12)$$

where $\varepsilon \ll 1$.

We have:

$$\begin{aligned} &\|u(t)\|^2 + \|v(t)\|^2 + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\ &\leq C(T) (\|u_0\|^2 + \|v_0\|^2 + (\int_0^T \|f(s)\| ds)^2). \end{aligned} \quad (5.13)$$

Furthermore we get:

$$\begin{aligned} &\sup_{0 \leq t \leq T} (\|u(t)\|^2 + \|v(t)\|^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\ &\leq C(T) (\|u_0\|^2 + \|v_0\|^2 + (\int_0^T \|f(s)\| ds)^2). \end{aligned} \quad (5.14)$$

5.2 The solution for a linear problem with $f \in L^1([0, T], L^2(R^2))$

Theorem 5.2.1 *In the initial value problem (5.1)-(5.3), we assume that*

- $u_0 \in L^2(R^2)$ and $v_0 \in L^2(R^2)$ and $f \in L^1([0, T], L^2(R^2))$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (5.1)-(5.3) are smooth functions and are constants outside a bounded domain.

Then, $\forall T > 0$ (5.1)-(5.3) has a unique solution (u, v) with $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ that satisfies the estimate (5.4).

Proof. The proof of Theorem (5.2.1) proceeds in the following two steps.

1. Step 1. Assume

- u_0, v_0 be functions in the space $C_0^\infty(R^2)$;
- f be a function in $C_0^\infty([0, T], R^2)$.

By Theorem 3.1.1, $\forall T > 0$, there exists a unique solution (u, v) to the problem (5.1)-(5.3), where $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$.

2. Step 2. Assume

- $u_0 \in L^2(R^2)$ and $v_0 \in L^2(R^2)$;
- $f \in L^1([0, T], L^2(R^2))$.

Since $C_0^\infty([0, T], R^2)$ is dense in the space $L^1([0, T], L^2(R^2))$ and $C_0^\infty(R^2)$ is dense in the space $L^2(R^2)$, we can construct a sequence of functions (f_j, u_{0j}, v_{0j}) in the product space $C_0^\infty([0, T], R^2) \times C_0^\infty(R^2) \times C_0^\infty(R^2)$ which are convergent to (f, u_0, v_0) in their corresponding spaces.

Consider the following system:

$$(u_j)_t + \Delta(u_j)_x + \alpha(u_j)_{xx} + \gamma\Delta^2(u_j) - \epsilon_1(v_j)_x = f_j \quad (5.15)$$

$$(v_j)_t + a_1(v_j)_x - \Gamma\Delta(v_j) - \epsilon_2(u_j)_x = 0 \quad (5.16)$$

$$(u_j)(x, y, 0) = u_{0j}, (v_j)(x, y, 0) = v_{0j}. \quad (5.17)$$

According to Theorem (3.1.1), for each set (u_{0j}, v_{0j}, f_j) , there exists a unique solution (u_j, v_j) with $u_j \in C([0, T], H^2(R^2)) \cap L^2([0, T], H^4(R^2))$ and $v_j \in C([0, T], H^1(R^2)) \cap L^2([0, T], H^2(R^2))$ satisfying (5.4).

Based on Theorem (5.1.1), we have that the difference of (u_j, v_j) and (u_{j-1}, v_{j-1}) satisfies:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u_j - u_{j-1}\|^2 + \|v_j - v_{j-1}\|^2) + \int_0^T (\|u_j - u_{j-1}\|_2^2 + \|v_j - v_{j-1}\|_1^2) ds \\ & \leq C(T) (\|u_{0j} - u_{0(j-1)}\|^2 + \|v_{0j} - v_{0(j-1)}\|^2 + (\int_0^T \|f_j - f_{j-1}\| ds)^2). \end{aligned} \quad (5.18)$$

From the convergence of (u_{0j}, v_{0j}, f_j) , it is readily seen that the sequence of solutions (u_j, v_j) is a Cauchy sequence in the spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$. From the completeness of these spaces, we have (u_j, v_j) converges to (u, v) with $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$. Obviously, (u, v) is the solution of (5.1)-(5.3) in the sense of distribution and satisfies (5.4).

Next we show that the solution of (5.1)-(5.3) is unique.

If there are two solutions (u_1, v_1) and (u_2, v_2) of problem (5.1)-(5.3), the difference $(u_1 - u_2, v_1 - v_2)$ also satisfies the problem (5.1)-(5.3). By Theorem (5.1.1), we have:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|(u_1 - u_2)(t)\|^2 + \|(v_1 - v_2)(t)\|^2) + \int_0^T (\|(u_1 - u_2)(s)\|_2^2 + \|(v_1 - v_2)(s)\|_1^2) ds \\ & \leq C(T) (\|(u_1 - u_2)(x, y, 0)\|^2 + \|(v_1 - v_2)(x, y, 0)\|^2) = 0. \end{aligned} \tag{5.19}$$

So $u_1 = u_2$ and $v_1 = v_2$ are in the spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ respectively.

Therefore, we finish the proof of Theorem (5.2.1).

Chapter 6

The Global Weak Solution for the Non-linear Problem

In this chapter, we establish a unique global weak solution (u, v) in $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ of the following non-linear problem under the assumption $(u_0, v_0) \in L^2(R^2) \times L^2(R^2)$. Here, (u, v) is called a weak solution of the following problem in the sense of distribution.

$$u_t + uu_x + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x = 0 \quad (6.1)$$

$$v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x = 0 \quad (6.2)$$

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y). \quad (6.3)$$

6.1 Global estimate

Theorem 6.1.1 *Any solution (u, v) in $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ of (6.1)-(6.3) satisfies the estimate:*

$$\sup_{0 \leq t \leq T} (\|u(t)\|^2 + \|v(t)\|^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \leq C(T) (\|u_0\|^2 + \|v_0\|^2), \quad (6.4)$$

where $C(T)$ denotes a constant depending on T and the coefficients of the non-linear system (6.1)-(6.3).

Proof. Notice that $L^\infty([0, T], L^2(R^2))$ is the dual space of $L^1([0, T], L^2(R^2))$, and $C([0, T], L^2(R^2)) \subseteq L^\infty([0, T], L^2(R^2))$. Since $uu_x = \frac{1}{2}(u^2)_x \in L^1([0, T], L^2(R^2))$, the dual of uu_x with u is well defined and

$$\langle uu_x, u \rangle_{-2}^* = \frac{1}{2} \langle u^2, u_x \rangle = 0, \quad (6.5)$$

since we have:

$$\langle u^2, u_x \rangle = -2 \langle uu_x, u \rangle = -2 \langle u^2, u_x \rangle. \quad (6.6)$$

From Theorem (5.1.1) and above equation, we can get the estimate (6.4).

Next, we use an η -weighted norm to derive a global estimate to the non-linear system (6.1)-(6.3).

Theorem 6.1.2 $\forall T > 0$, any solution $(u, v) \in C^\infty([0, T], S(R^2))$ of (6.1)-(6.3) satisfies the estimate:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|e^{-\eta t} u(t)\|^2 + \|e^{-\eta t} v(t)\|^2) + \int_0^T (\|e^{-\eta s} u(s)\|_2^2 + \|e^{-\eta s} v(s)\|_1^2) ds \\ & \leq C_\eta (\|u_0\|^2 + \|v_0\|^2), \end{aligned} \quad (6.7)$$

where C_η denotes a constant depending upon $\eta \gg 1$ and the coefficients of the non-linear system (6.1)-(6.3). In particular, C_η is independent of T .

Proof. Replace u, v in (6.1)-(6.3) with $e^{\eta t} \tilde{u}, e^{\eta t} \tilde{v}$ respectively. We have the following system:

$$\tilde{u}_t + \eta \tilde{u} + e^{\eta t} \tilde{u} \tilde{u}_x + \Delta \tilde{u}_x + \alpha \tilde{u}_{xx} + \gamma \Delta^2 \tilde{u} - \epsilon_1 \tilde{v}_x = 0 \quad (6.8)$$

$$\tilde{v}_t + \eta \tilde{v} + a_1 \tilde{v}_x - \Gamma \Delta \tilde{v} - \epsilon_2 \tilde{u}_x = 0 \quad (6.9)$$

$$\tilde{u}(x, y, 0) = u_0(x, y), \tilde{v}(x, y, 0) = v_0(x, y). \quad (6.10)$$

To get estimate (6.7), we notice the facts that:

$$\int_0^t \langle \tilde{u}_s, \tilde{u} \rangle ds = \frac{1}{2} (\|\tilde{u}(t)\|^2 - \|u_0\|^2). \quad (6.11)$$

$$\int_0^t \langle \eta \tilde{u}, \tilde{u} \rangle ds = \eta \int_0^t \|\tilde{u}(s)\|^2 ds. \quad (6.12)$$

$$\begin{aligned} \int_0^t \langle e^{\eta s} \tilde{u} \tilde{u}_x, \tilde{u} \rangle ds &= \int_0^t e^{\eta s} \langle \tilde{u} \tilde{u}_x, \tilde{u} \rangle ds \\ &= 0. \end{aligned} \quad (6.13)$$

Since

$$\begin{aligned} \langle \tilde{u} \tilde{u}_x, \tilde{u} \rangle &= \langle \tilde{u}_x, \tilde{u}^2 \rangle \\ &= -\langle 2\tilde{u} \tilde{u}_x, \tilde{u} \rangle, \end{aligned} \quad (6.14)$$

it follows

$$\langle \tilde{u} \tilde{u}_x, \tilde{u} \rangle = 0. \quad (6.15)$$

$$\int_0^t \langle \Delta \tilde{u}_x, \tilde{u} \rangle ds \leq \int_0^t (\varepsilon_1 + \frac{\varepsilon}{4\varepsilon_1}) \|\tilde{u}_{xx}\|^2 ds + \int_0^t (\frac{1}{16\varepsilon\varepsilon_1} + \frac{1}{16\varepsilon\varepsilon_2}) \|\tilde{u}\|^2 ds + \int_0^t \varepsilon_2 \|\tilde{u}_{xy}\|^2 ds + \int_0^t \frac{\varepsilon}{4\varepsilon_2} \|\tilde{u}_{yy}\|^2 ds. \quad (6.16)$$

