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Direct Determination of the Dynamic Elastic Modulus and Poisson's Ratio of Timoshenko Prisms and Rods

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Direct Determination of the Dynamic Elastic Modulus and Poisson's Ratio of Timoshenko
Prisms and Rods

Guadalupe Leon

Thesis submitted

**to the Benjamin M. Statler College of Engineering and Mineral Resources
at West Virginia University**

in partial fulfillment of the requirements for the degree of

**Master of Science
in
Civil and Environmental Engineering**

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ABSTRACT

Direct Determination of the Dynamic Elastic Modulus and Poisson's Ratio of Timoshenko Prisms and Rods

Guadalupe Leon

In this study, the exact solution of the Timoshenko beam vibration frequency equation under free-free boundary conditions was determined with an accurate shear shape factor. The exact solution was compared with a 3-D finite element calculation using ABAQUS program, and the difference between the exact solution and the 3-D FEM was within 0.15% for both the transverse and torsional modes. Furthermore, relationships between the resonance frequencies and Poisson's ratio were proposed that can directly determine the elastic constants, unlike the equations provided by ASTM C215. The frequency ratio between the 1st bending mode and the 1st torsional mode, or the frequency ratio between the 1st bending mode and the 2nd bending mode for any rectangular prism or rod can be directly estimated. Likewise, the bending and torsional modes can be used to determine the elastic constants of any rectangular prism or rod. The proposed equations were used to verify the elastic constants of a steel rod and prism with less than 0.36% error. The transverse and torsional frequencies of concrete, aluminum and steel rods were tested. Rectangular concrete beam specimens with three different mix designs were produced, and the transverse and torsional frequencies of these beams were tested. Results show that using the equations proposed in this study, the Young's modulus and Poisson's ratio of a Timoshenko prism and rod can be determined more directly than those obtained from ASTM C215 and with excellent accuracy.

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Chapter 1 Introduction

1.1 Background/Overview

Non-destructive testing (NDT) methods are often used to estimate the elastic properties of a material such as the dynamic elastic modulus and Poisson's ratio. There are several types of non-destructive testing (NDT) methods available. These include ultrasonic pulse velocity methods, resonance frequency methods, and other wave propagation techniques. Material properties measured with these methods are called dynamic elastic constants. ASTM C215 (2014) and ASTM E1876 (2015) standards are widely used to determine the dynamic elastic modulus using the fundamental resonance frequency. The equations provided in these standards were formulated by simplifying the free-free vibration solution of a Timoshenko beam with the effects of shear and rotational inertia reduced to a simple correction constant. However, the value of the elastic modulus cannot be directly calculated without knowing the material's Poisson's ratio. Therefore, to calculate the dynamic modulus and Poisson's ratio a lengthy iteration process is needed. To develop a direct method to determine the elastic constants, the frequency ratios between the bending modes and torsional mode were studied.

1.2 Project Scope & Objectives

The scope of this thesis was to develop a method to directly determine the dynamic elastic modulus and Poisson's ratio of a rectangular prism and rod without the need for iterations. Additionally, the method should employ the resonance frequencies and be

applicable to any homogenous material. These objectives were completed using the following steps:

- A brief discussion of prior research and available methods such as ASTM C215 and ASTM E1876.
- A detailed analysis of the free-free vibration of Timoshenko beams with rectangular or circular cross-sections.
- A study of the frequency ratios between the bending modes and 1st torsional mode.
- Propose empirical equations that allow an accurate and simple method to determine the elastic constants.
- An explanation of the experimental methods used to verify the proposed method.
- A summary of the results and conclusions with a comparison with ASTM C215 and ASTM E1876.

1.3 Thesis Organization

A brief overview of the organization of this thesis is as follows:

- Chapter 2:
 - This chapter will provide a brief overview of non-destructive techniques available to measure the dynamic modulus and Poisson's ratio of rectangular and circular Timoshenko beams, specifically techniques that use vibration methods.
- Chapter 3:

- This chapter contains the derivation of the vibration of a free-free Timoshenko beam including an accurate shear shape factor. Finite-element analysis using a commercial software was compared to the derived exact solution. This chapter also contains an investigation of the frequency ratios of the 2nd and 1st bending modes, and the 1st bending and 1st torsional modes.
- Chapter 4:
 - This chapter presents empirical equations to estimate the coefficients presented in Chapter 3. These equations allow for a direct calculation of the dynamic elastic modulus and Poisson's ratio of rectangular prisms and rods of any dimension. The proposed equations were used to calculate the frequency ratio of a steel prism and rod.
- Chapter 5:
 - This chapter discusses the experimental program employed during this study. A detailed explanation of the experiments and results is provided. The proposed methods were compared to ASTM C215 for the rectangular prism case.
- Chapter 6:
 - This chapter discusses the results obtained during the experimental program and limitations of the proposed methods. A sensitivity analysis of the methods is included in this chapter. This chapter provides a summary of the research study and recommendations for future work.

Chapter 2 Literature Review

2.1 Introduction

Concrete specifications usually require a specific value for the modulus of elasticity (Popovics 2008). Formwork removal time and pre-stressing of concrete members are often determined from this property at early-age (Zhou et al. 2015). Young's modulus is also an indicator of the degradation of concrete, hence it is used frequently to determine the damage due to freeze-thaw or fire (Lee et al. 2015; Gucunski 2013). An empirical relationship between the compressive strength and the modulus of elasticity of the concrete is given in ACI 318-14, however, the ACI formula usually provides overly conservative results. An alternative technique is to use a non-destructive testing (NDT) method to estimate the concrete properties. There are several types of non-destructive testing (NDT) methods available for measurements of concrete Young's modulus. Material properties measured with these methods are called dynamic elastic constants. Many researchers have found that the dynamic modulus, E_d , is often higher than that of the static modulus, E_s and proposed relationships to correlate the static and dynamic modulus (Zhou et al. 2015; Chavhan and Vyawahare 2015; Lee et al. 2017; Lydon and Balendran 1986; Yildirim and Sengul 2011; Weiss 2006). In this chapter an overview of the current research regarding the measurement of the Young's elastic modulus and Poisson's ratio using NDT methods is presented.

2.2 Literature

2.2.1 Wave Propagation Techniques

Most wave propagation techniques are based on the propagation of either longitudinal waves (P-wave), Rayleigh waves or transverse waves (S-wave). In a typical experiment, a piezoelectric transducer is used to generate a wave at one end of a specimen while another transducer captures the arrival at the opposite end. The wave speed is then calculated by dividing the length of the specimen by the time it took for the wave to be received by the piezoelectric transducer. Since the longitudinal wave is the fastest it will arrive before the other wave propagations, so the calculated wave speed corresponds to the longitudinal wave speed. For an isotropic and homogenous material, the longitudinal wave speed can be used to calculate the dynamic elastic modulus if the Poisson's ratio is known. Similarly, the S-wave and Rayleigh wave speed can also be used.

Lee et al. found the dynamic elastic modulus using both the P-wave and S-wave. Based on their experimental measurements, they stated that the wave propagation techniques produced slightly higher values for the dynamic modulus compared to the vibration techniques (Lee et al. 2017). Similarly, another research also concluded that the wave propagation method produced a concrete dynamic modulus approximately 10% higher than those obtained through vibration methods (Logan 2017). However, in their analysis a Poisson's ratio of 0.2 was assumed since it was not measured independently. Herein lies a disadvantage of using a single wave propagation method because the Poisson's ratio must be known in advance. However, it should be noted, that since the P-wave and S-wave both depend on the elastic modulus and Poisson's ratio a simultaneous measurement of each will allow the determination of the elastic constant.

2.2.2 Vibration Techniques

Researchers have used forced vibration and impulse vibration to measure the dynamic elastic constants of steel plates (Alfano and Pagnotta 2007) as well as impact-echo resonance and Rayleigh wave velocity (Medina and Bayón 2010). Kolluru et al. proposed a technique to determine the elastic material constants of a standard test cylinder using the longitudinal resonance frequencies of concrete, steel, and aluminum (Kolluru, Popovics, and Shah 2000). The ratio between the 2nd and 1st longitudinal resonance frequencies was used to determine the Poisson's ratio for a variety of length-to-diameter ratios (L/D) and afterward the elastic modulus was estimated. They observed a parabolic relationship between the longitudinal resonance as shown in **Figure 2-1**. Kolluru et al. proposed Eq. (1) to determine the Poisson's ratio.

Figure 2-1. Poisson's ratio vs. longitudinal resonance frequency ratio (Kolluru, Popovics, and Shah 2000)

$$\nu = A_1 \left(\frac{f_2}{f_1} \right)^2 + B_1 \left(\frac{f_2}{f_1} \right) + C_1 \quad (1)$$

where f_2 and f_1 are the measured 1st and 2nd longitudinal resonance frequency; A_1 , B_1 and C_1 are empirical constants that depend on the length to diameter, L/D , ratio. These empirical constants were found using the best-fit parabolic curve shown in **Figure 2-2**. It should be noted that only a small range of L/D ratios were included since their focus was on cored cylinders of intermediate lengths. Therefore, its applicability on shorter or longer cylinders is uncertain. Additionally, in longer cylinders it may be difficult to accurately excite the longitudinal mode due to concrete's high attenuation.

Figure 2-2. Empirical constants: (a) quadratic term, (b) linear term, and (c) constant term (Kolluru, Popovics, and Shah 2000)

Nieves et al. estimated the shear modulus and Poisson's ratio for short cylinders using the two lowest axisymmetric natural frequencies (Nieves, Gascón, and Bayón 1998). Verification of their methods was made using various cylindrical specimens with an L/D less than 3. Wang et al. proposed a method to evaluate Poisson's ratio of a rod with only two experimental impact-echo values (fundamental longitudinal and cross-sectional

resonant frequencies). However, their methods require an $L/D \geq 25\nu$ for the dynamic Poisson's ratio to be valid (Wang et al. 2012).

ASTM C215 (2014) and ASTM E1876 (2015) are widely used to determine the dynamic elastic modulus using the fundamental resonance frequency. The equations provided in these standards were formulated by simplifying the free-free vibration solution of a Timoshenko beam with the effects of shear and rotational inertia reduced to a simple correction constant. However, the value of the elastic modulus cannot be directly calculated without knowing the material's Poisson's ratio. Therefore, in order to calculate the dynamic modulus and Poisson's ratio, a lengthy iteration process is needed.

In this study, the exact solution of the Timoshenko beam vibration frequency equation under free-free boundary conditions was determined. The fundamental transverse and torsional frequencies were used to determine the dynamic elastic modulus and Poisson's ratio of rods with an $L/D \geq 2$. Then, relationships between the resonance frequencies and Poisson's ratio were proposed that can directly determine the elastic modulus and the Poisson's ratio, simultaneously without the need for iteration, using the frequency ratio between the 1st bending mode and the 1st torsional mode, or the frequency ratio between the 2nd bending mode and the 1st bending mode.

2.3 ASTM C215-14 Method

Following the ASTM Standard Test Method for Fundamental Transverse, Longitudinal, and Torsional Resonant Frequencies of Concrete Specimens (ASTM C215-

14), the dynamic modulus of elasticity can be obtained. The equations presented in ASTM C215-14 can be derived using the Timoshenko beam vibration. However, the solution shown in ASTM C215-14 was simplified by Goens (Goens 1931), and Pickett (Pickett 1945) later using numerical solutions for a correction factor, T , for different values of Poisson's ratio and K/L (the ratio of the radius of gyration, K , and the length, L), assuming only one-dimensional motion (Pickett 1945).

