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Supereulerian Properties in Graphs and Hamiltonian Properties in Line Graphs

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Supereulerian Properties in Graphs and Hamiltonian Properties in Line Graphs

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ABSTRACT

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Following the trend initiated by Chvátal and Erdös, using the relation of independence number and connectivity as sufficient conditions for hamiltonicity of graphs, we characterize supereulerian graphs with small matching number, which implies a characterization of hamiltonian claw-free graph with small independence number.

We also investigate strongly spanning trailable graphs and their applications to hamiltonian connected line graphs characterizations for small strongly spanning trailable graphs and strongly spanning trailable graphs with short longest cycles are obtained. In particular, we have found a graph family $\mathcal{F}$ of reduced nonsupereulerian graphs such that for any graph G with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$, G is supereulerian if and only if the reduction of G is not in $\mathcal{F}$.

We proved that any connected graph G with at most 12 vertices, at most one vertex of degree 2 and without vertices of degree 1 is either supereulerian or its reduction is one of six exceptional cases. This is applied to show that if a 3-edge-connected graph has the property that every pair of edges is joined by a longest path of length at most 8, then G is strongly spanning trailable if and only if G is not the wagner graph.

Using charge and discharge method, we prove that every 3-connected, essentially 10-connected line graph is hamiltonian connected. We also provide a unified treatment with short proofs for several former results by Fujisawa and Ota in [20], by Kaiser et al in [24], and by Pfender in [40]. New sufficient conditions for hamiltonian claw-free graphs are also obtained.

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DEDICATION

To

my father Xinhua Wang , my mother Aifang Xu

and

my friend Jian Cheng

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Chapter 1

Preliminaries

1.1 Notation and Terminology

We consider finite graphs which may have multiple edges, use [5] for terminology and notations not defined here, and unless otherwise stated, graphs are loopless. Let G be a graph, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. The **independence number** $\alpha(G)$ of a graph G is the cardinality of the largest independent vertex subset. $\kappa(G)$, $\kappa'(G)$, $\alpha'(G)$ and $\tau(G)$ represent the connectivity, the edge connectivity, the matching number of a graph G, and the maximum number of edge-disjoint trees in G respectively. A graph is trivial if it contains no edges.

Given a graph G, and an integer $k > 0$, a cycle (or a bond, respectively) of length k is a k**-cycle** (or a k-bond, respectively). A 2-cycle is also denoted as $2K_2$. A minimal edge cut is a bond. Let $\mathcal{C}_k(G)$ and $\mathcal{C}_k^*(G)$ denote the set of k-cycles of G and the set of k-bonds of G, respectively. The **circumference** of G, denoted by $c(G)$, is the length of a longest cycle of G. The **girth** of G, denoted by $g(G)$, is the length of a shortest cycle of G.

If $X \subseteq E(G)$ is an edge subset, then $V(X)$ denotes the set of vertices of G that are incident with an edge in X. For a vertex $v \in V(G)$, $E_G(v)$ denotes the set of edges incident with v in G, and $N_G(v)$ denotes the set of vertices adjacent to v in G. For a subset $W \subseteq V(G)$, define $N_G(W)$ to be the set of vertices in $V(G) - W$ that are adjacent to a vertex in W. For an edge subset X, $N_G(X) = N_G(V(X))$. For any integer $i \geq 1$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}, \text{ and } D_{\geq i}(G) = \bigcup_{k \geq i} D_k(G)$$

Let A, B be the subsets of $V(G)$ with $A \cap B = \emptyset$. Denote

$$E(A, B) = \{ab \in E(G) | a \in A, b \in B\} \text{ and } e(A, B) = |E(A, B)|.$$

An edge cut Y of G is **essential** if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is essentially k-edge-connected if G does not have an essential edge cut Y with $|Y| < k$.

For a set $\mathcal{F} = \{F_1, F_2, ...\}$ of graphs, a graph G is $\mathcal{F}$**-free** if G does not have an induced subgraph isomorphic to any member in $\mathcal{F}$. In particular, a $\{K_{1,3}\}$-free graph is referred as a *claw-free* graph.

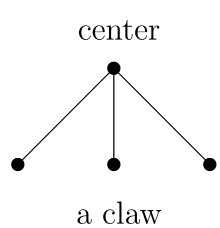


Figure 1.1

The **line graph** of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. See Figure 1.2 as an example. Graph G consists of sold vertices and sold edges. Empty vertices and dash edges are the corresponding $L(G)$.

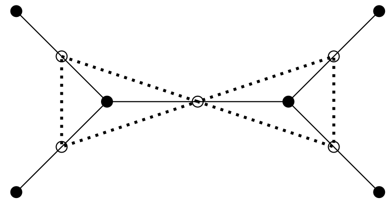


Figure 1.2

Beineke ([2]) and Robertson ([22] and [42]) showed that every line graph is also a claw-free graph. By definition, if a line graph $L(G)$ is not a complete graph, then a subset $X \subset V(L(G))$ is a k-vertex cut of $L(G)$ if and only if X is an essential k-edge-cut of G.

Conjecture 1.1.1 *(Matthews and Sumner [37]) Every 4-connected claw-free graph is hamiltonian.*

Conjecture 1.1.2 *(Thomassen [46]) Every 4-connected line graph is hamiltonian.*

In [43], Ryjáček pointed out that these two conjectures are equivalent. Towards these conjectures, there has been efforts in investigating 3-connected hamiltonian claw-free graphs, as can be seen in the survey paper by Fardree, Flandrin, and Ryjáček [18].

A subgraph H of a graph G is dominating if $E(G - V(H)) = \emptyset$. Let $O(G)$ denote the set of odd degree vertices in a graph G. If G is connected with $O(G) = \emptyset$, then G is **eulerian**. If G has a spanning eulerian subgraph, then G is **supereulerian**.

Boesch et al [3] first posed the problem of characterizing supereulerian graphs. Then Pulleyblank [41] proved that determining whether a 3-edge-connected planar graph is supereulerian is NP-complete. Catlin [11] gave a survey on supereulerian graphs, which was supplemented and updated in [15, 30]. Characterizations of supereulerian graphs for certain classes of graphs have been widely invetigated. See [[6], [13], [28], [33]].

A spanning cycle of G is a Hamiltonian cycle of G. If G has a **Hamiltonian cycle**, then G is **hamiltonian**. A graph G is **Hamiltonian-connected** if for any two vertices $u, v \in V(G)(u \neq v)$, there exists a (u, v)-path containing all vertices of G.

A trail of G as a vertex-edge alternating sequence

$$v_0, e_1, v_1, e_2, \cdots, e_k, v_k \tag{1.1}$$

such that all e_i's are distinct, $i = 1, 2, \cdots, k$, and e_i is incident with both v_{i-1} and v_i. Here vertices in $\{v_1, v_2, \cdots, v_{k-1}\}$ are **internal vertices** of the trail in (1.1). If a closed trail C of G satisfies $E(G - V(C)) = \emptyset$, then C is called a **dominating eulerian subgraph**.

Theorem 1.1.3 *(Li et al., [31])Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(G)$, G has a dominating (e_1, e_2)-trail.*

Let G be a graph such that $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. The **core** of the graph G, denoted by G_0, is obtained by deleting all pendant edges and contracting exactly one edge xy or yz for each path $P = xyz$ in G with $d_G(y) = 2$, where $d(x), d(z) > 2$ since $\kappa(L(G)) \geq 3$. The remaining edge of P will be referred as a nontrivial edge in the contraction. Shao [45] proved Theorem 1.1.4 (a)-(c). In a similar way as Theorem 1.1.4 (c), we can prove Theorem 1.1.4 (d).

Theorem 1.1.4 *(Shao, [45]) Let G_0 be the core of graph G, then each of the following holds:*

(a) G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$;

(b) G_0 is well defined;

(c) If G_0 has a spanning eulerian subgraph, then G has a dominating eulerian subgraph;

(d) If G_0 has a dominating eulerian subgraph containing all nontrivial vertices and both endvertices of each nontrivial edges, then G has a dominating eulerian subgraph.

Spanning trailable graphs are a special class of supereulerian graphs. Let $e, e' \in E(G)$. A trail from e to e' is called an (e, e')**-trail**. An (e, e')-trail is **dominating** if each edge of G

is incident with at least one internal vertex of the trail, it is **spanning** if it is a dominating trail which contains all the vertices of G. A graph is **spanning trailable** if for any pair of edges $e, e' \in E(G)$, G has a spanning (e, e')-trail. As $e = e'$ is possible, spanning trailable graphs are supereulerian. Luo et al [35] first studied spanning trailable graphs (called eulerian-connected graphs in [35]). They showed that every 4-edge-connected graph is spanning trailable, improved the former result of Caltin [8] and Jaeger [23] that every 4-edge-connected graph is supereulerian. Thus it is natural to study whether 3-edge-connected graphs are spanning trailable.

Suppose that $e = u_1v_1, e' = u_2v_2 \in E(G)$ denote two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ by a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$ is also denoted by $G(e)$ and is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$. Given $u, v \in V(G)$, a (u, v)-trail is a trail from u to v. A graph G is **strongly spanning trailable** if for any $e, e' \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. By definition,

$$\text{every strongly spanning trailable graph is also spanning trailable.} \tag{1.2}$$

As shown explicitly in [35] (see Theorem 4.1.2 below) and implicitly in Theorem 4 of [9], every 4-edge-connected graph is strongly spanning trailable. However, it is routine to see that the Wagner graph H_8 depicted in Figure 1.3 below is spanning trailable but not strongly spanning trailable. Thus strongly spanning trailable and spanning trailable are not equivalent concepts in graphs with edge-connectivity at most 3. As $e = e'$ is possible, strongly spanning trailable graphs are supereulerian. The following Catlin-Jaeger Theorem indicates that it suffices to study supereulerian graphs with edge-connectivity at most 3.

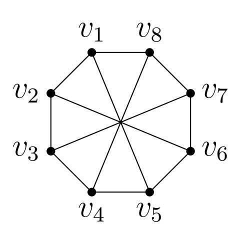


Figure 1.3

Theorem 1.1.5 *(Catlin [8] and Jaeger [23]) Every 4-edge-connected graph is supereulerian.*

The four cycle is an example that a supereulerian graph may not be spanning trailable. Luo, Chen and Chen [35] first explicitly studied spanning trailable graphs (called eulerian-connected graphs in [35]). The following theorem improves Theorem 1.1.5.

Theorem 1.1.6 *(Luo, Chen and Chen [35]) Every 4-edge-connected graph is spanning trailable.*

1.2 Catlin's Reduction Method

Let G be a graph and $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If $H \subseteq G$, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H, and is denoted by $PI_G(v_H)$.

Collapsible graphs were introduced by Catlin [8]. A graph G is **collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph G_R such that $O(G_R) = R$. Catlin showed in [8] that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The **reduction** of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is **reduced** if it does not contain nontrivial collapsible subgraphs.

Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 1.2.1 *Let G be a connected simple graph on $n \geq 1$ vertices.*
(ii) (Chen and Lai, Theorem 2.4 of [16]) If $n \leq 11$, $d_1(G) = 0$, $d_2(G) \leq 1$ and $F(G) \leq 3$, then the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{2,3}, P(10)\}$.

(iii) (Chen [14], see also Theorem 3.2 of [16]) If $n \leq 13$ and $\delta(G) \geq 3$, then either $G \in \mathcal{S}_{12}$, or the reduction of G is in $\{K_1,\ K_2,\ K_{1,2},\ K_{1,3},\ P(10)\}$.

Theorem 1.2.2 *Let G be a connected graph, H be a collapsible subgraph of G, v_H the vertex in G/H with $PI_G(v_H) = H$, and G' the reduction graph of G. Then each of the following holds.*
(i) (Theorems 3 and 8 of [8]) G is collapsible if and only if G/H is collapsible; and G is supereulerian if and only if G' is supereulerian. In particular, G is collapsible if and only if the reduction $G' = K_1$.
(ii) (Theorem 5 of [8]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(iii) (Theorem 8 of [8]) G' is simple, $girth(G') \geq 4$ and $\delta(G') \leq 3$.
(iv) (Lemma 1 of [7]) Every subdivision of K_4 with at most 6 vertices is collapsible. In particular, $K_{3,3}^-$ is collapsible, where $K_{3,3}^-$ is the graph obtained from $K_{3,3}$ by deleting an edge.
(v) (Theorem 1.3 of [12]) If G is connected and if $F(G) \leq 2$, then $G' \in \{K_1, K_2\} \cup \{K_{2,t} | t \geq 1\}$.
(vi) If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.
(vii) (Theorem 11 of [7]) For an integer $t > 0$, let $G(t)$ be the graph with $2t+3$ vertices and $3t+3$ edges with $V(G(t)) = \{x_0, x_1, ..., x_t, y_0, y_1, ..., y_t, v\}$, where $E(G(t))$ consists of the edges $\{x_0y_0, x_0v\} \cup \left(\bigcup_{i=1}^{t}\{x_iy_i, x_{i-1}x_i, y_{i-1}y_i\}\right)$ and exactly one edge in $\{vx_t, vy_t\}$. Then $G(t)$ is collapsible if and only if $G(t)$ is not bipartite.

Lemma 1.2.3 *Let G be a connected simple graph with $n \geq 3$ vertices.*
(i) (Li et al., Lemma 2.1 of [31]) If $n \leq 8$ and if $d_1(G) = 0$ and $d_2(G) \leq 2$, then the reduction of G is in $\{K_1, K_2, K_{2,3}\}$.
(ii) (Chen [14], also Theorem 2.5 of [16]) If G is reduced with $|V(G)| \leq 11$ and $\kappa'(G) \geq 3$, then $G = K_1$ or G is the Petersen graph.
(iii) (Theorem 2.4 of [16]) If G is a connected reduced graph with $6 \leq |V(G)| \leq 11$ and $F(G) \leq 3$, then either G is the Petersen graph, or G must have at least 3 vertices of degree at most 2.

1.3 Main Results

In the coming several chapters, we will present the following main results.

(1) Motivated by the Chinese Postman Problem, Boesch, Suffel, and Tindell in [3] proposed the supereulerian graph problem which seeks the characterization of graphs with a spanning eulerian subgraph. Pulleyblank in [41] showed that the supereulerian problem, even within planar graphs, is NP-complete. In this paper, we settle an open problem raised by An and Xiong in [1] on characterization of supereulerian graphs with small matching numbers. A well known theorem by Chvátal and Erdös in [17] states that if G satisfies $\alpha(G) \leq \kappa(G)$, then G is hamiltonian. Flandrin and Li in [19] showed that every 3-connected claw-free graph G with $\alpha(G) \leq 2\kappa(G)$ is hamiltonian. Our characterization is also applied to show that every 2-connected claw-free graph G with $\alpha(G) \leq 3$ is hamiltonian, with only one well characterized exceptional class.

(2) A graph is supereulerian if it has a spanning eulerian subgraph, and is spanning trailable if for any pair of edges $e, e' \in E(G)$, G has a spanning trail from e to e'. Luo et al in [35] proved that every 4-edge-connected graphs are trailable. In this paper, we show that the Wagner graph is the smallest 3-edge-connected non-spanning trailable graph, and we prove a characterization of the reduced graphs of an almost 3-edge-connected graph with at most 12 vertices, extending a former result of Chen in [14]. This characterization help us to prove that, under a condition on bounded length of longest paths, a 3-edge-connected graph is spanning trailable graphs if and only if it is not contractible to the Wagner graph.

(3) In Chapter 4, for a graph G and edges $e = u_1v_1, e' = u_2v_2 \in E(G)$, the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ by a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. A graph G is strongly spanning trailable if for any $e = u_1v_1, e' = u_2v_2 \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. Luo et alin [35] proved that every 4-edge-connected graph is spanning trailable. In this paper, we show that, for a 3-edge-connected graph G which is not the Wagner graph, if every pair of edges is joined by a longest path of length at most 8, then G is strongly spanning trailable.

(4) In Chapter 5, we use a discharge method to prove that every 3-connected, essentially 10-connected line graph is hamiltonian connected.

(5) In Chapter 6, we develop a cycle chain method to prove that every 3-edge-connected graph G is supereulerian if every 3-edge-cut of G intersects with short cycles in G. This is applied to the study of hamiltonian claw-free graphs, and provides a unified treatment with short proofs for several former results by Fujisawa and Ota in [20], by Kaiser et al in [24], and by Pfender in [40]. New sufficient conditions for hamiltonian claw-free graphs are also obtained.

Chapter 2

Supereulerian graphs with small matching number

2.1 Introduction

The main purpose of this chapter is to characterize supereulerian graphs with small matching number. For graph G with $\alpha'(G)$ small, the following have been proved.

Theorem 2.1.1 *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 2$. Following holds.*
(i) (Lai and Yan, [29]) Graph G is supereulerian if and only if G is not contractible to a $K_{2,t}$ for some odd integer $t \geq 3$.
(ii) (An and Xiong, [1]) Either G is collapsible, or G has a nontrivial collapsible subgraph H such that for some integer $t \geq 2$, $G/H \cong K_{2,t}$.
(iii) (An and Xiong, [1]) If $\kappa'(G) \geq 3$ and $\alpha'(G) \leq 5$, then G is supereulerian if and only if G is not contractible to the Petersen graph.

An and Xiong proposed a conjecture (Conjecture 12 in [1]), which can be restated as the following open problem. Chapter 2's main result is motivated by their conjecture.

Problem 2.1.2 *(An and Xiong, [1]) If $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$, determine the collection of graphs such that G is supereulerian if and only if G is not contractible to a member in this collection.*

A well known theorem by Chvátal and Erdös [17] states that if G satisfies $\alpha(G) \leq \kappa(G)$, then G is hamiltonian. Flandrin and Li in [19] showed that for 3-connected claw-free graphs, this assumption can be relaxed.

Theorem 2.1.3 *(E. Flandrin and H. Li, [19]) Every claw-free graph G with connectivity $\kappa(G) \geq 3$ and independence number $\alpha(G) \leq 2\kappa(G)$ is hamiltonian.*

In this chapter, we have determined a graph family $\mathcal{F}'$ and prove the following main results.

Theorem 2.1.4 *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$. Then G is supereulerian if and only if the reduction of G is not a member in $\mathcal{F}'$.*

Theorem 2.1.4 has an application to hamiltonian line graphs and hamiltonian claw-free graphs. Let $K_{2,3}$ be the complete bipartite graph with vertex bipartition $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$. For integers $s_1, s_2, s_3 \geq 1$, the graph $K_{2,3}^{s_1,s_2,s_3}$ is obtained from $K_{2,3}$ by attaching s_i pendant vertices adjacent to y_i, $(1 \leq i \leq 3)$. Theorem 2.1.4 implies the following.