Since we have:

$$\begin{aligned} \langle \Delta \tilde{u}_x, \tilde{u} \rangle &= -\langle \tilde{u}_{xx}, \tilde{u}_x \rangle - \langle \tilde{u}_{yy}, \tilde{u}_x \rangle \\ &= -\langle \tilde{u}_{xx}, \tilde{u}_x \rangle + \langle \tilde{u}_{xy}, \tilde{u}_y \rangle \\ &\leq \varepsilon_1 \|\tilde{u}_{xx}\|^2 + \frac{1}{4\varepsilon_1} \|\tilde{u}_x\|^2 + \varepsilon_2 \|\tilde{u}_{xy}\|^2 + \frac{1}{4\varepsilon_2} \|\tilde{u}_y\|^2 \\ &\leq (\varepsilon_1 + \frac{\varepsilon}{4\varepsilon_1}) \|\tilde{u}_{xx}\|^2 + (\frac{1}{16\varepsilon\varepsilon_1} + \frac{1}{16\varepsilon\varepsilon_2}) \|\tilde{u}\|^2 + \varepsilon_2 \|\tilde{u}_{xy}\|^2 + \frac{\varepsilon}{4\varepsilon_2} \|\tilde{u}_{yy}\|^2. \end{aligned} \quad (6.17)$$

Here we estimate $\|\tilde{u}_x\|^2$ and $\|\tilde{u}_y\|^2$ by means of the inequalities:

$$\frac{1}{4\varepsilon_1} \|\tilde{u}_x\|^2 \leq \frac{1}{4\varepsilon_1} (\varepsilon \|\tilde{u}_{xx}\|^2 + \frac{1}{4\varepsilon} \|\tilde{u}\|^2), \quad (6.18)$$

$$\frac{1}{4\varepsilon_2} \|\tilde{u}_y\|^2 \leq \frac{1}{4\varepsilon_2} (\varepsilon \|\tilde{u}_{yy}\|^2 + \frac{1}{4\varepsilon} \|\tilde{u}\|^2), \quad (6.19)$$

where $\varepsilon \ll \varepsilon_1 \ll 1$ and $\varepsilon \ll \varepsilon_2 \ll 1$.

$$\int_0^t \langle \gamma \Delta^2 \tilde{u}, \tilde{u} \rangle ds = \gamma \int_0^t (\|\tilde{u}_{xx}\|^2 + 2\|\tilde{u}_{xy}\|^2 + \|\tilde{u}_{yy}\|^2) ds. \quad (6.20)$$

$$\int_0^t \langle \alpha \tilde{u}_{xx}, \tilde{u} \rangle ds \leq \int_0^t (\alpha \varepsilon_3 \|\tilde{u}_{xx}\|^2 + \frac{\alpha}{4\varepsilon_3} \|\tilde{u}\|^2) ds. \quad (6.21)$$

$$\int_0^t \langle \varepsilon_1 \tilde{v}_x, \tilde{u} \rangle ds \leq \varepsilon_1 \varepsilon_4 \int_0^t \|\tilde{v}_x\|^2 ds + \frac{\varepsilon_1}{4\varepsilon_4} \int_0^t \|\tilde{u}\|^2 ds. \quad (6.22)$$

$$\int_0^t \langle \tilde{v}_s, \tilde{v} \rangle ds = \frac{1}{2} (\|\tilde{v}(t)\|^2 - \|\tilde{v}(0)\|^2). \quad (6.23)$$

$$\int_0^t \langle \eta \tilde{v}, \tilde{v} \rangle ds = \eta \int_0^t \|\tilde{v}(s)\|^2 ds. \quad (6.24)$$

$$\int_0^t \langle a_1 \tilde{v}_x, \tilde{v} \rangle ds \leq a_1 \varepsilon_5 \int_0^t \|\tilde{v}_x\|^2 ds + \frac{a_1}{4\varepsilon_5} \int_0^t \|\tilde{v}\|^2 ds. \quad (6.25)$$

$$- \int_0^t \langle \Gamma \Delta \tilde{v}, \tilde{v} \rangle ds = \Gamma \int_0^t (\|\tilde{v}_x\|^2 + \|\tilde{v}_y\|^2) ds. \quad (6.26)$$

$$\begin{aligned} \int_0^t \langle \varepsilon_2 \tilde{u}_x, \tilde{v} \rangle ds &\leq \varepsilon_2 \varepsilon_6 \int_0^t \|\tilde{v}\|^2 ds + \frac{\varepsilon_2}{4\varepsilon_6} \int_0^t \|\tilde{u}_x\|^2 ds \\ &\leq \varepsilon_2 \varepsilon_6 \int_0^t \|\tilde{v}\|^2 ds + \frac{\varepsilon_2}{4\varepsilon_6} \int_0^t (\varepsilon \|\tilde{u}_{xx}\|^2 + \frac{1}{4\varepsilon} \|\tilde{u}\|^2) ds, \end{aligned} \quad (6.27)$$

where $\varepsilon \ll \varepsilon_6 \ll 1$.

Therefore, we have:

$$\begin{aligned}
 & \frac{1}{2}(\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2) + \left(\eta - \frac{1}{16\varepsilon\varepsilon_1} - \frac{1}{16\varepsilon\varepsilon_2} - \frac{\alpha}{4\varepsilon_3} - \frac{\varepsilon_1}{4\varepsilon_4} - \frac{\varepsilon_2}{16\varepsilon\varepsilon_6}\right) \int_0^t \|\tilde{u}\|^2 ds \\
 & + \left(\gamma - \frac{\varepsilon_2\varepsilon}{4\varepsilon_6} - \varepsilon_1 - \alpha\varepsilon_3 - \frac{\varepsilon}{4\varepsilon_1}\right) \int_0^t \|\tilde{u}_{xx}\|^2 ds + (2\gamma - \varepsilon_2) \int_0^t \|\tilde{u}_{xy}\|^2 ds + \left(\gamma - \frac{\varepsilon}{4\varepsilon_2}\right) \int_0^t \|\tilde{u}_{yy}\|^2 ds \\
 & + \left(\eta - \frac{a_1}{4\varepsilon_5} - \varepsilon_2\varepsilon_6\right) \int_0^t \|\tilde{v}\|^2 ds + (\Gamma - a_1\varepsilon_5 - \varepsilon_1\varepsilon_4) \int_0^t \|\tilde{v}_x\|^2 ds + \Gamma \int_0^t \|\tilde{v}_y\|^2 ds \\
 & \leq \frac{1}{2}(\|u_0\|^2 + \|v_0\|^2).
 \end{aligned} \tag{6.28}$$

Notice that:

$$\|\tilde{u}\|_2 \sim \|\tilde{u}_{xx}\| + \|\tilde{u}_{yy}\| + \|\tilde{u}_{xy}\| + \|\tilde{u}\|, \tag{6.29}$$

we have the following inequality:

$$\begin{aligned}
 & \|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2 + \int_0^t (\|\tilde{u}(s)\|_2^2 + \|\tilde{v}(s)\|_1^2) ds \\
 & \leq C_\eta(\|u_0\|^2 + \|v_0\|^2).
 \end{aligned} \tag{6.30}$$

Furthermore, we can get the estimate of $(\tilde{u}, \tilde{v})$ as follows:

$$\begin{aligned}
 & \sup_{0 \leq t \leq T} (\|\tilde{u}(t)\|^2 + \|\tilde{v}(t)\|^2) + \int_0^T (\|\tilde{u}(s)\|_2^2 + \|\tilde{v}(s)\|_1^2) ds \\
 & \leq C_\eta(\|u_0\|^2 + \|v_0\|^2),
 \end{aligned} \tag{6.31}$$

which yields the estimate (6.7).

Here we get the estimate (6.7) under the assumption that all coefficients of (6.1)-(6.3) are constants. This estimate is also true for the cases that all coefficients of (6.1)-(6.3) are smooth and constants outside a bounded domain.

Remark 6.1.1 By the same argument as that in Theorem (6.1.1), we know that Theorem(6.1.2) is also true for the case that $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$.

6.2 The uniqueness of the global solution for the non-linear problem with $(u_0, v_0) \in L^2 \times L^2$

Theorem 6.2.1 *In the initial value problem (6.1)-(6.3), we assume that*

- $u_0 \in L^2(R^2)$ and $v_0 \in L^2(R^2)$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- *all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (6.1)-(6.3) are smooth functions and are constants outside a bounded domain.*

$\forall T > 0$, if there exists a solution (u, v) with $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$, then the solution (u, v) is unique.

