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Lylia Benhacine

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OPTICAL ABERRATIONS AND THEIR EFFECT ON THE CENTROID LOCATION OF UNRESOLVED OBJECTS

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Thesis submitted to the
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ABSTRACT

Optical Aberrations and their Effect on the Centroid Location of Unresolved Objects

Lydia Benhacine

Optical data from point sources of light plays a key role in systems from many different industries. In real optical systems, the photons from point sources are distributed in a pattern on the focal plane. This pattern is described by the Point Spread Function (PSF). Centroiding, defined here as calculating the center of a PSF, provides data that can subsequently be used in various state estimation problems. Thus, it is crucial to determine expected centroid accuracy so that uncertainty can be accounted for. Manufacturing defects and real-world effects affect the structure of a PSF. Much of this structure is often well-captured by the five classic optical aberrations. This thesis presents a novel framework for simulating PSFs affected by optical aberrations, then uses this simulation to quantify the accuracy of various centroiding algorithms in the presence of optical aberrations. The method of PSF simulation leverages Johnson distributions, which is a method of simulating distributions as a function of the desired mean, variance, skewness, and kurtosis. This method is chosen because of the flexibility and ease-of-visualization it provides the analyst. In addition, the analytic true centers of the Johnson distributions have been derived for use in analysis as the true centroid. The three centroiding algorithms analyzed are the Center of Intensity (COI) algorithm, the Cross-Correlation (CC) algorithm, and the Model Matching (MM) algorithm. It is found that in the case of symmetric, Gaussian PSFs, all three algorithms find the true centroid exactly. However, performance deteriorates for an asymmetric PSF, especially for the CC and MM algorithm. This is because both of these algorithms assume a 2D Gaussian structure. When that assumption no longer holds, the algorithms do not perform as well. This work concludes that PSFs highly affected by asymmetric optical aberrations should expect lower centroiding accuracy when using the CC and MM algorithms.

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Acronyms

ASEL	Applied Space Exploration Laboratory
NASA	National Aeronautics and Space Administration
WVU	West Virginia University
PSF	Point Spread Function
FOV	Field of View
LOS	Line-of-Sight
COI	Center of Intensity
CC	Cross-Correlation
MM	Model Matching
PDF	Probability Density Function

Nomenclature

Symbol	Units	Description
μ		mean of a distribution
σ^2		variance of a distribution
μ_3		skewness of a distribution
μ_4		kurtosis of a distribution

Chapter 1

Introduction

Optical data from point sources of light is key to successful systems in many different industries. In real optical systems, the photons from such point sources are not perfectly focused onto a single point on the focal plane. Instead, real world effects cause the photons to be distributed in a pattern that is described by the Point Spread Function (PSF). Centroiding, defined here as the task of finding the center of these PSFs, provides a data point for various state estimation problems. During system design, it is crucial to determine expected centroid accuracy so that this uncertainty can be accounted for in the estimation process.

In real optical systems, real-world effects and manufacturing defects affect the structure of a PSF. These effects are often well described by the five classic optical aberrations: spherical aberrations, coma, astigmatism, curvature of field, and distortion. These structural effects in the PSF shape may change the accuracy of centroiding algorithms if they are not considered.

Therefore, the need to estimate centroid accuracy in the presence of optical aberrations is the motivation for this work. A method of PSF simulation is presented that allows the analyst to model PSFs affected by optical aberrations. Then, three common types of centroiding algorithms in use today are presented. Finally, the accuracy of the centroiding algorithms is explored for different cases of both symmetric and asymmetric PSFs.

There are many applications today which use centroids from point sources as part of their optical data. Some sources of point sources are of natural origin, such as centroiding stars for use in celestial navigation. Since stars are located many light-years away from spacecraft, they appear as point sources in an image. Centroiding these point sources gives line-of-sight (LOS) vectors in the sensor frame. These LOS vectors can be compared to a star catalog to identify which stars are being

viewed. Once this is known, the pointing, or attitude, of the spacecraft can be calculated [19, 23, 26]. Clearly, this process rests on having accurate centroids.

Other point sources are placed in systems purposely using fiducial markers. Fiducial markers are placed in locations of interest or on objects of interest. When an image is taken, they appear as a point source. Knowledge of their location in the image as well as their known location on an object can be used for state estimation. Fiducial markers have multiple applications, such as precision robotic surgery [6, 21], object tracking [31], or proximity operations using LIDAR imagery [4, 5]. Again, it is crucial for these estimation problems that the PSF centroids are accurate.

Point sources appear in the focal plane of an optical system as a 2D distribution of light intensities. For cases in which PSFs are minimally affected by optical aberrations, a PSF can be well-modeled by a symmetric, 2D Gaussian distribution [13, 16]. This is the basis for many simple simulations of point sources. In addition, some optical aberrations only effect the PSF in the radial direction. In this case, a PSF may still be well-modeled by a 2D Gaussian distribution. When considering aberrations that cause asymmetric behavior in PSFs, there is particular interest in coma and astigmatism. Both of these can cause PSFs to become non-Gaussian in appearance by adding a blur or a “tail” on one side of the PSF. This can cause the distribution of the PSF to become skewed enough that a Gaussian distribution no longer approximates the PSF well. This thesis analyzes centroiding accuracy of these non-ideal PSFs.

In order to perform this analysis, a new framework for simulating PSFs must be developed. Simple PSF simulations often assume 2D Gaussians, which excludes the effects of asymmetric optical aberrations such as coma and astigmatism. A new framework is desired that is intuitive, easy to implement, and provides the analyst the flexibility to simulate any PSF of interest. Many options were considered, and ultimately Johnson distributions were chosen as the method of simulation. Johnson distributions are uniquely defined given the first four central moments of a distribution. The central moments are easily defined given any existing continuous or discrete distribution [14, 15]. Furthermore, the mean, standard deviation, skewness, and kurtosis of a distribution are intuitive physical parameters, making it simple to simulate the needed PSF. To form a 2D distribution, the joint probability function of two independent 1D distributions is calculated and discretized, giving a simulated PSF that can be analyzed with different centroiding algorithms.

This thesis is organized as follows. Section 2 will go into detail on the state-of-the-art in all areas of research that this thesis touches on. Section 3 explains the theory and the implementation of the PSF simulation method selected for this work. Next, Section 4 outlines the methodology for the three categories of centroiding algorithms in use today. Finally, Section 5 presents results and

discussion, and Section 6 presents conclusions based on the work of this thesis.

Chapter 2

Literature Review

2.1 Camera Model and Optical Aberrations

2.1.1 Thin Lens Model

Many optical systems are equipped with lenses. One of the simplest lens models that exists is the thin lens model.

The thin lens is a lens with two spherical surfaces of radius r . Parallel rays of light, also known as collimated light, travel through the lens. One of these rays travels through the center of the lens, O . This ray of light is not refracted as it passes through the lens and reaches the focal plane at a point P' . All other rays are refracted so that they also meet at point P' [8].

This model is shown in Figure 2.1 [8].

2.1.2 Perfect Perspective Projection

The geometrical model for the projection of a point (X, Y, Z) in 3D space onto a 2D image point (x, y) is perfect perspective projection. This can be described by the pinhole camera model, shown in Figure 2.2 [1, 3, 8, 30].