Corollary 2.1.5 *Let G be a connected simple graph. If $\kappa(L(G)) \geq 2$ and $\alpha(L(G)) \leq 3$, then $L(G)$ is hamiltonian if and only if G is not a member in $\{K_{2,3}^{s_1,s_2,s_3} : s_1 \geq s_2 \geq s_3 > 0\}$.*

It has been a question whether Theorem 2.1.3 holds for 2-connected line graphs. A consequence of Theorem 2.1.4 answers this question.

Definition 2.1.6 *(**Ryjáček closure** [43]) If G is a claw-free graph, then there is a graph $cl(G)$ such that*

(i) G is a spanning subgraph of $cl(G)$
(ii) $cl(G)$ is a line graph of a triangle-free graph, and
(iii) the length of a longest cycle in G and in $cl(G)$ is the same.

Corollary 2.1.7 *Let G be a claw-free graph with $\kappa(G) \geq 2$ and $\alpha(G) \leq 3$. Then G is hamiltonian if and only if the Ryjáček closure of G is not isomorphic to $L(H)$, for some $H \in \{K_{2,3}^{s_1,s_2,s_3} : s_1 \geq s_2 \geq s_3 > 0\}$.*

2.2 Preliminaries

The following theorem summarizes the useful results on collapsible graphs and reduced graphs needed in our arguments.

Theorem 2.2.1 *Let G be a connected graph. Then each of the following holds.*
(i) (Catlin, Theorem 3 of [8]) Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.
(ii) (Lemma 2.3 of [12]) If $G \neq K_1$ is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.
(iii) (Catlin, Han and Lai, Theorem 1.3 of [12]) If $F(G) \leq 2$, then G is collapsible if and only if the reduction of G is not isomorphic to a K_2 or to a $K_{2,t}$ for some integer $t \geq 1$.
(iv) (Catlin, [8]) The reduction of G is reduced. In particular, the reduction of G is simple and contains no cycles of length 3.
(v) (Catlin, Lemma 3 of [8]) If G is collapsible, then any contraction of G is also collapsible.

To answer the question in Problem 2.1.2, we first describe the graph families $\mathcal{F}$ and $\mathcal{F}'$, where $\mathcal{F}'$ is the excluded graph family stated in Theorem 2.1.4.

Definition 2.2.2 *(The families $\mathcal{F}$ and $\mathcal{F}'$): Let $i, s_1, s_2, s_3, m, l, t$ be natural numbers with $t \geq 2$ and $i, m, l \geq 1$. Let C^i denote the cycle of length i. Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding*

s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3$ (mod 3). Define $C^6(s_1, s_2, s_3) = K_{1,3}(s_1, s_2, s_3) - a$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to u' only. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u'')$. Let $K''_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K''_{2,t}(u, u'')$. We shall use $K'_{2,t}$ and $K''_{2,t}$ for a $K'_{2,t}(u, u', u'')$ and a $K''_{2,t}(u, u', u'')$, respectively. Let $S_{m,l}$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K'_{2,l}(w, w', w'')$ by identifying u with w, and w'' with u'. Let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; and $J'(m, l) = J(m, l) - ww''$. Let $K_{2,3}(1, 2, 2)$ be the union of three internally disjoint (u, w)-paths of lengths 2, 3 and 3, respectively; and let $K^*_{2,3}(1, 2, 2)$ be obtained from $K_{2,3}(1, 2, 2)$ by adding a chord e to the 6-cycle joining two vertices of degree 2 so that no 3-cycle is resulted. Let $C^7 = v_1v_2v_3v_4v_5v_6v_7v_1$ denote a cycle of length 7. Define $J^7_1 = C^7 + v_1v_4$ and $J^7_2 = J^7_1 + v_2v_5 = C^7 + \{v_1v_4, v_2v_5\}$. See Figure 1 for examples of these graphs. Let

$$\begin{aligned}\mathcal{F} &= \{K_1\} \bigcup \Big(\{C^7, J^7_1, J^7_2, K_{2,3}(1,2,2), K^*_{2,3}(1,2,2)\} \bigcup \{K_{2,t} | t \geq 1\} \\ &\quad \bigcup \{K_{1,3}(s, s', s''), C^6(s, s', s'') | s, s', s'' \geq 0\} \bigcup \{S_{m,l} : m, l \geq 1\} \Big) \bigcap \{G | \kappa(G) \geq 2\},\end{aligned}$$

and define

$$\mathcal{F}' = \{G \in \mathcal{F} : \ G \text{ is non supereulerian.}\}$$

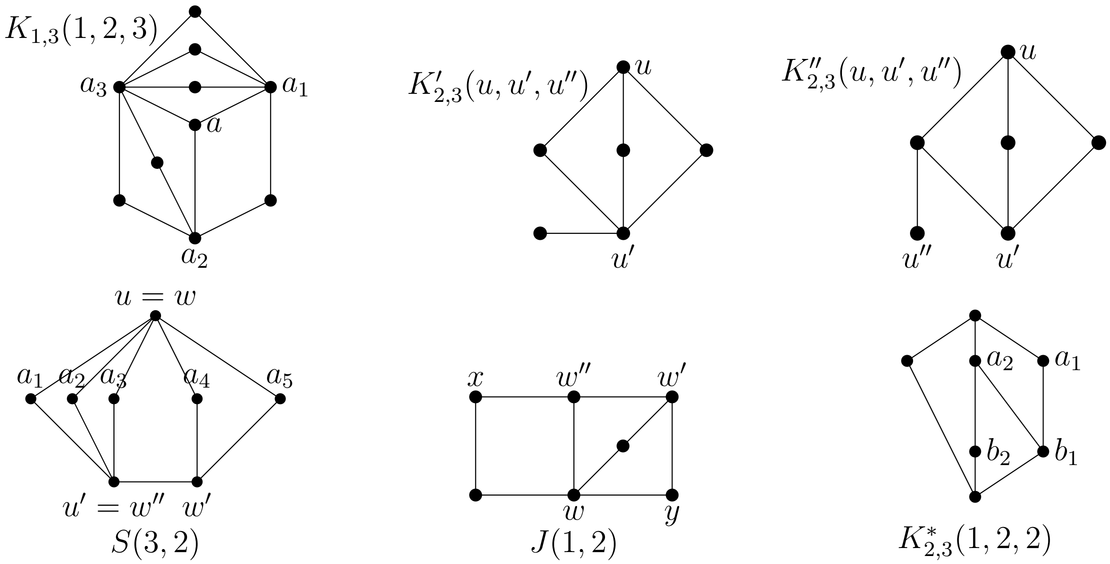


Figure 1: Some graphs in $\mathcal{F}$ with small parameters

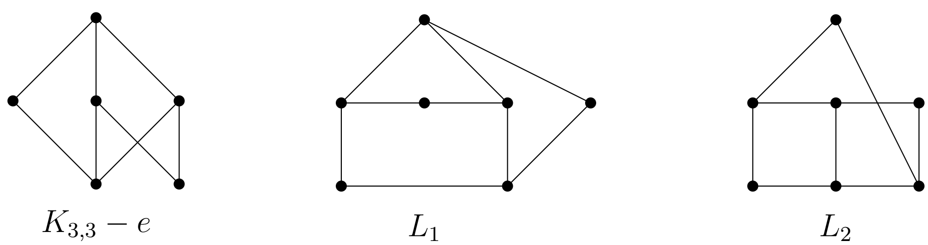


Figure 2: Some collapsible graphs: $K_{3,3}$, L_1 and L_2

Define $L_3 = K^*_{2,3}(1,2,2) + \{a_1b_2\}$.

Lemma 2.2.3 *The graphs $K_{3,3} - e$, L_1, L_2 and L_3 are collapsible.*

Proof: The graph $K_{3,3} - e$ is proved to be collapsible in Lemma 1 of [7]. By Theorem 11 in [7] we can directly get L_1 and L_2 are collapsible. L_3 is also collapsible by Theorem 3.3.2 (i).

Definition 2.2.4 *(Families $\mathcal{F}_1$ and $\mathcal{F}_2$) Define*

$\mathcal{F}_1 = \{C^7, K_{2,3}(1,2,2)\} \cup \{C^6(s,s',s''), K_{1,3}(s,s',s'') | s \geq s' > 0, s'' \geq 0\} \cup \{S_{m,l} | m \geq l \geq 1\}$.

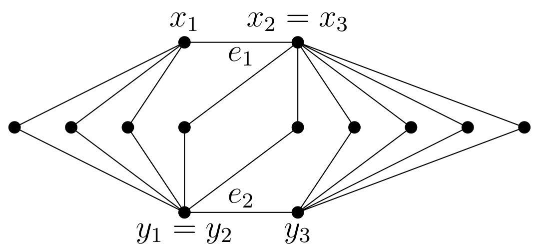


Figure 3: The graph $J(3,2,4)$

For integers $s_1, s_2, s_3 \geq 2$, Let $K_{2,s_1}(x_1,y_1)$, $K_{2,s_2}(x_2,y_2)$, $K_{2,s_3}(x_3,y_3)$ be three disjoint graphs such that for $i \in \{1,2,3\}$, $K_{2,s_i}(x_i,y_i)$ is isomorphic to K_{2,s_i} with x_i and y_i being the two nonadjacent vertices of degree s_i. The graph $J(s_1,s_2,s_3)$ is obtained by identifying y_1 with y_2 and x_2 with x_3, and by adding new edges $e_1 = x_1x_3$ and $e_2 = y_1y_3$. (See Figure 3 for an example). Note that $J(m,0,l) = J'(m,l)$.

Define

$\mathcal{F}_2 = \{K_1\} \cup (\mathcal{F} \cap \{\Gamma :\ \Gamma \text{ is essentially 4-edge-connected}\}) \cup \{J(s_1,s_2,s_3) | s_1 \geq s_3 \geq 3, s_2 \geq 2\}$.

Lemma 2.2.5 below is a key lemma in the proof of Theorem 2.1.4. It indicates that once a certain type of subgraph appears in G, then G must be in $\mathcal{F}$. The family $\mathcal{F}_2$ will be needed in Theorem 2.3.2 of the next section.

Lemma 2.2.5 *Let G be a reduced graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$. If G has a subgraph $H \in \mathcal{F}_1 - \{K_1\}$, then $G \in \mathcal{F}$.*

Proof. We first observe that if $H \in \mathcal{F}_1 - \{K_1, S_{1,1}\}$, then $\alpha'(H) \geq 3$. By contradiction, we assume that G is a counterexample to the lemma such that $|V(G)|$ is minimized.

Claim 1. $\kappa(G) \geq 2$.

By contradiction, assume that G has a cut vertex z. Then G has nontrivial connected subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cup V(G_2) = \{z\}$. Since $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$, both $\kappa'(G_1) \geq 2$ and $\kappa'(G_2) \geq 2$, and both $\alpha'(G_1) \leq 3$ and $\alpha'(G_2) \leq 3$. Since every graph in $\mathcal{F}_1$ is 2-connected, we may assume that H is a subgraph of G_1. If $H = S_{1,1}$, which is a 5-cycle, then H has a matching M_1 of size 2 such that M_1 is not incident with the vertex z. Since G_2 is a 2-edge-connected nontrivial reduced graph, G_2 has a cycle of length at least 4, and so G_2 has a matching M_2 of size at least 2. As M_1 is not incident with z, it follows that $M_1 \cup M_2$ is a matching of size at least 4, contrary to $\alpha'(G) \leq 3$. This contradiction proves Claim 1.

Since G is a counterexample, G has a subgraph $H \in \mathcal{F}_1 - \{K_1\}$, but $G \notin \mathcal{F}$. We assume that H is maximal, in the sense that H is not properly contained in another subgraph of G in $\mathcal{F}_1$. We have the following observations.

Observation 2.2.6 *Let H be a subgraph of G.*
(i) If $H \in \{C^7, K_{2,3}(1,2,2)\} \cup \{K_{1,3}(s,s',s'') | s \geq s' \geq s'' > 0\}$, and if G has an edge with exactly one end in H, then $\alpha'(G) \geq 4$.
(ii) If $H = K_{1,3}(s,s',s'')$ with $s \geq s' \geq s'' > 0$, then adding any additional edge to join two distinct vertices in H will result in a collapsible graph. Since G is reduced, we conclude that in this case $G = K_{1,3}(s,s',s'')$.
(iii) If G is spanned by $H = K_{2,3}(1,2,2)$, then by Lemma 6.2.1 and by the assumption that G is reduced, G cannot have L_3 or a 3-cycle as a subgraph. By inspection, $G \in \{K_{2,3}(1,2,2),\ K^*_{2,3}(1,2,2)\}$.
(iv) If G is spanned by C^7, then as G is reduced, C^7 can have at most 2 chords in G. (This is because, if C^7 with 3 chords, $F(G[V(C^7)]) \leq 2(7) - 10 - 2 = 2$, and so by Theorem 6.2.2(iii), $G[V(C^7)]$ is not reduced.) As G has no cycles of length at most 3, $G \in \{C^7, J^7_1, J^7_2\}$.

Only Observation 2.2.6 (i) when $H = K_{1,3}(s,s',s'')$ needs an explanation. We use the notations in Figure 1. Let xy denote an edge incident with a vertex $x \in V(H)$ and $y \notin V(H)$. If x has degree 2 in H or if $x = a$, then $G[E(H) \cup \{xy\}]$ has 4 independent edges. Therefore, we assume that any edges in G incident with exactly one vertex in H must be incident with one in $\{a_1, a_2, a_3\}$. Since $\kappa(G) \geq 2$, and since $y \in V(G) - V(H)$, we may assume that G has a path P with $y \in V(P)$ such that $V(P) \cap V(H) = \{a_i, a_j\}$ for some $i \neq j$ and $1 \leq i, j \leq 3$. Since H is maximal, $|E(P)| \geq 3$, and so $G[E(H) \cup E(P)]$ has 4 independent edges. This verifies the observation.

Recall that $c(G)$ is the length of a longest cycle in G. By Observation 2.2.6, and since any cycle of length at least 8 has 4 independent edges, we may assume that

$$G \text{ has no subgraph in } \{C^7, K_{2,3}(1,2,2)\} \cup \{K_{1,3}(s, s', s'') : s \geq s' \geq s'' > 0\} \text{ and } c(G) \leq 6. \tag{2.1}$$

By (2.1), we only need to examine the cases when $H \in \{C^6(s, s', s'')|s \geq s' \geq s'' \geq 0\} \cup \{K_{1,3}(s, s', 0)|s \geq s' > 0\} \cup \{S_{m,l}, |m \geq l \geq 1\}$. We make another observation.

Observation 2.2.7 *Let e' be an edge in $E(G) - E(H)$ joining two distinct vertices in H. Let $H' = G[E(H) \cup \{e'\}]$ be the edge induced subgraph of G. Each of the following holds.*
(i) If $H \in \{K_{1,3}(s, s', 0) : s \geq s' > 0\} \cup \{S_{m,l}, |m \geq l \geq 1\}$, then H' has a K_3 or a $K_{3,3} - e$, and so G is not reduced.
(ii) If $H = C^6(s, s', s'')$ for some $s \geq s' \geq s'' \geq 0$ either with $s'' > 0$, or with $s'' = 0$ and $s' \geq 2$, then either H' has a K_3, or H' is a $K_{1,3}(t, t', t'')$ with $t \geq t' > 0$ and $t'' \geq 0$, or is an $S_{m,l}$, with $m \geq l \geq 2$, contrary to the maximality of H.

By (2.1) and by Observation 2.2.7, we proceed the proof of the lemma by examining the following cases.

Case 1. $H = K_{1,3}(s, s', 0)$ with $s \geq s' > 0$.

Since $G \neq H$ and by Observation 2.2.7 (i), $V(G) - V(H)$ has a vertex z. By $\kappa(G) \geq 2$ and by $\alpha'(G) \leq 3 = \alpha'(H)$, there exist distinct vertices $u, v \in N_G(z) \cap V(H)$. Since $u, v \in N_G(z)$, the edge induced subgraph $H' = G[E(H) \cup \{uz, vz\}]$ of G either has one of $\{K_3, K_{3,3} - e, L_2\}$ as a subgraph, contrary to the assumption that G is reduced; or is a $K_{1,3}(t, t', t'')$ with $t \geq t' > 0$ and $t'' \geq 0$ properly containing H, contrary to the maximality of H; or $\alpha'(G) \geq \alpha'(H') \geq 4$. These contradictions complete the proof for Case 1.

Case 2. $H = S_{m,l}$ for some $m \geq l \geq 1$ and with $m + l \geq 3$ maximized.

Since $G \neq H$ and by Observation 2.2.7 (i), $V(G) - V(H)$ has a vertex z. By $\kappa(G) \geq 2$ and by $\alpha'(G) \leq 3 = \alpha'(H)$, there exist distinct vertices $u, v \in N_G(z) \cap V(H)$. Since $u, v \in N_G(z)$, the edge induced subgraph $H' = G[E(H) \cup \{uz, vz\}]$ of G either has one of $\{C^7, K_3, L_1, L_2\}$ as a subgraph, contrary to (2.1) or the assumption that G is reduced; or is an $S_{m',l'}$ with $m \geq l' > 0$ properly containing H, contrary to the maximality of H. These contradictions complete the proof for Case 2.

Case 3. $H = C^6(s, s', s'')$ is a subgraph of G for some $s \geq s' \geq s'' \geq 0$ with either $s'' > 0$ or $s'' = 0$ and $s' \geq 2$.

Since $G \neq H$ and by Observation 2.2.7 (i), $V(G) - V(H)$ has a vertex z. By $\kappa(G) \geq 2$ and by $\alpha'(G) \leq 3 = \alpha'(H)$, there exist distinct vertices $u, v \in N_G(z) \cap V(H)$. Since $u, v \in N_G(z)$, the edge induced subgraph $H' = G[E(H) \cup \{uz, vz\}]$ of G either has one of $\{C^7, K_{2,3}(1, 2, 2), K_3\}$ as a subgraph, contrary to (2.1) or the assumption that G is reduced; or is $C^6(t, t', t'')$ with $t \geq t' \geq t'' \geq 0$ and with either $t'' > 0$ or both $t'' = 0$ and $t' \geq 2$, properly containing H, contrary to the maximality of H; or $\alpha'(G) \geq \alpha(H') \geq 4$. These contradictions complete the proof for Case 3.

Case 4. $H = C^6(s, 1, 0)$ is a subgraph of G with either $s > 0$, and G does not have a subgraph in Cases 1, 2 or 3.