Proof. If there exist two solutions (u_1, v_1) and (u_2, v_2) of problem (6.1)-(6.3), then $(u_1 - u_2, v_1 - v_2)$ satisfies the following problem:

$$(u_1 - u_2)_t + u_1 u_{1x} - u_2 u_{2x} + \Delta(u_1 - u_2)_x + \alpha(u_1 - u_2)_{xx} + \gamma \Delta^2(u_1 - u_2) - \epsilon_1(v_1 - v_2)_x = 0 \quad (6.32)$$

$$(v_1 - v_2)_t + a_1(v_1 - v_2)_x - \Gamma \Delta(v_1 - v_2) - \epsilon_2(u_1 - u_2)_x = 0 \quad (6.33)$$

$$(u_1 - u_2)(x, y, 0) = 0, (v_1 - v_2)(x, y, 0) = 0. \quad (6.34)$$

Since we have:

$$\begin{aligned} u_1 u_{1x} - u_2 u_{2x} &= u_1 u_{1x} - u_2 u_{1x} + u_2 u_{1x} - u_2 u_{2x} \\ &= u_{1x}(u_1 - u_2) + u_2(u_1 - u_2)_x, \end{aligned} \quad (6.35)$$

it follows that $(u_1 - u_2, v_1 - v_2)$ satisfies the following linear problem:

$$(u_1 - u_2)_t + u_{1x}(u_1 - u_2) + u_2(u_1 - u_2)_x + \Delta(u_1 - u_2)_x + \alpha(u_1 - u_2)_{xx} + \gamma \Delta^2(u_1 - u_2) - \epsilon_1(v_1 - v_2)_x = 0. \quad (6.36)$$

$$(v_1 - v_2)_t + a_1(v_1 - v_2)_x - \Gamma \Delta(v_1 - v_2) - \epsilon_2(u_1 - u_2)_x = 0. \quad (6.37)$$

$$(u_1 - u_2)(x, y, 0) = 0, (v_1 - v_2)(x, y, 0) = 0. \quad (6.38)$$

Notice that:

$$\langle u_{1x}(u_1 - u_2), (u_1 - u_2) \rangle = -2\langle u_1(u_1 - u_2)_x, (u_1 - u_2) \rangle. \quad (6.39)$$

$$\begin{aligned} \langle u_1(u_1 - u_2)_x, (u_1 - u_2) \rangle &\leq \|u_1\| \|(u_1 - u_2)(u_1 - u_2)_x\| \\ &\leq \|u_1\| \sup_{x,y} |u_1 - u_2| \|(u_1 - u_2)_x\| \\ &\leq C \|u_1\| (\varepsilon \|u_1 - u_2\|_2^2 + \frac{1}{4\varepsilon} \|u_1 - u_2\|_1^2) \\ &\leq C(\varepsilon \|u_1\| + \frac{\varepsilon_1}{4\varepsilon} \|u_1\|) \|u_1 - u_2\|_2^2 + \frac{C}{16\varepsilon\varepsilon_1} \|u_1\| \|u_1 - u_2\|^2. \end{aligned} \quad (6.40)$$

$$\begin{aligned} \langle u_2(u_1 - u_2)_x, (u_1 - u_2) \rangle &\leq C(\varepsilon \|u_2\| + \frac{\varepsilon_1}{4\varepsilon} \|u_2\|) \|u_1 - u_2\|_2^2 + \frac{C}{16\varepsilon\varepsilon_1} \|u_2\| \|u_1 - u_2\|^2, \end{aligned} \quad (6.41)$$

where $\varepsilon_1 \ll \varepsilon \ll 1$.

Since u_1, u_2 are in the space $C([0, T], L^2(R^2))$, there exists a positive constant K such that:

$$\sup_{0 \leq t \leq T} \|u_1(t)\|^2 \leq K, \quad (6.42)$$

$$\sup_{0 \leq t \leq T} \|u_2(t)\|^2 \leq K. \quad (6.43)$$

According to Theorem (5.2.1) and (6.39)-(6.43), we have the estimate of $(u_1 - u_2, v_1 - v_2)$:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\| (u_1 - u_2)(t) \|^2 + \| (v_1 - v_2)(t) \|^2) + \int_0^T (\| (u_1 - u_2)(s) \|_2^2 + \| (v_1 - v_2)(s) \|_1^2) ds \\ & \leq C(T) (\| (u_1 - u_2)(x, y, 0) \|^2 + \| (v_1 - v_2)(x, y, 0) \|^2) = 0. \end{aligned} \tag{6.44}$$

So $u_1 = u_2$ and $v_1 = v_2$ are in the spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ respectively.

Hence, we finish the proof of Theorem (6.2.1).

6.3 The local solution for the non-linear problem with $(u_0, v_0) \in L^2 \times L^2$

Theorem 6.3.1 *In the initial value problem (6.1)-(6.3), we assume that*

- $u_0 \in L^2(R^2)$ and $v_0 \in L^2(R^2)$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- *all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (6.1)-(6.3) are smooth functions and are constants outside a bounded domain.*

Then, there exists a $T > 0$ such that (6.1)-(6.3) has a unique solution (u, v) with $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ that satisfies the estimate (6.7).

Proof. The proof of Theorem (6.3.1) consists of two steps.

1. Step 1. We consider the following linear problem under the assumption that u_0, v_0 are functions in the space $L^2(R^2)$.

$$u_{at} + \Delta u_{ax} + \alpha u_{axx} + \gamma \Delta^2 u_a - \epsilon_1 v_{ax} = 0 \quad (6.45)$$

$$v_{at} + a_1 v_{ax} - \Gamma \Delta v_a - \epsilon_2 u_{ax} = 0 \quad (6.46)$$

$$u_a(x, y, 0) = u_0(x, y), v_a(x, y, 0) = v_0(x, y). \quad (6.47)$$

According to Theorem (5.2.1), there exists a solution (u_a, v_a) with $u_a \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v_a \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ that satisfies the estimate:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u_a(t)\|^2 + \|v_a(t)\|^2) + \int_0^T (\|u_a(s)\|_2^2 + \|v_a(s)\|_1^2) ds \\ & \leq C(T) (\|u_0\|^2 + \|v_0\|^2). \end{aligned} \quad (6.48)$$

2. Step 2. We look for the solution of (6.1)-(6.3) with the form

$$(u, v) = (u_a + \dot{u}, v_a + \dot{v}). \quad (6.49)$$

Obviously (u, v) is the solution of (6.1)-(6.3) if and only if $(\dot{u}, \dot{v})$ is the solution of the following problem:

$$\dot{u}_t + u_a \dot{u}_x + u_{ax} \dot{u} + \dot{u} \dot{u}_x + \Delta \dot{u}_x + \alpha \dot{u}_{xx} + \gamma \Delta^2 \dot{u} - \epsilon_1 \dot{v}_x = -u_a u_{ax} \triangleq f \quad (6.50)$$

$$\dot{v}_t + a_1 \dot{v}_x - \Gamma \Delta \dot{v} - \epsilon_2 \dot{u}_x = 0 \quad (6.51)$$

$$\dot{u}(x, y, 0) = 0, \dot{v}(x, y, 0) = 0. \quad (6.52)$$

To find the solution $(\dot{u}, \dot{v})$, we use linear iteration method by solving the following linear problem:

$$\dot{u}_{jt} + u_a \dot{u}_{jx} + u_{ax} \dot{u}_j + \dot{u}_{j-1} \dot{u}_{jx} + \Delta \dot{u}_{jx} + \alpha \dot{u}_{jxx} + \gamma \Delta^2 \dot{u}_j - \epsilon_1 \dot{v}_{jx} = -u_a u_{ax} \triangleq f \quad (6.53)$$

$$\dot{v}_{jt} + a_1 \dot{v}_{jx} - \Gamma \Delta \dot{v}_j - \epsilon_2 \dot{u}_{jx} = 0 \quad (6.54)$$

$$\dot{u}_j(x, y, 0) = 0, \dot{v}_j(x, y, 0) = 0. \quad (6.55)$$

Notice the following inequalities:

$$\begin{aligned}
 \langle \dot{u}_{j-1} \dot{u}_{jx}, \dot{u}_j \rangle &\leq \|\dot{u}_{j-1}\| \|\dot{u}_{jx} \dot{u}_j\| \\
 &\leq \|\dot{u}_{j-1}\| (\sup_{x,y} |\dot{u}_j|) \|\dot{u}_{jx}\| \\
 &\leq C \|\dot{u}_{j-1}\| \|\dot{u}_j\|_2 \|\dot{u}_{jx}\| \\
 &\leq C \|\dot{u}_{j-1}\| (\varepsilon \|\dot{u}_j\|_2^2 + \frac{1}{4\varepsilon} \|\dot{u}_{jx}\|^2).
 \end{aligned} \tag{6.56}$$

Here we estimate $\|\dot{u}_{jx}\|^2$ by means of the inequality:

$$\begin{aligned}
 \|\dot{u}_{jx}\|^2 &= |-\langle \dot{u}_{jxx}, \dot{u}_j \rangle| \\
 &\leq \|\dot{u}_j\| \|\dot{u}_j\|_2 \\
 &\leq \varepsilon_1 \|\dot{u}_j\|_2^2 + \frac{1}{4\varepsilon_1} \|\dot{u}_j\|^2,
 \end{aligned} \tag{6.57}$$

where $\varepsilon_1 \ll \varepsilon \ll 1$. So we have:

$$\begin{aligned}
 \langle \dot{u}_{j-1} \dot{u}_{jx}, \dot{u}_j \rangle &\leq C\varepsilon \|\dot{u}_{j-1}\| \|\dot{u}_j\|_2^2 + \frac{C}{4\varepsilon} \|\dot{u}_{j-1}\| (\varepsilon_1 \|\dot{u}_j\|_2^2 + \frac{1}{4\varepsilon_1} \|\dot{u}_j\|^2) \\
 &\leq C(\varepsilon \|\dot{u}_{j-1}\| + \frac{\varepsilon_1}{4\varepsilon} \|\dot{u}_{j-1}\|) \|\dot{u}_j\|_2^2 + C \frac{1}{16\varepsilon\varepsilon_1} \|\dot{u}_{j-1}\| \|\dot{u}_j\|^2.
 \end{aligned} \tag{6.58}$$