This model can be visualized by imagining a thin lens whose size collapses down to a point (the pinhole) such that only the ray of light that passes through the center of the lens, O , needs to be considered. Thus, no refraction affects the rays of light before they reach the focal plane. Because of this, the 3D location of a point P can be related to its location on the focal plane by similar

Figure 2.1: The thin lens model.

Figure 2.2: The pinhole camera model. Figure adapted by author from [12].

triangles.

$$x = f \frac{X}{Z} \quad \text{and} \quad y = f \frac{Y}{Z} \quad (2.1)$$

In other words, each point $P = P(X, Y, Z)$ in the field of view is projected through the optical center of the optical system and onto the focal plane. Note that the optical center is the origin of the camera sensor frame with the X-axis, Y-axis, and Z-axis shown in Figure 2.2. Thus, each point (x, y) on the image corresponds to a unique line-of-sight vector with its origin at the optical center.

Note that due to the geometry described above, the focal plane will contain a flipped image. For ease of visualization and calculation, the image plane is used in practice instead. The image plane is parallel to the focal plane and is located f in front of the principal point of the camera. The image plane displays the image as it would normally be shown. Note that the image plane does not represent a physical surface in the optical system, but is simply used for ease of visualization and calculations. Figure 2.2 only shows the image plane.

In the pinhole camera model, no lens effects are considered. Under perspective projection, the Point Spread Function (PSF) of a defocused point source will appear on the focal plane as a symmetric distribution of intensities that is usually well-modeled by a 2D Gaussian distribution. Optical aberrations, which are introduced in optical systems with lenses, cause deviations from this ideal PSF. Aberrations are inherent to the projection of 3D point onto a 2D image, though they may also be exacerbated by imperfections that occur during manufacturing or lens assembly [20]. They can be accounted for in the design process of an optical system, but ultimately will always exist.

2.1.3 Point Spread Functions

Unresolved objects, or point sources, reach the optical system as collimated light. Collimated light refers to light whose rays are parallel. When centroiding point sources, Point Spread Functions (PSFs) are utilized in order to obtain a centroid with subpixel precision. Note that because of real world effects, a point source appears as a 2D function of intensities instead of appearing as a single pixel of high intensity [9, 19, 28].

The PSF is the impulse response of the optical system, and thus describes how collimated light is distributed on the focal plane. When this continuous response reaches the focal plane, the detector array discretizes the signal and outputs the discrete PSF. Some example PSFs from real images are shown in Figure 2.3. Note that these examples are a mixture of bright and dim stars, and a mixture of symmetric and asymmetric PSFs.

Figure 2.3: Examples of point source PSFs.

For some applications, such as using star trackers for attitude determination of a spacecraft, the camera is intentionally defocused in order to increase the size of the PSF. This allows the centroids to reach sub-pixel accuracy by virtue of the hyperacuity theorem [19].

In an optical system with the ideal thin lens, the PSF is well-modeled by a symmetric, 2D Gaussian distribution [13, 16]. However, this does not always hold for real optical systems.

2.1.4 Optical Aberrations

Types of Optical Aberrations

No real imaging system behaves like the ideal thin lens model. To capture these deviations of behavior, the concept of optical aberrations is introduced.

There are five classic types of optical aberrations: spherical aberration, coma, astigmatism, curvature of field, and distortion. These aberrations are represented mathematically by the five Seidel sums, which are shown in Equations 2.2-2.6. Note that (x, y) are the coordinates of a point in the object plane and (x', y') are the coordinates of a point in the image plane.

$$S = \sum_{i=1}^{\infty} 2(x'^2 + y'^2)^{i+1} \quad (2.2)$$

$$C = \sum_{i=1}^{\infty} 4(x'^2 + y'^2)^i (x'x + y'y) \quad (2.3)$$

$$A = \sum_{i=1}^{\infty} 6(x^2 + y^2)^{i-1} (x'x + y'y)^2 \quad (2.4)$$

$$F = \sum_{i=1}^{\infty} 4(x^2 + y^2)^i (x'^2 + y'^2) \quad (2.5)$$

$$D = \sum_{i=1}^{\infty} 4(x^2 + y^2)^i (x'x + y'y) \quad (2.6)$$

where the constants in front of the terms correspond to the order of the corresponding term [12].

The first term of these expressions represent the spherical aberration, coma, astigmatism, field curvature, and distortion, respectively. Each of these contribute a different effect to the wave aberrations. The magnitude of these effects depends on the design of the optical system, the focal length, and the locations of objects in the Field of View (FOV). Note that, as indicated by the summations in these expressions, higher order aberrations do exist. However, attempting to correct for them adds much complexity to the system with only minimal image quality improvements. For this reason, only the first terms of the expressions are considered in the classical aberrations [11].

The spherical aberration is a natural side effect of any optical system that contains a spherical

Figure 2.4: Representation of spherical aberration in an optical system.

Figure 2.5: Representation of coma in an optical system.

lens. Note that in the ideal thin lens model, all rays meet at the optical center. In real optical systems, rays of light farther from the principal point tend to meet along the optical axis before the optical center. This is illustrated in Figure 2.4. Clearly, the location at which a ray intersects the camera boresight depends on the location where it crosses the lens [10, 11].

Coma can be a result of lens imperfection or design of the optical system. It refers to the aberration that occurs when different regions of the lens have different magnifications. In an image, this appears as a blur (or a tail) on one side of a point source [10, 11]. See Figure 2.5, and note the ‘tail’ effect that appears. This is one of the aberrations that introduces an asymmetric effect to PSFs.

Astigmatism occurs in optical systems when the principal points of the vertical and horizontal

Figure 2.6: Representation of astigmatism in an optical system.

planes are located at different points along the optical axis, or camera boresight. In other words, the focal length is different in the x and y directions. In practice, a point source with astigmatism will appear to be ‘football-shaped’. Figure 2.6 shows this phenomenon. This is also an aberration that introduces asymmetric effects to PSFs [10, 11].

Field curvature is the optical aberration that comes from the having a curved image surface rather than a planar image surface. When this curved image surface is mapped onto a planar focal plane, one sees the effects of field curvature. This is shown in Figure 2.7 [11].

Finally, distortion is also caused by non-uniform magnification of the lens. This leads to a radial displacement of a point from the ideal image to the actual image. There are two types of radial distortion. If the magnification of the lens is greatest at the center and decreases as the radius increases, then the system is said to have a barrel distortion. If the magnification is least at the center and increases as the radius increases, the optical system is said to have pincushion distortion [11]. These are shown in Figure 2.8.

The derivation of the seidel sums described above are shown in the next section using first principles of optics and wavefronts.

Derivation of Classic Aberrations

Wavefronts are used to derive the power series representations, or the *Seidel sums*, of the optical aberrations. Consider a point source of light, P , in 3D space. Rays of light R of equal magnitude emanate from P . The wavefront of P can be thought of as the surface drawn such that it is perpendicular to R . For an ideal image, this wavefront will be a sphere. Any deviations of this

Figure 2.7: Representation of field curvature in an optical system.

Figure 2.8: Representation of distortion in an optical system.

Figure 2.9: A 2D representation of a wavefront.

wavefront are due to the optical aberrations in the imaging system [12]. A 2D representation of this phenomenon is shown in Figure 2.9. Quantifying the distorted wavefront provides a mathematical expression for the optical aberrations. Let (x, y, z) be the coordinates of P in the sensor reference frame, and let (x', y') be the coordinates of P in the image plane. The wavefront of P is thus a function of these four variables: $W = W(x, y, x', y')$ [12]. Note that W is not dependent on z because the wavefront is identical at all distances from the optical center.