Let $P = v_1v_2v_3v_4v_5$ be a path of length 4 in H such that $d_H(v_1) = 1$, and $N_H(v_3) \cap N_H(5)$ has s vertices of degree 2 in H. Since $\kappa(G) \geq 2$, $N_G(v_1) - \{v_2\}$ has a vertex z. Since G does not have a subgraph in Cases 1, 2 or 3, and by (2.1), $z \notin V(H)$. Since $\kappa(G) \geq 2$, the edges v_1z and v_3v_4 are in a cycle of G, and so $G - v_1$ has a path Q from z to a vertex $w \in V(H) - \{v_1, v_2\}$ such that $V(H) \cap V(Q) = \{w\}$. If $|E(Q)| \geq 3$, then G has a cycle of length at least 6, contrary to (2.1) or to the assumption that G does not have subgraph in Cases 1, 2 or 3. Hence $|E(Q)| = 2$ and so we must have $s = 1$,

$d_H(v_5) = 1$, $H = P$ and $w = v_4$. By Symmetry, there must be a vertex $z' \in V(G) - V(H)$ such that $z'v_5, z'v_2 \in E(G)$. Thus $G[V(P) \cup \{z, z'\}]$ contains a $K_{2,3}(1, 2, 2)$, contrary to (2.1). These contradictions prove Case 4, and the proof for the lemma is done. □

2.3 Proof of Theorem 2.1.4

In this section we will prove the following theorem, which together with Theorem 2.2.1(i), implies Theorem 2.1.4.

Theorem 2.3.1 *Let G be a graph with $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$. Then the reduction of G is in $\mathcal{F}$.*

The proof of Theorem 2.3.1 needs a useful tool stated as Theorem 2.3.2 below. We shall need the graphs introduced in Definitions 2.2.2 and 2.2.4. By Definition 2.2.4, $\mathcal{F}_2 = \{K_1\} \cup \{K_{2,t} : t \geq 3\} \cup K_{1,3}(s, s', s'') \ |s \geq s' \geq 2, s'' \geq 0\} \cup \{S_{m,l} | m \geq l \geq 2\}$ $\cup \{C^6(s, s', s'')|$ either $s \geq s' \geq 2$ and $s'' \geq 1$ or $s \geq s' \geq 3$ and $s'' = 0\} \cup \{K^*_{2,3}(1, 2, 2)\}$ $\cup \{J(s_1, s_2, s_3)| \ s_1 \geq s_3 \geq 3, s_2 \geq 2\}$.

Theorem 2.3.2 *Let G be a 2-edge-connected graph. Each of the following holds.*
(i) Suppose that $c(G) \leq 5$. Then G is collapsible if and only if the reduction of G is not a member in $\{K_{2,t}, S_{m,l}\}$, where $l, m \geq 1$ and $t \geq 2$ are integers.
(ii) Suppose that G is essentially 4-edge-connected graph with $c(G) \leq 6$. Then G is collapsible if and only if the reduction of G is not in $\mathcal{F}_2$.

Proof. Since any graph in $\{K_{2,t}, S_{m,l}\}$ is not collapsible, by Theorem 6.2.2 (v), if the reduction of G is in $\{K_{2,t}, S_{m,l}\}$, then G is not collapsible.

To prove the necessity, we argue by contradiction to assume that $G \neq K_1$ is reduced, but $G \notin \{K_{2,t}, S_{m,l}\}$. By Theorem 2.2.1 ($iv$), G has no cycles of length at most 3. Suppose first that $c(G) = 4$. Then G contains a $K_{2,2}$ as a subgraph. Let $t \geq 2$ be the maximum number such that G has $K_{2,t}$ as a subgraph. Since $G \neq K_{2,t}$, $V(G) - V(K_{2,t})$ has a vertex v. Since $\kappa(G) \geq 2$, G has a path P from a vertex x to a vertex y with $v \in V(P)$ such that $V(P) \cap V(K_{2,t}) = \{x, y\}$. As $v \in V(P) - V(K_{2,t})$, $|E(P)| \geq 2$. If x and y are adjacent in $K_{2,t}$, then G has a 5-cycle, contrary to $c(G) = 4$. Therefore, we must have $x, y \in N_G(v)$, and x and y are of distance 2 in $K_{2,t}$. It then follows that either $c(G) \geq 5$, contrary to $c(G) = 4$; or G has a $K_{2,t+1}$, contrary to the maximality of t.

Hence $c(G) = 5$, and so G contains a $C^5 = S_{1,1}$ as a subgraph. Thus G has $S_{m,l}$ as a subgraph with $m \geq l \geq 1$ such that $m + l$ is maximized. Since $G \neq S_{m,l}$, $V(G) - V(S_{m,l})$ has a vertex v'. By $\kappa(G) \geq 2$, G has a path P' from a vertex x' to a vertex y' with $v' \in V(P')$ such that $V(P') \cap V(S_{m,l}) = \{x', y'\}$.

If $x'y' \in E(S_{m,l})$, then G has a cycle of length at least 6, contrary to $c(G) = 5$. Therefore, the distance between x' and y' in $S_{m,l}$ is 2. It follows that $G[V(S_{m,l}) \cup \{v'\}]$ either has a cycle of length at least 6, contrary to $c(G) = 5$; or is isomorphic to an $S_{m+1,l}$ or an $S_{m,l+1}$, contrary to the maximality of $m + l$. This completes the proof for Theorem 2.3.2 (i).

To prove Theorem 2.3.2(ii), we argue by contradiction to assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{2.2}$$

By (2.2), by the assumption that G is essentially 4-edge-connected, and by Theorem 2.3.2 (i), we further assume that

$$G \text{ is reduced, } \kappa(G) \geq 2,\ D_2(G) \text{ is an independent set, and } c(G) = 6. \tag{2.3}$$

Let $C^6 = v_1v_2v_3v_4v_5v_6v_1$ be a longest cycle of G. Since G is reduced and by Lemma 6.2.1, G contains no K_3 or $K_{3,3} - e$ as a subgraph. Thus if C^6 has chords, then C^6 has exactly one chord, isomorphic to a $J(1,1) = K_{1,3}(1,1,0)$. Note that $C^6 = J'(1,1)$. Hence G has a subgraph $H \in \{J(m,l)|m \geq l \geq 1\} \cup \{J'(m,l)|m \geq l \geq 1\} \cup \{C^6(s,s',s'')$ $|s \geq s' \geq s'' > 0\} \cup \{K_{1,3}(s,s',s'')|s \geq s' > 0, s'' \geq 0\}$. Choose such an H so that

$$|V(H)| + |E(H)| \text{ is maximized.} \tag{2.4}$$

If $G = H$, then as G is essentially 4-edge-connected, $G \neq J'(m,l)$ with $m \geq l \geq 1$. Since $\alpha'(G) \leq 3$, $G \neq J(m,l)$ for $m \geq l \geq 2$. As $J(m,1) = K_{1,3}(m,1,0)$, we conclude that if $G = H$, then $G \in \mathcal{F}_2$, contrary to (2.2).

Hence $G \neq H$. By (2.4), $V(G) - V(H)$ has a vertex z. As $\kappa(G) \geq 2$, G has a path Q with $z \in V(Q)$, and $V(Q) \cap V(H) = \{u, v\}$ for some distinct u and v. Since $\alpha'(H) = 3 = \alpha'(G)$ and since G is reduced, $u, v \in N_G(z)$ and u and v are not adjacent in H. In the arguments below, we will use the notation in Figure 1.

If $H \in \{K_{1,3}(s,s',s'')|s \geq s' > 0, s'' \geq 0\} \cup \{C^6(s,s',s'')|s \geq s' \geq s'' > 0\}$, then either $u, v \in \{a_1, a_2, a_3\}$, whence (2.4) is violated; or (by symmetry) $H = K_{1,3}(s,1,0)$, $u = a$ and $v \in D_2(H)$, whence (2.4) is violated; or $\{u, v\} - \{a_1, a_2, a_3\} \neq \emptyset$, and $G[V(H) \cup \{z\}]$ contains a cycle of length at least 7, contrary to (2.3).

Assume that $H \in \{J(m,l)|m \geq l \geq 1\}$. Since $J(m,1) = K_{1,3}(m,1,0)$, we assume that $m \geq l \geq 2$. If $u, v \in D_2(H)$, then $G[V(H) \cup \{z\}]$ contains a cycle of length at least 7, contrary to (2.3). Hence we assume that $u \notin D_2(H)$. Then $G[V(H) \cup \{z\}]$ either violates (2.4), or contains a cycle of length at least 7, contrary to (2.3).

Finally we assume that $H \in \{J'(m,l)|m \geq l \geq 1\}$. As $J'(m,1) = C^6(m,1,1)$, we may assume $m \geq l \geq 2$. Since $J'(m,l) = J(m,0,l)$, we may assume that $H = J(s,s',s'')$ with $s \geq s'' \geq 2$ and $s' \geq 0$, and $s+s'+s''$ maximized. If $\{u,v\} \cap D_2(H) \neq \emptyset$, then $G[V(H) \cup \{z\}]$ also contains a cycle of length at least 7, contrary to (2.3). Hence $u, v \in V(H) - D_2(H)$. It follows that $G[V(H) \cup \{z\}]$ contains a $J(t,t',t'')$ with $t + t' + t'' = s + s' + s'' + 1$, contrary to the maximality of H. This completes the proof of Theorem 2.3.2 (ii). □

Proof of Theorem 2.3.1. By contradiction, we assume that

$$G \text{ is a counterexample to Theorem 2.3.1 with } |V(G)| \text{ minimized.} \tag{2.5}$$

By Theorem 4.1.2 and by (2.5), G is reduced. By the assumption $\alpha'(G) \leq 3$, $c(G) \leq 7$. If G has a C^7 or a C^6 as a subgraph, then by Lemma 2.2.5, $G \in \mathcal{F}$, contrary to (2.5). Therefore, we must have $c(G) \leq 5$. By Theorem 2.3.2 (i), $G \in \mathcal{F}$, contrary to (2.5). This completes the proof. □

2.4 Proofs of Corollaries 2.1.5 and 2.1.7

To prove Corollary 2.1.5, we also need the following theorem of Harary and Nash-Williams, which reveals a close relationship between eulerian subgraphs in G and Hamilton cycles in $L(G)$.

Theorem 2.4.1 *(Harary and Nash-Williams [21]) Let G be a connected graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph H such that $E(G - V(H)) = \emptyset$.*

Let G be a graph such that $\kappa(L(G)) \geq 2$, $E_1(G)$ denote the set of pendant edges (edges incident with a vertex in $D_1(G)$) of G, and let $\Gamma = G/E_1(G)$. Let Γ' denote the reduction of Γ, and define $\Lambda(\Gamma') = \{v \in V(\Gamma')$ such that v is the contraction image of a nontrivial connected subgraph of $G\}$. Using Theorem 2.4.1, Shao proved the following.

Proposition 2.4.2 *(Shao, Section 1.4 of [45]) If Γ' has an eulerian subgraph H with $\Lambda(\Gamma') \subseteq V(H)$, then $L(G)$ is hamiltonian.*

Proof of Corollary 2.1.5 Let G be a graph with $\kappa(L(G)) \geq 2$ and $\alpha(L(G)) \leq 3$. Since $\kappa(L(G)) \geq 2$ and $\alpha(L(G)) \leq 3$, $\kappa'(\Gamma) \geq 2$ and $\alpha'(\Gamma) \leq 3$. Let Γ' be the reduction of Γ. If Γ' is supereulerian, then by Proposition 2.4.2, $L(G)$ is hamiltonian. Thus by Theorem 2.1.4, we may assume that $\Gamma' \in \mathcal{F}'$. By the definition of $\mathcal{F}'$, we observe that

$$\forall F \in \mathcal{F}', \text{ and } \forall v \in D_2(F),\ F \text{ has an eulereian subgraph } H \text{ such that } V(F) - v \subseteq V(H). \tag{2.6}$$

By (2.6) and Proposition 2.4.2, if $\Gamma' \in \mathcal{F}'$ such that $D_2(\Gamma') - \Lambda(\Gamma') \neq \emptyset$, then $L(G)$ is hamiltonian. Thus $L(G)$ is not hamiltonian only if $D_2(\Gamma') \subseteq \Lambda(\Gamma')$. Therefore, each vertex in $D_2(\Gamma')$ contains an edge of G, and these edges are independent. Hence $|D_2(\Gamma')| \leq$

$\alpha'(\Gamma) \leq 3$, and so as $\Gamma' \in \mathcal{F}'$, we conclude that $L(G)$ is not hamiltonian only if $\Gamma' = K_{2,3}$ with $D_2(\Gamma') \subseteq \Lambda(\Gamma')$. Suppose that one vertex v in $D_2(\Gamma')$ is the contraction image of a nontrivial collapsible graph H. Let $A_G(H)$ denote the vertices of H that are adjacent to vertices in $V(G)-V(H)$ in G. Thus $|A_G(H)| \leq d_{\Gamma'}(v) \leq 3$. Since H is a simple collapsible graph, $|E(H)| \geq 3$, and so there must be an edge $e_1 \in E(H)$ and an edge $e_2 \in E_{\Gamma'}(v)$ such that $\{e_1, e_2\}$ is a matching in G. Let e_3, e_4 be two edges in the preimages of the two vertices of $D_2(\Gamma') - \{v\}$. Then $\{e_1, e_2, e_3, e_4\}$ would be a matching of G, contrary to $\alpha'(G) \leq 3$. With a similar argument, the two vertices of degree 3 in Γ' must be trivial, and so $G \cong K_{2,3}^{s_1,s_2,s_3}$ for some $s_1, s_2, s_3 > 0$. This proves Corollary 2.1.5. □.

A vertex $v \in V(G)$ is **locally connected** if $G[N_G(v)]$ is connected. Following the definition given by Ryjácěk ([43]), a graph H is the **closure** of a claw-free graph G, denoted by $H = cl(G)$, if both of the following hold.
(A) There is a sequence of graphs $G_1, ..., G_t$ such that $G_1 = G, G_t = H, V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \bigcup \{uv | u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected non-complete $G_i[N_{G_i}(x_i)]$, for $i = 1, ..., t-1$
(B) No vertex of H has a connected non-complete neighborhood.

Theorem 2.4.3 *(Ryjáček, [43]) Let G be a claw-free graph. Then*
(i) $cl(G)$ is uniquely determined.
(ii) $cl(G)$ is the line graph of a triangle-free graph.
(iii) G is hamiltonian if and only if $cl(G)$ is hamiltonian

Proof of Corollary 2.1.7 By Theorem 2.4.3, we may assume that for some simple graph H, $cl(G) = L(H)$. As adding edge to a graph does not increase the independence number α and does not decrease the connectivity κ, both $\kappa(cl(G)) \geq \kappa(G) \geq 2$ and $\alpha(cl(G)) \leq \alpha(G) \leq 3$ hold. By Corollary 2.1.5, $cl(G) = L(H)$ is hamiltonian if and only if $H \notin \{K_{2,3}^{s_1,s_2,s_3} : s_1 \geq s_2 \geq s_3 > 0\}$. □

Chapter 3

On 3-edge-connected Strongly Spanning Trailable Graphs

3.1 Introduction

In order to apply Catlin's reduction method by contracting collapsible subgraphs, identifying small reduced graphs are of particular importance ([11, 15]). Let $P(10)$ denote the Petersen graph, and let $\mathcal{S}_{12}$ denote the family of supereulerian graphs on 12 vertices. Chen proved the following useful results.

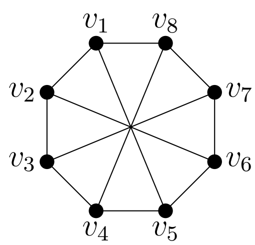
Theorem 3.1.1 *Let G be a connected simple graph on $n \geq 1$ vertices.*
(ii) (Chen and Lai, Theorem 2.4 of [16]) If $n \leq 11$, $d_1(G) = 0$, $d_2(G) \leq 1$ and $F(G) \leq 3$, then the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{2,3}, P(10)\}$.
(iii) (Chen [14], see also Theorem 3.2 of [16]) If $n \leq 13$ and $\delta(G) \geq 3$, then either $G \in \mathcal{S}_{12}$, or the reduction of G is in $\{K_1,\ K_2,\ K_{1,2},\ K_{1,3},\ P(10)\}$.

The graph H_8 and $K_{2,3}^+$, are defined in Figure 1. The graph H_8 is often known as the **Wagner graph**. A relaxation of Theorem 3.1.1(ii) is obtained in this chapter.

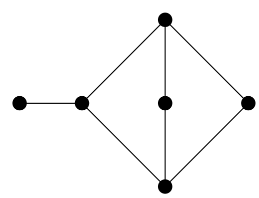
Theorem 3.1.2 *Let G be a connected simple graph with $n \leq 12$ vertices and with $d_1(G) = 0$ and $d_2(G) \leq 1$. Then the reduction of G is either in $\{K_1, K_2, K_{1,2}, K_{2,3}, K^+_{2,3}, P(10), P(10)(e)\}$ or G is a supereulerian graph on 12 vertices.*

By inspection, H_8 is not strongly spanning trailable. In fact, we have the following conclusion.

Proposition 3.1.3 *Let G be a 3-edge-connected non strongly spanning trailable graph. Each of the following holds.*
(i) $|V(G)| \geq 8$.
(ii) If $|V(G)| = 8$, then $G \cong H_8$.



H_8



$K^+_{2,3}$

Figure 1: The graph H_8 and $K^+_{2,3}$

Since spanning trailable graphs are supereulerian, and since deciding 3-edge-connected supereulerian graphs is NP-complete, characterizing 3-edge-connected spanning trailable graphs will also be difficult. In this chapter, we prove the following, which implies that under the given longest path length condition, G is not strongly spanning trailable if and only if G can be contractible to H_8 in which the preimage of every vertex of H_8 is nontrivial.

Theorem 3.1.4 *Let G be a graph with $\kappa(G) \geq 2$ and $\kappa'(G) \geq 3$ such that G is not contractible to H_8. For any distinct edges $e, e' \in E(G)$, if every longest $(v_e, v_{e'})$-path P in*

$G(e, e')$ is not spanning and satisfies $|E(P)| \leq 8$, then one of the following holds:
(i) $G(e, e')$ is not reduced.
(ii) $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail.

For an integer $n \geq 0$, let P_n denote the path on n vertices, and Z_7 denote the graph obtained by identifying a vertex of a K_3 and a P_8. Let $\mathcal{F}$ denote the family of graphs that are obtained from H_8 by attaching at least one pendant edge to each vertex of H_8; and let H_8' denote the graph obtained from H_8 by attaching exactly on pendant edge to each vertex of H_8. In a separate paper, Theorem 3.1.4 will be applied to show that every 3-connected line graph without an induced Z_7 is hamiltonian-connected if and only if it is not the line graph of H_8'; and that every 3-connected line graph without an induced P_{10} is hamiltonian-connected if and only if it is not the line graph of a member in $\mathcal{F}$.