Notice that $\langle u_{ax} \dot{u}_j, \dot{u}_j \rangle + \langle u_a \dot{u}_{jx}, \dot{u}_j \rangle = -\langle u_a \dot{u}_{jx}, \dot{u}_j \rangle$, we have the following inequality based on the same argument of (6.58) :

$$\langle u_a \dot{u}_{jx}, \dot{u}_j \rangle \leq C(\varepsilon \|u_a\| + \frac{\varepsilon_1}{4\varepsilon} \|u_a\|) \|\dot{u}_j\|_2^2 + C \frac{1}{16\varepsilon\varepsilon_1} \|u_a\| \|\dot{u}_j\|^2. \tag{6.59}$$

According to (5.4) of Theorem (5.1.1) and inequalities (6.56)-(6.59), $(\dot{u}_j, \dot{v}_j)$ satisfies the following estimate:

$$\begin{aligned}
 &\sup_{0 \leq t \leq T} (\|\dot{u}_j(t)\|^2 + \|\dot{v}_j(t)\|^2) + \int_0^T (\|\dot{u}_j(s)\|_2^2 + \|\dot{v}_j(s)\|_1^2) ds \\
 &\leq C_j(T) (\int_0^T \|f(s)\| ds)^2.
 \end{aligned} \tag{6.60}$$

In particular, we notice the constant $C_j(T)$ in (6.60) depends upon $\dot{u}_{j-1}$ only in its $C([0, T], L^2(R^2))$ norm, and it is uniform in T for small T (say, $T \leq 1$).

We show in the following that the sequence $(\dot{u}_j, \dot{v}_j)$ is uniformly bounded in spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ for small T .

Take $\dot{u}_0 = 0, \dot{v}_0 = 0$, then this conclusion is true for the case $j = 0$.

Assume there exists a constant K such that

$$\sup_{0 \leq t \leq T} \|\dot{u}_{j-1}(t)\|^2 + \int_0^T \|\dot{u}_{j-1}(s)\|_2^2 ds \leq K, \quad (6.61)$$

then we show in the following that (6.61) is also true for $\dot{u}_j$.

Notice that the constant $C_j(T)$ in (6.60) depends on K and is independent of specific $\dot{u}_{j-1}$, we take $T_0 \ll 1$ such that:

$$C(T_0) \left(\int_0^{T_0} \|f(s)\| ds \right)^2 \leq K. \quad (6.62)$$

Then (6.60) implies that :

$$\sup_{0 \leq t \leq T_0} \|\dot{u}_j(t)\|^2 + \int_0^{T_0} \|\dot{u}_j(s)\|_2^2 ds \leq K. \quad (6.63)$$

So, for such T_0 the sequence $(\dot{u}_j, \dot{v}_j)$ is uniformly bounded in spaces $C([0, T_0], L^2(R^2)) \cap L^2([0, T_0], H^2(R^2))$ and $C([0, T_0], L^2(R^2)) \cap L^2([0, T_0], H^1(R^2))$ respectively.

Second, we show in the following that the sequence $(\dot{u}_j, \dot{v}_j)$ in (6.53)-(6.55) converges to $(\dot{u}, \dot{v})$ which is the solution of (6.50)-(6.52).

The argument is the same as that in the proof of Theorem (4.1.1). We use the same notation $(w_j, z_j) = (\dot{u}_{j+1} - \dot{u}_j, \dot{v}_{j+1} - \dot{v}_j)$ in our proof as that in Theorem (4.1.1). According to (5.4) of Theorem (5.1.1) we have:

$$\begin{aligned}
 & \sup_{0 \leq t \leq T_0} (\|w_j(t)\|^2 + \|z_j(t)\|^2) + \int_0^{T_0} (\|w_j(s)\|_2^2 + \|z_j(s)\|_1^2) ds \\
 & \leq C(T_0) \left(\int_0^{T_0} \|w_{j-1}(s) \dot{u}_{jx}(s)\| ds \right)^2 \\
 & \leq C(T_0) \left[\int_0^{T_0} (\sup_{x,y} |w_{j-1}|) \|\dot{u}_{jx}(s)\| ds \right]^2.
 \end{aligned} \tag{6.64}$$

Here we estimate $\|\dot{u}_{jx}\|$ by means of the inequality:

$$\begin{aligned}
 \|\dot{u}_{jx}\| &= | - \langle \dot{u}_j, \dot{u}_{jxx} \rangle |^{\frac{1}{2}} \\
 &\leq \|\dot{u}_j\|^{\frac{1}{2}} \|\dot{u}_j\|_2^{\frac{1}{2}} \\
 &\leq \varepsilon \|\dot{u}_j\|_2 + \frac{1}{4\varepsilon} \|\dot{u}_j\|,
 \end{aligned} \tag{6.65}$$

where $\varepsilon < 1$ is a small positive number. So we have:

$$\begin{aligned}
 & \sup_{0 \leq t \leq T_0} (\|w_j(t)\|^2 + \|z_j(t)\|^2) + \int_0^{T_0} (\|w_j(s)\|_2^2 + \|z_j(s)\|_1^2) ds \\
 & \leq C(T_0) \left[\int_0^{T_0} (\sup_{x,y} |w_{j-1}|) (\varepsilon \|\dot{u}_j\|_2 + \frac{1}{4\varepsilon} \|\dot{u}_j\|) ds \right]^2 \\
 & \leq C(T_0) (\varepsilon \int_0^{T_0} \|w_{j-1}\|_2 \|\dot{u}_j\|_2 ds)^2 + C(T_0) \left(\frac{1}{4\varepsilon} \int_0^{T_0} \|w_{j-1}\|_2 \|\dot{u}_j\| ds \right)^2 \\
 & \leq C(T_0) \varepsilon^2 \int_0^{T_0} \|\dot{u}_j\|_2^2 ds \int_0^{T_0} \|w_{j-1}\|_2^2 ds + \frac{C(T_0)}{16\varepsilon^2} \int_0^{T_0} \|\dot{u}_j\|^2 ds \int_0^{T_0} \|w_{j-1}\|_2^2 ds \\
 & \leq C(T_0) \varepsilon^2 \int_0^{T_0} \|\dot{u}_j\|_2^2 ds \int_0^{T_0} \|w_{j-1}\|_2^2 ds + \frac{C(T_0)T_0}{16\varepsilon^2} (\sup_{0 \leq t \leq T_0} \|\dot{u}_j\|^2) \int_0^{T_0} \|w_{j-1}\|_2^2 ds \\
 & = (C(T_0)K\varepsilon^2 + \frac{C(T_0)KT_0}{16\varepsilon^2}) \int_0^{T_0} \|w_{j-1}\|_2^2 ds.
 \end{aligned} \tag{6.66}$$

Choose $\varepsilon \ll 1, T_0 \ll 1$ such that $(C(T_0)K\varepsilon^2 + \frac{C(T_0)KT_0}{16\varepsilon^2}) < 1$. Hence the sequence $(\dot{u}_j, \dot{v}_j)$ converges to $(\dot{u}, \dot{v})$ which is the solution of (6.50)-(6.52). One can readily recover the solution (u, v) of (6.1)-(6.3) by (6.49).

6.4 The global solution for the non-linear problem with $(u_0, v_0) \in L^2 \times L^2$

Theorem 6.4.1 *In the initial value problem (6.1)-(6.3), we assume that*

- $u_0 \in L^2(R^2)$ and $v_0 \in L^2(R^2)$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (6.1)-(6.3) are smooth functions and are constants outside a bounded domain.

Then, $\forall T > 0$, (6.1)-(6.3) has a unique solution (u, v) with $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ that satisfies the estimate (6.7).

Remark 6.4.1 Since $T > 0$ in the Theorem (6.4.1) is arbitrary and independent of L^2 norms of the initial data (u_0, v_0) , Theorem (6.4.1) establishes the global existence of solution to problem (6.1)-(6.3).

Proof. We denote E the subset of $[0, T]$ such that for $\tilde{T} \in E$, (6.1)-(6.3) has a unique solution $(u(x, y, t), v(x, y, t))$ for $t \in [0, \tilde{T}]$, where $u \in C([0, \tilde{T}], L^2(R^2)) \cap L^2([0, \tilde{T}], H^2(R^2))$ and $v \in C([0, \tilde{T}], L^2(R^2)) \cap L^2([0, \tilde{T}], H^1(R^2))$.

1. E is not empty, because we know $T_0 \in E$ from Theorem (6.3.1).
2. E is open. Let $\tilde{T}_0 \in E$, $\tilde{T} \in [0, T]$ and $|\tilde{T} - \tilde{T}_0| \leq \varepsilon$. We show in the following that for $\varepsilon \ll 1$, $\tilde{T} \in E$.

Since $\tilde{T}_0 \in E$, (6.1)-(6.3) has a unique solution $(u(x, y, t), v(x, y, t))$ for $t \in [0, \tilde{T}_0]$ with $u \in C([0, \tilde{T}_0], L^2(R^2)) \cap L^2([0, \tilde{T}_0], H^2(R^2))$ and $v \in C([0, \tilde{T}_0], L^2(R^2)) \cap L^2([0, \tilde{T}_0], H^1(R^2))$. We can use $(u(\tilde{T}_0), v(\tilde{T}_0))$ as our initial value. From Theorem (6.3.1), we know there exists $(u^*(x, y, t), v^*(x, y, t))$ which is the solution of (6.1)-(6.3) for $t \in [\tilde{T}_0, \tilde{T}_0 + \tilde{t}]$. By Theorem (6.2.1), there exists a unique solution (u, v) to (6.1)-(6.3) for $t \in [0, \tilde{T}_0 + \tilde{t}]$.