Let us perform a change of variables on W . If $W = W(x, y, x', y')$, $\mathbf{r} = \mathbf{r}(x, y)$, and $\mathbf{r}' = \mathbf{r}'(x', y')$, then W can be written in terms of $W = W(x, y, x', y') = W(\mathbf{r}^T \mathbf{r}, \mathbf{r}'^T \mathbf{r}', \mathbf{r}^T \mathbf{r}')$. The new variables are defined as:

$$\mathbf{r}^T \mathbf{r} = r^2 = x^2 + y^2 \quad (2.7)$$

$$\mathbf{r}'^T \mathbf{r}' = r'^2 = x'^2 + y'^2 \quad (2.8)$$

$$\mathbf{r}^T \mathbf{r}' = \|\mathbf{r}\| \|\mathbf{r}'\| \cos(\phi) = xx' + yy' \quad (2.9)$$

This change of variables reduces the degree-of-freedom of the wavefront from four to three. This can be done because it is also assumed that the optical system is rotationally symmetric about the optical axis. This approximation is valid for most real-world optical systems. As long as the point (x', y') is located anywhere equidistant from the principal point, its wavefront remains the same. This reduces the degree-of-freedom of the wavefront by one, which is why W can be fully described by three variables.

So, W can be written as a function of $W = W(x^2 + y^2, x'^2 + y'^2, xx' + yy')$. This can be

expanded using a power series based on these three variables. Rearranging the power series gives the mathematical representations of the five types of optical aberrations [12]. The power series expansion with grouped terms is given by

$$W(x^2 + y^2, x'^2 + y'^2, xx' + yy') = a_0 + b_1(x'^2 + y'^2) + b_2(xx' + yy') + S + C + A + F + D \quad (2.10)$$

where S , C , A , F , and D are as defined in Equations 2.2-2.6.

Any terms with no dependence on the image coordinates (x', y') are equal to 0, since the optical aberrations are dependent on the image coordinates. Thus, a_0 can be neglected [12].

Additionally, the b_1 and b_2 terms can be neglected. These terms can be adjusted by the choice of reference point on the image plane. It is common practice to select an image focus such that $(x', y') = (0, 0)$, in which case both terms are equal to 0 [12].

The simplification of the power series expansion is thus:

$$W(x^2 + y^2, x'^2 + y'^2, xx' + yy') = S + C + A + F + D \quad (2.11)$$

This expression mathematically describes the monochromatic optical aberrations that appear in an optical system through the concept of wave aberrations.

2.2 Centroiding Algorithms

Given a PSF associated with a point source of light, it is often desirable to calculate the coordinates of the PSF center. This is done using centroiding algorithms. Though there are multiple types of centroiding algorithms, this thesis focuses on the most common ones.

The three most common types of centroiding algorithms are the Center of Intensity (COI) algorithm, the Cross-Correlation (CC) algorithm, and the Model Matching (MM) algorithm. All three algorithms use the intensity values of the pixels in the Point Spread Function (PSF) to find the centroid of a point source. Note that PSFs are explained in detail in Section 2.1.3.

The COI algorithm is analogous to the concept of center of mass for a 2-dimensional object, using each pixel's intensity as its weight. The main advantage of this algorithm is that it does not require any prior knowledge of the structure of the PSF.

The CC algorithm convolves the PSF of interest with some template PSF to calculate a cross-correlation function. Before performing this procedure, both the PSF of interest and the template

PSF are normalized so that they sum to 1. Doing this results in the normalized cross-correlation function. This is done to ensure that peaks in the cross-correlation does indeed occur from a high correlation, and not from a high star brightness [2]. The maximum of this function is the point at which the two PSFs are most correlated, and is the centroid of the PSF. Finding the peak of the cross-correlation function is non-trivial - in this thesis, it is done using parabola fitting. Note that the CC method assumes that the template PSF is a good model of the observed PSFs. Normalized cross-correlation is performed in the time domain. Alternatively, since the convolution process can be slow and inefficient in the spatial domain, cross-correlation can be done in the frequency domain using the Fast Fourier Transform (FFT) of the non-normalized PSFs [27]. Both methods are presented, though only the normalized cross-correlation is performed for this thesis.

The MM algorithm also uses a template PSF to find the centroid. The template PSF is often chosen to be a 2D Gaussian, and is parameterized by the means of the x and y directions, the variances in the x and y directions, the amplitude, and a rotation term. A non-linear least squares solver is used to find the optimal state vector to minimize the residuals between the model and the observed PSF. The centroid is simply the final result for the means in the x and y directions [13].

Note that both the CC and MM algorithms rely on some assumption on the structure of the observed PSF, while the COI algorithm does not. However, the COI algorithm is the only algorithm that directly uses the intensity values of the pixels in the calculation of the centroid, which makes it more sensitive to noise in the image than the other two algorithms.

2.3 Centroiding Applications

Centroiding of point sources finds applications in a variety of industries. For all of these industries, the centroid locations are key to estimating some state of interest. Some common applications are presented.

One such application is robotic surgery guided by fiducial markers [6,21]. These robotic systems are equipped with an image guidance system which capture the fiducial markers in their field of view. The centroids of the fiducial markers are found, and the system uses these locations and the known locations of the fiducial markers to find the sensor location in a relevant coordinate frame. For precision surgery, the concept of centroiding is crucial to successful outcomes.

Another application is centroiding star field images in order to find the attitude of a spacecraft [19,23,26]. An image of a star field is taken and the centroids of visible stars are found. Then, a star matching algorithm uses these centroid locations and identifies the stars that are being viewed.

Using the known location of the stars in inertial space in combination with the line-of-sight vectors of the stars in the image, a spacecraft attitude can be calculated.

Centroiding can also be used to track an object of interest. Given a series of frames, potential points sources of interest are identified and centroided. Then, given two frames, the point correspondence problem must be solved to identify exactly which points of interest have been observed. This way, an object can be tracked from frame to frame [\[31\]](#).

Chapter 3

PSF Simulation

3.1 Optical Aberrations and the PSF

For well-built cameras with a circular aperture, the Point Spread Function (PSF) of a point source is often well-approximated by a symmetric 2D Gaussian distribution [13, 16]. However, real-world effects and manufacturing defects often introduce asymmetric optical aberrations to the image. This changes the structure of the PSF so that a 2D Gaussian is no longer a reasonable approximation for the PSF of a point source.

The rest of the chapter introduces a framework for intuitively modeling PSFs that can include the effects of optical aberrations, and provides a procedure for modeling the effects of specific optical aberrations.

3.2 Defining the Distribution

This section is based on “Systems of Frequency Curves Generated by Methods of Translation,” written by Johnson [14]. The work done in this thesis utilizes the Johnson Curve Toolbox for MATLAB [15].

Johnson distributions are chosen as the method of PSF simulation. These distributions take the first four central moments - mean μ , variance σ^2 , skewness μ_3 , and kurtosis μ_4 - as an input, and outputs a unique distribution with central moments equal to the inputs. The first four central moments are an intuitive way of describing any distribution, and are also easily calculated given a known continuous or discrete distribution. Equations 3.1-3.2 give the expressions for the n -th

central moment for a continuous distribution and a discrete distribution, respectively.

$$\mu_n = E[(X - E[X])^n] = \int_{-\infty}^{\infty} f(x)(x - \mu)^n dx \quad (3.1)$$

$$\mu_n = E[(X - E[X])^n] = \sum_{i=1}^m x_i^n P[X = x_i] \quad (3.2)$$

The central moments are used to calculate a set of four Johnson parameters that are used to define the variable transformation: the shape parameters γ and δ , a location parameter ξ , and a scale parameter λ . The expressions used to calculate these parameters differs between the distribution types and are described below.