This chapter is organized as follows: In Sections 2 and 3, we present needed tools in our proofs for the main results. In the last two sections, we will prove the main results.

3.2 Collapsible Graphs

We will present some basic properties of collapsible graph in this section. By $H \subseteq G$ we mean that H is a subgraph of a graph G. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) \mid vu \in E(G)\}$, and for $X \subseteq V(G)$, $N_G(X) = \cup_{x \in X} N_G(x)$. The subscript G might be omitted if G is understood from the context. The next theorem summarizes the properties needed in our arguments in the proofs.

Theorem 3.2.1 *(Catlin, [8]) Let G be a connected graph, H be a collapsible subgraph of G, v_H the vertex in G/H with $PI_G(v_H) = H$, and G' the reduction graph of G. Let $K_{3,3}^-$ denote the graph obtained from $K_{3,3}$ by deleting an edge. Then each of the following holds.*
(i) (Theorem 3 of [8]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction $G' = K_1$.
(ii) (Theorem 5 of [8]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(iii) (Theorem 8 of [8]) G' is simple, girth$(G') \geq 4$ and $\delta(G') \leq 3$.

(iv) (Theorem 8 of [8]) G is supereulerian if and only if G' is supereulerian.
(v) (Theorem 8 of [8]) If L' if an open (or closed, respectively) trail of G/H such that $v_H \in V(L')$, then G has an open (or closed, respectively) trail L with $E(L') \subseteq E(L)$ and $V(H) \subseteq V(L)$.
(vi) (Lemma 1 of [7]) Every subdivision of K_4 with at most 6 vertices is collapsible. In particular, $K_{3,3}^-$ is collapsible.
(vii) (Theorem 1.3 of [12]) If G is connected and if $F(G) \leq 2$, then the reduction of G must be in $\{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$.

The **symmetric difference** of two sets X and Y, is

$$X \oplus Y = (X \cup Y) - (X \cap Y).$$

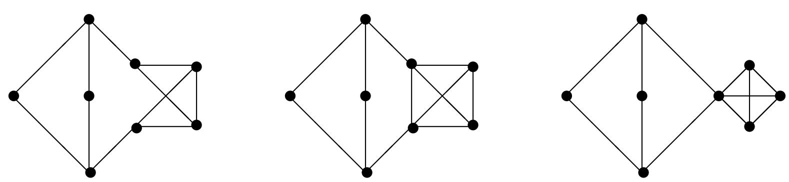


Figure 2: Non collapsible graphs in Lemma 1.2.2

Lemma 3.2.2 *Each of the following holds.*
(i) (Li et al, Lemma 2.1 of [31]) Let G be a connected simple graph with $n \leq 8$ vertices and with $D_1(G) = \emptyset$, $|D_2(G)| \leq 2$. Then either G is one of three graphs depicted in Figure 2, or the reduction of G is K_1 or K_2.
(ii) Let $u, w \in V(G)$, H be a collapsible subgraph of G, and $G' = G/H$. Let v_H denote the vertex in G' onto which H is contracted, and

$$u' = \begin{cases} u & \text{if } u \notin V(H) \\ v_H & \text{if } u \in V(H). \end{cases}, \text{ and } w' = \begin{cases} u & \text{if } w \notin V(H) \\ v_H & \text{if } w \in V(H). \end{cases}.$$

If G' has a (u', w')-trail L' containing v_H, then G has a (u, w)-trail L such that $(V(L') - \{v_H\}) \cup V(H) \subseteq V(L)$.

Proof. (ii). Let L' be a (u', w')-trail of G'. By the definition of contractions, $E(L') \subseteq E(G)$. Let L'' be the spanning subgraph of G with edge set $E(L')$ and define

$$R = \{v \in V(H) \mid \ d_{L''}(v) \text{ is odd } \} \oplus \{u, w\}.$$

Since L' is a (u', w')-trail of G', $d_{L'}(v_H)$ is odd if and only if $|V(H) \cap \{u, w\}| = 1$. It follows that $|R| \equiv 0 \pmod 2$. Since H is collapsible, H has a spanning connected subgraph H_R such that $O(H_R) = R$. Let $L = G[E(L') \cup E(H_R)]$ be an edge induced subgraph of G. Since L' contains v_H, L is connected. By the definition of R and the choice of H_R, $O(L) = \{u, w\}$, and so L satisfies the conclusion of (ii). □

Lemma 3.2.3 *Let G be a graph with $\kappa'(G) \geq 3$, and let $G_1, G_2, \cdots, G_k$ be the blocks of G. Then the following are equivalent.*
(i) G is strongly spanning trailable.
(ii) For every $i = 1, 2, \cdots, k$, G_i is strongly spanning trailable.

Proof. Since each block of G is also 3-edge-connected, (i) implies (ii). To prove (ii) implies (i), we argue by induction on k, the number of blocks of G. As (ii) trivially implies (i) when $k = 1$, we assume that $k > 1$ and for any graph with fewer than k blocks, (ii) implies (i).

Since $k \geq 2$, G has two connected subgraphs H and L and a vertex z_0 such that $G = H \cup L$ and $V(H) \cap V(L) = \{z_0\}$. Let $e, e' \in E(G)$. If $\{e, e'\} \cap E(L) = \emptyset$, then by induction, $H(e, e')$ has a spanning $(v_e, v_{e'})$-trail Q_1. By induction, for any edge $e'' \in E(L)$, $L(e'')$ has a spanning $(v_{e''}, v_{e''})$-trial, and so L has a spanning closed trail Q_2. It follows that $Q = G[E(Q_1) \cup E(Q_2)]$ is a spanning $(v_e, v_{e'})$-trail of G. The proof for the case when $\{e, e'\} \subseteq E(L)$ is similar, and will be omitted. Hence we may assume that $e \in E(H)$ and $e' \in E(L)$.

Since $\kappa'(H) \geq \kappa'(G) \geq 3$, and so H has an edge $e'' \in E_H(z_0) - \{e\}$. By induction, H has a spanning closed $(v_e, v_{e''})$-trail T_1'. Assume that $e'' = z_0 w$. Define

$$T_1 = \begin{cases} T_1' - z_0 v(e'') & \text{if } z_0 v(e'') \text{ is the last edge in } T_1' \\ H[E(T_1' - v(e'')) \cup \{e''\}] & \text{if } w v(e'') \text{ is the last edge in } T_1' \end{cases}.$$

Thus T_1 is a spanning $(v(e), z_0)$-trail of H. Similarly, L has a spanning $(z_0, v(e'))$-trail T_2. It follows $T = T_1 \cup T_2$ is a spanning $(v_e, v_{e'})$-trail. □

Lemma 3.2.4 *Let G be a graph with $\kappa'(G) \geq 3$ and $\kappa(G) \geq 2$. Then for any connected subgraph W of G with $|V(W)| \geq 2$, and for any $v \in V(G) - V(W)$, G has three edge-disjoint paths (v, w_i)-paths Q_i $(1 \leq i \leq 3)$ such that*

$$V(Q_i') \cap V(P) = \{w_i\}, \ \text{and} \ |\{w_1, w_2, w_3\}| \geq 2. \tag{3.1}$$

Proof. Since $\kappa'(G) \geq 3$, by Menger Theorem (Theorem 9.7 of [5]), for some vertices $w_1', w_2', w_3' \in V(W)$, G has edge-disjoint (v, w_i')-paths Q_i such that $V(Q_i) \cap V(W) = \{w_i'\}$, $i \in \{1, 2, 3\}$. If $|\{w_1', w_2', w_3'\}| \geq 2$, then we are done with $\{w_1, w_2, w_3\} = \{w_1', w_2', w_3'\}$. Therefore, we assume that $|\{w_1', w_2', w_3'\}| = 1$. Thus $w_1' = w_2' = w_3'$. Since $|V(W)| \geq 2$, $W - w_1'$ has a vertex w. Since $\kappa(G) \geq 2$, $G - w_1'$ is also connected, and so $G - w_1'$ has a (v, w)-path Q_4 in $G - w_1'$. Since $v \in V(Q_1 \cup Q_2 \cup Q_3)$ and since $w \notin V(Q_1 \cup Q_2 \cup Q_3)$, Q_4 has a vertex w' which is the last vertex of Q_4 in $V(Q_1 \cup Q_2 \cup Q_3)$. Without lose of generality, we assume that $w' \in V(Q_3)$. Let $Q_3[v, w']$ denote the subpath of Q_3 from v to w', and let $Q_4[w', w]$ denote the subpath $Q_4[w', w]$ of Q_4 from w' to w. Then $Q_3' = Q_3[v, w'] \cup Q_4[w', w]$ is a path edge-disjoint from Q_1 and Q_2. Therefore, with $w_1 = w_2 = w_1'$ and $w_3 = w$, Q_1, Q_2, Q_3' are edge disjoint (v, w_i)-paths from v to W satisfying $|\{w_1, w_2, w_3\}| \geq 2$. □

3.3 π-collapsible Graphs

We in this section will introduce the concept of π-collapsible graphs, which was first introduced by Catlin.

Definition 3.3.1 *Let $C = w_1 w_2 w_3 w_3 w_4$ be a 4-cycle in G with a partition $\pi(C) = \langle \{w_1, w_3\}, \{w_2, w_4\} \rangle$. Following [7], we define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying w_1 and w_3 to form a vertex w', by identifying w_2 and w_4 to form a vertex w'', and by adding an edge $e_{\pi(C)} = w'w''$.*

Theorem 3.3.2 *(Caltin, [7]) Let G be a graph containing a 4-cycle C and let $G/\pi(C)$ be defined as in Definition 3.3.1. Each of the following holds.*
(i) If $G/\pi(C)$ is collapsible, then G is collapsible.
(ii) If $G/\pi(C)$ has a spanning eulerian subgraph, then G has a spanning eulerian subgraph.
(iii) If G is reduced with a 4-cycle C, then $F(G/\pi) \leq F(G) - 1$.

Theorem 3.3.2 (ii) can be directly verified by definition. Let J_i, $1 \leq i \leq 7$, denote the graphs depicted in Figure 3, where J_7 denotes any graph in the family of graphs such that each of the vertices w_1 and v_3 can be adjacent to any (identical of distinct) vertices in v_6, v_7 or v_8. Applying Theorem 3.3.2(i) to each of these graphs with the indicated partition of a given 4-cycle $C = w_1w_2w_3w_4w_1$, it is routine to verify that all these are collapsible graphs.

Lemma 3.3.3 *For each i with $1 \leq i \leq 7$, J_i is collapsible.*

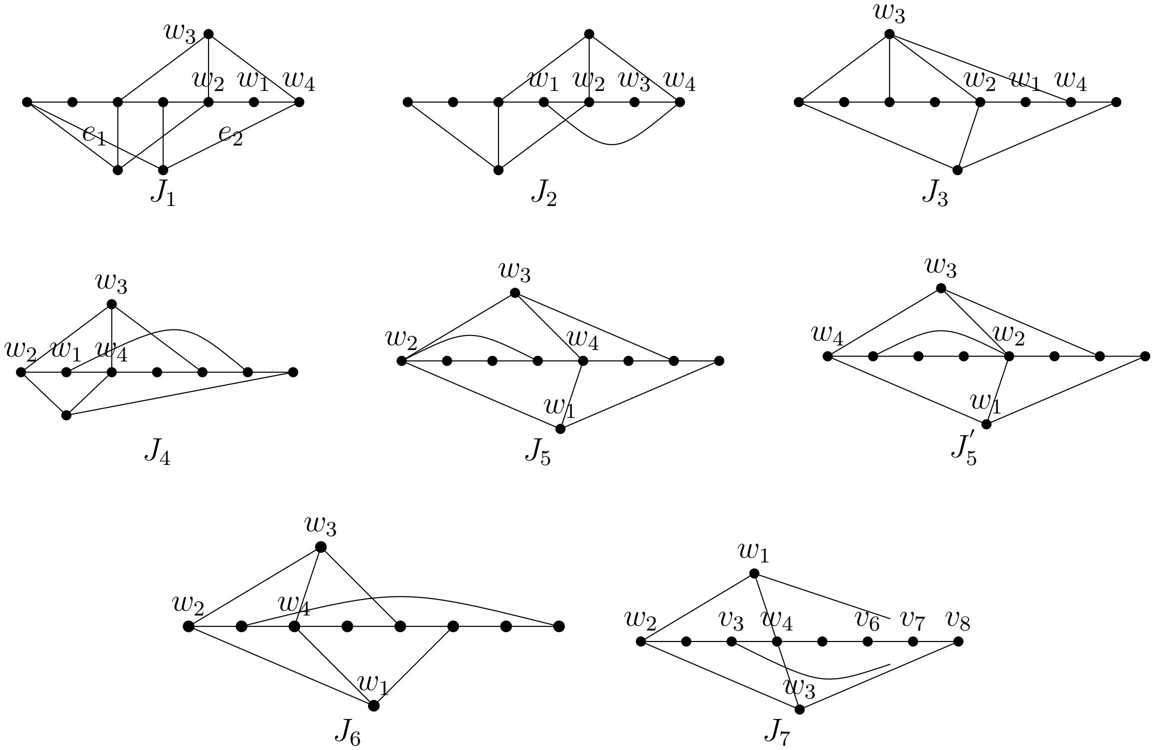


Figure 3: A collection of collapsible graphs

3.4 Proof of Theorem 3.1.2 and Proposition 3.1.3

The main purpose of this section is to prove Theorem 3.1.2 and proposition 3.1.3. For a graph G and for an integer k, define

$$D_{\leq k}(G) = \cup_{i=1}^{k} D_i(G).$$

Define the graphs Φ_1, Φ_2, Φ_3 as depicted in Figure 4 below. Note that $\Phi_1 = P(10)(e)$. It is routine to verify that the graphs Φ_2, Φ_3 are in $\mathcal{S}_{12}$.

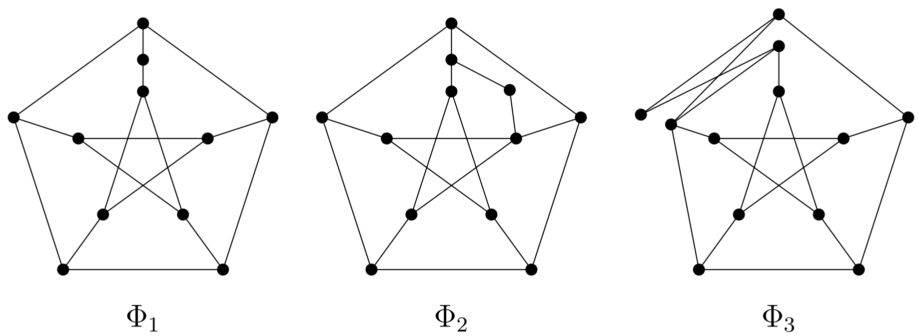


Figure 4: The graphs Φ_1, Φ_2, Φ_3

We shall prove the following useful tool, which is a restatement of Theorem 3.1.2.

Theorem 3.4.1 *Let G be a connected simple graph on n vertices. If*

$$n \leq 12, d_1(G) = 0 \text{ and } d_2(G) \leq 1, \tag{3.2}$$

then either the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{2,3}, K_{2,3}^{+}, P(10), \Phi_1\}$, or $G \in \mathcal{S}_{12}$.

Proof. Let G' be the reduction of G. We argue by contradiction and assume that

$$G \notin \mathcal{S}_{12} \text{ and } G' \notin \{K_1, K_2, K_{1,2}, K_{2,3}, P(10), \Phi_1\}. \tag{3.3}$$

We shall frequently use the facts that K_3 is the smallest nontrivial collapsible simple graph and the nontrivial reduced graphs with at most 5 vertices are either a tree, or a 4-cycle, or $K_{2,3}$ or $K_{2,3}$ minus an edge. Since $d_1(G) = 0$ and $d_2(G) \leq 1$,

$$\text{both } |\cup_{v \in D_1(G')} PI_G(v)| \geq 4d_1(G') \text{ and } |\cup_{v \in D_2(G')} PI_G(v)| \geq 4(d_2(G') - 1) + 1. \quad (3.4)$$

By (3.2) and (3.4), we must have $d_1(G') \leq 2$ and $d_1(G') + d_2(G') \leq 3$. Let $m' = |E(G')|$ and $n' = |V(G')|$. We first indicate that

$$G \text{ is reduced, with } 11 \leq n' \leq 12,\ d_1(G') = 0 \text{ and } d_2(G') \leq 1. \quad (3.5)$$

If $d_2(G') = 3$, then $d_1(G') = 0$. By (3.4), $|\cup_{v \in D_2(G')} PI_G(v)| \geq 9$. Hence $n' \leq 12 - 9 + 3 = 6$, and so by (3.2) and by $d_2(G') = 3$, we must have $G' = K_{2,3}$, contrary to (3.3). Hence $d_2(G') \leq 2$. Assume that $d_1(G') = 2$. By (3.4), $|\cup_{v \in D_1(G')} PI_G(v)| \geq 8$. Thus $n' \leq 12 - 8 + 2 = 4$, and so by $d_1(G') = 2$, we have $G' \in \{K_2, K_{1,2}\}$, contrary to (3.3). Hence $d_1(G') \leq 1$.

Suppose that $D_1(G') = \{v\}$. Let $H_1 = G - PI_G(v)$, and $v_1 \in V(H_1)$ be the vertex adjacent to a vertex in H_1. By (3.4), $|V(H_1)| \leq 12 - 4 = 8$. If $d_G(v_1) \geq 3$, then by (3.2), $d_1(H_1) = 0$ and $d_2(H_1) \leq 2$. By Lemma 3.2.2(i), the reduction of H_1 is in $\{K_1, K_2, K_{2,3}\}$. It follows that the the reduction of G is in $\{K_2, K_{1,2}, K_{2,3}^+\}$, contrary to (3.3). Hence $d_G(v_1) = 2$. Let $N_{H_1}(v_1) = \{v_2\}$. Let $H_2 = G - (PI_G(v) \cup \{v_1\})$. Since $d_2(G) \leq 1$, we have $d_G(v_2) \geq 3$, and so $d_1(H_2) = 0$ and $d_2(H_1) \leq 1$. By Lemma 3.2.2(i), the reduction of H_2 is in $\{K_1, K_2\}$. Accordingly, the reduction of G is in $\{K_2, K_{1,2}\}$, contrary to (3.3). Hence we must have $d_1(G') = 0$.

As $d_1(G') = 0$ and $d_2(G') \leq 2$, by Lemma 3.2.2(i) and by (3.3), we assume that $9 \leq n' \leq 12$. By (3.4) and by $n \leq 12$, we have $9 \leq n' \leq 12 - (4d_2(G') - 3)$, and so $d_2(G') \leq 1$.