We can choose $\varepsilon \leq \tilde{t}$, then it follows that $\tilde{T} \in E$ for $|\tilde{T} - \tilde{T}_0| \leq \varepsilon$.

3. E is closed. Let $T_n \in E$ and $T_n \rightarrow \alpha$. we show in the following that $\alpha \in E$.

Since $T_n \in E$, there are $(u_n(x, y, t), v_n(x, y, t))$ which are solutions of (6.1)-(6.3) in $[0, T_n]$. Also, it follows that $\forall T \in [0, \alpha)$, there exists a T_n such that $T_n \geq T$. By Theorem (6.2.1), we know that the solution of (6.1)-(6.3) is unique in $[0, \alpha)$. Therefore $(u, v) = (u_n, v_n)$ is well defined in $[0, T]$ and (u, v) is the unique solution of (6.1)-(6.3) in $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ for each $T \in [0, \alpha)$.

By the global estimate (6.7) in Theorem (6.1.2), particularly for $\forall T \in [0, \alpha)$ and $t \in [0, T]$, we have:

$$\begin{aligned}
 & e^{-\eta T}(\|u(t)\|^2 + \|v(t)\|^2) + e^{-\eta T} \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\
 & \leq \sup_{0 \leq t \leq T} (\|e^{-\eta t} u(t)\|^2 + \|e^{-\eta t} v(t)\|^2) + \int_0^T (\|e^{-\eta s} u(s)\|_2^2 + \|e^{-\eta s} v(s)\|_1^2) ds \\
 & \leq C_\eta (\|u_0\|^2 + \|v_0\|^2).
 \end{aligned} \tag{6.67}$$

This implies that for $t \in [0, T]$,

$$\begin{aligned}
 & \|u(t)\|^2 + \|v(t)\|^2 + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\
 & \leq C_\eta e^{\eta T} (\|u_0\|^2 + \|v_0\|^2) \\
 & \leq C_\eta e^{\eta \alpha} (\|u_0\|^2 + \|v_0\|^2) \triangleq M.
 \end{aligned} \tag{6.68}$$

Furthermore we have:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} (\|u(t)\|^2 + \|v(t)\|^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\
& \leq C_\eta e^{\eta T} (\|u_0\|^2 + \|v_0\|^2) \\
& \leq C_\eta e^{\eta \alpha} (\|u_0\|^2 + \|v_0\|^2) \triangleq M.
\end{aligned} \tag{6.69}$$

Now we can use $(u(T), v(T))$ as our initial value, by Theorem (6.3.1) and (6.69), $\forall T \in [0, \alpha)$, there exists a $T_1 > 0$ which depends only on M and is independent of specific initial data $(u(T), v(T))$, such that (6.1)-(6.3) has a unique solution (u, v) in $[T, T + T_1]$.

Notice that for each $T \in [0, \alpha)$, (6.1)-(6.3) has a solution (u, v) in $[0, T]$, therefore we obtain a solution (u, v) in $[0, T + T_1]$. In particular, we can choose $T = \alpha - \frac{1}{2}T_1$, then (u, v) is the unique solution of (6.1)-(6.3) in $[0, \alpha]$ which implies $\alpha \in E$.

Chapter 7

The Global Classical Solution for the Non-linear Problem

In this chapter, we study the same non-linear problem (6.1)-(6.3) as that in chapter 6. With the initial data $(u_0, v_0) \in H^{k+2}(R^2) \times H^{k+1}(R^2)$, we obtain a unique global solution (u, v) with $u \in C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+4}(R^2))$ and $v \in C([0, T], H^{k+1}(R^2)) \cap L^2([0, T], H^{k+2}(R^2))$. Here, k is a nonnegative integer. According to Sobolev imbedding theorem [19], for $k \geq 4$, (u, v) is a classical solution to problem (6.1)-(6.3). For the convenience of reference, we relabel (6.1)-(6.3) as follows:

$$u_t + uu_x + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x = 0 \quad (7.1)$$

$$v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x = 0 \quad (7.2)$$

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y). \quad (7.3)$$

7.1 The global solution for the linear problem with

$$u^0 \in C([0, T], L^2(R^2))$$

$$u_t + u^0 u_x + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x = 0 \quad (7.4)$$

$$v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x = 0 \quad (7.5)$$

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y). \quad (7.6)$$

The problem (7.4)-(7.6) is very similar to the linear problem (2.4)-(2.6) studied in chapter 3, except for the requirement of u^0 . It turns out that we have the same conclusion as that in Theorem (3.1.1) for the case $k = 0$ under the assumption $u^0 \in C([0, T], L^2(R^2))$.

Theorem 7.1.1 *In the initial value problem (7.4)-(7.6), we assume that*

- $u_0 \in H^2(R^2)$ and $v_0 \in H^1(R^2)$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (7.4)-(7.5) are smooth functions and are constants outside a bounded domain, $u^0 \in C([0, T], L^2(R^2))$.

Then, $\forall T > 0$, (7.4)-(7.6) has a unique solution (u, v) with $u \in C([0, T], H^2(R^2)) \cap L^2([0, T], H^4(R^2))$ and $v \in C([0, T], H^1(R^2)) \cap L^2([0, T], H^2(R^2))$ that satisfies the estimate:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u\|_2^2 + \|v\|_1^2) + \int_0^T (\|u\|_4^2 + \|v\|_2^2) ds \\ & \leq C(T) (\|u_0\|_2^2 + \|v_0\|_1^2), \end{aligned} \quad (7.7)$$

where $C(T)$ denotes a constant depending on T and the coefficients of the linear system (7.4)-(7.6).

Proof. To get the estimate (7.7) in Theorem (7.1.1), we need to make the following modifications from Theorem (2.2.1) to Theorem (2.4.1).

As in the proof of Theorem (2.2.1), we take L^2 inner product of equation (7.4) with u and equation (7.5) with v over R^2 respectively, we obtain:

$$\begin{aligned} & \langle u_t + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x, u \rangle + \langle v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x, v \rangle \\ & \geq \partial_t (\|u\|^2 + \|v\|^2) + \|u\|_2^2 + \|v\|_1^2 - C(\|u\|^2 + \|v\|^2) \end{aligned} \quad (7.8)$$

exactly in the same way as that in Theorem (2.2.1). The only difference in (7.4) is the term $u^0 u_x$ which can be estimated as follows:

$$\begin{aligned}
 \langle u^0 u_x, u \rangle &\leq \|u^0\| \|u_x u\| \\
 &\leq \|u^0\| (\sup_{x,y} |u|) \|u_x\| \\
 &\leq C \|u^0\| \|u\|_2 \|u\|_1 \\
 &\leq C \|u^0\| (\varepsilon \|u\|_2^2 + \frac{1}{4\varepsilon} \|u\|_1^2) \\
 &\leq C \|u^0\| (\varepsilon \|u\|_2^2 + \frac{\varepsilon_1}{4\varepsilon} \|u\|_2^2 + \frac{1}{16\varepsilon_1\varepsilon} \|u\|^2),
 \end{aligned} \tag{7.9}$$

where $\varepsilon_1 \ll \varepsilon \ll 1$.

Hence we have (u, v) satisfies:

$$\begin{aligned}
 &\sup_{0 \leq t \leq T} (\|u(t)\|^2 + \|v(t)\|^2) + \int_0^T (\|u(s)\|_2^2 + \|v(s)\|_1^2) ds \\
 &\leq C(T) (\|u_0\|^2 + \|v_0\|^2).
 \end{aligned} \tag{7.10}$$

Similarly, we take L^2 inner product of equation (7.4) with Δu and equation (7.5) with Δv over R^2 respectively, combining with (7.10), we obtain:

$$\begin{aligned}
 &\langle u_t + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x, \Delta u \rangle + \langle v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x, \Delta v \rangle \\
 &\geq \partial_t (\|u\|_1^2 + \|v\|_1^2) + \|u\|_3^2 + \|v\|_2^2 - C (\|u\|_1^2 + \|v\|_1^2)
 \end{aligned} \tag{7.11}$$

exactly in the same way as that in Theorem (2.3.1) except the following term:

$$\begin{aligned}
 \langle u^0 u_x, \Delta u \rangle &\leq \|u^0\| \|u_x \Delta u\| \\
 &\leq \|u^0\| (\sup_{x,y} |u_x|) \|\Delta u\| \\
 &\leq C \|u^0\| \|u\|_3 \|u\|_2 \\
 &\leq C \|u^0\| (\varepsilon \|u\|_3^2 + \frac{1}{4\varepsilon} \|u\|_2^2) \\
 &\leq C \|u^0\| (\varepsilon \|u\|_3^2 + \frac{\varepsilon_1}{4\varepsilon} \|u\|_3^2 + \frac{1}{16\varepsilon_1\varepsilon} \|u\|_1^2).
 \end{aligned} \tag{7.12}$$

So we have (u, v) satisfies:

$$\begin{aligned}
 &\sup_{0 \leq t \leq T} (\|u(t)\|_1^2 + \|v(t)\|_1^2) + \int_0^T (\|u(s)\|_3^2 + \|v(s)\|_2^2) ds \\
 &\leq C(T) (\|u_0\|_1^2 + \|v_0\|_1^2).
 \end{aligned} \tag{7.13}$$

At last, we take L^2 inner product of equation (7.4) with $\Delta^2 u$ and equation (7.5) with Δv over R^2 , combining with (7.13), we obtain:

$$\begin{aligned}
 &\langle u_t + \Delta u_x + \alpha u_{xx} + \gamma \Delta^2 u - \epsilon_1 v_x, \Delta^2 u \rangle + \langle v_t + a_1 v_x - \Gamma \Delta v - \epsilon_2 u_x, \Delta v \rangle \\
 &\geq \partial_t (\|u\|_2^2 + \|v\|_1^2) + \|u\|_4^2 + \|v\|_2^2 - C (\|u\|_2^2 + \|v\|_1^2)
 \end{aligned} \tag{7.14}$$

exactly in the same way as that in Theorem (2.4.1) except the following term:

$$\begin{aligned}
 \langle u^0 u_x, \Delta^2 u \rangle &\leq \|u^0\| \|u_x \Delta^2 u\| \\
 &\leq \|u^0\| (\sup_{x,y} |u_x|) \|\Delta^2 u\| \\
 &\leq C \|u^0\| \|u\|_3 \|u\|_4 \\
 &\leq C \|u^0\| (\varepsilon \|u\|_4^2 + \frac{1}{4\varepsilon} \|u\|_3^2) \\
 &\leq C \|u^0\| (\varepsilon \|u\|_4^2 + \frac{\varepsilon_1}{4\varepsilon} \|u\|_4^2 + \frac{1}{16\varepsilon_1\varepsilon} \|u\|_2^2),
 \end{aligned} \tag{7.15}$$

where $\varepsilon_1 \ll \varepsilon \ll 1$.