In addition, analytic expressions for the means, or expected values, of the distribution in terms of the Johnson parameters are derived.

$$x_c = \int_{-\infty}^{\infty} x f(x) dx \quad (3.3)$$

These are used to calculate the true, analytic centroid locations of a simulated PSF, which can then be compared to the calculated centroid location in analysis. Note that the Johnson parameters have a one-to-one mapping to the central moments of the distribution μ , σ^2 , μ_3 , and μ_4 .

3.2.1 Normal Distribution

For the normal distributions, the following equations govern the calculation of the Johnson parameters.

$$\gamma = -\mu/\sigma \quad (3.4)$$

$$\delta = 1/\sigma \quad (3.5)$$

$$\xi = \mu \quad (3.6)$$

$$\lambda = 1 \quad (3.7)$$

Use these to calculate the variables $f(u)$, D , and Z . Note that X is the interval over which the

Figure 3.1: Normal distribution produced with Johnson Toolbox.

distribution will be defined.

$$u_i = (X_i - \xi)/\lambda \quad (3.8)$$

$$ratio = \delta/\lambda \quad (3.9)$$

$$f(u) = u \quad (3.10)$$

$$D = ratio \quad (3.11)$$

$$Z_i = \gamma + \delta f(u_i) \quad (3.12)$$

Finally, the distribution is defined over X by Equation 3.13. Note that both X and Y are discrete.

$$Y_i = \frac{D}{\sqrt{2\pi}} \exp\left(\frac{-1}{2} Z_i^2\right) \quad (3.13)$$

Essentially, Y is defined over interval X by calculating the PDF values of a Gaussian PDF with $\mu = 0$ and $\sigma = 1$ at the values of Z , and then weighting those values by the constant D .

Figure 3.1 is an example distribution calculated using the equations described in this section.

The expected value of the distribution is calculated to find the expression for the mean.

$$\begin{aligned}
x_c &= \int_{-\infty}^{\infty} xD(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{Z(x)^2}{2}} dx \\
&= \left[-\frac{1}{2} \gamma \lambda \operatorname{erf} \left(\frac{\frac{\delta(X-\xi)}{\lambda} + \gamma}{\sqrt{2}} \right) + \frac{1}{2} \xi \operatorname{erf} \left(\frac{\frac{\delta(X-\xi)}{\lambda} + \gamma}{\sqrt{2}} \right) - \frac{c\lambda e^{-\frac{1}{2} \left(\frac{\delta(X-\xi)}{\lambda} + \gamma \right)^2}}{\delta} \right] \Bigg|_{-\infty}^{\infty} \\
&= \xi - \gamma\lambda
\end{aligned} \tag{3.14}$$

where $\operatorname{erf}(\cdot)$ is the error function and c is a numerical constant.

So, given any set of Johnson parameters for a normal distribution, the true center, or centroid, can be easily calculated using the analytic expression above. This will be crucial in performing the analysis of the centroiding algorithms using Johnson distributions.

3.2.2 Log-Normal Distribution

For the log-normal distributions, the following equations govern the calculation of the Johnson parameters.

First, define

$$x = 0.5\mu_3^2 + 1 \tag{3.15}$$

$$y = |\mu_3| \sqrt{0.25\mu_3^2 + 1} \tag{3.16}$$

$$u = (x + y)^{1/3} \tag{3.17}$$

$$w = u + \frac{1}{u} - 1 \tag{3.18}$$

Then, we may calculate the Johnson parameters.

$$\lambda = \begin{cases} 1, & \text{if } \mu_3 > 0 \\ -1, & \text{if } \mu_3 < 0 \end{cases} \tag{3.19}$$

$$\delta = \frac{1}{\sqrt{\log(w)}} \tag{3.20}$$

$$\gamma = 0.5\delta \log\left(\frac{w(w-1)}{\sigma^2}\right) \tag{3.21}$$

$$\xi = \lambda(\lambda\mu - \exp\left(\frac{0.5/\delta - \gamma}{\delta}\right)) \tag{3.22}$$

Use these to calculate the variables $f(u)$, D , and Z . Note that X is the interval over which the

Figure 3.2: Log-Normal distribution produced with Johnson Toolbox.

distribution will be defined.

$$u_i = (X_i - \xi)/\lambda \quad (3.23)$$

$$ratio = \delta/\lambda \quad (3.24)$$

$$f(u) = \log(u) \quad (3.25)$$

$$D_i = ratio/u_i \quad (3.26)$$

$$Z_i = \gamma + \delta f(u_i) \quad (3.27)$$

Finally, the distribution is again defined over X by Equation 3.28. Note that both X and Y are discrete. The expression is reproduced below for convenience.

$$Y_i = \frac{D_i}{\sqrt{2\pi}} \exp\left(\frac{-1}{2} Z_i^2\right) \quad (3.28)$$

So, Y is defined over interval X by calculating the PDF values of a Gaussian PDF with $\mu = 0$ and $\sigma = 1$ at the values of Z , and then weighting those values by the array D .

Figure 3.2 is an example distribution calculated using the equations described in this section.

The expected value of the distribution is calculated to find the expression for the mean.

$$\begin{aligned}
x_c &= \int_{\xi}^{\infty} x D(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{z(x)^2}{2}} dx \\
&= \frac{1}{2} \left[e^{-\frac{2\gamma\delta-1}{2\delta^2}} \lambda \operatorname{erf} \left(\frac{\gamma\delta + \delta^2 \log(\frac{X-\xi}{\lambda}) - 1}{\sqrt{2}\delta} \right) + \xi \operatorname{erf} \left(\frac{\delta \log(\frac{X-\xi}{\lambda})}{\sqrt{2}} \right) \right] \Bigg|_{\xi}^{\infty} \\
&= e^{-\frac{2\gamma\delta-1}{2\delta^2}} \lambda + \xi
\end{aligned} \tag{3.29}$$

Note that the bounds of the integral change from $[-\infty, \infty]$ to $[\xi, \infty]$. This is because the natural log is undefined for $x < 0$. So, $x \geq \xi$, and the lower bound of the interval is thus ξ .

Again, notice that the true center of a log-normal distribution can be calculated analytically with the Johnson parameters of the distribution. This true center will be used in error analysis.

3.3 Point Spread Function Creation

A 2D discretized distribution is used to simulate the Point Spread Functions (PSFs). First, two discrete 1D distributions are defined over the interval X . Then, a 2D PSF is created by finding the joint probability distribution function (PDF) of the two 1D distributions. This is the PSF that is used for testing. If desired, noise can be added to the PSF as well.