If $n' \in \{9, 10\}$, then $2m' \geq 2 + 3(n' - 1) = 3n' - 1$. $F(G') = 2n' - m' - 2 \leq 2n' - \frac{3}{2}n' - \frac{3}{2} = \frac{n'-3}{2}$, implying that $F(G') \leq 3$. As $d_2(G') \leq 1$, by Theorem 3.1.1 (ii), $G' = P(10)$, contrary to (3.3). Therefore, we have $11 \leq n' \leq 12$. Since every simple collapsible graph must have at least 3 vertices, G is reduced. This verifies (3.5).

Claim 1. $\kappa'(G) \geq 2$ and $d_2(G) = 1$.

Assume that e is an cut-edge of G, and let H_1 and H_2 be the components of $G - e$. Since G is reduced and by (3.2), $5 \leq |V(H_i)| \leq 7$, for $i = 1, 2$. By Lemma 3.2.2(i), $H_i \in \{K_1, K_2, K_{2,3}\}$, contrary to (3.5). Hence $\kappa'(G) \geq 2$. Suppose that $\delta(G) \geq 3$. Then by $\kappa'(G) \geq 2$, and by Theorem 3.1.1(iii), either $G \in \mathcal{S}_{12}$, contrary to (3.3), or G is contracted to $P(10)$, whence by (3.5), $G = P(10)$, contrary to (3.3). Hence $d_2(G) = 1$. This proves Claim 1.

By Claim 1, we denote $D_2(G) = \{v\}$.

Claim 2. $d_3(G) = 10$. Moreover, if $n = 12$, then $d_4(G) = 1$.

If $d_{\geq 4}(G) \geq 1$, then $2|E(G)| \geq 3n + d_{\geq 5}(G)$. $F(G) = 2n - m - 2 \leq 2n - \frac{3n + d_{\geq 5}(G)}{2} - 2 = \frac{n - 4 - d_{\geq 5}(G)}{2}$. If $n = 11$, then $F(G) \leq 3$. By Theorem 3.1.1(ii), $G = P(10)$, contrary to $n = 11$. Hence if $n = 11$, then $d_3(G) = 10$.

Let $n = 12$. Since $n = 12$, and since the number of odd degree vertices is even, $d_3(G) \leq 10$, and so $d_{i \geq 4}(G) \geq 1$. If $d_{i \geq 5}(G) \geq 1$, then $F(G) \leq 24 - 19 - 2 = 3$. Let $H = G - v$. As $\kappa'(G) \geq 2$, H is connected and reduced with $|V(H)| = 11$, $d_2(H) \leq 2$ and $F(H) = 3$. By Theorem 3.1.1(ii), $H \in \{K_1, K_2, K_{1,2}, K_{2,3}, P(10)\}$, contrary to the fact $|V(H)| = 11$. Thus we have $d_2(G) = 1, d_3(G) = 10$ and $d_4(G) = 1$. Claim 2 holds.

Claim 3. If $girth(G) \geq 5$, then $G \in \{\Phi_1, \Phi_2\}$.

If $n = 11$, then by $girth(G) \geq 5$, there must be a vertex $w \in V(G)$ such that the distance between v and w is at least 3. If $n = 12$, then by Claim 2, we denote $D_4(G) = \{w\}$. In any case, for integer $i \geq 0$, define

$$T_i = \{u \in V(G)| \text{ the distance from } u \text{ to } w \text{ in } G \text{ is } i\}.$$

Suppose first that $n = 11$. Then $|T_0| = 1, |T_1| = 3, |T_2| = 6$ and $|T_3| = 1$. Let $T_1 = \{u_1, u_2, u_3\}$ and let $T_2 = \{v_1, v_2, \cdots, v_6\}$. Since the degree of each vertex of $T_2 \cup T_3$ in the induced subgraph $G[T_2 \cup T_3]$ is two, and since G is reduced, the induced subgraph $G[T_2 \cup T_3]$ is a C_7. Without loss of generality, we assume that $C_7 = v_1 v_2 \cdots v_6 v v_1$. Also we assume that $N_G(u_1) \cap T_2 = \{v_1, v_4\}$ and $u_2v_2 \in E(G)$. Then $N_G(u_2) = \{v_2, v_5\}$ and $N_G(u_3) = \{v_3, v_6\}$. Thus $G = \Phi_1$.

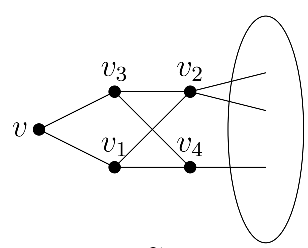
Next we assume that $n = 12$. Then $|T_0| = 1, |T_1| = 4, |T_2| = 7$ and $u_1 \in T_1$. Let $T_1 = \{u_1, u_2, u_3, u_4\}$ and let $T_2 = \{v_1, v_2, \cdots, v_7\}$. Since the degree of each vertex of T_2 in the induced subgraph $G[T_2]$ is two, and since G is reduced, the induced subgraph $G[T_2]$ is a C_7. Without loss of generality, we assume that $C_7 = v_1v_2\cdots v_7v_1$. By symmetry, we assume that $u_1v_7 \in E(G)$, $N_G(u_2) \cap T_2 = \{v_1, v_4\}$, $N_G(u_3) = \{v_2, v_5\}$ and $N_G(u_3) = \{v_3, v_6\}$. Thus $G = \Phi_2$. This completes the proof for Claim 3.

By Claim 3, we may assume that G has a 4-cycle $C = v_1v_2v_3v_4v_1$. Let $\pi(C) = \langle\{v_1, v_3\}, \{v_2, v_4\}\rangle$ be a partition of $V(C)$. Form the graph G/π with the new edge $e_\pi = e_{\pi(C)}$ as in Definition 3.3.1. Hence we identify v_1 and v_3 to get u_1 and identify v_2 and v_4 to get u_2.

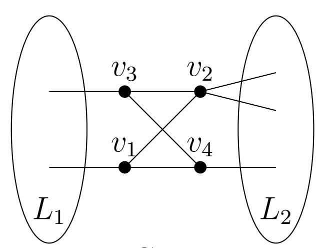
Claim 4. $\kappa'(G/\pi) \geq 2$.

By Claim 1, $\kappa'(G) \geq 2$. If G/π has a cut edge, then it must be e_π. Assume that $e_\pi = u_1u_2$ is an cut edge in G/π and let H_1 and H_2 be the components of $G/\pi - e$ with $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$.

Case 1. $N_G(v_1) = N_G(v_3)$. Let $N_G(v_1) = N_G(v_3) = \{v_2, v_4, x\}$. By Claim 1, we must have $x = v$. (See Figure 5).



Case 1.



Case 2.

Figure 5: The two cases in the proof of Claim 4

Let $K = G - \{v, v_1, v_2, v_3, v_4\}$. Then K is connected, and $|V(K)| = \begin{cases} 6, & \text{if } n = 11 \\ 7, & \text{if } n = 12 \end{cases}$.

By Claims 3 and 4, $|E(K)| = \begin{cases} 8, & \text{if } n = 11 \\ 9 \text{ or } 10, & \text{if } n = 12 \end{cases}$.

Thus $F(K) = \begin{cases} 2, & \text{if } n = 11, \text{ or } n = 12 \text{ and } |E(K)| = 10 \\ 3, & \text{if } n = 12 \text{ and } |E(K)| = 9 \end{cases}$.

Furthermore, $d_1(K) = 0$ and $d_2(K) = \begin{cases} 2, & \text{if } n = 11, \text{ or } n = 12 \text{ and } |E(K)| = 10 \\ 3, & \text{if } n = 12 \text{ and } |E(K)| = 9 \end{cases}$.
If $F(K) = 2$, then by Theorem 3.2.1 (vii), $K = K_{2,n-2}$, contrary to fact $d_2(K) \leq 3$. If $F(K) = 3$, then $d_2(K) = 3$ and $|V(K)| = 7$. By Theorem 3.1.1(ii), $K = K_1$, contrary to (3.3). This proves Case 1.

Case 2. $N_G(v_1) \neq N_G(v_3)$ and $N_G(v_2) \neq N_G(v_4)$.

Let L_1 and L_2 be components of $G - \{v_1, v_2, v_3, v_4\}$. (See Figure 5). Since $\kappa'(G) \geq 2$, L_1 and L_2 are connected. As G is reduced and $d_2(G) \leq 1$, $|V(L_1)| + |V(L_2)| \geq 9$, and so $n \geq 13$, contrary to the hypothesis that $n \leq 12$. Claim 4 holds.

If $n = 11$, then $|V(G/\pi)| = 9$, and so by Theorem 3.1.1(ii), the reduction of G/π is in $\{K_1, K_2, K_{1,2}, K_{2,3}\}$. By Claim 4, $\kappa'(G/\pi) \geq 2$, and by Claim 1, $d_2(G) = 1$. It follows that the reduction of G/π is K_1. By Theorem 3.3.2, G is collapsible, contrary to (3.3).

Hence we have $n = 12$. By (3.5), G is reduced. By Claims 1 and 2, $d_2(G) = 1$, $d_3(G) = 10$ and $d_4(G) = 1$, $F(G) = 4$, and so by Theorem 3.3.2 (iii), $F(G/\pi) \leq F(G) - 1 = 3$. By Definition 3.3.1, $|V(G/\pi)| = n - 2 \in \{9, 10\}$. By Theorem 3.1.1(ii) and by Claim 4, the reduction of G/π is in $\{K_1, P(10)\}$. By Theorem 3.3.2, if the reduction of G/π is K_1, then G is collapsible, contrary to (3.3). Therefore, $n = 12$ and $G/\pi = P(10)$. This implies $G = \Phi_3$. The proof of Theorem 3.4.1 is now complete. □

Proof of Proposition 3.1.3. Let G be a non strongly spanning trailable graph with $\kappa'(G) \geq 3$. Then for some pair of edges e, e', $G(e', e'')$ does not have a spanning $(v_e, v_{e'})$-

trail. Let H be the graph obtained from $G(e', e'')$ by adding a new vertex z_0 and new edges $z_0v_{e'}$, $z_0v_{e''}$. Then H cannot be supereulerian.

Suppose that $|V(G)| \leq 8$, then $|V(H)| \leq 11$. By Theorem 3.4.1, we must have $H = P(10)(e)$, which forces that $G = H_8$. This proves the proposition. □

3.5 Proof of Theorem 3.1.4

The follow lemma is verified using case analysis. Detailed proofs will be in Chapter 4.

Lemma 3.5.1 *(Chapter 4 Theorem 4.3.1) Let G be a graph with $\kappa(G) \geq 2$ and $\kappa'(G) \geq 3$, $e = v_0v_1$ and $e' = v_{c-1}v_c$ be distinct edges in G, and $P = v_ev_1 \cdots v_{c-1}v_{e'}$ is a longest $(v_e, v_{e'})$-path in $G(e, e')$ with $c = |E(P)| \leq 8$. If $G(e, e')$ is reduced, then $V(G) = V(P) \cup \{v_0, v_c\}$.*

Lemma 3.5.2 *Let G be a graph with $\kappa'(G) \geq 3$ and let $e = uv \in E(G)$. If $|V(G)| \leq 11$ and if the reduction of $G(e)$ is $P(10)$, then G has a (u, v)-path of length at least 9.*

Proof. Let v_e denote the vertex so that $G(e)$ is obtained from $G - e$ by adding new edges uv_e, v_ev. Since the the reduction of $G(e)$ is $P(10)$, v_e, u, v must be in a maximal collapsible subgraph of $G(e)$, which is the preimage of a vertex w (say) of $P(10)$. Since $P(10)$ has a cycle of length 9 passing w, this cycle of length 9 can be lifted to a cycle of length 11 containing v_e in $G(e)$. Therefore, G has a (u, v)-path with length at least 9. □

Proof of Theorem 3.1.4. We argue by contradiction and assume that

$$G \text{ is counterexample to the theorem.} \tag{3.6}$$

Thus for some edges $e, e' \in E(G)$, every longest $(v_e, v_{e'})$-path in $G(e, e')$ has length $c \leq 8$. By Lemma 3.2.2 (ii) and by (3.6), we may assume that $G(e, e')$ is reduced. Hence $G(e, e')$ does not have a spanning $(v_e, v_{e'})$-trail.

Let $P = v_e v_1 \cdots v_{c-1} v_{e'}$ be a longest $(v_e, v_{e'})$-path in $G(e, e')$ and $V(P) \neq V(G(e, e'))$. Let $e = v_0 v_1$ and $e' = v_{c-1} v_p$ for some vertices $v_0, v_c \in V(G)$.

Obtain a new graph L from $G(e, e')$ by adding a new vertex z_0 not in $G(e, e')$ and by adding two new edges $z_0 v_e, z_0 v_{e'}$. Thus L has a spanning eulerian subgraph. This would imply that $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail, contrary to the assumption that $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. Since $\kappa'(G) \geq 3$, we have $\kappa'(L) \geq 2$ and L has exactly one edge cut of size 2. By Lemma 3.5.1, $V(G(e, e')) = V(P) \cup \{v_0, v_c\}$, and so $|V(G(e, e'))| \leq c + 3 = 11$. Thus $|V(L)| \leq 12$. Hence one of the conclusions of Theorem 3.1.2 must hold. Since $\kappa'(L) \geq 2$ and L has exactly one edge cut of size 2, the reduction of L cannot be $K_2, K_{1,2}$, $K_{2,3}$ or $K_{2,3}^+$. If L is collapsible or supereulerian, then Theorem 3.1.4 (ii) holds. If $L = \Phi_1$, then since $\kappa'(G) \geq 3$, the only vertex of degree 2 in Φ_1 must be the newly added vertex z_0 in L. It follows that $G = H_8$. contrary to the assumption that G is not contractible to H_8.

Therefore, by Theorem 3.1.2, the reduction of L must be $P(10)$. By Lemma 3.5.2, $G(e, e')$ has a $(v_e, v_{e'})$-path with length at least 9, contrary to the assumption that every longest such path has length at most 8. This completes the proof of the theorem. □

Chapter 4

Strongly Spanning Trailable Graphs with Short Longest Paths

4.1 Introduction

Catlin in [8] and Jaeger in [23] proved that every 4-edge-connected graph is supereulerian. In fact, Catlin's proof implies a stronger result stated below.

Theorem 4.1.1 *(Catlin [8]) Every 4-edge-connected graph is strongly spanning trailable.*

The four cycle is an example that a supereulerian graph may not be spanning trailable. Luo, Chen and Chen [35] first explicitly studied spanning trailable graphs (called eulerian-connected graphs in [35]). The following theorem improves Theorem 4.1.1.

Theorem 4.1.2 *(Luo, Chen and Chen [35]) Every 4-edge-connected graph is spanning trailable.*

Theorem 4.1.2 was implicitly proved in Theorem 4.1.3 below.

Theorem 4.1.3 *(Catlin and Lai [9]) If G has two edge-disjoint spanning trees, then G is strongly spanning trailable if and only if G is essentially 3-edge-connected.*

Spanning trailable graphs have several useful applications. Shao [45] indicated that spanning trailable graphs have applications in the investigation of hamiltonian-connected line graphs. For fixed distinct edges $e, e' \in E(G)$, $G^*(e, e')$ is obtained from $G(e, e')$ by adding a new vertex z and new edges $zv_e, zv_{e'}$. Then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail if and only if $G^*(e, e')$ is supereulerian. As Pulleyblank [41] indicated that determining if a 3-edge-connected graph is supereulerian is NP-complete, determining if a 3-edge-connected graph is strongly spanning trailable is at least as hard as the supereulerian graph problem.

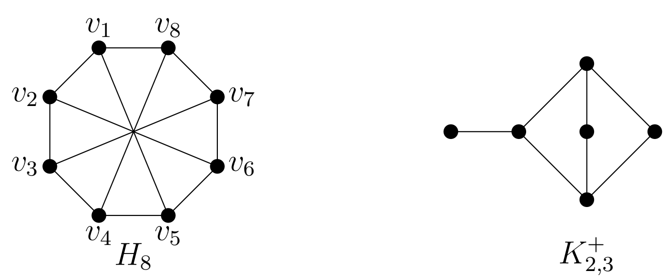


Figure 1. The graph W_8 and $K^+_{2,3}$

Let H_8 denote the **Wagner graph** as depicted in Figure 1. For $e = v_1v_5$, $e' = v_3v_7 \in E(H_8)$, it is routine to check that $H_8(e, e')$ does not have a spanning $(v_e, v_{e'})$-trail, and so any longest $(v_e, v_{e'})$-path in $H_8(e, e')$ has length 8. Later in this chapter, we verified that H_8 is the 3-edge-connected non-spanning trailable graph with fewest number of vertices. Our main results of this paper are the following.

Theorem 4.1.4 *Let G be a 3-edge-connected graph. Let $e, e' \in E(G)$ be two edges. If the length of a longest $(v_e, v_{e'})$-path in $G(e, e')$ is at most 8, then either G has a spanning $(v_e, v_{e'})$-trail or $G = H_8$ with, up to isomorphism, $e = v_1v_5$ and $e' = v_3v_7$.*

Corollary 4.1.5 *Let G be a 3-edge-connected graph. If for any edges $e, e' \in E(G)$, the length of a longest $(v_e, v_{e'})$-path in $G(e, e')$ is at most 8, then either G is strongly spanning trailable.*

Corollary 4.1.5 follows from Theorem 4.1.4 immediately as for edges $e_1 = v_1v_2$ and $e_2 = v_7v_8$, a longest (v_{e_1}, v_{e_2})-path $v_{e_1}v_2v_3v_4v_8v_1v_5v_6v_7v_{e_2}$ has length 9. Hence H_8 does not satisfy the hypothesis of Corollary 4.1.5. This chapter is organized as the proof of Theorem 4.1.4 will be given in the last two sections.

4.2 Preliminaries

Lemma 4.2.1 *Let $u, w \in V(G)$, H be a collapsible subgraph of G, and let v_H denote the vertex in G/H onto which H is contracted, and*

$$u' = \begin{cases} u, & \text{if } u \notin V(H) \\ v_H, & \text{if } u \in V(H) \end{cases} \quad \text{and } w' = \begin{cases} w, & \text{if } w \notin V(H) \\ v_H, & \text{if } w \in V(H) \end{cases}.$$

If G/H has a (u', w')-trail L' containing v_H, then G has a (u, w)-trail L such that $(V(L') - \{v_H\}) \cup V(H) \subseteq V(L)$.