Therefore we have the same conclusion as that in Theorem (2.4.1) with the weaker assumption on u^0 .

To show the existence of the solution of (7.4)-(7.6), we use the same method as that in Theorem (3.1.1) for the case $k = 0$. Also, we use the same notations as those in Theorem (3.1.1).

As in step 2 (to show Υ is closed) in the proof of Theorem (3.1.1), we apply the estimate (7.7) to the solution $(\tilde{u}_j, \tilde{v}_j)$ of (3.9). Where $(\tilde{u}_j, \tilde{v}_j) \equiv (u_j - u_{j-1}, v_j - v_{j-1})$ ($j = 2, 3, \dots$). We notice that in the first equation of (3.9) there is a term $\mathcal{L}_1(u_{j-1}, v_{j-1})$ which equals $\Delta u_{(j-1)x} + u^0 u_{(j-1)x} + \alpha u_{(j-1)xx} + \gamma \Delta^2 u_{j-1} - \epsilon_1 v_{(j-1)x}$. The norm $\|\mathcal{L}_1(u_{j-1}, v_{j-1})\|^2$ can be estimated in the following two steps.

First, we can get

$$\|\Delta u_{(j-1)x} + \alpha u_{(j-1)xx} + \gamma \Delta^2 u_{j-1} - \epsilon_1 v_{(j-1)x}\|^2 \leq C(\|u_{j-1}\|_4^2 + \|v_{j-1}\|_2^2) \quad (7.16)$$

exactly the same way as that in the proof of Theorem (3.1.1).

Second, the term $u^0 u_{(j-1)x}$ in $\mathcal{L}_1(u_{j-1}, v_{j-1})$ can be estimated as follows:

$$\begin{aligned} \|u^0 u_{(j-1)x}\|^2 &\leq (\sup_{x,y} |u_{(j-1)x}|)^2 \|u^0\|^2 \\ &\leq C \|u^0\|^2 \|u_{j-1}\|_3^2. \end{aligned} \quad (7.17)$$

So, we have:

$$\|\mathcal{L}_1(u_{j-1}, v_{j-1})\|^2 = \|\Delta u_{(j-1)x} + u^0 u_{(j-1)x} + \alpha u_{(j-1)xx} + \gamma \Delta^2 u_{j-1} - \epsilon_1 v_{(j-1)x}\|^2 \leq C(\|u_{j-1}\|_4^2 + \|v_{j-1}\|_2^2), \quad (7.18)$$

it follows that we get (3.10) in Theorem (3.1.1).

Same argument also applies to the estimate of (3.16) in step 3 (to show Υ is open) in the proof of Theorem (3.1.1). Therefore, by Theorem (3.1.1), $\forall T > 0$, there exists a unique solution (u, v) of (7.4)-(7.6) which satisfies the estimate (7.7).

7.2 The global solution for the non-linear problem

with $(u_0, v_0) \in H^2 \times H^1$

Theorem 7.2.1 *In the initial value problem (7.1)-(7.3), we assume that*

- $u_0 \in H^2(R^2)$ and $v_0 \in H^1(R^2)$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- *all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (7.1)-(7.3) are smooth functions and are constants outside a bounded domain.*

Then, $\forall T > 0$, (7.1)-(7.3) has a unique solution (u, v) with $u \in C([0, T], H^2(R^2)) \cap L^2([0, T], H^4(R^2))$ and $v \in C([0, T], H^1(R^2)) \cap L^2([0, T], H^2(R^2))$ that satisfies the estimate:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u(t)\|_2^2 + \|v(t)\|_1^2) + \int_0^T (\|u(s)\|_4^2 + \|v(s)\|_2^2) ds \\ & \leq C(T) (\|u_0\|_2^2 + \|v_0\|_1^2), \end{aligned} \quad (7.19)$$

where $C(T)$ denotes a constant depending on T and the coefficients of the non-linear system (7.1)-(7.3).

Proof. From Theorem (6.4.1), $\forall T > 0$ (7.1)-(7.3) has a solution (u, v) with $u \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $v \in C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$. We use this u to construct the following linear problem for (u^*, v^*) :

$$u_t^* + uu_x^* + \Delta u_x^* + \alpha u_{xx}^* + \gamma \Delta^2 u^* - \epsilon_1 v_x^* = 0 \quad (7.20)$$

$$v_t^* + a_1 v_x^* - \Gamma \Delta v^* - \epsilon_2 u_x^* = 0 \quad (7.21)$$

$$u^*(x, y, 0) = u_0(x, y), v^*(x, y, 0) = v_0(x, y). \quad (7.22)$$

By Theorem (7.1.1), $\forall T > 0$, there exists a unique solution (u^*, v^*) of the initial value problem (7.20)-(7.22) with $u^* \in C([0, T], H^2(R^2)) \cap L^2([0, T], H^4(R^2))$ and $v^* \in C([0, T], H^1(R^2)) \cap L^2([0, T], H^2(R^2))$ that satisfies the estimate (7.19).

Next, we show that the solution (u, v) of the initial value problem (7.1)-(7.3) is in the spaces $C([0, T], H^2(R^2)) \cap L^2([0, T], H^4(R^2))$ and $C([0, T], H^1(R^2)) \cap L^2([0, T], H^2(R^2))$ and satisfies the estimate (7.19).

Take the differences of equations (7.1)-(7.3) and (7.20)-(7.22) respectively, we have:

$$(u - u^*)_t + u(u - u^*)_x + \Delta(u - u^*)_x + \alpha(u - u^*)_{xx} + \gamma\Delta^2(u - u^*) - \epsilon_1(v - v^*)_x = 0 \quad (7.23)$$

$$(v - v^*)_t + a_1(v - v^*)_x - \Gamma\Delta(v - v^*) - \epsilon_2(u - u^*)_x = 0 \quad (7.24)$$

$$(u - u^*)(x, y, 0) = 0, (v - v^*)(x, y, 0) = 0. \quad (7.25)$$

Notice that:

$$\langle u(u - u^*)_x, u - u^* \rangle \leq C\|u\|(\varepsilon\|u - u^*\|_2^2 + \frac{\varepsilon_1}{4\varepsilon}\|u - u^*\|_2^2 + \frac{1}{16\varepsilon_1\varepsilon}\|u - u^*\|^2), \quad (7.26)$$

where $\varepsilon_1 \ll \varepsilon \ll 1$.

According to Theorem (5.2.1) and (7.26), we have:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|(u - u^*)(t)\|^2 + \|(v - v^*)(t)\|^2) + \int_0^T (\|(u - u^*)(s)\|_2^2 + \|(v - v^*)(s)\|_1^2) ds \\ & \leq C(T) (\|(u - u^*)(x, y, 0)\|^2 + \|(v - v^*)(x, y, 0)\|^2) = 0. \end{aligned} \quad (7.27)$$

Therefore, $u = u^*$ and $v = v^*$ are in the spaces $C([0, T], L^2(R^2)) \cap L^2([0, T], H^2(R^2))$ and $C([0, T], L^2(R^2)) \cap L^2([0, T], H^1(R^2))$ respectively.

According to the uniqueness of the solution to problem (7.1)-(7.3) in Theorem (6.2.1), the solution (u, v) of the initial value problem (7.1)-(7.3) is in the spaces $C([0, T], H^2(R^2)) \cap L^2([0, T], H^4(R^2))$ and $C([0, T], H^1(R^2)) \cap L^2([0, T], H^2(R^2))$ respectively and satisfies the estimate (7.19).

7.3 The regularity of the global solution for the non-linear problem with $(u_0, v_0) \in H^{k+2} \times H^{k+1}$

Theorem 7.3.1 *In the initial value problem (7.1)-(7.3), k is a non-negative integer, we assume that*

- $u_0 \in H^{k+2}(R^2)$ and $v_0 \in H^{k+1}(R^2)$;
- $\gamma \geq \delta > 0$ and $\Gamma \geq \delta > 0$;
- all the coefficients $\xi \equiv (\alpha, a_1, \gamma, \Gamma, \epsilon_1, \epsilon_2)$ in (7.1)-(7.3) are smooth functions and are constants outside a bounded domain.

Then, $\forall T > 0$, (7.1)-(7.3) has a unique solution (u, v) in the product space Π_T^k that satisfies the estimate :

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u(t)\|_{k+2}^2 + \|v(t)\|_{k+1}^2) + \int_0^T (\|u(s)\|_{k+4}^2 + \|v(s)\|_{k+2}^2) ds \\ & \leq C_k(T) (\|u_0\|_{k+2}^2 + \|v_0\|_{k+1}^2). \end{aligned} \quad (7.28)$$

Proof. We will prove this theorem by induction.