Using the impact resonance frequency method, the fundamental resonant frequency was used to calculate the dynamic modulus. This method was first applied on concrete samples by Powers (Powers 1938). **Figure 2-3** shows the schematic drawing of the test setup for a rectangular concrete prism with b = width, and t = height, and L = length. The location of the boundary conditions, accelerometer and impact can be found in ASTM C215. It should be noted, ASTM C215 specifies that the supports should be placed at the nodal locations to allow free vibration in the fundamental transverse mode with free-free boundary condition.

Figure 2-3. Schematic of transverse mode (Adopted from ASTM C215)

ASTM C215-14 provides the following equation to calculate the dynamic modulus:

$$E = CMn^2 \quad (2)$$

where, E is the dynamic modulus of elasticity (Pa); $C = 0.9464 (L^3T/bt^3)$ (m^{-1}); M is the mass of the specimen (kg); n is the fundamental transverse frequency (Hz); L is the length of the specimen (m); t , b are the height and the width of the cross section of the prism (m); and T is a correction factor that depends on the ratio of the radius of gyration, K , and the length, L , of the specimen. The parameter T in C was developed by Pickett (Pickett 1945) to account for shear and rotational inertia, and numerical values for T at different Poisson's ratio and K/L values can be found in a table shown in ASTM C215.

The torsional frequency can also be obtained. A schematic drawing of the test setup can be found in **Figure 2-4**. The support is placed at the center of the specimen which corresponds to the nodal point of the specimen. Details of the testing set-up and procedures will be explained in Chapter 4.

Figure 2-4. Schematic of torsional mode testing (Adopted from ASTM C215)

The fundamental torsional frequency was used to calculate the dynamic modulus of rigidity (shear modulus). ASTM C215 provides the equation below to calculate the dynamic modulus of rigidity:

$$G = BM(n'')^2 \quad (3)$$

where G is the dynamic modulus of rigidity (Pa); $B = 4LR/A$ (m^{-1}); M is the mass of the specimen (kg); n'' is the fundamental torsional frequency (Hz); R is the ratio between the polar moment of inertia and the equivalent moment of inertia.

The solution for the modulus of rigidity was derived using a Euler-Bernoulli beam in torsion (Augustyn and Kozien 2015; Pickett 1945; Vet 1962). Poisson's ratio can be calculated using the ratio between the modulus of elasticity and modulus of rigidity if the concrete specimen is assumed to be a homogeneous linear-elastic isotropic solid. The equation below can be used to calculate Poisson's ratio:

$$\nu = \left(\frac{E}{2G} \right) - 1 \quad (4)$$

where ν is Poisson's ratio; E is the elastic modulus; and G is the shear modulus. Concrete Poisson's ratio typically ranges from 0.1 for dry specimens to 0.25 for saturated specimens (ASTM C215 2014).

2.3 Summary

In this chapter a brief overview of NDT methods was presented. Two of the most common non-destructive techniques were discussed, wave propagation and vibration methods. From literature it was found that the wave propagation method produces slightly larger values of the dynamic elastic modulus than the vibration methods. It was also found that researchers have successfully used the ratio between the resonance frequencies to directly determine the elastic constants. Therefore, in this study the ratio between the bending and torsional modes will be utilized.

Chapter 3 Vibration of a Timoshenko Beam

3.1 Introduction

The vibration of a beam can be simplified using a Euler-Bernoulli beam. For the vibration of a Euler-Bernoulli beam the effects from shear are assumed neglectable. This simplification is only true when the beam is considered slender and produces large errors for shorter beams. In the early 20th century, Timoshenko developed a fourth-order differential equation to consider shear and rotary effects. Using this beam vibration equation, the vibration of thick beams could be predicted with more accuracy. Since it is not feasible to always test slender beams the effect of shear must be considered. Therefore, in this study the Timoshenko beam formulation will be used to determine the bending modes. Furthermore, it will be shown that the Euler-Bernoulli torsional mode is accurate and therefore applicable in this study.

3.2 Analysis

3.2.1 Bending

For a homogenous and isotropic material, the Timoshenko beam vibration equation can be expressed as:

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} - \rho I \left(1 + \frac{E}{k'G} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{k'G} \frac{\partial^4 y}{\partial t^4} = 0 \quad (5)$$

The third and fourth terms were added by Timoshenko to consider the shear and rotary effects in the beam (Timoshenko 1937). In the equation, y is the transverse displacement; t is time; k' is a shape factor; G is the shear modulus; E is the Young's

modulus; I is the moment of inertia of the cross section; ρ is the mass density and m is the mass per unit length. In Timoshenko's publication (Timoshenko 1937), a shape factor of $\frac{3}{4}$ was assumed for a circular cross-section and $\frac{2}{3}$ was assumed for a rectangular cross-section. In 1966, Cowper also derived the shape factor for different geometries (Cowper 1966). However, Hutchinson derived more accurate shape factors for a variety of geometries which match more closely with experimental results (Hutchinson 2000). In this study, Hutchinson's shape factor, Eq. (6) and Eq. (7), were used to calculate the shape factor for a circular and rectangular cross-section respectively. The shape factor for a rod is solely dependent on Poisson's ratio while for a rectangular prism it depends on the dimensions. In Eq. (7), t is the height of the beam and b is the width.

$$k' = \frac{6(1 + \nu)^2}{7 + 12\nu + 4\nu^2} \quad (6)$$

$$k' = -\frac{2(1 + \nu)}{\left[\frac{144}{t^5 b} C_4 + \nu \left(1 - \frac{b^2}{t^2}\right)\right]} \quad (7)$$

where:

$$C_4 = \frac{1}{720} t^3 b (-12t^2 - 15\nu t^2 + 5\nu b^2) + \sum_{n=1}^{\infty} \frac{\nu^2 b^5 \left(n\pi t - b \tanh\left(\frac{n\pi t}{b}\right)\right)}{4(n\pi)^5(1 + \nu)} \quad (8)$$

For a Timoshenko beam with free-free boundary conditions, the following can be applied:

$$y''(x)|_{x=0} = 0; \quad y''(x)|_{x=L} = 0; \quad y'''(x)|_{x=0} = 0; \quad y'''(x)|_{x=L} = 0$$

Using these boundary conditions for a nontrivial solution, the frequency equation can be written as (Goens E 1931; Chen and Kiriakidis 2005):

$$2 - 2 \cos h(\alpha) \cos(\beta) + \frac{\gamma^2(\gamma^4 r^2(r^2 - s^2)^2 + (3r^2 - s^2))}{\sqrt{1 - \gamma^4 r^2 s^2}} \sinh(\alpha) \sin(\beta) = 0 \quad (9)$$

where $r^2 = \frac{I}{AL^2}$, $s^2 = \frac{EI}{k'GAL^2}$, $\gamma^4 = \frac{\rho AL^4}{EI} \omega^2$ and A is the cross-sectional area; α and β represent the eigenvalues of the frequency equation. The eigenvalues α and β can be expressed as:

$$\alpha_i = \frac{\gamma_i^2}{\sqrt{2}} \sqrt{-(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + \frac{4}{\gamma_i^4}}} \quad (10)$$

and

$$\beta_i = \frac{\gamma_i^2}{\sqrt{2}} \sqrt{(r^2 + s^2) + \sqrt{(r^2 - s^2)^2 + \frac{4}{\gamma_i^4}}}$$

A MATLAB program was used to solve for the eigenvalues of the frequency equation. The natural frequencies (Hz) can be expressed using the eigenvalues (Huang 1961):

$$n_i = \frac{\gamma_i^2}{2\pi} \sqrt{\frac{EI}{\rho AL^4}} \quad (11)$$

Let $i = 1$ to obtain the 1st bending mode and $i = 2$ for the 2nd bending mode.

3.2.2 Torsional

In this study, the torsional mode obtained from the free-free Euler-Bernoulli beam solution was employed. If the cross-section of the beam has two axes of symmetry, the torsional vibration of the beam can be expressed with Eq. (12) and is independent of the bending modes (Augustyn and Marek Kozien 2015):

$$\rho J_0 \frac{\partial^2 \varphi}{\partial t^2} - G J_s \frac{\partial^2 \varphi}{\partial x^2} = 0 \quad (12)$$

In the equation, φ is the angle of twist; J_0 is the polar moment of inertia and J_s is the equivalent moment of inertia due to torsion. For a Euler-Bernoulli beam with free-free conditions, the following can be applied:

$$\varphi'(x)|_{x=0} = 0; \quad \varphi'(x)|_{x=L} = 0;$$

Using these boundary conditions for a nontrivial solution, the torsional frequency equation can be written as:

$$\sin(kL) = 0 \quad (13)$$

where

$$k_i^2 = \frac{\rho J_0 \omega_i^2}{G J_s} \quad (14)$$

The solutions of the torsional frequency equation can be expressed as $k_i L = N_i \pi$, where N_i can be any positive integer. Finally, the 1st free-free torsional mode of a beam can be calculated as:

$$n'' = \frac{1}{2L} \sqrt{\frac{GJ_s}{\rho J_0}} = \frac{1}{2L} \sqrt{\frac{G}{\rho R}} \quad (15)$$

where R is the ratio between the polar moment of inertia and the equivalent moment of inertia. For a rectangular prism whose cross-sectional dimensions are b and t with $b < t$, R can be calculated with Eq. (16). For a rod, the equivalent moment of inertia is equal to the polar moment of inertia therefore $R=1$.

$$R = (b/t + t/b)/[(4b/t - 2.52(b/t)^2 + 0.21(b/t)^6)] \quad (16)$$

3.3 Finite-Element Analysis

3.3.1 Rectangular Prism

Frequency analysis using 3-D finite element method (FEM) was carried out and compared to the exact solution for the transverse and torsional modes of the rectangular prism. The density of the specimen was assumed to be $2,400 \text{ kg m}^{-3}$. Six different sets of elastic modulus and Poisson's ratio were used in the FEM as shown in **Table 3-1**.

Table 3-1. Elastic modulus and Poisson's ratio assumed in FEM

Test	Elastic Modulus (GPa)	Poisson's ratio
1	25.0	0.1
2	25.0	0.15
3	25.0	0.2
4	10.0	0.2
5	20.0	0.2
6	30.0	0.2

A total of 192,000 elements of 2.5 mm (0.1-in) cubes, 8-node linear brick with reduced integration and hourglass control elements (C3D8R) from ABAQUS program

were utilized. The 75 mm x 100 mm x 400 mm concrete prism was modeled as a free-free beam shown in **Figure 3-1**.

Figure 3-1. Mesh of 75 mm (b) x 100 mm (t) x 400 mm (L) prism

Figure 3-2 shows the 1st and 2nd bending modes as well as the torsional mode. In the prism experimental testing, the 2nd bending mode was not observed, however with a longer specimen it should be possible to experimentally measure the 2nd mode. Additionally, the experimental set-up can be modified to capture the 2nd mode (i.e. placement of the accelerometer).