Let $p_{Y_1}(X)$ and $p_{Y_2}(X)$ be the marginal PDFs of the 1D distributions over an interval X . The PSF is simply the joint PDF $p_{Y_1, Y_2}(y_1, y_2)$ of the marginal PDFs. The joint PDF is defined in Equation 3.30 [17].

$$\begin{aligned}
p_{Y_1, Y_2}(y_{1,i}, y_{2,j}) &= P[\{Y_1 = y_{1,i}\} \cap \{Y_2 = y_{2,j}\}] \\
&= P[Y_1 = y_{1,i}, Y_2 = y_{2,j}] \\
&= p_{Y_1}(Y_1 = y_{1,i}) p_{Y_2}(Y_2 = y_{2,j})
\end{aligned} \tag{3.30}$$

In other words, simply multiplying the value of the two marginal PDFs at the $(y_{1,i}, y_{2,j})$ term gives the value of the joint PDF at that location. This simulates a point source as it would appear on the pixel plane, and so the centroiding algorithms can be run on the simulated PSF for analysis and testing. Note that before processing, the PSF is normalized so that intensity values range from $[0,1]$.

3.3.1 Modeling Optical Aberrations

This method also provides the option of simulating PSFs with specific optical aberrations. Instructions to do this are given in this section.

First, select a real PSF that it is desired to simulate. This PSF should come from the optical system of interest and contain the optical aberrations of interest. Next, the principal axes of the distribution are calculated. Then, these are used to define two 1D distributions. The four central moments of each of these distributions are calculated in order to define their equivalent Johnson distribution. Then, the joint PDF that simulates the PSF is calculated. This give a good approximation of the measured star, as well as providing the true centroid.

Principal Component Analysis is performed to identify the principal axes of the PSF. This process is analogous to the mass moment of inertia, where the pixel intensities are used in place of mass. First, the weighted mean in the u - and v - directions are calculated.

$$\mu_u = \frac{1}{I_{tot}} \sum_S uI(u, v) \quad \mu_v = \frac{1}{I_{tot}} \sum_S vI(u, v) \quad (3.31)$$

where

$$I_{tot} = \sum_S I(u, v) \quad (3.32)$$

and S is the set of pixels belonging to the PSF and I is the image of intensity values.

Then, the weighted covariance matrix is found.

$$cov(PSF) = \frac{1}{I_{tot}} \sum_S I(u, v) \begin{bmatrix} (u - \mu_u)^2 & (u - \mu_u)(v - \mu_v) \\ (u - \mu_u)(v - \mu_v) & (v - \mu_v)^2 \end{bmatrix} \quad (3.33)$$

The eigenvectors of this matrix are calculated. These are the principal axes of the PSF, and are used to define two 1D distributions of interest. The angle of rotation between the principal axes and the x - and y -axes needs to be calculated in order to define the sampling locations along the principal axes. To calculate the angle, calculate the angle between the principal axis and the y -axis. Select the principal axis that corresponds to the largest eigenvalue. So,

$$\theta = \cos^{-1} \left([0 \quad 1] \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right) \quad (3.34)$$

where e is the eigenvector corresponding to the largest eigenvalue.

Figure 3.3: A measured PSF overlaid with its principal axes.

Plug the value for θ into the equation for a 2D rotation matrix.

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (3.35)$$

Then, calculate the rotated coordinates of sample locations along the x - and y -axes. In addition, the mean is subtracted from the coordinates so that the transformation is centered. The pixel intensity remains unchanged.

$$(u, v)_{i,rot} = \mathbf{R} \begin{bmatrix} u_i - \mu_u \\ v_i - \mu_v \end{bmatrix} \quad (3.36)$$

This gives sample locations along the principal axes. An example of a real star with its associate principal axes is shown in Figure 3.3.

Now, the 1D distributions of interest can be defined. This is done by sampling the PSF along the calculated principal axes, which now correspond to the vertical and horizontal axes. This gives two PDFs along the principal axes of the original PSF. These are the distributions that are used to

Figure 3.4: The two PDFs found by sampling the PSF along its principal axes.

Figure 3.5: Corresponding Johnson distributions of the sampled distributions.

simulate the PSF. Two such sampled PDFs are shown in Figure 3.4.

Given these two discrete distributions, it is possible to calculate the first four central moments - the inputs for the Johnson distribution simulation - using Equation 3.2.

$$\mu_n = E[(X - E[X])^n] = \sum_{i=1}^m x_i^n P[X = x_i] \quad (3.37)$$

Use these value to define an equivalent Johnson distribution as explained in Section 3.2. Figure 3.5 show the correlated Johnson distributions overlaid on the sampled distributions.

Finally, using Equation 3.30, calculate the joint PDF that simulates the PSF.

$$p_{Y_1, Y_2}(y_{1,i}, y_{2,j}) = p_{Y_1}(Y_1 = y_{1,i})p_{Y_2}(Y_2 = y_{2,j}) \quad (3.38)$$

Figure 3.6: A side-by-side comparison of the real PSF with its corresponding simulated PSF.

In Figure 3.6, the real PSF is shown side-by-side with the simulated PSF. While not exactly the same, the simulated PSF is a good approximation that also provides the researcher a true centroid for use in analysis.

This gives a good approximation of the measured star, as well as providing the true centroid for use in analysis. A PSF is analyzed in Section 5 using this procedure.

3.3.2 Sensor Noise

This thesis also considers sensor noise from the detector. In real images, the sensor noise affects the intensities of a PSF. Thus, a method of simulating the effects of detector noise is necessary. To simulate noisy PSFs, an m -by- n matrix of values randomly selected from a zero-mean normal distribution is generated. Here m is the number of rows in the simulated PSF and n is the number of columns in the simulated PSF. Recall that the pixel intensities of a simulated PSFs is represented as a matrix of numerical values. So, the noise matrix is simply added to the simulated PSF to give a noisy PSF that can be further analyzed.

Chapter 4

Image Centroiding

4.1 Algorithms

The following sections describe three types of centroiding algorithms in use today. Note that before the algorithms can be performed for real images, various image pre-processing steps must occur. In practice, this includes image noise reduction and finding the Region of Interest (ROI) of the stars that will be centroided. For the PSFs simulated for this thesis, such pre-processing is unnecessary.

4.1.1 Center of Intensity

The Center of Intensity (COI) algorithm acts by calculating the center of mass of the PSF in the u and v directions. Instead of mass, the intensity values of the individual pixels in the PSF are used as the weight of that pixel.

Before centroiding in a real image, a threshold must be defined to identify which pixels of a Region of Interest (ROI) are in S_i and which are part of the background. Methods of adaptive thresholding do exist [22, 25, 29]. Alternatively, a fixed pixel intensity threshold value can be set [3]. For this thesis, however, the entire simulated ROI is used in calculations since background pixels will have an intensity value of zero. Thus, they don't have any effect on the calculation of the centroid.

The PSF centroid may then be computed by taking the weighted center of intensity within the ROI,

$$u'_i = \frac{1}{I_{tot}} \sum_{S_i} uI(u, v) \quad v'_i = \frac{1}{I_{tot}} \sum_{S_i} vI(u, v) \quad (4.1)$$

where

$$I_{tot} = \sum_{S_i} I(u, v) \quad (4.2)$$

and S_i is the set of pixels belonging to the i^{th} PSF and I is the image of intensity values.

The COI algorithm is a straight-forward algorithm to implement. However, because the intensity value of each pixel is used directly in the calculation of the centroid, this algorithm is very susceptible to noise.

4.1.2 Cross-Correlation

The cross-correlation (CC) method requires calculating the cross-correlation function of some template PSF with the observed PSF. Two possible methods of achieving this are performing normalized cross-correlation in the time domain, or performing cross-correlation in the frequency domain using Fast Fourier Transforms (FFTs). Both methods are presented. This work uses the normalized cross-correlation.