According to Theorem (7.2.1), this theorem is true for the case $k = 0$.

We assume that (7.1)-(7.3) has a solution (u, v) with $u \in C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+4}(R^2))$ and $v \in C([0, T], H^{k+1}(R^2)) \cap L^2([0, T], H^{k+2}(R^2))$ that satisfies the estimate (7.28), and $u_0 \in H^{k+3}(R^2)$, $v_0 \in H^{k+2}(R^2)$. We use this u to construct the following linear problem for (u^*, v^*) :

$$u_t^* + uu_x^* + \Delta u_x^* + \alpha u_{xx}^* + \gamma \Delta^2 u^* - \epsilon_1 v_x^* = 0 \quad (7.29)$$

$$v_t^* + a_1 v_x^* - \Gamma \Delta v^* - \epsilon_2 u_x^* = 0 \quad (7.30)$$

$$u^*(x, y, 0) = u_0(x, y), v^*(x, y, 0) = v_0(x, y). \quad (7.31)$$

By Theorem (2.4.2), the solution (u^*, v^*) of (7.29)-(7.31) satisfies:

$$\begin{aligned} & \partial_t(\|u^*\|_{k+2}^2 + \|v^*\|_{k+1}^2) + \|u^*\|_{k+4}^2 + \|v^*\|_{k+2}^2 \\ & \leq C_k(\|u^*\|_{k+2}^2 + \|v^*\|_{k+1}^2). \end{aligned} \quad (7.32)$$

We show in the following that the solution (u^*, v^*) is also in the spaces $C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$ and $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$ respectively and satisfies the following estimate:

$$\begin{aligned} & \sup_{0 \leq t \leq T}(\|u^*(t)\|_{k+3}^2 + \|v^*(t)\|_{k+2}^2) + \int_0^T (\|u^*(s)\|_{k+5}^2 + \|v^*(s)\|_{k+3}^2) ds \\ & \leq C_{k+1}(T)(\|u_0\|_{k+3}^2 + \|v_0\|_{k+2}^2). \end{aligned} \quad (7.33)$$

As in the proof of Theorem (2.4.2), we take $(k+1)$ -th partial derivative of equations (7.29)-(7.30) with respect to x and taking L^2 inner product of (7.29) with $\Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} u^*$ and (7.30) with $\Delta \frac{\partial^{k+1}}{\partial x^{k+1}} v^*$, we obtain:

$$\begin{aligned} & \frac{1}{2} \partial_t (\| \frac{\partial^{k+1}}{\partial x^{k+1}} u_{xx}^* \|^2 + \| \frac{\partial^{k+1}}{\partial x^{k+1}} u_{yy}^* \|^2) + \partial_t \| \frac{\partial^{k+1}}{\partial x^{k+1}} u_{xy}^* \|^2 + \frac{\gamma_0}{2} \| \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} u^* \|^2 + \frac{\Gamma_0}{2} \| \Delta \frac{\partial^{k+1}}{\partial x^{k+1}} v^* \|^2 \\ & \leq C_{k+1}(\|u^*\|_{k+3}^2 + \|v^*\|_{k+2}^2). \end{aligned} \quad (7.34)$$

exactly the same way as that in the proof of Theorem (2.4.2) except for the following term. Notice that $u \in C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+4}(R^2))$. Applying Cauchy-Schwarz inequality and Sobolev imbedding theorem, we have:

$$\begin{aligned} \langle \frac{\partial^{k+1}}{\partial x^{k+1}} u u_x^*, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} u^* \rangle & \leq \| \frac{\partial^{k+1}}{\partial x^{k+1}} u u_x^* \| \| \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} u^* \| \\ & \leq C \|u\|_{k+1} \|u^*\|_{k+4} \|u^*\|_{k+5} \\ & \leq C \|u\|_{k+1} (\frac{1}{16\varepsilon_1} \|u^*\|_{k+3}^2 + \varepsilon \|u^*\|_{k+5}^2 + \frac{\varepsilon_1}{4\varepsilon} \|u^*\|_{k+5}^2). \end{aligned} \quad (7.35)$$

where $\varepsilon_1 \ll \varepsilon \ll 1$.

Similarly, for any non-negative integers α, β , with $\alpha + \beta = k + 1$, we have the following

inequalities:

$$\begin{aligned} & \frac{1}{2} \partial_t (\|\partial_x^\alpha \partial_y^\beta u_{xx}^*\|^2 + \|\partial_x^\alpha \partial_y^\beta u_{yy}^*\|^2) + \partial_t \|\partial_x^\alpha \partial_y^\beta u_{xy}^*\|^2 + \frac{\gamma_0}{2} \|\Delta^2 \partial_x^\alpha \partial_y^\beta u^*\|^2 + \frac{\Gamma_0}{2} \|\Delta \partial_x^\alpha \partial_y^\beta v^*\|^2 \\ & \leq C_{k+1} (\|u^*\|_{k+3}^2 + \|v^*\|_{k+2}^2). \end{aligned} \quad (7.36)$$

Taking summation for all $\alpha + \beta = k + 1$ and combining (7.36) with (7.32), applying the Gronwall's inequality, we obtain the estimate (7.33).

For the convenience of expression in the following proof, we need to generalize the equations (7.29)-(7.31) into the following form:

$$u_t^* + uu_x^* + \Delta u_x^* + \alpha u_{xx}^* + \gamma \Delta^2 u^* - \epsilon_1 v_x^* = f \quad (7.37)$$

$$v_t^* + a_1 v_x^* - \Gamma \Delta v^* - \epsilon_2 u_x^* = g \quad (7.38)$$

$$u^*(x, y, 0) = u_0(x, y), v^*(x, y, 0) = v_0(x, y). \quad (7.39)$$

Obviously (7.29)-(7.31) is a special case of (7.37)-(7.39) with $f = g = 0$.

Since we have:

$$\left\langle \frac{\partial^{k+1}}{\partial x^{k+1}} f, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} u^* \right\rangle \leq \|f\|_{k+1} \|u^*\|_{k+5}. \quad (7.40)$$

.

$$\left\langle \frac{\partial^{k+1}}{\partial x^{k+1}} g, \Delta^2 \frac{\partial^{k+1}}{\partial x^{k+1}} u^* \right\rangle \leq \|g\|_{k+1} \|u^*\|_{k+5}. \quad (7.41)$$

.

Combining the estimate (7.33) with (7.40) and (7.41) we have that the solution (u^*, v^*) of (7.37)-(7.39) satisfies:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|u^*(t)\|_{k+3}^2 + \|v^*(t)\|_{k+2}^2) + \int_0^T (\|u^*(s)\|_{k+5}^2 + \|v^*(s)\|_{k+3}^2) ds \\ & \leq C_{k+1}(T) (\|u_0\|_{k+3}^2 + \|v_0\|_{k+2}^2 + \int_0^T (\|f\|_{k+1}^2 + \|g\|_{k+1}^2) ds). \end{aligned} \quad (7.42)$$

We show in the following that there exists a unique solution (u^*, v^*) of (7.29)-(7.31) satisfying the estimate (7.33). The proof is similar to the proof of Theorem (3.1.1). We

consider the following one-parameter family of initial value problems ($\lambda \in [0, 1]$):

$$\begin{cases} u_t^* + \lambda \mathcal{L}_1(u^*, v^*) + (1 - \lambda) \Delta^2 u^* = 0 \\ v_t^* + \lambda \mathcal{L}_2(u^*, v^*) - (1 - \lambda) \Delta v^* = 0 \\ u^*(x, y, 0) = u_0(x, y), \quad v^*(x, y, 0) = v_0(x, y), \end{cases} \quad (7.43)$$

where

$$\begin{cases} \mathcal{L}_1(u^*, v^*) = uu_x^* + \Delta u_x^* + \alpha u_{xx}^* + \gamma \Delta^2 u^* - \epsilon_1 v_x^* \\ \mathcal{L}_2(u^*, v^*) = a_1 v_x^* - \Gamma \Delta v^* - \epsilon_2 u_x^*. \end{cases} \quad (7.44)$$

We denote Υ the subset of $[0, 1]$ such that for parameter $\lambda \in \Upsilon$ the initial value problem (7.29)-(7.31) has a unique solution in the spaces $C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$ and $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$ satisfying (7.33). We need to show that $\Upsilon = [0, 1]$.

As in step 2 (to show Υ is closed) in the proof of Theorem (3.1.1), let $\lambda_j \in \Upsilon$, and $\lambda_j \rightarrow \lambda_0$. We need to show $\lambda_0 \in \Upsilon$.