Figure 3-2. (a) 1st bending mode, (b) 2nd bending mode, and (c) 1st torsional mode

A comparison between the FEM and the analytical solutions is shown in **Table 3-2**. Results show that the percent error for all the cases is less than 0.05%. This indicates that the elastic modulus and Poisson's ratio obtained from the roots of the Timoshenko beam frequency equation in the analysis section above is accurate and the 1st free-free torsional mode from the Euler-Bernoulli beam solution is also close to exact.

Table 3-2. Comparison of FEM and the analytical solutions

Test	Transverse Mode (Hz)						Torsional Mode (Hz)		
	1 st			2 nd					
	Exact	FEM	Error (%)	Exact	FEM	Error (%)	Exact	FEM	Error (%)
1	1728.8	1728.1	0.041%	3899.5	3898.1	0.036%	2361.6	2360.5	0.047%
2	1722.3	1721.7	0.033%	3860.3	3859.4	0.023%	2261.1	2260.3	0.033%
3	1719.0	1718.5	0.031%	3841.0	3840.0	0.027%	2215.4	2214.7	0.031%
4	1089.3	1088.9	0.033%	2441.5	2440.9	0.023%	1430.0	1429.5	0.036%
5	1540.4	1539.9	0.035%	3452.7	3451.9	0.024%	2022.4	2021.6	0.037%
6	1886.7	1886.0	0.035%	4228.7	4227.8	0.020%	2476.9	2479	0.035%

3.3.2 Rods

Similarly, a frequency analysis using 3-D finite element method (FEM) was carried out and compared to the exact solution for the transverse and torsional modes of a rod. A similar methodology was applied by other researches for circular cylinders (Gladwell and Vijay 1975). The same sets, shown in **Table 3-1**, of elastic modulus and Poisson's ratio were used in the FEM simulations.

A total of 199,950 elements of 2.0-mm (0.08-in) cubes, 8-node linear brick with reduced integration and hourglass control elements (C3D8R) from ABAQUS program were utilized. A 75 mm (diameter) x 300 mm rod was modeled as a free-free beam. The 1st and 2nd bending modes as well as the 1st torsional mode are shown in **Figure 3-3**. A comparison between the FEM and the analytical solutions is shown in **Table 3-3**. Results show that the percent error for all the cases is less than 0.15%.

Figure 3-3. Mesh of 75 mm (D) x 300 mm (L) rod: (a) 1st bending mode: 2101.5 Hz, (b) 2nd bending mode: 4910.6 Hz, (c) 1st torsional mode: 3622.1 Hz

Table 3-3. Comparison of FEM and the analytical solutions.

Test	Transverse Mode (Hz)						Torsional Mode (Hz)		
	1 st			2 nd					
	Exact	FEM	Error (%)	Exact	FEM	Error (%)	Exact	FEM	Error (%)
1	2103.083	2101.5	0.08%	4914.196	4910.6	0.07%	3626.66	3622.1	0.13%
2	2100.394	2098.9	0.07%	4896.9	4894	0.06%	3546.943	3542.4	0.13%
3	2097.75	2096.3	0.07%	4879.969	4877.4	0.05%	3472.262	3467.8	0.13%
4	1326.551	1325.6	0.07%	3085.938	3084.3	0.05%	2195.749	2193	0.13%
5	1876.349	1875	0.07%	4364.927	4362.6	0.05%	3105.793	3101.8	0.13%
6	2297.917	2296.3	0.07%	5345.615	5342.8	0.05%	3803.585	3798.8	0.13%

The percent error is slightly higher for the rod than the rectangular prism. This might be caused by the non-uniformity of the mesh used in the rod FEM analysis. Since FEM usually requires large amounts of computations using commercial software, it would be more efficient to use the analytical solutions to determine the transverse and torsional vibration frequencies.

3.4 Frequency Analysis

3.4.1 Ratio Between 1st Bending and Torsional Mode

The ratio between the fundamental torsional frequency and the fundamental transverse frequency, χ_t can be expressed as the following relationship:

$$\chi_t = 2 \left(\frac{n''}{n_1} \right)^2 (v + 1) \quad (17)$$

where, n'' is the fundamental torsional frequency (Hz) and n_1 is the fundamental transverse frequency (Hz). For a Euler-Bernoulli beam, it can be shown that, χ_t approaches a constant value which only depends on the dimensions of the beam and its boundary conditions

(Augustyn and Marek Koziem 2015). Using the first torsional mode and the first bending mode, the relationship shown in Eq. (18) and Eq. (19) can be derived to express the ratio, χ_t for a rectangular and circular Euler-Bernoulli beam with free-free boundary conditions, which only depends on the dimensions of the beam and is independent of the Poisson's ratio.

$$\chi_t = \frac{12\pi^2 L^2}{(4.73)^4 R t^2} \quad (18)$$

$$\chi_t = \frac{16\pi^2 \left(\frac{L}{D}\right)^2}{(4.73)^4} \quad (19)$$

However, that is not the case for the vibration of a Timoshenko Beam. Using the roots of the Timoshenko frequency equation, Eq. (9), and the solution for the torsional mode, Eq. (15), the frequency ratio, χ_t is expressed as Eq. (20).

$$\chi_t = \pi^2 A L^2 / R I \gamma_1^4 \quad (20)$$

Eq. (20) was evaluated for a variety of Poisson's ratios and beam dimensions as shown in **Figure 3-4** and **Figure 3-5**. From these figures, a linear relationship between the Poisson's ratio, ν , and the ratio, χ_t can be observed. The y-intercept, B_1 , corresponds to the value of χ_t when $\nu = 0.0$. A clear linear trend can be seen after normalizing the frequency ratio, χ_t , with χ_t at $\nu = 0$ as shown in **Figure 3-5c**. It was found that the slope of the line and the y-intercept are independent of the elastic modulus and the mass density. The frequency ratio, χ_t only depends on the dimensions of the prism and rod as shown in Eq. (20). Hence, the ratio, χ_t can be expressed as Eq. (21).

$$\chi_t = 2 \left(\frac{n''}{n_1} \right)^2 (v + 1) = A_1 v + B_1 \quad (21)$$

Figure 3-4. Relationship between χ_t and Poisson's ratio with different prism dimensions (width x height x length): (a) 75 mm x 100 mm x 400 mm, (b) 10 mm x 20 mm x 160 mm and (c) 50 mm x 50 mm x 100 mm

(c)

Figure 3-5. Relationship between χ_t and Poisson's ratio with different rod dimensions (diameter x length): (a) 100 mm x 400 mm, (b) 20 mm x 160 mm and (c) normalized χ_t for different L/D ratios ($L/D = 2, 4, 8$ and 10)

Table 3-4 shows the calculated values of γ_1 and χ_t due to different Poisson's ratio and prism length/depth (L/t) ratio while maintaining a constant width/height ratio (b/t) of 0.75. As expected, **Table 3-4** shows that as the length/depth ratio becomes large the value of γ_1 approaches 4.73 which is the same value shown in Eq. (18) for a Euler-Bernoulli beam, and in this case, χ_t is no longer dependent of Poisson's ratio and approaches a constant value.

Table 3-4. Calculation of γ_1 and χ_t for different length/height ratios ($b/t = 0.75$)

Poisson's Ratio, ν	k'	$L/t = 4$		$L/t = 20$		$L/t = 50$		$L/t = 100$		$L/t = 200$	
		γ_1	X_t	γ_1	X_t	γ_1	X_t	γ_1	X_t	γ_1	X_t
0.0	0.8333	4.3609	4.0739	4.7108	74.7944	4.7269	461.1327	4.7293	1840.8997	4.729846	7359.966
0.1	0.8464	4.3526	4.1053	4.7103	74.8316	4.7268	461.1702	4.7292	1840.9374	4.72984	7360.003
0.2	0.8581	4.3443	4.1365	4.7097	74.8685	4.7267	461.2075	4.7292	1840.9746	4.729834	7360.041
0.3	0.8686	4.3362	4.1676	4.7091	74.9053	4.7266	461.2445	4.7292	1841.0117	4.729828	7360.078

0.4	0.8781	4.3281	4.1988	4.7085	74.9419	4.7266	461.2814	4.7292	1841.0486	4.729822	7360.115
0.5	0.8866	4.3202	4.2299	4.7080	74.9781	4.7265	461.3179	4.7291	1841.0852	4.729816	7360.151

The Poisson's ratio can then be calculated using the equation below:

$$v = \frac{2 \left(\frac{n''}{n_1} \right)^2 - B_1}{A_1 - 2 \left(\frac{n''}{n_1} \right)^2} \quad (22)$$

Therefore, the Poisson's ratio can be readily calculated using the experimentally measured torsional vibration frequency, n'' , and the transverse vibration frequency, n_1 . Furthermore, after knowing the Poisson's ratio, the elastic modulus can be determined using the experimental torsional frequency with Eq. (23):

$$E = 8\rho RL^2 n''^2 (1 + v) \quad (23)$$

3.4.2 Ratio Between 2nd and 1st Bending Modes

Although the method presented above can be used to determine the dynamic elastic constants easily and quickly, it has two disadvantages. First, it requires a precise measurement of the torsional mode which is difficult to excite in a cylindrical specimen (ASTM C215 2014; Wang et al. 2012). Additionally, it requires two separate experimental tests. Therefore, it would be beneficial to develop a method based on only the transverse bending modes which can be measured simultaneously and accurately. Similar to χ_t , the ratio between the 2nd bending mode and the 1st bending mode was used to directly calculate Poisson's ratio and the dynamic elastic modulus.

The ratio between the 2nd and 1st fundamental transverse frequency, χ_b can be expressed with the following relationship:

$$\chi_b = \left(\frac{n_2}{n_1}\right)^2 = \frac{\gamma_2^4}{\gamma_1^4} \quad (24)$$

where, n_2 is the 2nd fundamental transverse frequency (Hz). γ_1 and γ_2 are the roots from the frequency equation, shown in Eq. (5), for the 1st and the 2nd bending mode. For a very slender beam, the relationship shown in Eq. (25) can be derived to express χ_b for a Euler-Bernoulli beam with free-free boundary conditions.

$$\chi_b = \frac{(7.853)^4}{(4.73)^4} = 7.598 \quad (25)$$

However, for the vibration of a non-slender beam, χ_b can only be expressed with the roots of the Timoshenko frequency equation using Eq. (5). Hence, Eq. (24) was evaluated for a variety of Poisson's ratios and beam dimensions as shown in **Figure 3-6** and **Figure 3-7**. From the figures, a slightly parabolic relationship between the Poisson's ratio, ν , and the ratio, χ_b can be observed. A clear trend can be seen after normalizing the frequency ratio, χ_b , with χ_b at $\nu = 0$ as shown in **Figure 3-7c**. The y-intercept, E_1 , corresponds to the value of χ_b when $\nu = 0$. It was found that the frequency ratio, χ_b is independent of the elastic modulus and the mass density, and χ_b only depends on the roots of Eq. (5). Hence, the ratio, χ_b can be expressed as Eq. (26)

$$\chi_b = \left(\frac{n_2}{n_1}\right)^2 = C_1 \nu^2 + D_1 \nu + E_1 \quad (26)$$

C_1, D_1 and E_1 are the coefficients of the 2nd order polynomial describing the relationship between χ_b and Poisson's ratio. **Table 3-5** shows the calculated values of γ_1 .