The template PSF used can be anything representative of the PSFs that are expected to be seen in the image. For the analysis done in this thesis, a 2D Gaussian is used for this template. Note that a 2D Gaussian is also commonly used in practice. [13]

Normalized Cross-Correlation

Cross-correlation is equivalent to the convoluion of the m -by- n observed PSF with the template PSF in the spatial domain, and is found by

$$c(i, j) = \sum_{u=1}^n \sum_{v=1}^m f(u, v) g(u - i, v - j) \quad (4.3)$$

where $c(i, j)$ is the value of the cross-correlation at the (i, j) pixel, $f(u, v)$ is the normalized observed PSF, and $g(u, v)$ is the normalized template PSF. Both PSFs are normalized so that the sum of all elements is 1.

Finding the peak of the cross-correlation function is non-trivial. There are many potential methods. For the work in this thesis, interpolation via parabola is used to find the maximum of the cross-correlation with sub-pixel level accuracy [13]. This maximum is the centroid of the observed PSF.

For interpolation via parabola, the pixel of maximum value of the cross-correlation is found. Then, a second-order polynomial function is fitted to both the row data and the column data

containing the pixel of maximum value. Finding the coefficients of these two polynomials is done using the *fit()* function in MATLAB. To find the maximum of these polynomials, the derivative is taken and set equal to zero, then solved. The solutions of these two derivative equations result in the centroid for the observed PSF in the u - and the v - direction, respectively.

Cross-Correlation with the Fast Fourier Transform

Alternatively, calculating the cross-correlation function can be implemented in the frequency domain using the Fast Fourier Transforms (FFTs) of the non-normalized PSFs, and then converted back to the spatial domain [27]. Performing correlation in the frequency domain rather than the spatial domain is much more computationally efficient. Instead of performing the integral operation shown in Equation 4.3, the equation in the frequency domain is simply

$$C(x, y) = F(x, y)G(x, y)^* \quad (4.4)$$

where $F(u, v)$ and $G(u, v)$ are the Fourier Transforms of their corresponding functions in the spatial domain, and the $*$ indicates the complex conjugate.

Furthermore, the Fast Fourier Transform (FFT) is used rather than the Discrete Fourier Transform (DFT). This is done because solving for the DFT has an order of $(mn)^2$ operations, whereas using the FFT reduces this to $mn\log_2(mn)$ operations, where m is the number of rows and n is the number of columns in the ROI [9]. Thus, implementing the cross-correlation using the FFTs is more computationally efficient.

While both approaches are valid, this thesis performs the normalized cross-correlation in the time domain. This allows the use of the normalized PSFs, which ensures that peaks in the cross-correlation do not occur from high intensity values in the observed PSF [2]. Otherwise, particularly bright stars might register as a peak in the cross-correlation function that does not in actuality indicate a centroid.

4.1.3 Model Matching

Model matching also requires a template PSF. In this method, the template PSF is defined by some set of parameters. As the template PSF is chosen to be a 2D Gaussian, these parameters are the mean in the u direction μ_u , the mean in the v direction μ_v , the standard deviation in the u direction σ_u , the standard deviation in the v direction σ_v , the amplitude of the PSF A , and the rotation of the template PSF θ . Let β be the vector of parameters, so that $\beta = [\mu_u, \mu_v, \sigma_u, \sigma_v, A, \theta]^T$.

The equation that governs the Probability Density Function (PDF) of a 2D Gaussian is

$$g(u, v) = A \exp(-(a(u - \mu_u)^2 - 2b(u - \mu_u)(v - \mu_v) + c(v - \mu_v)^2)) \quad (4.5)$$

where

$$a = \frac{\cos(\theta)^2}{2\sigma_u^2} + \frac{\sin(\theta)^2}{2\sigma_v^2} \quad b = -\frac{\sin(2\theta)}{4\sigma_u^2} + \frac{\sin(2\theta)}{4\sigma_v^2} \quad c = \frac{\sin(\theta)^2}{2\sigma_u^2} + \frac{\cos(\theta)^2}{2\sigma_v^2} \quad (4.6)$$

This model is used to find the optimal estimate for the parameters by minimizing Equation 4.7. Let $f(u, v)$ be the observed PSF and $\hat{f}(u, v|\beta)$ be the value of a PDF defined by substituting Equation 4.5 with β .

$$\text{Minimize } J(\beta) = \sum_{all(u,v)} |f(u, v) - \hat{f}(u, v|\beta)|^2 \quad (4.7)$$

The solution to this optimization problem can be found using any non-linear least squares estimator. For this thesis, the Levenberg-Marquardt Algorithm (LMA) is selected for its robustness and computational efficiency [3]. There are many variations of the LMA, and the algorithm used for this thesis is outlined below [7, 18, 24].

1. Initialize β . Let $\mu_u = \mu_v = 0$, let A equal the intensity of the brightest pixel of the observed PSF, let $\sigma_u^2 = \sigma_v^2 = 2$, and let $\theta = 0$.
2. Calculate the initial $J(\beta)$.
3. Calculate $\mathbf{J}$, the jacobian of the system. The jacobian is derived analytically and is defined by:

$$\mathbf{J}_{ij} = \frac{\delta f_i}{\delta \beta_j} \quad (4.8)$$

for the i -th pixel in the ROI and the j -th parameter in β . So, for the pixel $(u, v)_i$, the partial derivatives for the jacobian are given by

$$\frac{\delta f_i}{\delta \mu_u} = A \exp \left(-a(u - \mu_u)^2 + 2b(u - \mu_u)(v - \mu_v) - c(v - \mu_v)^2 \right) \left(2a(u - \mu_u) - 2b(v - \mu_v) \right) \quad (4.9)$$

$$\frac{\delta f_i}{\delta \mu_v} = A \exp \left(-a(u - \mu_u)^2 + 2b(u - \mu_u)(v - \mu_v) - c(v - \mu_v)^2 \right) \left(2c(v - \mu_v) - 2b(u - \mu_u) \right) \quad (4.10)$$

$$\frac{\delta f_i}{\delta \sigma_u} = A \exp \left(-a(u - \mu_u)^2 + 2b(u - \mu_u)(v - \mu_v) - c(v - \mu_v)^2 \right) \left(\frac{\cos(\theta)^2(u - \mu_u)^2}{\sigma_u^3} + \frac{\sin(\theta)^2(v - \mu_v)^2}{\sigma_u^3} + \frac{\sin(2\theta)(u - \mu_u)(v - \mu_v)}{\sigma_u^3} \right) \quad (4.11)$$

$$\frac{\delta f_i}{\delta \sigma_v} = A \exp \left(-a(u - \mu_u)^2 + 2b(u - \mu_u)(v - \mu_v) - c(v - \mu_v)^2 \right) \left(\frac{\cos(\theta)^2(v - \mu_v)^2}{\sigma_v^3} + \frac{\sin(\theta)^2(u - \mu_u)^2}{\sigma_v^3} + \frac{\sin(2\theta)(u - \mu_u)(v - \mu_v)}{\sigma_v^3} \right) \quad (4.12)$$

$$\frac{\delta f_i}{\delta A} = \exp \left(-a(u - \mu_u)^2 + 2b(u - \mu_u)(v - \mu_v) - c(v - \mu_v)^2 \right) \quad (4.13)$$

$$\frac{\delta f_i}{\delta \theta} = -A \exp \left(-a(u - \mu_u)^2 + 2b(u - \mu_u)(v - \mu_v) - c(v - \mu_v)^2 \right) \left(-2b(v - \mu_v)^2 + 2b(u - \mu_u)^2 + \left[\frac{\cos(2\theta)}{\sigma_u^2} - \frac{\cos(2\theta)}{\sigma_v^2} \right] (u - \mu_u)(v - \mu_v) \right) \quad (4.14)$$