Proof. By the definition of contraction, $E(L') \subseteq E(G)$. Let L'' be the subgraph of G with edge set $E(L')$ and define

$$\begin{aligned} R &= [\{v \in V(H) \mid d_{L''}(v) \text{ is odd }\} \cup \{u, w\}] \\ &\quad -[\{v \in V(H) \mid d_{L''}(v) \text{ is odd }\} \cap \{u, w\}]. \end{aligned}$$

Since L' is a (u', w')-trail of G/H, $d_{L'}(v_H)$ is odd if and only if $|V(H) \cap \{u, w\}| = 1$. It follows that $|R| \equiv 0 \pmod 2$. Since H is collapsible, H has a spanning connected subgraph H_R such that $O(H_R) = R$. Let $L = G[E(L') \cup E(H_R)]$ be an edge induced subgraph of G. Since L' contains v_H, L is connected. By the definition of R and the choice of H_R, $O(L) = \{u, w\}$, and so L satisfies $(V(L') - \{v_H\}) \cup V(H) \subseteq V(L)$. □

Lemma 4.2.2 *Let G be a graph with $\kappa'(G) \geq 3$, and let $G_1, G_2, \cdots, G_k$ be the blocks of G. Then the following are equivalent.*
(i) G is strongly spanning trailable.
(ii) For every $i = 1, 2, \cdots, k$, G_i is strongly spanning trailable.

Proof. Since each block of G is also 3-edge-connected, (i) implies (ii). To prove (ii) implies (i), we argue by induction on k, the number of blocks of G. As (ii) trivially implies (i) when $k = 1$, we assume that $k > 1$ and for any graph with fewer than k blocks, (ii) implies (i).

Since $k \geq 2$, G has two connected subgraphs H and L and a vertex z_0 such that $G = H \cup L$ and $V(H) \cap V(L) = \{z_0\}$. Let $e, e' \in E(G)$. If $\{e, e'\} \cap E(L) = \emptyset$, then by induction, $H(e, e')$ has a spanning $(v_e, v_{e'})$-trail Q_1. By induction, for any edge $e'' \in E(L)$, $L(e'')$ has a spanning $(v_{e''}, v_{e''})$-trial, and so L has a spanning closed trail Q_2. It follows that $Q = G[E(Q_1) \cup E(Q_2)]$ is a spanning $(v_e, v_{e'})$-trail of G. The proof for the case when $\{e, e'\} \subseteq E(L)$ is similar, and will be omitted. Hence we may assume that $e \in E(H)$ and $e' \in E(L)$.

Since $\kappa'(H) \geq \kappa'(G) \geq 3$, and so H has an edge $e'' \in E_H(z_0) - \{e\}$. By induction, H has a spanning closed $(v_e, v_{e''})$-trail T_1'. Assume that $e'' = z_0 w$. Define

$$T_1 = \begin{cases} T_1' - z_0 v(e''), & \text{if } z_0 v(e'') \text{ is the last edge in } T_1' \\ H[E(T_1' - v(e'')) \cup \{e''\}], & \text{if } w v(e'') \text{ is the last edge in } T_1' \end{cases} .$$

Thus T_1 is a spanning $(v(e), z_0)$-trail of H. Similarly, L has a spanning $(z_0, v(e'))$-trail T_2. It follows $T = T_1 \cup T_2$ is a spanning $(v_e, v_{e'})$-trail. □

In the rest of this section, we will discuss properties of a graph H as specified in the next definition.

Definition 4.2.3 *Let $k > 0$ be an integer. A connected graph H with two distinct vertices $w_1, w_2 \in V(H)$ is said to have* **Property R(k)** *if each of the following holds.*
(i) For any $v \in V(H) - \{w_1, w_2\}$, $d_H(v) \geq 3$.
(ii) For any longest (w_1, w_2)-path P and for any $u \in V(H) - V(P)$, H has three edge-disjoint paths Q_1, Q_2, Q_3 from u to $V(P)$ with $|V(Q_i) \cap V(P)| = 1$, $(1 \leq i \leq 3)$ and with

$|\cup_{i=1}^{3} V(Q_i) \cap V(P)| \geq 2$.
(iii) every (w_1, w_2)*-path in* H *has length at most* k.

Throughout the rest of this section in Lemmas 4.2.4-4.2.7, H always denotes a 2-connected graph with Property R(6) and with the same notations in Definition 4.2.3, and

$$P = z_0 z_1 ... z_h \text{ is a longest } (w_1, w_2)\text{-path in } H. \tag{4.1}$$

Lemma 4.2.4 *If for a pair of distinct vertices* z_i, z_j $(0 \leq i < j \leq h \leq 6)$, $H - E(P)$ *has a longest* (z_i, z_j)*-path* $Q = x_0 x_1 ... x_k$ *with* $x_0 = z_i$ *and* $x_k = z_j$ *such that* $k = |E(Q)| \geq 4$ *and* $V(P) \cap V(Q) = \{z_i, z_j\}$, *then either*
(i) H *is not reduced, or*
(ii) $H[V(P) \cap V(Q)]$ *is spanned by the graph* L *depicted in Figure 2 below.*

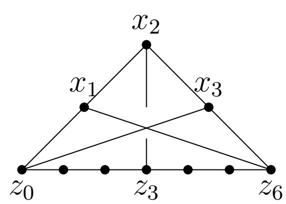


Figure 2: The graph L

Proof. We assume that H is reduced to prove Lemma 4.2.4(ii). By Theorem 1.2.2 (iii), $girth(H) \geq 4$. Since H is connected, for each i $(1 \leq i < k)$, H has a shortest path T_i from x_i to P with $V(T_i) \cap V(P) = \{z_{t_i}\}$. Then we have the following observations: either (A0), or one of (A1) or (A2) must hold, and either (B0) or (B1) must hold. Moreover, if $k = 6$, (C0) holds.

(A0) $V(T_1) \cap V(Q) - (V(P) \cup \{x_1\} - \{z_i, z_j\}) \neq \emptyset$.

Note that if (A0) fails, then $t_1 \leq j$. Otherwise, assume that $t_1 > j$. By (4.1), $j \geq i + |E(Q)| \geq i + 4$, and so $6 \geq t_1 \geq j + 4 \geq i + 8$, a contradiction.

(A1) If (A0) fails and $i < t_1 < j$, then by $girth(H) \geq 4$, $t_1 \geq i+2$. Hence $6 \geq j \geq t_1 + |E(Q)| \geq t_1 + 4 \geq i+6$. It follows that we must have $i = 0$, $t_1 = 2$, $j = 6$, $x_1 z_2 \in E(H)$ and $|E(Q)| = 4$.

(A2) If (A0) fails and $t_1 < i$, then by $girth(H) \geq 4$, $i \geq t_1 + 2$. Hence $6 \geq j \geq i + |E(Q)| \geq i+4 \geq t_1+6$. It follows that we must have $t_1 = 0$, $i = 2$, $j = 6$, $x_1 z_0 \in E(H)$ and $|E(Q)| = 4$.

(B0) $V(T_2) \cap V(Q) - (V(P) \cup \{x_2\} - \{z_i, z_j\}) \neq \emptyset$.

Note that if (B0) fails, then $i \leq t_2 \leq j$. Otherwise, if $t_2 < i$, then by (4.1), $i \geq t_2 + 3$, whence $6 \geq j \geq i + |E(Q)| \geq i + 4 \geq t_1 + 7$, a contradiction. Thus $i \leq t_2$. By symmetry, $t_2 \leq j$.

(B1) If (B0) fails and $i < t_2 < j$, then by (4.1), $t_2 \geq i + 3$ and so $6 \geq j \geq t_2 + |E(Q[x_2, x_k])| \geq t_2 + 3 \geq i + 6$. It follows that we must have $i = 0$, $t_2 = 3$, $j = 6$, $x_2 z_3 \in E(F)$ and $|E(Q)| = 4$.

(C0) If $k = 6$, then $V(T_3) \cap V(Q) - (V(P) \cup \{x_3\} - \{z_i, z_j\}) \neq \emptyset$. Otherwise, symmetrically, we may assume that $j \geq t_3 \geq 3$ and so $P[z_0, z_{t_3}]x_3 Q[x_3, z_j]\ P[z_j, z_h]$ is longer than P, contrary to (4.1).

By symmetry, these observations can also be applied to x_{k-1} (as symmetric to x_1) and to x_{k-2} (as symmetric to x_2).

Claim 1. (A0) holds for x_1.

If not, then assume first that (A1) holds for x_1. Thus (B1) does not hold as otherwise $t_1 = 2$ and $t_2 = 3$, and so $P[z_0, z_2]T_1^-[x_1, z_2]T_2[x_2, z_3]\ P[z_4, z_h]$ is longer than P. Hence (B0) must hold. By $|E(Q)| = 4$, $t_2 \in \{0, 6\}$. If $t_2 = 0$, then by $girth(H) \geq 4$, $|E(T_2)| \geq 2$, and so $T_2^-[x_2, z_0]x_1 T_1[x_1, z_2]P[z_2, z_6]$ is longer than P, contrary to (4.1). Hence $t_2 = 6$ and $T_2 = x_2 x_3' z_6$. Applying the observations of (A0)-(A2) to x_3 (by the symmetry between x_1 and x_3), we conclude that $t_3 = 0$, whence $z_0 x_3 x_2 x_1 P[z_2, z_6]$ is longer than P, contrary to (4.1). This proves that (A1) does not hold. The proof for (A2) does not hold is similar.

This proves Claim 1.

Since $girth(H) \geq 4$, and since Q is longest, $T = T_1[x_1, x_s]$ satisfies $V(T) \cap V(Q) = \{x_1, x_s\}$ such that if $s = 3$, then $|E(T)| \geq 2$. As T_1 is shortest, $s \geq 3$.

Claim 2. If $x_s \neq z_j$, then (B0) holds for x_2.

If (B0) does not hold for x_2, then (B1) holds for x_2, implying $s = 3$. As Q is longest, $T = x_1x_2'x_3$, and so (B0) or (B1) must also hold for x_2'. If (B1) holds for x_2', then $x_2'z_3 \in E(H)$, and so $P[z_0, z_3]x_x'x_1x_2x_3z_6$ is longer than P, contrary to (4.1). Hence (B0) holds for x_2'. By symmetry, and since $girth(H) \geq 4$ and since Q is longest, we may assume that H has a vertex $x_1' \notin V(P) \cup V(Q)$ such that $z_0x_1', x_1'x_2' \in E(H)$. (See Figure 3). Thus $z_0x_1'x_2'x_1x_2z_3P[z_3, z_6]$ is longer than P, contrary to (4.1). This proves Claim 2.

(A1) and (A2)

Figure 3: (A0),(B1) and (B0) for $x_{2'}$

If $t_2 = i$, then by $girth(H) \geq 4$ and as Q is longest, $T_2 = x_2x_1'z_i$ for some $x_1' \notin V(P) \cup V(Q)$. By the symmetry between x_1 and x_1', we must also have (A0) holds for x_1'. Let T' be an $(x_1', x_{s'})$-path with $V(T') \cap V(Q) = \{x_1', x_{s'}\}$ and $V(T') \cap V(P) = \emptyset$.

Claim 3. If $x_s \neq z_j$ and $t_2 = i$, then $s = s'$. Moreover, $s > 3$.

If $s' \neq s$, then as $girth(H) \geq 4$, H would have an (z_i, z_j)-path of length at least 7 (See Figure 4). Hence we assume that $s = s' \geq 3$. If $s = s' = 3$, then by $girth(H) \geq 4$ and by (4.1), $|E(T')| = 2$ and $E(T_1[x_1, x_3]) = 2$. It follows that $H[\{x_0, x_2\} \cup V(T') \cup V(T_1[x_1, x_3])] \cong G(2)$ defined in Theorem 1.2.2 (vii), and so by Theorem 1.2.2 (vii), H is not reduced, contrary to the assumption that H is reduced. This proves Claim 3.

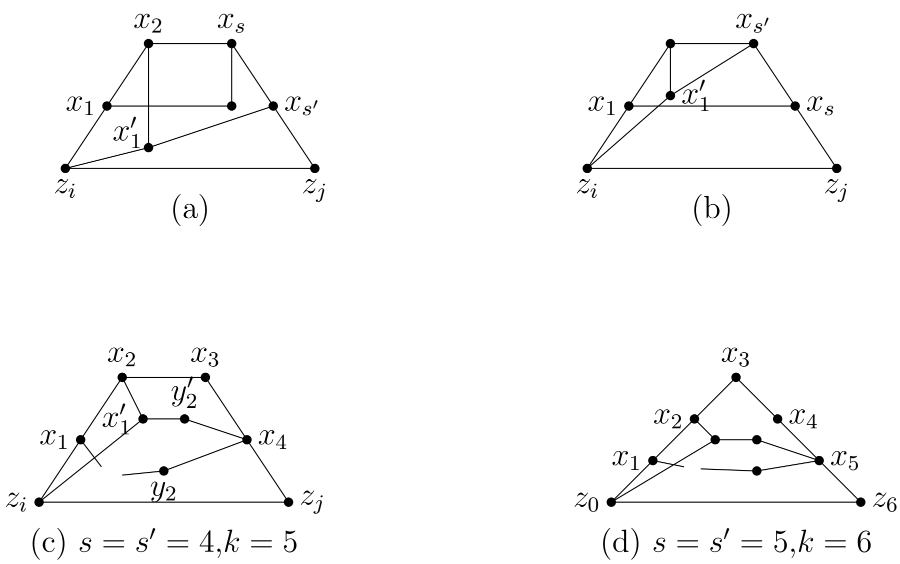


Figure 4

Claim 4. If $x_s \neq z_j$ and $t_2 = i$, then $s = s' = 5$ (and so $k = 6$).

If $s = s' = 4$, and if $min\{|E(T')|, |E(T_1[x_1, x_4])|\} = 1$, then by Theorem 1.2.2 (vii), H is not reduced. Hence when $s = s' = 4$, by (4.1), we must have $|E(T')| = |E(T_1[x_1, x_4])| = 2$, and so we may denote $T' = x'_1y'_2x_4$ and $T_1[x_1, x_4] = x_1y_2x_4$. If $k = 5$, then by symmetry with x_2, either (B0) or (B1) holds (with x_3 replacing x_2). If $i < t_3 < j$, then $z_ix_1x_2T'[x'_1, x_4]x_3T_3[x_3, z_{t_3}]P[z_{t_3}, z_j]$ is longer than $P[z_i, z_j]$, contrary to (4.1). Therefore (B0) must hold for x_3, and so H has an (x_3, z)-path T'_3 such that $V(T'_3) \cap (V(Q) \cup V(T') \cup V(T_2)) = \{x_3, z\}$. Table 1 below shows that for any values of x, either H has a nontrivial collapsible subgraph, or (4.1) is violated, and so Claim 4 is proved.

z	longer (z_0, z_h)-path or H is not reduced	Explanations
x_1	$P[z_0, z_i](T_3')^-[x_1, x_3]x_2x_1'x_4P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 2$.
x_2	$P[z_0, z_i]x_1(T_3')^-[x_2, x_3]x_4P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 3$.
x_4	$P[z_0, z_i]Q[x_1, x_3]T'[x_3, x_4]P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 3$.
x_1'	$P[z_0, z_i]x_1x_2x_3(T_3')^-[x_1', x_3]y_2'x_4P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 2$.
y_2	$P[z_0, z_i]x_1y_2x_3x_2x_1'y_2'x_4P[z_j, z_h]$.	
z_i	$H[V(Q[x_0, x_4]) \cup x_1', y_2, y_2'\}]$ is not reduced,	by Lemma 1.2.3(i), if $\|E(T_3')\| = 1$.
z_i	$P[z_0, z_i](T_3')^-[z_i, x_3]x_2x_1T_1[x_1, x_4]P[z_j, z_h]$,	if $\|E(T_3')\| \geq 2$.
z_j	$P[z_0, z_i]T_1[x_1, x_4](T')^-[x_4, x_1']x_2T_3'[x_3, z_j]P[z_j, z_h]$.	

Table 1: When $s = s' = 4$, a contradiction is found

Claim 5. If $x_s \neq z_j$, then $t_2 \neq i$, and so $t_2 = j$.

If not, then by Claim 4, we have $s = s' = 5$ and $k = 6$. Thus $i = 0$ and $j = 6$. By Definition 4.2.3 (ii), H has an (y_2, y)-path T_4' such that $x_1'y_2, y_2x_5 \notin E(T_4')$ and $V(T_4') \cap (V(P) \cap V(Q) \cup \{x_1', y_2, y_2'\}) = \{y_2, y\}$. If $y = z_t$ for some $0 < t < 6$, then either $P[z_0, z_t](T_4')^-[y_2, z_t]x_1'Q[x_2, z_6]$ or $z_0x_1x_2x_1'T_4'[y_2, z_t]P[z_t, z_6]$ is longer than P, contrary to (4.1). Hence $y \in V(Q) \cup \{x_1', y_2, y_2'\}$. Table 2 below indicates that for any possibilities of y, a violation to (4.1) always occurs, and so Claim 5 follows.

y	longer (z_0, z_h)-path in H	Explanations
x_1	$z_0x_1'T_4'[y_2, x_1]Q[x_1, z_6]$.	
x_2	$z_0x_1'T_4'[y_2, x_2]Q[x_2, z_6]$.	
x_3	$z_0x_1x_2x_1'T_4'[y_2, x_3]Q[x_4, z_6]$.	
x_4	$z_0x_1x_2x_1'T_4'[y_2, x_4]x_5z_6$.	
x_5	$z_0x_1x_2x_1'T_4'[y_2, x_5]z_6$,	by $girth(H) \geq 4$, $\vert E(T_4')\vert \geq 3$.
x_1'	$z_0x_1x_2x_1'(T_4')[y_2, x_1']x_5z_6$,	by $girth(H) \geq 4$, $\vert E(T_4')\vert \geq 3$.
y_2'	$z_0x_1x_2x_1'T_4'[y_2, y_2']x_5z_6$,	by $girth(H) \geq 4$, $\vert E(T_4')\vert \geq 2$.
z_0	$(T_4')^-[z_0, y_2]x_1'x_2Q[x_2, z_6]$.	
z_6	$Q[z_0, z_5]y_2T_4'[y_2, z_6]$.	

Table 2: When $s = s' = 5$, a contradiction to (4.1) is found.

Claim 6. $x_s = z_j$, and so $t_1 = j$.