γ_2 and χ_b due to different Poisson's ratio and length/diameter (L/D) for a rod. As expected, as the L/D ratio becomes very large, the values of γ_1 and γ_2 approach 4.73 and 7.853. They are the same values shown in Eq. (25) and correspond to a Euler-Bernoulli beam. For a very slender rod ($L/D = 200$), χ_b is no longer dependent on Poisson's ratio and approaches a constant value.

Figure 3-6. Relationship between χ_b and Poisson's ratio with different prism dimensions (width x height x length): (a) 75 mm x 100 mm x 400 mm, (b) 10 mm x 20 mm x 160 mm and (c) 50 mm x 50 mm x 100 mm

(a)

(b)

(c)

Figure 3-7. Relationship between χ_b and Poisson's ratio with different rod dimensions (diameter x length): (a) 100 mm x 400 mm, (b) 20 mm x 160 mm and (c) normalized χ_b for different L/D ratios ($L/D = 2, 4, 8$ and 10)

Table 3-5. Calculation of γ_1 , γ_2 and χ_b for different length/diameter ratios

Poisson's Ratio, ν	L/D =4			L/D = 20			L/D = 50			L/D = 100			L/D = 200		
	γ_1	γ_2	χ_b	γ_1	γ_2	χ_b	γ_1	γ_2	χ_b	γ_1	γ_2	χ_b	γ_1	γ_2	χ_b
0	4.4389	6.801	5.511	4.7157	7.789	7.441	4.7277	7.843	7.572	4.7295	7.851	7.592	4.7299	7.853	7.597
0.1	4.4331	6.776	5.460	4.7154	7.786	7.435	4.7277	7.842	7.571	4.7294	7.850	7.592	4.7299	7.853	7.597
0.2	4.4274	6.753	5.412	4.7150	7.784	7.429	4.7276	7.842	7.570	4.7294	7.850	7.591	4.7299	7.852	7.597
0.3	4.4220	6.730	5.365	4.7147	7.782	7.423	4.7276	7.842	7.569	4.7294	7.850	7.591	4.7299	7.852	7.597
0.4	4.4166	6.708	5.321	4.7143	7.780	7.418	4.7275	7.841	7.568	4.7294	7.850	7.591	4.7299	7.852	7.597
0.5	4.4114	6.686	5.278	4.7140	7.778	7.412	4.7275	7.841	7.568	4.7294	7.850	7.591	4.7299	7.852	7.597

The Poisson's ratio can then be calculated using the equation below:

$$\nu = \frac{-D_1 - \sqrt{D_1^2 - 4C_1(E_1 - \left(\frac{n_2}{n_1}\right)^2)}}{2C_1} \quad (27)$$

Therefore, Poisson's ratio can be readily calculated using the experimentally measured transverse vibration frequencies, n_2 and n_1 . Furthermore, after knowing Poisson's ratio, the elastic modulus can be determined using the 1st bending mode as shown in Eq. (28). To find γ_1 , one can use Eq. (22) and Eq. (23) and the calculated Poisson's ratio from Eq. (27). Hence, the elastic modulus can be calculated directly as:

$$E = 4\rho L^2 n_1^2 R(A_1 \nu + B_1) \quad (28)$$

The equation can also be used to determine the 1st bending mode solely from the dimensions of the beam and assumed elastic constants without solving the Timoshenko frequency equation. Using the equations presented above the dynamic elastic modulus and Poisson's ratio can be calculated using the experimental resonance frequencies if the coefficients A_1 , B_1 , C_1 , D_1 and E_1 are known beforehand.

3.5 Summary

This chapter discusses the derivation of the bending modes using the Timoshenko beam vibration equation and the torsional mode using the Euler-Bernoulli beam simplification. The equation of motion was solved using the boundary conditions of a free-free beam. In the case of the Timoshenko beam vibration a closed form solution cannot be obtained therefore the natural frequencies are expressed in terms of the eigenvalues which can be solved with numerical methods. It was shown that the exact solution matches closely with a commercial FEM software. After solving the bending and torsional modes, the ratio between the 1st bending mode and 1st torsional mode was investigated. Likewise, an examination of the ratio between the 1st and 2nd bending modes is described. Furthermore, equations are presented to allow a direct determination of the dynamic elastic modulus and Poisson's ratio.

Chapter 4 Empirical Relationships

4.1 Introduction

In order to determine the elastic constants, the coefficients A_1 , B_1 , C_1 , D_1 and E_1 are needed. In this chapter, the coefficients were tabulated for a variety of dimensions. For the rectangular prism the width/height (b/t) and the length/height (L/t) ratios were considered. For the rod, only the length/diameter (L/D) ratio must be considered. Based on the tabulated values, empirical equations will be proposed to determine the coefficients based on the stated ratios. This chapter contains two separate sections. The first describes the empirical equations for A_1 and B_1 which are used with χ_t . Likewise the second section describes the empirical equations to obtain C_1 , D_1 and E_1 .

4.2 A_1 and B_1

4.2.1 Rectangular Prism

The values of A_1 and B_1 for different specimen dimensions were found by calculating χ_t for different width/height and length/height ratios as well as varying Poisson's ratio. It is noted from Eq. (20) that the frequency ratio χ_t is only dependent on the dimensional properties of the beam and is independent of the elastic modulus, Poisson's ratio and mass density of the beam, therefore, any set of material properties would produce the same values of A_1 and B_1 . Note, that the values of χ_t and its derived A_1 and B_1 shown in this study are only applicable to the 1st bending and 1st torsional mode of a Timoshenko beam with free-free boundary conditions. The constant B_1 was found by calculating χ_t when Poisson's ratio is equal to zero. The calculated values for χ_t are shown in **Table 4-1**;

$B_1 = \chi_t$ (when $\nu = 0$). B_1 increases when either the width/height or length/height ratio increases.

Table 4-1. The values of B_1 for different length/height and width/height ratios

Width/ height	Length/height						
	2	4	8	10	20	50	100
0.100	0.08393	0.19443	0.61853	0.93502	3.56964	22.00805	87.85889
0.200	0.30414	0.70452	2.24123	3.38804	12.93459	79.74610	318.35648
0.300	0.60594	1.40364	4.46524	6.75005	25.76982	158.87963	634.26746
0.400	0.93411	2.16384	6.88359	10.40585	39.72664	244.92812	977.78387
0.500	1.24246	2.87812	9.15585	13.84081	52.84035	325.77858	1300.54908
0.600	1.49940	3.47332	11.04929	16.70310	63.76778	393.14986	1569.50370
0.700	1.68946	3.91356	12.44981	18.82025	71.85045	442.98227	1768.44095
0.750	1.75868	4.07392	12.95992	19.59138	74.79440	461.13270	1840.89974
0.800	1.81196	4.19735	13.35257	20.18494	77.06048	475.10382	1896.67420
0.900	1.87866	4.35185	13.84409	20.92797	79.89715	492.59289	1966.49279
1.000	1.91126	4.42737	14.08433	21.29113	81.28360	501.14079	2000.61709

It is found from **Table 4-1** that when B_1 values are normalized, it becomes independent of length/height ratio. In this case, B_1 was normalized using the value at a width/height ratio of 1.0, representing a square cross-section. **Figure 4-1** shows the relationship between the normalized B_1 , called $\overline{B_1}$, and the width/height, b/t , ratio which is independent of length/height ratios. A sixth order polynomial can be used to express the normalized B_1 with an $R^2 = 1.000$. The equation is shown below:

$$\overline{B}_1 = -1.430658 \left(\frac{b}{t}\right)^6 + 4.569006 \left(\frac{b}{t}\right)^5 - 2.577872 \left(\frac{b}{t}\right)^4 \quad (29)$$

$$- 4.773641 \left(\frac{b}{t}\right)^3 + 5.283568 \left(\frac{b}{t}\right)^2 - 0.073874 \left(\frac{b}{t}\right) + 0.003463$$

$$B_1 = \overline{B}_1 * B_1 \text{ at } \left(\frac{b}{t}\right) = 1.0 \quad (30)$$

After calculating $\overline{B}_1$, B_1 can be found by multiplying its respective normalization value. For example, B_1 for a width/height ratio of 0.75 and length/height ratio of 20 can be calculated as follows:

1. Calculate $\overline{B}_1$ using Eq. (29); ($b/t = 0.75$) to obtain $\overline{B}_1 = 0.92015$
2. Using **Table 4-2** obtain B_1 for a square specimen of the same length/height ratio ($L/t = 20$):
 $B_1 = 81.2836$
3. Multiply the values from steps 1 and 2 to get $B_1 = 74.7931$

Table 4-2. Value of B_1 for a width/height ratio of 1.0

Width/ height	Length/height						
	2	4	8	10	20	50	100
1.000	1.91126	4.42737	14.08433	21.29113	81.28360	501.14079	2000.61709

Figure 4-1. (a) B_1 versus width/height (b/t) ratio for the cases with different length/height ratios
(b) Normalized B_1 by B_1 at ($b/t = 1.0$) for all the cases with different width/height ratios

A similar procedure was carried out to determine the slope constant A_1 . The transverse mode and torsional mode were determined using Poisson's ratio ranging from 0.0 to 0.5 in increments of 0.05. After plotting χ_t , the best fit line was used to find A_1 . The coefficient of determination, R^2 , was above 0.999 for all the cases. **Table 4-3** shows the calculated value of A_1 for different specimen dimensions. Like B_1 , the slope, A_1 , increases when the width/height ratio and length/height ratio are enlarged. As the beam gets longer, the value of A_1 approaches a constant value, indicating that the beam behaves more like a Euler-Bernoulli slender beam. For larger length/height ratios ($L/t > 20$), when compared to B_1 , the variation of A_1 is very small. Therefore, χ_t approaches a constant value equal to only B_1 .

Table 4-3. The value of A_1 for different width/height and length/height ratios

Width/ height	Length/height						
	2	4	8	10	20	50	100

0.100	0.011936	0.015104	0.017019	0.017343	0.017827	0.017974	0.017995
0.200	0.042568	0.054725	0.061662	0.062837	0.064590	0.065124	0.065203
0.300	0.086132	0.108994	0.122812	0.125152	0.128642	0.129707	0.129863
0.400	0.132667	0.167886	0.189171	0.192776	0.198151	0.199792	0.200032
0.500	0.176157	0.222933	0.251199	0.255986	0.263125	0.265303	0.265622
0.600	0.211951	0.268255	0.302273	0.308034	0.316624	0.319249	0.319628
0.700	0.237695	0.300878	0.339043	0.345506	0.355141	0.358082	0.358511
0.750	0.246657	0.312251	0.351866	0.358573	0.368573	0.371626	0.372071
0.800	0.253177	0.320540	0.361216	0.368102	0.378369	0.381503	0.381961
0.900	0.259994	0.329266	0.371066	0.378142	0.388690	0.391909	0.392380
1.000	0.261149	0.330844	0.372879	0.379992	0.390594	0.393829	0.394303

After normalization by the square section ($b/t = 1.0$), the normalized A_1 value is plotted in **Figure 4-2**; the equation below was found from the best fit curve to obtain the normalized value, $\overline{A_1}$, for any width/height ratio.

$$\overline{A_1} = 3.19490 \left(\frac{b}{t}\right)^4 - 8.58348 \left(\frac{b}{t}\right)^3 + 6.61445 \left(\frac{b}{t}\right)^2 - 0.23720 \left(\frac{b}{t}\right) + 0.01152 \quad (31)$$

$$A_1 = \overline{A_1} * A_1 \text{ at } \left(\frac{b}{t}\right) = 1.0 \quad (32)$$

After calculating $\overline{A_1}$, A_1 can be found by multiplying by its respective normalization value. For example, A_1 for a width/height ratio of 0.75 and length/height ratio of 20, can be calculated as follows:

1. Calculate $\overline{A_1}$ using Eq. (31); ($b/t = 0.75$) to obtain $\overline{A_1} = 0.94398$
2. Using **Table 4-4**, determine A_1 for a square specimen of the same length/height ratio ($L/t = 20$): $A_1 = 0.390594$
3. Multiply the values from Steps 1 and 2 to calculate $A_1 = 0.368712$

Therefore, $\chi_t = A_1 v + B_1 = 0.368712 * 0.2 + 74.7931 = 74.867$, and the frequency ratio square can be calculated as $\frac{74.867}{2(0.2+1)} = 31.195$ for a beam with $b/t = 0.75$ and $L/t = 20$.