Since $\lambda_j \in \Upsilon$, let (u_j^*, v_j^*) be the solution of the following initial value problem:

$$\begin{cases} u_{jt}^* + \lambda_j \mathcal{L}_1(u_j^*, v_j^*) + (1 - \lambda_j) \Delta^2 u_j^* = 0 \\ v_{jt}^* + \lambda_j \mathcal{L}_2(u_j^*, v_j^*) - (1 - \lambda_j) \Delta v_j^* = 0 \\ u_j^*(x, y, 0) = u_0(x, y), \quad v_j^*(x, y, 0) = v_0(x, y). \end{cases} \quad (7.45)$$

We apply the estimate (7.42) to the solution (u_j^*, v_j^*) of (7.45). Then we have that (u_j^*, v_j^*) is uniformly bounded in the spaces $C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$

and $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$. In particular:

$$\begin{aligned} & \int_0^T (\|u_j^*(s)\|_{k+5}^2 + \|v_j^*(s)\|_{k+3}^2) ds \\ & \leq C_{k+1}(T) (\|u_0\|_{k+3}^2 + \|v_0\|_{k+2}^2) \leq K. \end{aligned} \quad (7.46)$$

Let $(\tilde{u}_j, \tilde{v}_j) \equiv (u_j^* - u_{j-1}^*, v_j^* - v_{j-1}^*)$ ($j = 2, 3, \dots$) which satisfies:

$$\left\{ \begin{aligned} & \tilde{u}_{jt} + \lambda_j \mathcal{L}_1(\tilde{u}_j, \tilde{v}_j) + (1 - \lambda_j) \Delta^2 \tilde{u}_j \\ & = -(\lambda_j - \lambda_{j-1}) [\mathcal{L}_1(u_{j-1}^*, v_{j-1}^*) - \Delta^2 u_{j-1}^*] \\ & \tilde{v}_{jt} + \lambda_j \mathcal{L}_2(\tilde{u}_j, \tilde{v}_j) - (1 - \lambda_j) \Delta \tilde{v}_j \\ & = -(\lambda_j - \lambda_{j-1}) (\mathcal{L}_2(u_{j-1}^*, v_{j-1}^*) - \Delta v_{j-1}^*) \\ & \tilde{u}_j(x, y, 0) = 0, \quad \tilde{v}_j(x, y, 0) = 0. \end{aligned} \right. \quad (7.47)$$

By (7.42), we have:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\tilde{u}_j(t)\|_{k+3}^2 + \|\tilde{v}_j(t)\|_{k+2}^2) + \int_0^T (\|\tilde{u}_j(s)\|_{k+5}^2 + \|\tilde{v}_j(s)\|_{k+3}^2) ds \\ & \leq C_{k+1}(T) |(\lambda_j - \lambda_{j-1})|^2 \int_0^T (\|u_{j-1}^*(s)\|_{k+5}^2 + \|v_{j-1}^*(s)\|_{k+3}^2) ds \\ & \leq C_{k+1}(T) |(\lambda_j - \lambda_{j-1})|^2 K. \end{aligned} \quad (7.48)$$

In particular, we notice that $uu_{(j-1)x}^*$ is one of the terms of $\mathcal{L}_1(u_{j-1}^*, v_{j-1}^*)$ in (7.47), we have the estimate of $uu_{(j-1)x}^*$ as follows:

$$\begin{aligned} \|uu_{(j-1)x}^*\|_{k+1}^2 & \leq C \|uu_{(j-1)}^*\|_{k+2}^2 \\ & \leq C \|u\|_{k+2}^2 \|u_{(j-1)}^*\|_{k+2}^2. \end{aligned} \quad (7.49)$$

Since $\lambda_j \rightarrow \lambda_0$, it follows that (u_j^*, v_j^*) is a Cauchy sequence in the spaces $C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$ and $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$, and its limit (u^*, v^*) is obviously the solution of (7.43) for λ_0 . This shows Υ is closed in $[0, 1]$.

We have the similar modifications to prove Υ is open in $[0, 1]$. Therefore, $\forall T > 0$, there exists a unique solution (u^*, v^*) of the initial value problem (7.29)-(7.31) with $u^* \in C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$ and $v^* \in C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$ that satisfies the estimate (7.33).

Now, we show in the following that the solution (u, v) of (7.1)-(7.3) in the spaces $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+4}(R^2))$ and $C([0, T], H^{k+1}(R^2)) \cap L^2([0, T], H^{k+2}(R^2))$ is also in the spaces $C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$ and $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$ respectively and satisfies the estimate (7.33).

Take the differences of equations (7.1)-(7.3) and (7.29)-(7.31) respectively, we have :

$$(u - u^*)_t + u(u - u^*)_x + \Delta(u - u^*)_x + \alpha(u - u^*)_{xx} + \gamma\Delta^2(u - u^*) - \epsilon_1(v - v^*)_x = 0 \quad (7.50)$$

$$(v - v^*)_t + a_1(v - v^*)_x - \Gamma\Delta(v - v^*) - \epsilon_2(u - u^*)_x = 0 \quad (7.51)$$

$$(u - u^*)(x, y, 0) = 0, (v - v^*)(x, y, 0) = 0. \quad (7.52)$$

According to Theorem (3.2.1), we have:

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|(u - u^*)(t)\|_{k+2}^2 + \|(v - v^*)(t)\|_{k+1}^2) + \int_0^T (\|(u - u^*)(s)\|_{k+4}^2 + \|(v - v^*)(s)\|_{k+2}^2) ds \\ & \leq C_k(T) (\|(u - u^*)(x, y, 0)\|_{k+2}^2 + \|(v - v^*)(x, y, 0)\|_{k+1}^2) = 0. \end{aligned} \quad (7.53)$$

Therefore, $u = u^*$ and $v = v^*$ are in the spaces $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+4}(R^2))$ and $C([0, T], H^{k+1}(R^2)) \cap L^2([0, T], H^{k+2}(R^2))$ respectively.

According to the uniqueness of the solution to problem (7.1)-(7.3) in Theorem (6.2.1), the solution (u, v) of the initial value problem (7.1)-(7.3) is in the spaces $C([0, T], H^{k+3}(R^2)) \cap L^2([0, T], H^{k+5}(R^2))$ and $C([0, T], H^{k+2}(R^2)) \cap L^2([0, T], H^{k+3}(R^2))$ respectively and satisfies the estimate (7.33).

Hence, we finish the proof of Theorem (7.3.1).

Remark 7.3.1 The conclusion of Theorem (7.3.1) is also true if the initial data (u_0, v_0) are constants outside a bounded domain. The proof will involve the use of uniformly local Sobolev space $H_{ul}^s(R^2)$, and the details are omitted here.

Bibliography

Bibliography

- [1] D.J. Benney, *Long waves on liquid films*, J. Math. Phys. 45(1966) 150-155.
- [2] H.-C. Chang, E.A. Demekhin and D.I. Kopelevich, *Stability of a Solitary Pulse against Wave Packet Disturbances in an Active Medium*, Phys. Rev. Lett. 75(1995) 1747-1750.
- [3] H.-C. Chang, *Travelling waves on fluid interfaces: normal form analysis of the Kuramoto-Sivashinsky equation*, Phys. Fluids 29(1986) 3142-3147.
- [4] C.I. Christov and M.G. Velarde, *Dissipative Solitons*, Physica D 86(1995) 323-347.
- [5] C. Elphick, G.R. Ierley, O. Regev and E.A. Spiegel, *Interacting localized structures with Galilean invariance*, Phys. Rev. A 44(1991) 1110-1122.
- [6] B.-F. Feng, B.A. Malomed and T. Kawahara, *Stable periodic waves in coupled Kuramoto-Sivashinsky-Korteweg-de Vries equations*, J. Phys. Soc. Jpn. 71 (2002) 2700-2707.
- [7] B. F. Feng, B. A. Malomed and T. Kawahara, *Cylindrical solitary pulses in a two-dimensional stabilized Kuramoto-Sivashinsky system*, Physica D, 3035 (2002) 1-12.
- [8] J.A. Gear and R. Grimshawde, *Weak and strong interactions between internal solitary waves*, Stud. Appl. Math. 70 (1984) 235-258.
- [9] A. Jeffrey and T. Kakutani, *Weakly nonlinear dispersive waves: A discussion centered around the Korteweg-de Vries equation*, Siam Rev. 14(1972) 582-643.

- [10] T. Kato, *The Cauchy problem for quasilinear symmetric hyperbolic systems*, Arch. Rat. Mech. Anal., 58(1975) 181-205.
- [11] C.E. Lawrence, *Partial Differential Equations*, American Mathematical Society, USA(1998).
- [12] N.A. Larkin, *Korteweg-de Vries Equation in Bounded Domains*, Bol. Soc. Paran. Mat. 22(2001) 30-37.
- [13] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, 1985.
- [14] Y. Lenbury, D. Li and C. Rattanakul, *Stability of solution of Kuramoto- Sivashinsky-Korteweg-de Vries system*, Comp. Math. Appl., 52(2006), 497-508
- [15] B.A. Malomed, B.-F. Feng and T. Kawahara, *Stabilized Kuramoto-Sivashinsky system*, Phys. Rev. E 64, 046304(2001).
- [16] T.R. Marchant and N.F. Smyth, *The initial-boundary problem for the Korteweg-de Vries equation on the negative quarter-plane*, Proc. R. Soc. Lond. A 458(2002) 857-871.
- [17] R.M. Miura, *The Korteweg-de Vries equation: A survey of results*, Siam Rev. 18(1976) 412-459.
- [18] A.Oron and D.A. Edwards, *Stability of a falling liquid film in the presence of interfacial viscous stress*, Phys. Fluids 5(1993) 506-508.
- [19] M. Renardy and R. Rogers, *An introduction to partial differential equations*, Springer-Verlag, New York, 2004.
- [20] G.I. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations*, Acta Astronaut. 4(1977) 1177-1206.
- [21] G. I. Sivashinsky, *Instabilities, Pattern Formation, and Turbulence in Flames*. Publication Annual Review of Fluid Mechanics, vol. 15 (1983) 179-199.

- [22] S. Toh and T. Kawahara, *On the stability of soliton-like pulses in a nonlinear dispersive system with instability and dissipation*, J. Phys. Soc. Jpn. 54(1985) 1257-1269.
- [23] T. Yamada and Y. Kuramoto, *A reduced model showing chemical turbulence*, Prog. Theor. Phys. 56(1976) 681-683.
- [24] K. Yosida, *Functional Analysis*, Springer-Verlag, NewYork,1978.
- [25] N. Zabusky and M. Kruskal, *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett. 15(1965) 240-243.