If not, then by Claim 5, $t_2 = j$. If $k = 4$, then $s = 3$. By $girth(H) \geq 4$, both $|E(T[x_1, x_3])| \geq 2$ and $|E(T_2)| \geq 2$. It follows that $z_0T_1[x_1, x_3]x_2T_2[x_2, z_j]$ is longer than Q, contrary to the choice of Q. Suppose that $k = 5$ and $s = 4$. By symmetry, $t_3 = i$ and T_3 is internally disjoint from Q. If $|E(T[x_1, x_4])| = |E(T_2)| = |E(T_3)| = 1$, then $H[V(Q)]$ contains a $K_{3,3}^-$, contrary to the assumption that H is reduced. Hence $\max\{|E(T[x_1, x_4])|,\ |E(T_2)|,\ |E(T_3)|\} \geq 2$. Table 3 shows that a contradiction to the choice of Q always exists.

Cases	(z_i, z_j)-path longer than Q in $H - E(P)$
$\vert E(T[x_1, x_4])\vert \geq 2$	$z_iT_1[x_1, x_4]x_3'T_2[x_2, z_j]$.
$\vert E(T_2)\vert \geq 2$	$z_iT_1[x_1, x_4]x_3'T_2[x_2, z_j]$.
$\vert E(T_3)\vert \geq 2$	$T_3^-[z_i, x_3]x_2T_1[x_1, z_j]$.

Table 3: When $k = 5$ and $s = 4$, a, (z_i, z_j)-path longer than Q is found.

Therefore we must have $k = 5$ and $s = 3$. By $girth(H) \geq 4$, $|E(T_1)| \geq 2$. As Q is longest, $T_1 = x_1x_2'x_3$ and $T_2 = x_2z_j$. By (4.1), $t_4 = i$ and $T_4 = x_4z_i$. By Theorem 1.2.2 (vii), $H[V(Q)]$ is not reduced, contrary to the assumption that H is reduced. This proves Claim 6.

Claim 7. (B0) holds for x_2.

If not, then (B1) holds for x_2, whence $i = 0$, $t_2 = 3$, $j = 6$, $x_2z_3 \in E(F)$ and $|E(Q)| = 4$. If $|E(T_1)| \geq 2$, then $P[z_0, z_3]x_2T_1[x_1, z_6]$ is longer than P. Hence we have $T_1 = x_1z_6$. By symmetry to x_1, we also have $T_3 = x_3z_0$. Thus $H[V(P) \cap V(Q)]$ is spanned by the graph depicted in Figure 2, and so Claim 1 must hold. Hence Claim 7 follows.

By Claims 1 and 6, and by the symmetry between x_1 and x_{k-1}, we may assume that $V(T_1) \cap (V(P) \cup V(Q)) = \{z_j, x_1\}$ and $V(T_{k-1}) \cap (V(P) \cup V(Q)) = \{z_i, x_{k-1}\}$. By the maximality of Q, we have

$$T_1 = x_1z_j \text{ and } T_{k-1} = x_{k-1}z_i. \tag{4.2}$$

By Claim 7, T_2 contains a subpath $T_2' = T_2[x_2, y']$ such that $V(T_2') \cap V(Q) = \{x_2, y'\}$ and such that $x_1x_2, x_2x_3 \notin E(T_2')$. By the choice of Q, $y' \notin \{x_1, x_3\}$. By Claim 6 and by symmetry, $y' \neq x_{k-1}$.

If $k = 4$, then as Q is longest and by $girth(H) \geq 4$, we have

$$T_2 = x_2xz, \text{ where } z \in \{z_i, z_j\} \text{ and } x \notin V(P) \cup V(Q). \tag{4.3}$$

By the symmetry between x and x_1, we must have $\{x, z\} = \{z_i, z_j\}$. Thus $H[V(Q) \cup \{x\}] \cong K_{3,3}^-$, contrary to the assumption that H is reduced. Hence $k \geq 5$.

Assume that $k = 5$. As $y' \neq x_{k-1}$, $y' \in \{z_i, z_j\}$. If $y' = z_i$, then by maximality of Q, (4.3) holds. Hence by Lemma 1.2.3(i), $H[V(Q) \cup V(T_2)]$ is not reduced, contrary to the assumption that H is reduced. Thus we have $y' = z_j$. By the maximality of Q, $|E(T_2)| = 2$. By the symmetry between x_2 and x_3, we also have $t_3 = i$ and $|E(T_3)| = 2$. If follows by Lemma 1.2.3 (i) that $H[V(Q) \cup V(T_2) \cup V(T_3)]$ is not reduced, contrary to the assumption that H is reduced.

Hence $k = 6$. Then $y' \in \{z_i, x_4, z_j\}$. If $y' = z_i$, by the maximality of Q, (4.3) holds with $z = z_i$. By the symmetry between x_1 and x, and by (4.2), we have $xz_j \in E(H)$. It follows that $H[\{z_i, z_j, x_1, x, x_2, x_5\}] \cong K_{3,3}^-$, contrary to the assumption that H is reduced. Thus $y' \neq z_i$ ($t_2 \neq i$) and by symmetry, $t_4 \neq j$.

If $y' = x_4$, then $|E(T_2')| = 2$. By (C0), T_3 has a subpath $T_3' = T_3[x_3, y'']$ such that $V(T_3') \cap V(Q) = \{x_3, y''\}$ and such that $x_2x_3, x_3x_4 \notin E(T_3')$. By Claim 6 and by symmetry, $y'' \notin \{x_1, x_5\}$. Thus $y'' \in \{z_i, z_j\}$. Assume first $y'' = z_i$. If $|E(T_3)| \geq 2$, then $z_iT_2^-[z_i, x_3]x_4T_2^-[x_4, x_2]x_1z_j$ has length at least 7, contrary to (4.1); if $T_3 = x_3z_i$, then by Lemma 1.2.3 (i), $H[V(Q) \cup V(T_2)]$ is not reduced, contrary to the assumption that H is reduced. Thus $y'' \neq z_i$. By symmetry, we also have $y'' \neq z_j$.

Hence we must have $y' = z_j$. If $|E(T_2)| \geq 3$, then $z_iQ^-[x_5, x_2]T_2[x_2, z_j]$ has length at least 7, contrary to (4.1). Thus we must have $T_2x_2x'z_j$. By symmetry, we must also have $T_4 = x_4x''z_i$. Now applying (4.2) to the path $Q' = x_0x''x_4x_3x_2x'x_6$, we also have $x''z_j, x'z_i \in E(H)$. Thus by Lemma 1.2.3 (iii), $H[V(Q) \cup \{x', x''\}]$ is not reduced, contrary to the assumption that H is reduced. This completes the proof of the lemma. □

Denote $V(H) - V(P) = \{u_1, u_2, ...\}$. By Definition 4.2.3 (ii), for each $1 \leq i \leq 3$ and $u_j \in V(H) - V(P)$, let

$$Q_i^j \text{ denote a } (u_j, z_{t_i^j})\text{-path with } V(P) \cap V(Q_i^j) = \{z_{t_i^j}\} \tag{4.4}$$

such that Q_1^j, Q_2^j and Q_3^j are mutually edge-disjoint.

Lemma 4.2.5 *Let H be a reduced graph satisfying Property R(6). If for some j, i_1 and i_2 with $i_1 \neq i_2$, we have $V(Q_j^{i_1}) \cap V(Q_{i_2}^j) - (V(P) \cup \{u_1\}) \neq \emptyset$, then $\{t_{i_1}^j, t_{i_2}^j\} = \{0, 6\}$ and for a path Q in $Q_j^{i_1} \cup Q_j^{i_2}$, $P \cup Q$ is spanned by the graph L depicted in Figure 2.*

Proof. Let $Q_i^1 = v_0^iv_1^i...v_{n_i}^i$ with $u_1 = v_0^i$ and $z_{t_i^1} = v_{n_i}^i$. We assume that $p < n_1$ is the largest such that $v_p^1 \in V(Q_2^1)$ and $q < n_2$ is the largest such that $v_q^2 \in V(Q_1^1)$.

Figure 5. $V(Q_1^1) \cap V(Q_2^1) - (V(P) \cup \{u_1\}) \neq \emptyset$.

Suppose first that $v_p^1 \neq v_q^2$. By symmetry, we may assume that $v_q^2 \in V(Q_1^1[u_1, v_p^1])$. Hence $t_2^1 > t_1^1$. As $girth(H) \geq 4$, H has a $(z_{t_1^1}, z_{t_2^1})$-path Q_{12} with

$$V(P) \cap V(Q_{12}) = \{z_{t_1^1}, z_{t_2^1}\} \text{ and with } |E(Q_{12}| \geq 4. \tag{4.5}$$

Next we assume that $v_p^1 = v_q^2$. By (4.6), $H - v_p^1$ has a path Q' from v_{q-1}^2 to a vertex in $V(P)$. By $girth(H) \geq 4$, we also conclude that (4.5) must hold (see Figure 5). By Lemma 4.2.4, we must have $\{t_1^1, t_2^1\} = \{0, 6\}$ and $H[V(P) \cup V(Q_{12})]$ is spanned by L. □

Lemma 4.2.6 *Let H be a reduced graph satisfying Property R(6) with $h = 6$, $z_0 = w_1$, $z_6 = w_2$ and P in (4.1) being a longest (w_1, w_2)-path. Each of the following must hold.*
(A) H does not have L (depicted in Figure 2) as a subgraph.
(B) For any $u_j \in V(H) - V(P)$, if $i_1 \neq i_2$, then $V(Q_{i_1}^j) \cap V(Q_{i_2}^j) \subseteq \{u_j\} \cup V(P)$.
(C) There is no path Q satisfying the condition in Claim 1 in the proof of Lemma 4.2.4.
(Thus any path Q satisfying the condition of Claim 1 of Lemma 4.2.4 will be referred to a **forbidden path***).*
(D) $E(H - V(P)) = \emptyset$.

Proof. We shall use the same notations in Lemmas 4.2.4 and 4.2.5.

(A). Assume that H has L as a subgraph. Let Q be a path stated in Lemma 4.2.5 such that $H[V(P) \cup V(Q)]$ is spanned by L. If the neighbors of vertices in z_1, z_2, z_4 and z_5 are all on $V(P)$, then by Definition 4.2.3(i), we have $\kappa'(H[V(P) \cup V(Q)]) \geq 3$, and so by $|V(L)| = 10$ and by Lemma 1.2.3(ii), $H[V(P) \cup V(Q)]$ is not reduced, contrary to the assumption that H is reduced. Hence by symmetry we assume that z_1 or z_2 is adjacent to a vertex z' not on P. Let $z \in \{z_1, z_2\}$. By Claim 1 in the proof of Lemma 4.2.4, z is not adjacent to any vertex in $Q - V(P)$. Let $z' \in N_H(z) - V(P)$. By Definition 4.2.3 (ii), H has a (z', z_i)-path Q_z, for some $0 \leq i \leq 6$ such that $V(Q_z) \cap V(P) = \{z_i\}$. By Lemma 4.2.4, $V(Q_z) \cap V(Q) \subseteq \{z_0, z_6\}$. By (4.1), $z_i \notin N_H(z) \cap V(P)$. If $i \in \{4, 5\}$, then $P[z_0, z]Q_z[z, z_i]P^-[z_3, z_i]x_2x_1z_6$ is longer than P. Therefore, if $z = z_1$, then $i \neq 0$ and so $i = 6$. It follows that $z_0x_1x_2z_3z_2z_1zQ_z[z', z_6]$ is longer than P. Therefore, we must have $z = z_2$ and $N_H(z') = \{z_0, z_2, z_6\}$. It follows by Lemma 1.2.3 (i) that $H[V(Q) \cup \{z', z_2, z_3\}]$ is not reduced, contrary to the assumption that H is reduced. This proves (A).

(B). If not, we assume that for $j = 1$ and so $V(Q_1^1) \cap V(Q_2^1) - \{u_1\} \cup V(P) \neq \emptyset$. By Lemma 4.2.5, the graph L depicted in Figure 2 is a subgraph of H containing P, contrary to (A).

(C). If it does, then by Lemma 4.2.4, H contains L as a subgraph, contrary to (A).

(D). By contradiction, we assume that $u_1u_2 \in E(H - V(P))$. By (B), the paths in (4.4) are internally vertex disjoint. Without lose of generality, for $j \in \{1, 2\}$, we assume $t_1^j \leq t_2^j \leq t_3^j$. By Definition 4.2.3 (ii), we may assume $t_1^1 \leq t_2^1 < t_3^1$, and $t_2^1 < t_3^2$. By symmetry, we further assume that $t_1^1 \leq t_2^2, t_2^1 \leq t_3^2$. If $t_1^1 < t_2^1 < t_2^2 < t_3^2$, then since $girth(H) \geq 4$ and by (4.1), we must have $t_2^1 \geq t_1^1 + 2$, $t_2^2 \geq t_2^1 + 3$ and $t_3^2 \geq t_2^2 + 2$. Thus $6 \geq h \geq t_3^2 \geq t_1^1 + 7$, a contradiction. Hence we cannot have $t_1^1 < t_2^1 < t_2^2 < t_3^2$. Similarly, we cannot have $t_1^1 < t_2^2 < t_2^1 < t_3^2$. Hence we must have $|\{t_1^1, t_2^2, t_2^1, t_3^2\}| \leq \{2, 3\}$,

Case D1. $|\{t_1^1, t_2^2, t_2^1, t_3^2\}| = 3$

Thus either $t_1^1 < t_2^1 = t_2^2 < t_3^2$, or $t_1^1 = t_2^2 < t_2^1 < t_3^2$ or $t_1^1 < t_2^2 < t_2^1 = t_3^2$ must hold (see Figure 6 (a) and (b)). As $girth(H) \geq 4$, if $t_1^1 < t_2^1 = t_2^2 < t_3^2$, then $|E(Q_2^1)| + |E(Q_2^2)| \geq 3$, and so either $Q_1^1 \cup Q_2^2 \cup \{u_1u_2\}$ or $Q_2^1 \cup Q_3^2 \cup \{u_1u_2\}$ is a forbidden path; if $t_1^1 = t_2^2 < t_2^1 < t_3^2$, then $|E(Q_1^1)| + |E(Q_2^2)| \geq 3$, and so either $Q_1^1 \cup Q_3^2 \cup \{u_1u_2\}$ or $Q_2^1 \cup Q_2^2 \cup \{u_1u_2\}$ is a forbidden path. The case when $t_1^1 < t_2^2 < t_2^1 = t_3^2$ is similar to that for $t_1^1 = t_2^2 < t_2^1 < t_3^2$. Thus a contradiction to (C) always exists.

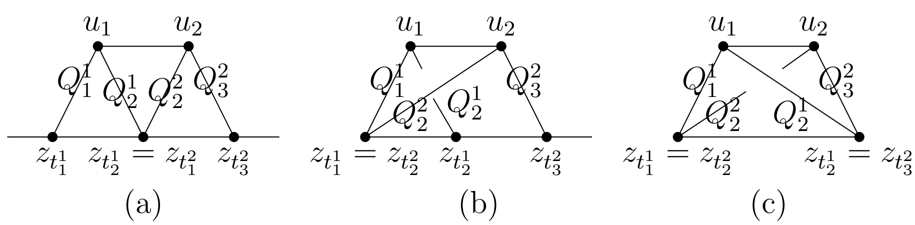


Figure 6. forbidden paths in Lemma 4.2.6(D).

Case D2. $|\{t_1^1, t_2^2, t_2^1, t_3^2\}| = 2$.

Thus $\{t_1^1, t_2^1\} = \{t_2^2, t_3^2\}$, and so $t_1^1 = t_2^2$ and $t_2^1 = t_3^2$. By $girth(H) \geq 4$, both $|E(Q_1^1)| + |E(Q_2^2)| \geq 3$ and $|E(Q_2^1)| + |E(Q_2^3)| \geq 3$ (see Figure 6(c)). It follows that a forbidden path must exist, contrary to (C).

This completes the proof for Case D2, and so (D) is justified. □

Lemma 4.2.7 *If H is a graph satisfying Property R(6), then H is not reduced.*

Proof. By contradiction, we assume that H is a smallest counterexample, and so H is reduced. Then by Theorem 1.2.2, $girth(H) \geq 4$. We shall assume and use the notations of (4.1).

If $|V(H)| \leq 8$, then by Lemma 1.2.3, H is not reduced. Hence we must have $|V(H) - V(P)| \geq 2$. If H has a cut vertex w, then H has two connected nontrivial subgraphs H_1 and H_2 such that $H = H_1 \cup H_2$ and $V(H_1) \cap V(H_2) = \{w\}$. If for some $i \in \{1, 2\}$, $w_1, w_2 \in V(H_i)$, then by the minimality of H, H_i is not reduced. If for $i \in \{1, 2\}$, $w_i \in V(H_i)$, then applying the minimality of H to w_1, w in H_1, we conclude that H_1 is not reduced. These contradictions show that

$$\kappa(H) \geq 2. \tag{4.6}$$

Claim 1. $|V(H) - V(P)| \geq 4$.

If $|V(H) - V(P)| < 4$, then as $|V(P)| \leq 7$, $|V(H)| \leq 10$. Construct a new graph G from H by adding a new vertex z, which is adjacent to both w_1 and w_2. Then z is the only vertex of degree 2 in G and $|V(G)| \leq 11$. By (4.6) and Lemma 1.2.3 (iv), the reduction of G is either $K_{2,3}$, whence H is not reduced; or is in $\{P(10), P(10)(e)\}$, whence a longest path connecting w_1 and w_2 in H is at least 9. These contradictions establish Claim 1.

By Lemma 4.2.6(D), for any $u_j \in V(H) - V(P)$, $z_{t_1^j}, z_{t_2^j}, z_{t_3^j} \in N_H(u_j) \subseteq V(P)$. By Claim 1, $|E(G)| = |E(P)| + |E(G) - E(P)| \geq |V(P)| - 1 + 3|V(H) - V(P)| = 3|V(H)| - 1 - 2|V(P)| \geq 2|V(H)| - 1 + |V(H) - V(P)| - |V(P)| \geq 2|V(H)| + 3 - 7 = 2|V(H)| - 4$. By Theorem 1.2.2(vi) $F(H) = 2|V(H)| - |E(H)| - 2 \leq 2$. By Theorem 1.2.2(vi), either H is not reduced, contrary to the assumption; or H is a $K_{2,t}$ for some integer $t \geq 11 - 2 = 9$, contrary to the fact that H has a path of length at least 6. This proves the lemma. □

4.3 Proof of Theorem 4.1.4

In order to prove Theorem 4.1.4, we need an auxiliary theorem as stated below. The proof of Theorem 4.3.1 will be given in Section 4.