Table 4-4. Value of A_1 for a width/height ratio of 1.0

Width	Length/height						
/height	2	4	8	10	20	50	100
1.0	0.261149	0.330844	0.372879	0.379992	0.390594	0.393829	0.394303

Figure 4-2. (a) A_1 versus width/height (b/t) ratio for the cases with different length/height ratios
(b) Normalized A_1 by A_1 at ($b/t = 1.0$) for all the cases with different width/height ratios

As an example, FEM was performed for a steel prism with an assumed elastic modulus, Poisson's ratio and density of 200 GPa, 0.3 and 7750 kg m^{-3} , respectively. The ratio between the 1st bending mode and 1st torsional mode was calculated using the resonance frequencies obtained from FEM. Using Eq. (17) with the values of A_1 and B_1 , obtained from Eq. (29) and Eq. (31), the ratio between the frequencies can be calculated as well. **Table 4-5** shows that the first bending mode and the first torsional mode frequencies

calculated using Eq. (21), Eq. (22) and Eq. (23) compare closely with the FEM results with a maximum difference of 0.11%.

Table 4-5. Steel prism with different length/height ratio

Specimen Size	FEM		Theoretical			
	Bending Mode (Hz)	Torsional Mode (Hz)	A_1	B_1	Bending Mode (Hz)	Torsional Mode (Hz)
75 mm x 100 mm x 2000 mm	127.3	682.98	0.368712	74.79311	127.39	683.74
75 mm x 100 mm x 400 mm	2699.3	3417.9	0.312310	4.073845	2700.22	3418.71

The method described above can be used to find A_1 and B_1 for any width/height ratio provided that the value of A_1 and B_1 are known for a $b/t = 1.0$. In **Table 4-2** and **Table 4-4**, the values of A_1 and B_1 at $b/t = 1.0$ were given for a set of length/height ratios. For those cases that the length/height ratio is not tabulated in **Table 4-2** and **Table 4-4**, a set of equations were developed to calculate A_1 and B_1 in terms of the length/height ratio. Eq. (33) and Eq. (34) were developed using the analytical results from Eq. (20) and Eq. (21). The results of A_1 and B_1 at $b/t = 1.0$ versus the length/height ratio are plotted in **Figure 4-3** and **Figure 4-4**. High order polynomials were chosen to minimize the maximum percent error within 0.1% from the exact frequency ratio calculated from Eq. (20). Hence, the frequency ratio of the 1st bending mode and the 1st torsional mode for any combination of specimen dimensions can now be estimated.

Figure 4-3. Plot of B_1 (at $b/t = 1.0$) versus length/height (L/t) ratio

$$\begin{aligned}
 B_1 \left(\frac{b}{t} = 1.0 \right) = & 5.2931 \times 10^{-8} \left(\frac{L}{t} \right)^5 - 9.8630 \times 10^{-6} \left(\frac{L}{t} \right)^4 \\
 & + 5.7429 \times 10^{-4} \left(\frac{L}{t} \right)^3 + 0.18707 \left(\frac{L}{t} \right)^2 + 0.11725 \left(\frac{L}{t} \right) \\
 & + 0.92607
 \end{aligned} \tag{33}$$

Figure 4-4. Plot of A_1 (at $b/t = 1.0$) versus height/length ratio, (t/L)

$$A_1 \left(\frac{b}{t} = 1.0 \right) = -3.3844 \left(\frac{t}{L} \right)^4 + 4.5900 \left(\frac{t}{L} \right)^3 - 2.0115 \left(\frac{t}{L} \right)^2 + 0.014948 \left(\frac{t}{L} \right) + 0.39433 \quad (34)$$

To illustrate the usefulness of the above equations, the measured bending frequencies, n_1 (Hz) of eighteen steel specimens from the paper by Spinner et.al. are shown in **Table 4-6** (Spinner, Reichard, and Tefft 1960). In that paper, each set of specimens were cut from the same stock piece to maintain uniformity of the material properties; the Young's modulus and Poisson's ratio were given as 21,130 MPa and 0.285. The frequencies predicted by the proposed equations are presented in the **Table 4-7**; it can be seen that the bending frequency predictions of all the specimens are mostly within 0.1 % difference, with a maximum difference of 0.223 %.

Table 4-6. Experimental data from Spinner et.al (1960).

Specimen	Length (cm)	Height (cm)	Width (cm)	Density (g/cm ³)	n_l (Hz)
II-1	15.202	3.1496	3.1496	7.817	6411
II-2	15.202	3.1433	2.5405	7.819	6399
II-3	15.202	3.1433	1.9055	7.814	6398
II-4	15.202	3.1433	1.5875	7.817	6400
II-5	15.202	3.1433	1.43	7.814	6397
II-6	15.202	3.1433	1.2708	7.816	6399
II-7	15.202	3.1433	1.112	7.812	6399
II-8	15.202	3.1433	0.953	7.812	6394
II-9	15.202	3.1433	0.7943	7.813	6400
II-10	15.202	3.1433	0.6363	7.815	6400
II-11	15.202	3.1433	0.4773	7.811	6400
II-12	15.202	3.1433	0.3172	7.814	6396
II-2a	7.882	3.1433	2.5405	7.814	19203
II-2b	7.01	3.1433	2.5405	7.817	22934
II-3a	7.546	3.1433	1.9055	7.818	20495
II-3b	7.249	3.1433	1.9055	7.817	21789
II-10a	15.202	2.0574	0.6363	7.816	4475
II-10b	15.202	0.6426	0.6363	7.814	-

Table 4-7. Analytical calculations and comparison with Spinner et.al. experimental data

Specimen	Ratios		B_I ($b/t=1.0$)	A_I ($b/t=1.0$)	$\overline{B}_1$	B_I	$\overline{A}_1$	A_I	R	n'' (Hz)	(n''/n_I)	n_I (Hz)	% Difference
	L/t	b/t	Eq (33)	Eq (34)	Eq (29)	Eq (30)	Eq (31)	Eq (32)		Eq (23)	Eq (21)		
II-1	4.827	1.000	5.909	0.346	1.000	5.909	1.000	0.346	1.183	9804.826	1.529	6412.752	0.027%
II-2	4.836	0.808	5.928	0.346	0.952	5.643	0.972	0.336	1.243	9564.862	1.494	6400.907	0.030%
II-3	4.836	0.606	5.928	0.346	0.792	4.694	0.818	0.283	1.495	8726.009	1.363	6401.902	0.061%
II-4	4.836	0.505	5.928	0.346	0.658	3.898	0.681	0.235	1.800	7950.976	1.242	6400.977	0.015%
II-5	4.836	0.455	5.928	0.346	0.580	3.438	0.601	0.208	2.041	7467.965	1.166	6402.290	0.083%
II-6	4.836	0.404	5.928	0.346	0.496	2.940	0.515	0.178	2.386	6905.851	1.079	6401.294	0.036%
II-7	4.836	0.354	5.928	0.346	0.409	2.427	0.425	0.147	2.891	6275.102	0.980	6402.441	0.054%
II-8	4.836	0.303	5.928	0.346	0.323	1.912	0.335	0.116	3.670	5569.300	0.870	6401.829	0.122%
II-9	4.836	0.253	5.928	0.346	0.239	1.417	0.248	0.086	4.953	4793.699	0.749	6401.329	0.021%
II-10	4.836	0.202	5.928	0.346	0.163	0.964	0.169	0.058	7.279	3953.955	0.618	6402.143	0.033%
II-11	4.836	0.152	5.928	0.346	0.096	0.571	0.100	0.034	12.266	3046.702	0.475	6408.042	0.126%
II-12	4.836	0.101	5.928	0.346	0.045	0.265	0.046	0.016	26.483	2073.045	0.324	6401.741	0.090%
II-2a	2.508	0.808	2.405	0.286	0.952	2.289	0.972	0.278	1.243	18453.633	0.960	19223.110	0.105%
II-2b	2.230	0.808	2.124	0.274	0.952	2.022	0.972	0.266	1.243	20745.168	0.903	22962.887	0.126%
II-3a	2.401	0.606	2.293	0.281	0.792	1.816	0.818	0.230	1.495	17574.722	0.856	20540.628	0.223%
II-3b	2.306	0.606	2.198	0.277	0.792	1.741	0.818	0.227	1.495	18295.949	0.838	21830.853	0.192%
II-10a	7.389	0.309	12.209	0.370	0.333	4.064	0.346	0.128	3.556	5656.620	1.263	4478.203	0.072%
II-10b	23.657	0.990	113.301	0.392	0.999	113.161	1.000	0.392	1.185	9800.375	6.639	1476.203	-

4.2.2 Rods

The values of A_1 and B_1 for different rod dimensions were found by calculating χ_t for different length/diameter ratios as well as varying Poisson's ratio. The constant B_1 was found by calculating χ_t when Poisson's ratio is equal to zero. The calculated values for B_1 are shown in **Table 4-8**. A similar procedure was carried out to determine A_1 . After plotting χ_t , the slope of the best fit line was used to find A_1 . **Table 4-8** shows the calculated value of A_1 for different length/diameter (L/D) ratio.

Table 4-8. The values of B_1 and A_1 for different length/diameter ratios

	Length/diameter (L/D)								
	2	3	4	5	8	10	20	50	100
B_1	2.59893	4.25642	6.50793	9.37206	21.7075	33.0728	127.7261	790.2198	3156.257
A_1	0.26573	0.30686	0.33084	0.34551	0.36556	0.37104	0.37898	0.38135	0.38617

As shown in **Figure 4-5** and **Figure 4-6**, polynomials can be used to express the values of B_1 and A_1 . Although only nine L/D points are shown in **Table 4-8**, a total of 97 L/D data points were used to find the best-fit polynomials. Due to the wide range of B_1 , shown in **Figure 4-5**, a single polynomial will have a 3% difference between the estimated values and the theoretical values found using Eq. (20) when $\frac{L}{D} < 10$. Therefore, to reduce the difference to less than 0.06%, two polynomials (sixth order) are needed to express B_1 for two separate intervals, Eq. (35) and Eq. (36).