4. Initialize the value for the tuning coefficient λ . For this thesis, $\lambda_o = 0.001$.

5. Solve for h , the parameter update vector, with the following linear equation.

$$[\mathbf{J}^T \mathbf{J} + \lambda \text{diag}(\mathbf{J}^T \mathbf{J})]h = \mathbf{J}^T(y - \hat{y}) \quad (4.15)$$

6. Let $\beta^+ = \beta + h$ and calculate y^+ and $\mathbf{J}^+$.

7. Calculate $J(\beta^+)$ and compare with the previous calculated $J(\beta)$. If $|J(\beta^+) - J(\beta)| \leq \epsilon$ or if the maximum number of iterations has been reached, then β^+ is the optimal estimate of β and the algorithm ends. If not, go to Step 8.

8. Compare $J(\beta^+)$ to $J(\beta)$.

- If $J(\beta^+) < J(\beta)$, set $J(\beta^+)$ to $J(\beta)$, set β^+ to β , and let $\lambda = \lambda/10$. Go back to Step 3.
- If $J(\beta^+) < J(\beta)$, set $\lambda = 10\lambda$ and go back to Step 3.

Running the LMA outputs an optimal $\hat{\beta}$. The centroid is defined as the μ_u and μ_v in this $\hat{\beta}$. Note that ϵ is a tolerance value that is set to 0.3 for this work, and the maximum number of allowed iterations is 500.

Chapter 5

Results and Discussion

5.1 Verification of Analytic Centers

We can verify the analytic expressions derived for the distribution true centers through a numerical integration method. For this analysis, the trapezoidal rule is used. This method evaluates the integral of a function by evaluating the function at evenly spaced points along the interval, forming trapezoids between consecutive pairs of points, and calculating the sum of the area of these trapezoids. So, the formula is approximated as:

$$x_c = \int_{-\infty}^{\infty} xf(x)dx \quad (5.1)$$

$$\approx \frac{h}{2} \sum_{i=1}^{N-1} [x_i f(x_i) + x_{i+1} f(x_{i+1})] \quad (5.2)$$

where N is the number of points and h is the spacing between each point. For this application, $f(x)$ is the value of the PDF of the distribution at x .

Consider the normal distribution shown in Figure 5.1.

The analytic center and numeric center are equivalent to working precision, proving that the expression for the analytic center of a normal distribution is correct. Thus, in further analysis, the analytic expression for the center can be used as the true center.

Again, the analytic center and the numeric center are equivalent to working precision. So, the

Table 5.1: Normal Distribution

Analytic	0.0
Numeric	0.0

Figure 5.1: Normal distribution and its analytic center and numerically calculated center.

Figure 5.2: Log-Normal distribution and its analytic center and numerically calculated center.

Table 5.2: Log-Normal Distribution

Analytic	2.0962e-17
Numeric	-5.6370e-10

expression for the analytic center of a log-normal distribution can be considered in analysis as the true center.

The center calculated from the analytic expression are the true centers of the distribution. Error analysis of the three centroiding algorithms will be performed by comparing the centroiding solutions to the true centers. This step is crucial in developing conclusions regarding the various centroiding algorithms in non-ideal, non-Gaussian cases.

5.2 Centroiding Simulated Point Spread Functions

Since the true centers for the considered types of Johnson distributions have been derived analytically and verified numerically, they can be used to perform error analysis on centroiding algorithms.

Simulated PSFs are generated using the processes outlined in Section 3, and the true centers are calculated using the analytic expressions. The centroid of the simulated PSF is calculated using the three centroiding algorithms described in Section 4. Finally, error analysis will be performed on the results of the algorithms. To calculate the error of a given case, the distance between the true center and the calculated centroid is found.

$$e_{centroid} = \|(u, v)_{calc} - (u, v)_{true}\|^2 \quad (5.3)$$

$$= \sqrt{(u_{calc} - u_{true})^2 + (v_{calc} - v_{true})^2} \quad (5.4)$$

Note that errors will be presented in units of pixels. Given specifications for an optical system, units of pixels can be converted to an angular distance.

Both non-noisy and noisy cases will be considered. When analyzing noisy simulated PSFs, a Monte Carlo is performed.

Symmetric PSF

The first case that will be looked at is a symmetric, 2D Gaussian PSF. This will be used to confirm the centroiding algorithms. The results are shown below in Figure 5.3 and Table 5.3. This case shows a PSF that is perfectly-modeled by the 2D Gaussian that is commonly used in simple simulations. It will be used as the basis of comparison against other, more asymmetric PSFs that one might encounter.

Clearly, this case shows that for the ideal case of a noiseless, symmetric PSF, all three centroids give perfect results. This is expected for all three algorithms, and especially for the CC and MM

Figure 5.3: Results of the three centroiding algorithms for a symmetric 2D Gaussian.

Table 5.3: Centroiding results for a non-noisy, symmetric 2D Gaussian.

	Centroid	Error
True Center	(0,0)	-
COI	(0,0)	0
CC	(0,0)	0
MM	(0,0)	0

Figure 5.4: Results of the three centroiding algorithms for a symmetric PSF with noise.

Table 5.4: Centroiding results for a noisy, symmetric PSF.

	Error, Average	Error, Standard Deviation
True Center	(0,0)	-
COI	0.0136	0.0070
CC	0.0274	0.0148
MM	0.0102	0.0052

algorithms. Both of these algorithms assume that the PSF being processed is a 2D Gaussian.

Next, this will be explored with added sensor noise. Though any real image will undergo pre-processing to remove as much sensor noise as possible, it is expected that a small amount of noise will remain. The process of adding white noise is explained in Section 3.3.2. Note that the true centers of the PSFs are calculated before sensor noise is added.

Consider the noisy, symmetric 2D Gaussian explored in Figure 5.4 and results shown in Table 5.4.

Even in the presence of some sensor noise, the three centroiding algorithms still perform well in the case of a symmetric PSF. In the presence of noise, the MM algorithm seems to be most accurate.

Figure 5.5: Results of the three centroiding algorithms for an asymmetric PSF.

Table 5.5: Centroiding results for an asymmetric PSF.

	Centroid	Error
True Center	(0,0)	-
COI	(-2.2333e-05,0)	2.2333e-05
CC	(-8.6739e-06,0)	8.6739e-06
MM	(0.0020,-0.0834)	8.3469e-02

This has also been shown in the literature [13], indicating that the MM algorithm is well-suited to identify centroids if the image is particularly noisy.

Asymmetric PSF

Next, an asymmetric case is explored. Consider the simulated PSF in Figure 5.5, which is a good example of a highly skewed, asymmetric PSF. This is of particular interest - through this example, the degradation of the centroiding algorithms in non-ideal cases will be explored. Results are shown in Table 5.5.

The errors for this asymmetric PSF are higher than that of the symmetric PSF. As predicted, the

Figure 5.6: Results of the three centroiding algorithms for an asymmetric PSF with noise.