(a)

(b)

Figure 4-5. Plot of B_1 versus the length/diameter ratio: (a) $\frac{L}{D} \leq 10$ and (b) $\frac{L}{D} > 10$

Figure 4-6. Plot of A_1 versus diameter/length (D/L) ratio

For $\frac{L}{D} \leq 10$

$$\begin{aligned}
 B_1 = & -2.20790 \cdot 10^{-6} \left(\frac{L}{D}\right)^6 + 9.69895 \cdot 10^{-5} \left(\frac{L}{D}\right)^5 - 1.78941 \cdot 10^{-3} \left(\frac{L}{D}\right)^4 \\
 & + 0.0179955 \left(\frac{L}{D}\right)^3 + 0.208881 \left(\frac{L}{D}\right)^2 + 0.368369 \left(\frac{L}{D}\right) \\
 & + 0.908491
 \end{aligned} \tag{35}$$

For $\frac{L}{D} > 10$

$$\begin{aligned}
 B_1 = & -2.39473 \cdot 10^{-13} \left(\frac{L}{D}\right)^6 + 1.38944 \cdot 10^{-10} \left(\frac{L}{D}\right)^5 - 3.08859 \cdot 10^{-8} \left(\frac{L}{D}\right)^4 \\
 & + 3.34001 \cdot 10^{-6} \left(\frac{L}{D}\right)^3 + 0.315287 \left(\frac{L}{D}\right)^2 + 4.89786 \\
 & \cdot 10^{-3} \left(\frac{L}{D}\right) + 1.49164
 \end{aligned} \tag{36}$$

Unlike B_1 , only a single polynomial expression is needed for A_1 . Eq. (37) was found from the best fit curve shown in **Figure 4-6**. Eq. (37) can be used to obtain A_1 for any diameter/length (D/L) ratio.

$$\begin{aligned}
 A_1 = & 9.44922 \left(\frac{D}{L}\right)^6 - 13.8392 \left(\frac{D}{L}\right)^5 + 5.60087 \left(\frac{D}{L}\right)^4 + 1.08665 \left(\frac{D}{L}\right)^3 \\
 & - 1.27914 \left(\frac{D}{L}\right)^2 + 5.47273 \cdot 10^{-3} \left(\frac{D}{L}\right) + 0.381757
 \end{aligned} \tag{37}$$

4.3 C_1, D_1 and E_1

4.3.1 Rods

The coefficients C_1, D_1 and E_1 for different specimen dimensions were found by calculating χ_b for different length/diameter ratios as well as varying Poisson's ratio. It is noted from Eq. (24) that the frequency ratio χ_b is only dependent on the roots of the Timoshenko frequency equation and is independent of the elastic modulus and mass density, therefore, any set of material properties would produce the same values of C_1, D_1 and E_1 . The values of χ_b and its derived C_1, D_1 and E_1 shown are only applicable to the 1st and 2nd bending mode of a Timoshenko rod with free-free

boundary conditions. The constant E_1 was found by calculating χ_b when Poisson's ratio is equal to zero. The calculated values for E_1 are shown in **Table 4-9**.

A similar procedure was carried out to determine the coefficients C_1 and D_1 . The transverse modes were determined using Poisson's ratio ranging from 0.0 to 0.5 in increments of 0.05. After plotting χ_b , the best fit second-order polynomial was used to find C_1 and D_1 . The coefficient of determination, R^2 , was above 0.999 for all the cases. **Table 4-9** shows the calculated values for different specimen dimensions. Unlike E_1 , the coefficients decrease when the length/diameter ratio is enlarged. As the rod gets longer, the value of C_1 and D_1 approach zero, indicating that the rod behaves more like a Euler-Bernoulli slender rod. For larger length/diameter ratios ($L/D > 20$), χ_b approaches a constant value equal to only E_1 .

Table 4-9. The values of C_1 , D_1 and E_1 for different length/diameter ratios

Coefficient	Length/diameter (L/D)								
	2	3	4	5	8	10	20	50	100
E_1	3.69886	4.282677	5.51093	5.99205	6.78228	7.03545	7.44104	7.57243	7.59195
D_1	-0.83478	-0.60134	-0.51647	-0.44455	-0.27264	-0.19948	-0.06117	-0.01043	-0.00263
C_1	0.22969	0.12914	0.10186	0.08175	0.04227	0.02862	0.00748	0.00120	0.0002986

As shown in **Figure 4-7**, **Figure 4-8** and **Figure 4-9**, the three coefficients can be estimated using polynomial equations, and to increase the accuracy of the proposed method, two intervals are specified for C_1 and D_1 . A total of 97 L/D data points were used to determine the polynomials. The proposed equations can be used to find the coefficients for any $L/D \geq 2$ and allows for the calculation of the elastic constants after knowing the 1st and 2nd bending mode as well as the rod's dimensions. The 6th order polynomials were found to have a maximum percent difference of 0.12% with the theoretical values. On average the percent difference was 0.01% indicating that the sixth-order polynomials can be used adequately to estimate the coefficients.

Figure 4-7. Plot of C_1 versus the diameter/length (D/L) ratio.

Figure 4-8. Plot of D_1 versus the diameter/length (D/L) ratio.

Figure 4-9. Plot of E_1 versus the diameter/length (D/L) ratio.

$$\text{For } \frac{L}{D} \leq 10$$

$$C_1 = 112.284 \left(\frac{D}{L}\right)^6 - 223.341 \left(\frac{D}{L}\right)^5 + 185.884 \left(\frac{D}{L}\right)^4 - 77.1737 \left(\frac{D}{L}\right)^3 \quad (38)$$

$$+ 15.7309 \left(\frac{D}{L}\right)^2 - 0.952765 \left(\frac{D}{L}\right) + 2.73114 \cdot 10^{-2}$$

$$D_1 = -340.388 \left(\frac{D}{L}\right)^6 + 717.045 \left(\frac{D}{L}\right)^5 - 588.110 \left(\frac{D}{L}\right)^4 + 224.277 \left(\frac{D}{L}\right)^3 \quad (39)$$

$$- 35.7879 \left(\frac{D}{L}\right)^2 - 0.603707 \left(\frac{D}{L}\right) + 0.0468$$

$$\text{For } \frac{L}{D} > 10$$

$$C_1 = 2541.87 \left(\frac{D}{L}\right)^6 - 920.988 \left(\frac{D}{L}\right)^5 + 72.8415 \left(\frac{D}{L}\right)^4 - 2.22046 \left(\frac{D}{L}\right)^3 \quad (40)$$

$$+ 3.02631 \left(\frac{D}{L}\right)^2 - 3.61314 \cdot 10^{-4} \left(\frac{D}{L}\right) + 1.08041 \cdot 10^{-6}$$

$$D_1 = 2744.60 \left(\frac{D}{L}\right)^6 - 4196.55 \left(\frac{D}{L}\right)^5 + 1132.34 \left(\frac{D}{L}\right)^4 - 11.5492 \left(\frac{D}{L}\right)^3 - 26.1754 \left(\frac{D}{L}\right)^2 - 1.98589 \cdot 10^{-3} \left(\frac{D}{L}\right) + 6.00293 \cdot 10^{-6} \quad (41)$$

A single sixth-order polynomial can be used to estimate E_1 . The equation is shown below:

$$E_1 = 39.7722 \left(\frac{D}{L}\right)^6 + 269.519 \left(\frac{D}{L}\right)^5 - 530.656 \left(\frac{D}{L}\right)^4 + 350.357 \left(\frac{D}{L}\right)^3 - 96.4206 \left(\frac{D}{L}\right)^2 + 1.08854 \left(\frac{D}{L}\right) + 7.58933 \quad (42)$$

As an example, FEM was performed for a steel rod with an assumed elastic modulus, Poisson's ratio and density of 200 GPa, 0.3 and 7750 kg m^{-3} , respectively. The values of A_1 and B_1 were obtained from Eq. (35) to Eq. (37) while C_1, D_1 and E_1 were obtained using Eq. (38) to Eq. (42). **Table 4-10** shows that the 1st and 2nd bending mode, and the 1st torsional mode frequencies calculated using Eq. (22), Eq. (23) and Eq. (28) compare closely with the FEM results with a maximum difference of 0.36%.

Table 4-10. Steel rod with different length/diameter ratio.

Size (mm x mm)	FEM			Theoretical							
	Bending (Hz)		Torsional (Hz)	A_I	B_I	C_1	D_1	E_1	Bending (Hz)		Torsional (Hz)
	1 st	2 nd							1 st	2 nd	
100 x2000	112.09	305.44	784.8	0.381786	127.7267	0.00748	-0.06116	7.4433	112.32	306.079	787.62
100 x400	2468.2	5718.0	3932.7	0.330818	6.50796	0.10188	-0.51670	5.5142	2470.39	5723.82	3938.11

4.3.2 Rectangular Prism

The coefficients for different b/t and L/t ratios for a rectangular prism are shown in **Table 4-11** to **Table 4-13**. The coefficients were found by evaluating Eq. (34) and fitting the best-fit

second-order polynomial. Like the rod case, the coefficients are solely dependent on the dimensions of the prism. From the tables as the beam gets slenderer, the value of C_1 and D_1 approach zero. Therefore, χ_b becomes a constant equal to only E_1 . It should be noted that E_1 is only dependent on the length/height ratio unlike the other two coefficients. Similar to the method shown in Section 4.2.1, the coefficients were normalized based on the $b/t=1.0$ values.

Table 4-11. Values of C_1 for different rectangular prism

Width/ height	Length/height						
	2	4	8	10	20	50	100
0.1	0.368982	0.0883	0.027942	0.015773	0.001645	4.98E-05	3.23E-06
0.2	0.369076	0.088384	0.027997	0.015815	0.001659	5.25E-05	3.89E-06
0.3	0.369468	0.088734	0.028226	0.015994	0.001721	6.34E-05	6.68E-06
0.4	0.370479	0.089639	0.028817	0.016454	0.00188	9.17E-05	1.39E-05
0.5	0.372523	0.091465	0.030008	0.017382	0.0022	0.000149	2.83E-05
0.6	0.376077	0.094636	0.032077	0.018993	0.002756	0.000248	5.35E-05
0.7	0.381673	0.099614	0.035322	0.021521	0.003627	0.000403	9.29E-05
0.75	0.385412	0.102931	0.037483	0.023203	0.004207	0.000506	0.000119
0.8	0.389876	0.106881	0.040055	0.025206	0.004897	0.000629	0.00015
0.9	0.401277	0.116923	0.046586	0.030289	0.006648	0.00094	0.000229
1	0.416492	0.130217	0.055217	0.037004	0.008959	0.001351	0.000334

Table 4-12. Values of D_1 for different rectangular prisms

Width/ height	Length/height						
	2	4	8	10	20	50	100
0.1	-1.15596	-0.67323	-0.39308	-0.29698	-0.09656	-0.01681	-0.00426
0.2	-1.15593	-0.67321	-0.39307	-0.29697	-0.09656	-0.01681	-0.00426
0.3	-1.1558	-0.67315	-0.39304	-0.29695	-0.09655	-0.01681	-0.00426
0.4	-1.15548	-0.673	-0.39296	-0.29689	-0.09653	-0.01681	-0.00425

0.5	-1.15483	-0.6727	-0.3928	-0.29677	-0.0965	-0.0168	-0.00425
0.6	-1.15371	-0.67218	-0.39253	-0.29657	-0.09644	-0.01679	-0.00425
0.7	-1.15195	-0.67136	-0.3921	-0.29625	-0.09634	-0.01677	-0.00425
0.75	-1.15078	-0.67082	-0.39181	-0.29604	-0.09627	-0.01676	-0.00424
0.8	-1.14939	-0.67018	-0.39147	-0.29579	-0.0962	-0.01675	-0.00424
0.9	-1.14589	-0.66854	-0.3906	-0.29516	-0.096	-0.01672	-0.00423
1	-1.1413	-0.6664	-0.38946	-0.29432	-0.09575	-0.01667	-0.00422

Table 4-13. Values of E_1 for different rectangular prisms

Width/ height	Length/height						
	2	4	8	10	20	50	100
0.1	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.2	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.3	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.4	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.5	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.6	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.7	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.75	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.8	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
0.9	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896
1	3.151485	5.153886	6.564285	6.871066	7.3881	7.5633	7.5896

The normalized values were plotted in **Figure 4-10** to **Figure 4-11**. Unlike A_1 and B_1 , no clear trend exists for C_1 and D_1 . Therefore, empirical equation cannot be formulated. However, the coefficients can be found for any specimen following the steps below:

1. Measure the specimen dimensions.
2. Assume a dynamic elastic modulus (Note: any combination of elastic constants would produce the same values for C_1 , D_1 and E_1).

3. Solve the Timoshenko frequency equation and determine the values of γ_1 and γ_2 for different Poisson's ratio (preferable in intervals of 0.05).
4. Calculate χ_b and plot it versus its respective Poisson's ratio.
5. Find the best-fit second order polynomial using χ_b at $\nu=0$ as the y-intercept.
6. The coefficients of the second-order polynomial are C_1 , D_1 and E_1 .

Figure 4-10. Normalized C_1 for rectangular prisms

Figure 4-11. Normalized D_1 for rectangular prism

4.4 Summary

In this chapter empirical equations were presented for the coefficients A_1 , B_1 , C_1 , D_1 and E_1 . It was found that they were independent of the elastic modulus and density and therefore any set would produce the same values. Additionally, the equations presented were non-dimensionalized based on the width/height, length/height and length/diameter ratios. Therefore, the coefficient could be found for any rectangular or circular Timoshenko beam. However, for the rectangular prism empirical equations for C_1 , D_1 and E_1 could not be found. Nevertheless, these coefficients can be determined based solely from the dimensions of the rectangular prism before conducting the experimental tests. It was found that the empirical equations could be used to verify the bending and torsional modes of a steel beam with a maximum

difference of 0.36%. Two separate methods were proposed, Method 1 uses χ_t while Method 2 employs χ_b . First, the dimensions and mass of the specimen must be measured. Next, the required coefficients should be determined based on the chosen method. Then, using the measured resonance frequencies, the Poisson's ratio and the elastic modulus can be calculated.

Chapter 5 Experiments

5.1 Introduction

The methods presented in Chapter 3 were used to measure the elastic modulus and Poisson's ratio of three rectangular prisms and rods. In this chapter the experimental setup will be explained in detail. First, an overview of the testing will be given followed by an explanation of the experimental data. This chapter also discusses the results of the experimental program.

5.2 Experimental Program

Three concrete mix designs used to produce the prism specimens for the experiments can be found in **Table 5-1**. The measured air content and slump can be found in

Table 5-2. The concrete specimens were moisture-cured in a 23 °C water tank concurrently to ensure equivalent maturity of the samples. The size of the concrete prism specimen was 75 mm x 100 mm x 400 mm. Three concrete prism specimens were cast for each mix design.

Table 5-1. Mix designs

Material	Unit Weight (kg m ⁻³)		
	50% Slag	30% Fly Ash	OPC
Cement	151	202	335
Slag	151	-	-
Fly Ash	-	100	-
Sand	809	807	845
#57 Aggregate	1065	-	969
#67 Aggregate	-	1056	-
Water	128	128	139
W/Cem Ratio	0.42	0.42	0.42

Table 5-2. Fresh properties for concrete batches

Concrete Type	Air Content (%)	Slump (mm)	Density (kg m ⁻³)
50% Slag	4.3%	170	2370
30% Fly ash	7.6%	130	2453
OPC	11.0%	210	2433

The dynamic modulus of elasticity was measured using the impact resonance frequency test. **Figure 5-1** shows the instrument used in the experiment. The supports were placed at the nodal locations to allow free vibration in the fundamental transverse mode with free-free boundary condition. The accelerometer was attached to the specimen centered at 40 mm from the edge of the specimen using a hot glue adhesive and the impact point was located at the center of the top face of the prism. For each test, the acceleration time history was plotted on a LabView waveform analyzer. Using fast Fourier transform (FFT) the time domain was converted to the frequency domain.

Figure 5-1. Impact hammer, accelerometer, and specimen

The torsional frequency was also tested on the concrete prisms. The support was placed at the center of the specimen which corresponds to the nodal point of the specimen. Similar to the

transverse frequency testing the accelerometer was placed with hot glue adhesive 15 mm from the top edge on the side surface as shown in **Figure 5-2**.

Figure 5-2. Typical setup of torsional mode testing

An aluminum, steel and concrete rod with different length/diameter ratio were experimentally tested to validate the proposed methods. The density and specimen dimensions can be found in **Table 5-3**.

Table 5-3. Material properties of rods.

Material	Properties		
	Diameter (mm)	Length (mm)	Density (kg m ⁻³)
Aluminum	63.50	736.60	2780.56
Steel	46.83	279.4	8143.13
Concrete	99.63	622.30	2288.75

The impact resonance frequency method was used to measure the fundamental transverse and torsional modes. The rods were suspended with soft flexible rubber tubing placed at

approximately $0.15L$ to allow free vibration in the fundamental transverse and torsional mode as shown in **Figure 5-3**. The accelerometer was attached to the specimen centered at $0.55L$ from the edge of the specimen using a hot glue adhesive. The torsional frequency was also tested for each rod. The supports were kept at the same location as the transverse mode experiment. The accelerometer was placed with hot glue adhesive on a rigid metal tab which was located $0.223L$ from the edge of the specimen. A tangential impact was made using another metal tab located $0.13L$ from the other edge. A schematic of the torsional mode testing can be found in ASTM C215. For each test, the acceleration time history was plotted on a LabView waveform analyzer. Using fast Fourier transform (FFT) the time domain was converted to the frequency domain.

Figure 5-3. Picture of experiment (a) Impact hammer, accelerometer, and specimens (b)

Experimental set-up.

5.3 Rectangular Prism

Typical acceleration time histories for the impact and response before and after performing the FFT are shown in **Figure 5-4** and **Figure 5-5**. The excitation frequency range for the transverse and torsional impact is approximately 4,500 Hz. To find the peak frequency, the frequency

response function, which is the ratio between the output response and the input excitation force in the frequency domain, was calculated (Chen and Kiriakidis 2000). An example of the peak amplitude is shown in **Figure 5-6**. As the concrete matures the resonance frequency shifts towards the right due to its gain in stiffness; the specimens shown below were tested at an age of about 12 months. Based on **Figure 5-6**, the 50% slag mix achieves a higher stiffness than both the OPC and fly ash mix designs.

Figure 5-4. (a) Typical acceleration time history and (b) Typical response power spectrum (transverse mode)

Figure 5-5. (a) Typical impact loading time history and (b) Typical impact power spectrum (transverse mode)

Figure 5-6. Example of amplitude vs. frequency (transverse mode)

Similarly, **Figure 5-7**, **Figure 5-8** and **Figure 5-9**, present the typical acceleration time histories and the frequency response function for the torsional mode. A clear peak can be seen in all the specimens. Likewise, the 50% slag replacement mix shows a higher torsional frequency than the two other mix designs.

Figure 5-7. (a) Typical acceleration time history and (b) Typical response power spectrum density (torsional mode)

Figure 5-8. (a) Typical impact loading time history and (b) Typical impact power spectrum (torsional mode)

Figure 5-9. Example of amplitude vs. frequency (torsional mode)

Table 5-4 presents the average measured frequencies for the fundamental transverse and torsional frequencies of three typical specimens. The average value was determined from at least four tests. The sampling frequency for both impact force and acceleration response was 25.6 kHz, and the resolution of the measured frequencies was 0.25 Hz. Due to the placement of the

accelerometer, the second bending mode was not measured. Therefore, only Method 1 could be used on the rectangular prism.

The values given for A_1 and B_1 can be obtained for the prism specimen with a width/height ratio of 0.75 and a length/height ratio of 4. In this case for a 75 mm x 100 mm x 400 mm prism specimen, the value of A_1 and B_1 are 0.312251 and 4.07392, respectively. Using Eq. (22), the Poisson's ratio for the three different mix designs was calculated and shown in **Table 5-4**. The calculated values for the dynamic elastic modulus using Eq. (23), are also shown in **Table 5-4**. As expected, the 50% slag replacement mix design had the largest elastic modulus. Since the measured natural frequencies had a resolution of 0.25 Hz; the resolution of the calculated Poisson's ratio and the elastic modulus are approximately 0.0005 and 1.5 MPa.

Table 5-4. Calculated properties of three different mix designs

Mix Design	Transverse Frequency (Hz)	Torsional Frequency (Hz)	ν		Shape Factor (k)		E (MPa)	
			Theoretical Method	ASTM C215	Theoretical Method	ASTM C215	Theoretical Method	ASTM C215
50% Slag	2,211.69±0.43	2,903.59±0.71	0.2000	0.2033	0.8581	0.889	42,150	42,267
30% Fly Ash	2,012.51±0.33	2,699.67±0.82	0.1445	0.1460	0.8518	0.889	34,468	34,513
OPC	2,038.40±1.07	2,714.89±0.94	0.1627	0.1645	0.8539	0.889	34,486	34,541

5.3 Rods

Typical acceleration time histories for the impact and response before and after performing the FFT are shown in **Figure 5-10** and **Figure 5-11**. The excitation frequency range for the transverse and torsional impact is approximately 5,000 Hz. To find the peak frequency, the frequency response function, which is the ratio between the output response and the input excitation force in

the frequency domain, was calculated (Chen Hung-Liang “Roger” and Kiriakidis Alejandro C. 2000). Four peaks can be seen in the output response; these peaks correspond to the first four fundamental bending modes. In this study, only the 1st and 2nd bending mode were analyzed however the 3rd and 4th bending modes could be used as verification of the measured elastic constants.

Figure 5-10. (a) Typical acceleration time history and (b) Typical response power spectrum (Transverse Mode).

Figure 5-11. (a) Typical impact loading time history and (b) Typical impact power spectrum (Transverse Mode).

Similarly, **Figure 5-12** and **Figure 5-13**, present the typical acceleration time histories and the frequency response function for the torsional mode. A clear peak can be seen in all the specimens

Figure 5-12. (a) Typical acceleration time history and (b) Typical response power spectrum density (Torsional Mode).

Figure 5-13. (a) Typical impact loading time history and (b) Typical impact power spectrum density (Torsional Mode).