Table 5.6: Centroiding results for a noisy, asymmetric PSF.

	Error, Average	Error, Standard Deviation
True Center	(0,0)	-
COI	0.0326	0.0175
CC	0.0733	0.0746
MM	0.0843	0.0045

COI algorithm performs better than both the CC and MM algorithm in this asymmetric case. This is expected, since the CC and MM both assume the structure of a 2D Gaussian in their computations. When this assumptions no longer holds, they perform worse.

Next, sensor noise is added to the PSF. Consider Figure 5.6 and Table 5.6.

Here, the trends in algorithm accuracy are different. The MM algorithm performs best in a Monte Carlo, followed by the COI algorithm, and then the CC algorithm. This indicates that MM is the algorithm most robust to noise, even if it is the algorithm most sensitive to this asymmetric PSF.

This indicates that a researcher must weigh both the asymmetry of expected PSFs and the

Figure 5.7: Results of the three centroiding algorithms for a simulated, asymmetric PSF.

Table 5.7: Centroiding results for a simulated, asymmetric PSF.

	Centroid	Error
True Center	(-0.2678,-0.2805)	-
COI	(-0.2682,-0.2815)	0.0010
CC	(-0.2400,-0.2732)	0.0287
MM	(-0.3248,-0.3076)	0.0631

expected level of sensor noise when selecting a centroiding algorithm. If the expected signal-to-noise ratio is low, and asymmetry is expected to be high, then the MM algorithm may be the best choice. However, if the expected signal-to-noise ratio is high, the COI algorithm may be the best choice.

Simulated PSF from Real PSF

The PSF simulated in Section 3.2 is now analyzed. Recall that this PSF was produced to simulate a real PSF. Results are shown in Figure 5.7 and Table 5.7.

Clearly, the COI algorithm performs the best out of the three explored algorithms. This again

Figure 5.8: Results of the three centroiding algorithms for a noisy PSF.

Table 5.8: Centroiding results for a noisy, simulated PSF.

	Error, Average	Error, Standard Deviation
True Center	(-0.2678,-0.2805)	-
COI	0.1497	0.0208
CC	0.0454	0.0266
MM	0.0617	0.0081

confirms that the COI algorithm performs the best for asymmetric PSFs. The CC algorithm performs the second best, and the MM algorithm performs the worst.

Then, noise is added to this PSF. One iteration of the Monte Carlo is shown in Figure 5.8, with results from the entire Monte Carlo shown in Table 5.8.

These results show trends similar to what is seen in the previous noisy case. Namely, even though the COI algorithm performs the best with no sensor noise, the MM algorithm is most accurate in the presence of sensor noise.

Again, this leads to the conclusion that the COI algorithm is least sensitive to asymmetry in the PSF, but most sensitive to sensor noise. On the other hand, the MM algorithm is most

sensitive to asymmetry in the PSF, but least sensitive to sensor noise. The centroid accuracy of the CC algorithm is between those of the other algorithms for both noisy and non-noisy PSFs. This indicates that expected level of sensor noise must be considered in addition to asymmetry of expected PSFs.

Chapter 6

Conclusion

This work explores the effects of optical aberrations on the centroiding of point sources. Specifically, the degradation in accuracy of various centroiding algorithms as asymmetric optical aberrations increase is explored and quantified. In addition, this work introduces a framework for simulating the Point Spread Functions (PSFs) of point sources on the focal plane, contributing a new simulation tool to the literature.

The state of the art in relevant areas is introduced in Chapter 2. First, the ideal thin lens model is introduced. This introduces the ideal mapping of collimated light onto the focal plane. Then, the pinhole camera model is introduced as the ideal image model. It assumes an optical system in which no lens exists and all rays of light pass through a single point with no refraction before they reach the focal plane. Next, the concept of PSFs of point sources are introduced. A point source of light reaches the optical system as collimated light and appears in the image of a defocused system as a discretized 2D distribution. Fourth, the five types of monochromatic optical aberrations are derived and defined. Their effects on PSFs are explained, with particular attention on optical aberrations such as coma and astigmatism that cause PSFs to become asymmetric in structure.

Continuing, three types of centroiding algorithms are explained. PSFs are processed through centroiding algorithms to find a centroid and, in turn, calculate a line-of-sight vector. The three algorithms that are focused on in this work are the Center of Intensity (COI) algorithm, the Cross-Correlation (CC) algorithm, and the Model Matching (MM) algorithm. Finally, some applications of centroiding in industry are presented. These include system pointing, object tracking, and precision robotic surgery.

Chapter 3 presents the method of PSF simulation developed for this work. The method is based

on a method of simulation distributions developed by Johnson [14]. By providing the first four central moments of a desired distribution, a discretized version of the distribution can be simulated. Using the first four central moments is an intuitive way of visualizing a distribution. In addition, the first four central moments of a given distribution is easily calculated. Thus, this method provides flexibility and ease-of-use to analysts. By simulating two such 1D distributions and finding their joint Probability Density Function (PDF), a PSF can be simulated. An existing MATLAB toolbox for Johnson distributions is leveraged for this process. Also, analytic solutions for the means of Johnson distributions are derived. In a simulated PSF, these means are the true centroids of a PSF. Having a true centroid solution allows accurate error analysis to be performed. In addition to simulating a PSF, a simple sensor noise model is produced for use in the PSF simulation.

In Chapter 4, the centroiding algorithms are explained in-depth with specific algorithms and governing equations presented. First, it is shown that the COI algorithm is analogous to Center of Mass calculations in physical systems. Instead of using the mass to weight the data points of the calculation, the light intensity values of the pixels in the PSF are use. Second, the CC algorithm finds the peak of the cross-correlation function of the observed PSF with a representative template PSF. The location of the peak is the centroid of the PSF. The template PSF is chosen to be a 2D Gaussian, which is often done in industry as well. The last presented algorithm is the MM algorithm. This uses a non-linear least squares algorithm to estimate the optimal parameters of a given PSF model. For this work, a 2D Gaussian is parameterized and the optimal mean in two directions, standard deviation in two directions, amplitude, and rotation is calculated. The optimal means give the centroid of the PSF.

Chapter 5 presents the analysis performed and discusses results. Several noisy and non-noisy PSF cases are simulated and then processed with the three presented centroiding algorithms. The first case is a perfect 2D Gaussian simulated PSF. As expected, all three algorithms find the centroid of the PSF with zero error in a case with no noise. As the PSF moves away from this initial case by becoming increasingly asymmetric, the CC and MM algorithms tend to perform worse than the COI algorithm. This is because both of these algorithms assume that the PSF is well-modeled by a 2D Gaussian. When this assumption no longer holds, these algorithms don't perform well compared to the COI algorithm. However, when sensor noise is added to the simulated PSFs, the MM algorithm tends to give the most accurate centroid results. This indicates that the MM algorithm is the least sensitive to sensor noise. An analyst making algorithm decisions must weigh the expected signal-to-noise ratio to the expected asymmetry of PSFs.

In conclusion, this work contributes a new method of PSF simulation that leverages existing

statistical techniques in order to intuitively and easily simulate PSFs. This simulation method is used to explore the accuracy of various centroiding algorithm, especially in the presence of optical aberrations that introduce asymmetric effects such as coma and astigmatism. This information is very valuable for analysts and system designers across a variety of industries.

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