Table 5-5 presents the average measured frequencies for the fundamental transverse and torsional frequencies of the three rods. The average value was determined from at least four tests. The values for A_1 , B_1 , C_1 , D_1 and E_1 can be obtained for each rod specimen using their respective L/D ratio. Using Eq. (22) and Eq. (27), the Poisson's ratio for the three rods was calculated and shown in **Table 5-5**. The calculated values for the dynamic elastic modulus using Eq. (23) and Eq. (28), are also shown in **Table 5-5**. **Table 5-5** shows a comparison between Method 1 and Method 2. Method 1 uses χ_t to calculate Poisson's ratio and elastic modulus. As expected, Method 2, which uses the 1st and 2nd bending modes, produces more reasonable elastic constants. This is mainly due to the inaccuracy of the experimental torsional mode frequency. Note, the largest difference occurs in the Poisson's ratio while the elastic modulus for both methods are almost identical.

Table 5-5. Calculated properties of an aluminum, steel and concrete rod.

Rod	Transverse Frequency		Torsional Frequency	ν		E (GPa)	
	(Hz)			Method 1	Method 2	Method 1	Method 2
	1 st	2 nd					
Aluminum	531.05±0.03	1415.85±0.07	2,083.02±0.45	0.434	0.364	75.12	75.08
Steel	2589.83±0.33	6460.95±0.43	5686.97±0.87	0.332	0.300	219.10	218.91
Concrete	847.02±0.20	2130.61±0.43	1972.06±0.77	0.283	0.241	35.38	35.35

Chapter 6 Discussion and Conclusion

6.1 Introduction

In this chapter, the results from the experimental program are discussed in depth. A comparison between ASTM C215 and Method 1 for the rectangular prism is made. Additionally, a sensitive analysis is conducted for Method 1 and Method 2 for the rod specimens. Finally, in the last section of the chapter future work is recommended.

6.2 Discussion

The frequency ratio between the 1st torsional mode and the 1st bending mode, χ_t , was found to be linearly proportional to the Poisson's ratio with a slope, A_1 , and a y-intercept, B_1 , whereas the frequency ratio between the first two bending modes, χ_b , was slightly parabolic with coefficients C_1 , D_1 , and E_1 . It was found that the constants A_1 , B_1 , C_1 , D_1 , and E_1 depend only on the dimensions of the specimen and can be tabulated for different specimen sizes. Values for A_1 , B_1 , C_1 , D_1 , and E_1 at different length/diameter ratios were found, allowing the use for any rod dimensions. As the length/diameter ratio increased, the value of A_1 could be ignored and χ_t approached to a constant. In other words, χ_t is independent of Poisson's ratio at very large length/diameter ratios. Likewise, as the length/diameter increases, the values of C_1 and D_1 approach to zero, and χ_b approaches a constant value of 7.598. Equations for the elastic modulus, E , and Poisson's ratio, ν , were proposed in this paper using the tabulated values of A_1 , B_1 , C_1 , D_1 , and E_1 , allowing for a quick calculation of both E and ν . The proposed methods do not require any iterations to solve for the material constants. Results show that, using the equations

proposed in this study, the Young's modulus and Poisson's ratio of a rod can be determined from the measured frequency ratio quickly and efficiently.

6.2.1 Comparison Between ASTM and Exact Solution

Through an iterative process the elastic modulus and Poisson's ratio can be calculated using ASTM C215. Iterations are necessary because the correction factor found in ASTM C215, T , must be found using an assumed Poisson's ratio. After which, Poisson's ratio can be calculated using the relationship between the elastic modulus and shear modulus. Both the elastic modulus and shear modulus can be calculated using the experimental resonance frequencies. The calculated Poisson's ratio can then be compared to the assumed value used to find the correction factor, T . Subsequently, T is recalculated using the calculated Poisson's ratio until the calculated and assumed values for Poisson's ratio converge. Typically, convergence is achieved after five iterations. Following the ASTM process, the elastic modulus and Poisson's ratio were calculated for the three different mix designs and the results are shown in **Table 5-4**. **Table 5-4** shows the comparison between the ASTM method results and the elastic modulus and Poisson's ratio obtained using the equations proposed by this paper. Both methods produce similar values, however, ASTM C215 method requires iterations and is based on an assumed constant shape factor, $k' = 8/9$. In fact, k' as shown in Eq. (6) and Eq. (7) is not a constant, and in the frequency equation k' is needed to calculate the roots. The shape factor for the different mix designs is also shown in **Table 5-4**. There is a slight variation for the shape factor since the mix designs have different Poisson's ratio.

6.2.2 Sensitivity Analysis

In this study, the diameter and length of the specimens were measured with a resolution of 0.01 mm, while the mass had a resolution of 0.01 kg. A sensitivity analysis of the proposed methods was conducted using the experimental resolution. The experimental frequencies were measured with a resolution of 0.25 Hz, which was achieved using a signal processing technique. It was found that Poisson's ratio and the elastic modulus had a maximum deviation of 0.3 ± 0.001 and 218.91 ± 0.2 GPa, respectively, when changing the diameter or length by 0.01 mm. Therefore, it is recommended to at least have a resolution of 0.1 mm in order to achieve an accuracy of Poisson's ratio with ± 0.01 deviation. It was also found that a mass change of 0.01 kg can influence the elastic modulus by ± 0.5 GPa. Furthermore, if only a 0.25 Hz frequency resolution is obtainable, the estimated Poisson's ratio can have a large deviation of ± 0.05 when a specimen has a large L/D like the aluminum specimen ($L/D = 11.6$). This is because, as shown in **Figure 3-7** and **Table 3-5**, Poisson's ratio is not sensitive to the change of the bending frequency ratio χ_b when L/D becomes large. During the experiment, the concrete and steel rods both showed high accuracy in the measurement of Poisson's ratio using the bending modes as displayed in **Table 6-1**. However, since the aluminum rod has an $L/D = 11.6$, a frequency resolution of ± 0.05 Hz was used to reduce the Poisson's ratio deviation to ± 0.009 . For an accurate measurement of the Poisson's ratio, it is recommended that an $L/D \cong 6$ be used when using χ_b .

Table 6-1. Sensitivity analysis of the aluminum, steel, and concrete specimens.

Measured Properties	Method	Aluminum		Steel		Concrete	
		E (GPa)	ν	E (GPa)	ν	E (GPa)	ν
Mass (± 0.01 kg)	1	75.12 ± 0.1	-	219.10 ± 0.5	-	35.38 ± 0.03	-
	2	75.08 ± 0.1	-	218.91 ± 0.5	-	35.35 ± 0.03	-
L (± 0.01 mm)	1	75.12 ± 0.003	0.434 ± 0.00004	219.10 ± 0.02	0.332 ± 0.00009	35.38 ± 0.002	0.283 ± 0.00004
	2	75.08 ± 0.003	0.364 ± 0.00008	218.91 ± 0.02	0.300 ± 0.0002	35.35 ± 0.002	0.241 ± 0.00009
	1	75.12 ± 0.05	0.434 ± 0.0004	219.10 ± 0.2	0.332 ± 0.0005	35.38 ± 0.001	0.283 ± 0.0002

D (± 0.01 mm)	2	75.08 ± 0.05	0.364 ± 0.0009	218.91 ± 0.2	0.300 ± 0.001	35.35 ± 0.001	0.241 ± 0.0005
n_1 (± 0.25 Hz)	1	75.12 ± 0.07	0.434 ± 0.001	219.10 ± 0.04	0.332 ± 0.0003	35.38 ± 0.02	0.283 ± 0.0008
	2	75.08 ± 0.1 *	0.364 ± 0.05 *	218.91 ± 0.06	0.300 ± 0.004	35.35 ± 0.03	0.241 ± 0.01
n_2 (± 0.25 Hz)	1	-	-	-	-	-	-
	2	75.08 ± 0.01 **	0.364 ± 0.02 **	218.91 ± 0.008	0.300 ± 0.001	35.35 ± 0.004	0.241 ± 0.004
n'' (± 0.25 Hz)	1	75.12 ± 0.07	0.434 ± 0.001	219.10 ± 0.001	0.332 ± 0.0001	35.38 ± 0.0003	0.283 ± 0.0003
	2	-	-	-	-	-	-

* For a frequency resolution of ± 0.05 Hz for n_1 , $E = 75.08 \pm 0.02$ GPa and $\nu = 0.364 \pm 0.009$.

** For a frequency resolution of ± 0.05 Hz for n_2 , $E = 75.08 \pm 0.002$ GPa and $\nu = 0.364 \pm 0.004$.

6.3 Conclusion

This thesis presents a method to calculate the Poisson's ratio and dynamic elastic modulus using experimentally measured transverse and torsional modes. The method is based on the exact solution to the frequency equation for a Timoshenko beam. Finite element method using an ABAQUS program was shown to match with less than 0.15% with the exact solution for both the transverse and torsional mode. The frequency ratio, χ_t , was found to be linearly proportional to the Poisson's ratio with a slope, A_1 , and a y-intercept, B_1 . It was found that the constants A_1 and B_1 depend only on the dimensions of the specimen and can be tabulated for different specimen sizes. Values for A_1 and B_1 at different width/height and length/height ratios were found allowing the use of different specimen geometries. As the length/height ratio increases, the value of A_1 could be ignored and χ approached a constant, and in other words, χ_t is independent from Poisson's ratio at very large length/height ratios. An equation for the elastic modulus, E , and Poisson's ratio, ν , was proposed in this paper using the tabulated values of A_1 and B_1 allowing for a quick calculation of both E and ν . Unlike ASTM C215, the proposed method does not require any iterations to solve for the material constants. The proposed equations can be used to calculate the frequency ratio of rectangular prisms with different dimension sizes and was used to predict the measured bending frequencies from seventeen steel specimens with different length/height and width/height ratios.

The results show the differences between the prediction and the measured bending frequencies of all the specimens are mostly within 0.1 %, with a maximum difference of 0.223 %.

The frequency ratio between the 1st bending mode and the 1st torsional mode or the frequency ratio between the 1st bending mode and the 2nd bending mode for any rod with a length-to-diameter ratio of $L/D \geq 2$ can be directly estimated using the proposed equations. The proposed method does not require any iterations to solve for the material constants and can be used to calculate the elastic constants of rods with different dimension sizes. The proposed equations were used to verify the elastic constants of a steel rod with less than 0.36% error percentage, and the equations were also used to measure elastic constants of three rods with different material properties (aluminum, steel, and concrete) using simple impact vibration. It was found that the torsional mode is more difficult to excite and not as reliable as the bending modes, whereas a more accurate measurement can always be obtained from using the 1st and 2nd bending modes which were measured simultaneously.

6.4 Future Work

This study focused exclusively on the free-free vibration of rectangular prisms and rods. In some cases, it may not be possible to excite the free-free transverse and torsional modes. Therefore, to have a wider range of implementation other boundary conditions should be investigated. Similarly, other cross-section can be investigated however prisms and rods are the most common specimens. Although not stated explicitly, this method can be used to estimate the bending and torsional mode with given specimen dimensions and elastic constants. Therefore, it may be used to estimate the free-free vibration of reinforced concrete beams. However, more work should be done to verify its applicability.

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Chen, H.-L.R.; Leon, G. Direct Determination of Dynamic Elastic Modulus and Poisson's Ratio of Rectangular Timoshenko Prisms. *Journal of Engineering Mechanics* **2019**, *145*(9), p. 04019071.

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