Theorem 4.3.1 *Suppose G is a graph such that $G \neq H_8$, and satisfies $\kappa(G) \geq 2$ and $\kappa'(G) \geq 3$. Let $e = v_0v_1$ and $e' = v_{c-1}v_c$ be edges in G, and $P = v_ev_1 \cdots v_{c-1}v_{e'}$ be a longest $(v_e, v_{e'})$-path in $G(e, e')$ with $c = |E(P)| \leq 8$. If $G(e, e')$ is reduced and contains no spanning $(v_e, v_{e'})$-trail, then $V(G) = \{v_i : 0 \leq i \leq c\}$.*

Proof of Theorem 4.1.4. Let G be a counterexample to Theorem **??** with

$$|V(G)| \text{ is minimized.} \tag{4.7}$$

Let $e, e' \in E(G)$ be edges such that the length of a longest $(v_e, v_{e'})$-path in $G(e, e')$ is at most 8 and such that $G(e, e')$ does not have a spanning $(v_e, v_{e'})$-trail. If $G = H_8$, then it is routine to verify that, up to isomorphism, $e = v_1v_5$ and $e' = v_3v_7$ (using notations in Figure 1 or Figure 7). Hence we assume that $G \neq H_8$. Denote $e = z_0z_1$ and $e' = z_{c-1}z_c$ and let $P = v_ez_1 \cdots z_{c-1}z_{e'}$ be a longest $(v_e, v_{e'})$-path in $G(e, e')$, By the assumption of Theorem 4.1.4, $c \leq 8$. By (4.7), by Lemmas 4.2.1 and 4.2.2, we may assume that

$$\kappa(G) \geq 2 \text{ and } G(e, e') \text{ is reduced.} \tag{4.8}$$

By Theorem 4.3.1, $V(G) = \{z_i : 0 \leq i \leq c\}$. Obtain a new graph L_w from $G(e, e')$ by adding a new vertex w and new edges wv_e and $wv_{e'}$. Then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail if and only if L_w is supereulerian. Since $|V(G)| = c + 1 \leq 9$, we have $|V(L_w)| \leq |V(G)| + 3 = 12$. As L_w has exactly one edge cut of size 2, it follows by Lemma 1.2.3(iv) that either L is supereulerian, or the reduction of L_z is $P(10)$ or $P(10)(f)$, for any edge $f \in E(P(10))$.

If the reduction of L_w is $P(10)$, then one vertex v_L (say) of $P(10)$ must be the image of a nontrivial collapsible subgraph of L_w containing the new vertex w. As $P(10)$ has a cycle of length 9 containing v_L, this implies that $G(e, e')$ has a $(v_e, v_{e'})$-path of length at least 9, contrary to the assumption of Theorem 4.1.4. If L_w is supereulerian, then

$G(e, e')$ has a spanning $(v_e, v_{e'})$-trail, contrary to (4.7). Hence the reduction of L_w must be isomorphic to $P(10)(f)$. (In Figure 7, we choose $f = v_5'v_7'$ as an illustration.) Thus w must be the only vertex of degree 2 in $P(10)(f)$.

If any vertex w' of $P(10)(f) - w$ is the contraction image of a nontrivial collapsible subgraph of $G(e, e')$, then $P(10)(f)$ has a cycle containing both w and w' with length 10. By definition of contraction, this cycle can be lifted to yield a (v_e, v_e')-path in $G(e, e')$ of length at least 9, contrary to the assumption of Theorem 4.1.4. Thus we must have $L_w = P(10)(f)$, implying that $G = H_8$ with, up to isomorphism, $e = v_1v_5$ and $e' = v_3v_7$. (See Figure 7. The labels indicates that the H_8 in Figure 7 is a different drawing of the H_8 in Figure 1.) This contradicts to the assumption that $G \neq H_8$, and completes the proof of the theorem. □

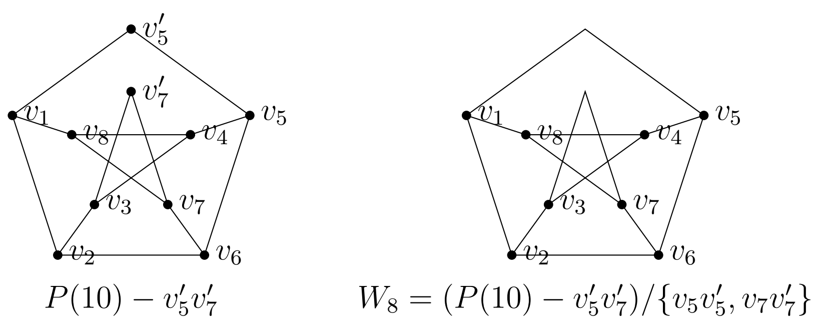


Figure 7. The graphs in the proof of Theorem 4.1.4.

4.4 Proof of Theorem 4.3.1

We denote $e = v_0v_1$ and $e' = v_{c-1}v_c$. For any longest (v_e, v_e')-path $P = v_ev_1...v_{c-1}v_{e'}$ in $G(e, e')$, define $P' = v_0v_1...v_{c-1}v_c$ and $J = J(P) = G(e, e') - (V(P) \cup \{v_0, v_c\}) = G - V(P')$. The strategy of the proof is to work on a counterexample G. First show that $E(J) = \emptyset$ for any longest (v_e, v_e')-path P in $G(e, e')$. Then we use case analysis to show that if $|V(J)| \geq 2$, then a longer (v_e, v_e')-path or a nontrivial collapsible subgraph can always be found. Finally, we prove that assuming $|V(J)| = 1$ will also yield a similar contradiction, which forces that $V(J) = \emptyset$, and completes the proof of Theorem 4.3.1.

If H is a subgraph of a graph Γ, we define the set of vertices of attachment of H in Γ as

$$A_\Gamma(H) = \{v \in V(H) \ : \ \ v \text{ is adjacent to a vertex in } V(\Gamma) - V(H)\}.$$

To prove Theorem 4.3.1, we assume $G \neq W_8$,

$$P = v_e v_1 \cdots v_{c-1} v_{e'} \text{ is a longest } (v_e, v_{e'})\text{-path in } G(e, e'), \tag{4.9}$$

that $G(e, e')$ has no spanning $(v_e, v_{e'})$-trail and that

$$c \leq 8 \text{ and } G(e, e') \text{ is reduced (and so } girth(G(e, e')) \geq 4). \tag{4.10}$$

If $e = e'$, then let $e = w_1 w_2$. Thus in $G - e$, every longest (w, w')-path has length at most 6, and so $G - e$ has Property $R(6)$. By Lemma 4.2.7, $G - e$ is not reduced, contrary to (4.10). Throughout the rest of this section, we assume that $e \neq e'$. For each vertex $u \in V(J)$, as $\kappa(G) \geq 2$ and $\kappa'(G) \geq 3$, we note that G has edge-disjoint (u, v_{i_j})-path Q_i with

$$V(Q_i) \cap V(P') = \{v_{i_j}\}(1 \leq j \leq 3) \text{ and } |\{v_{i_1}, v_{i_2}, v_{i_3}\}| \geq 2. \tag{4.11}$$

The following notation will be used in the proof:

$$Q_j = u z_1^j z_2^j ... z_{n_j}^j v_{i_j}, \text{ for } j = 1, 2, 3. \tag{4.12}$$

Claim 1. In each component of J, choose a vertex u and the related paths Q_i $(1 \leq i \leq 3)$ such that

$$|\{v_{i_1}, v_{i_2}, v_{i_3}\}| \text{ is maximized.} \tag{4.13}$$

Then $|\{i_1, i_2, i_3\}| = 3$.

Proof of Claim 1. By contradiction and without lose of generality, we assume that J has a component J_1 with a vertex $u \in V(J_1)$ satisfying (4.13) with $\{i_1, i_2, i_3\} = \{i_1, i_2\}$. Since u satisfies (4.13), we have $A_{G(e,e')}(J_1) = \{v_{i_1}, v_{i_2}\}$. It follows by $c \leq 8$ that J_1 has Property $R(6)$ with $\{w_1, w_2\} = \{v_{i_1}, v_{i_2}\}$. By Lemma 4.2.7, J_1 is not reduced, contrary to (4.10). This proves Claim 1.

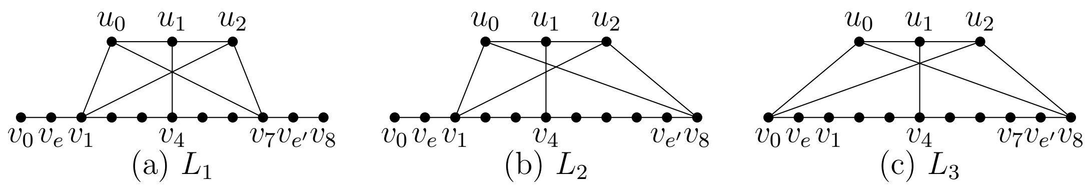


Figure 8. The subgraphs in Claim 2(B).

Claim 2. Let $u \in V(J)$ and define the Q_j's as in (4.11). Each of the following holds.
(A) J does not have an edge $u_1u_2 \in E(J)$ such that $N_G(u_1) \cup N_G(u_2) - \{u_1, u_2\} \subseteq V(P) \cup \{v_0, v_c\}$.
(B) If $v_0 \neq v_c$, and if a nontrivial component of J contains a vertex $u \in N_G(v_i)$ for some i with $1 \leq i \leq c-1$, then either $G(e, e')$ is not reduced or $G(e, e')$ has a $(v_e, v_{e'})$-path longer than P, or $G(e, e')$ has a subgraph isomorphic to one of the graphs depicted in Figure 8.
(C) $E(J) = \emptyset$.

Proof of Claim 2. **(A).** Assume that such u_1, u_2 exist. Let $v_{i_1}, v_{i_2} \in N_G(u_1) - \{u_2\}$ and $v_{j_1}, v_{j_2} \in N_G(u_2) - \{u_1\}$. By (4.10), i_1, i_2, j_1, j_2 are mutually distinct. By symmetry, we assume that $j_1 < j_2$ and $i_1 < \min\{i_2, j_1\} < j_2$.

Case A1. $i_1 < i_2 < j_1 < j_2$.

By (4.9), $i_2 \geq i_1 + 2$, $j_1 \geq i_2 + 3$ and $j_2 \geq j_1 + 2$. Hence $j_2 \geq i_1 + 7$. It follows that we must have $i_1 = 0$ or $j_2 = c = 8$. If $i_1 = 0$, then by (4.9), $i_2 \geq 3$, and so $j_2 \geq i_2 + 5 = 8$, forcing $j_1 = 6$ and $j_2 = 8$. But then, $P[v_e, v_6]u_2v_8v_{e'}$ is longer than P, contrary to (4.9).

Case A2. $i_1 < j_1 < i_2 < j_2$.

By (4.9), $j_2 \geq i_1 + 3$, $i_2 \geq j_1 + 3$ and $j_2 \geq i_2 + 3$. It follows that $8 = c \geq j_2 \geq i_1 + 9$, a contradiction.

Case A3. $i_1 < j_1 < j_2 < i_2$.

By (4.9), $j_1 \geq i_1 + 3$, $j_2 \geq j_1 + 2$ and $i_2 \geq j_2 + 3$. Hence $i_2 \geq i_1 + 7$. It follows that we must have $i_1 = 0$ or $i_2 = c = 8$. If $i_1 = 0$, then by (4.9), $j_1 \geq 4$, and so $i_2 \geq j_1 + 5 = 9$, forcing $j_2 = 6$ and $i_2 = 8$. But then, $P[v_e, v_6]u_2u_1v_8v_{e'}$ is longer than P, contrary to (4.9).

(B). Let H be a nontrivial component of J with a vertex $u \in V(H)$ satisfying the hypothesis of Claim 2(B). Since $E(H) \neq \emptyset$, H contains a longest (u, u')-path Q for some vertex $u' \in A_G(H) - \{u\}$ such that u' is adjacent to a vertex v_j with $j \neq i$. Since $\kappa(G) \geq 2$, such u' exists. By symmetry, we may assume that $i < j$. Denote $Q = u_0 u_1 u_2 ... u_q$ with $u = u_0$, $u' = u_q$. By (4.9), we have $4 \geq q \geq 2$. If every path from a vertex in Q to a vertex in $P \cup \{v_0, v_c\}$ must use v_i or v_j, then by Lemma 4.2.7 (with w_i, w_i replaced by v_i, v_j), $G[V(H) \cup \{v_i, v_j\}]$ is not reduced. Hence we may assume that for some h with $0 \leq h \leq q$, and a (u_h, v_k)-path Q' with $k \notin \{i, j\}$.

Case B1. $i < k < j$. By (4.9), we have

$$\begin{cases} k \geq i+2, j \geq k + |E(Q)| + 2 \geq i+6 & \text{if } h = 0 \\ k \geq i+3, j \geq k + |E(Q[u_h, u_q])| + 2 \geq i+6 & \text{if } 0 < h < q \\ k \geq i + |E(Q)| + 2, j \geq k+2 \geq i+6 & \text{if } h = q \end{cases}$$

Since $i \geq 1$, we must have $i = 1$, $j \in \{7, 8\}$, $k \in \{3, 4, 5\}$, $|E(Q')| = 1$ and $|E(Q)| = 2$. Let

$$z = \begin{cases} u_1 & \text{if } h = 0 \text{ (whence } k = 3) \text{ or if } h = 2 \text{ (whence } k = 5) \\ u_0 & \text{if } h = 1 \text{ (whence } k = 4) \end{cases}$$

As $\kappa'(G) \geq 3$, $G(e, e')$ has a (z, v_l)-path T_1 such that $V(Q) \cap V(T_1) = \{z\}$. Tables 4A and 4B below indicate that either a $(v_e, v_{e'})$-path longer than P always exists, or when $k = 3$ and $h = 0$, $l = j \in \{7, 8\}$. Thus we may assume that $k = 3$ and $h = 0$, either $u_0 v_7, u_2 v_7 \in E(G)$ or $u_0 v_8, u_2 v_8 \in E(G)$. By symmetry, either $u_0 v_0, u_2 v_0 \in E(G)$ or $u_0 v_1, u_2 v_1 \in E(G)$. Thus $G(e, e')$ contains a graph in Figure 8 as a subgraph.

k	h	j	z	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e, e')$)
3	0	7,8	u_1	0	$v_e v_0 u_1 u_0 P[v_1, v_{e'}]$
3	0	7,8	u_1	1	$v_e v_0 u_1 P[v_1, v_{e'}]$
3	0	7,8	u_1	$2, 3$	$v_e v_1 u_0 u_1 P[v_l, v_{e'}]$
3	0	7,8	u_1	$4, 5$	$v_e v_1 v_2 v_3 u_0 u_1 P[v_l, v_{e'}]$
3	0	7,8	u_1	6	$v_e P[v_1, v_6] u_1 u_2 v_7 v_{e'}]$
3	0 0	7(or 8)	u_1	7(or 8)	$G[\{u_1, u_2, v_7\}] \cong K_3$ or $G[\{u_1, u_2, v_8\}] \cong K_3$
3	0	7	u_1	8	$v_e P[v_1, v_7] u_2 u_1 v_8 v_{e'}]$
3	0	8	u_1	7	$v_e P[v_1, v_7] u_1 u_2 v_8 v_{e'}]$

Table 4A: Case B1 with $k = 3$ and $h = 0$

(The case $k = 5$ and $h = 2$ is symmetric to this).

k	h	j	z	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e, e')$)
4	1	7,8	u_0	0	$v_e v_0 u_0 P[v_1, v_{e'}]$
4	1	7,8	u_0	2(or 4)	$G[\{u_1, v_1, v_2\}] \cong K_3$ or $G[\{u_1, u_2, v_4\}] \cong K_3$
4	1	7,8	u_0	3	$P[v_e, v_3] u_0 u_1 P[v_4, v_{e'}]$
4	1	7,8	u_0	$5, 6$	$P[v_e, v_4] u_1 u_0 P[v_l, v_{e'}]$
4	1	7	u_0	8	$P[v_e, v_7] u_2 u_1 u_0 v_8 v_{e'}]$
4	1	8	u_0	7	$P[v_e, v_7] u_0 u_1 u_2 v_8 v_{e'}]$

Table 4B: Case B1 with $k = 3$ and $h = 0$.

Case B2. $k < i < j$. If $h > 0$, then by (4.9), $i \geq k + |E(Q[u_1, u_h])| + 2$ and $8 \geq j \geq i + |E(Q)| + 2 \geq k + |E(Q[u_1, u_h])| + 5$. It follows that either $k = 1$, $i = 4$, $|E(Q)| = 2$, $h = 1$, $u_1 v_1 \in E(G)$ and $j = 8$, or $k = 0$, $i = 4$, $|E(Q)| = 2$, $h = 1$, $u_1 v_0 \in E(G)$ and $j = 8$. In any case, $P[v_e, v_4] u_0 u_1 u_2 v_8 v_{e'}$ is longer than P. Hence we must have $h = 0$. By (4.9), $i \geq k + 2$ and $j \geq i + |E(Q)| + 2 \geq k + |E(Q[u_1, u_h])| + 4$. Thus $k \in \{0, 1\}$, $i = 3$,

$u_0v_k \in E(G)$, $j \in \{7,8\}$ and $|E(Q)| = 2$. As $\kappa'(G) \geq 3$, $G(e,e')$ has a (u_1, v_l)-path T_2 such that $V(Q) \cap V(T_2) = \{u_1\}$. Table 5 completes the proof of Case B2.

k	h	j	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e,e')$)
0	0	7, 8	0 (or 3, or j)	$G[\{u_0,u_1,v_0\}] \cong K_3$ or $G[\{u_1,u_2,v_3\}] \cong K_3$ or $G[\{u_1,u_2,v_j\}] \cong K_3$
0	0	7,8	1, 2	$v_ev_0u_0u_1P[v_l, v_{e'}]$
0	0	7,8	4, 5	$P[v_e, v_3]u_0u_1P[v_l, v_{e'}]$
0	0	7,8	6	$P[v_e, v_l]u_1u_2v_jv_{e'}$
0	0	7	8	$P[v_e, v_7]u_2u_1v_8v_{e'}$
0	0	8	7	$P[v_e, v_7]u_1u_2v_8v_{e'}$

Table 5: Case B2 .

Case B3. $i < j < k$. If $h < q$, then by (4.9), $j \geq i + |E(Q)| + 2$ and $8 \geq k \geq j+|E(Q[u_h, u_q])|+2 \geq i+7$. It follows that $i = 1$, $j = 4$, $|E(Q)| = 2$, $h = 1$, $u_1v_8 \in E(G)$ and $k = 8$, whence $v_ev_1u_0u_1u_2P[v_4, v_{e'}]$ is longer than P. Therefore, we must have $h = q$. By (4.9), $j \geq i+|E(Q)|+2$ and $k \geq j+2 \geq i+|E(Q)|+4$. Thus $i = 1$, $j = 5$, $k \in \{7,8\}$, $|E(Q)| = 2$ and $u_2v_k \in E(G)$. As $\kappa'(G) \geq 3$, $G(e,e')$ has a (u_1, v_l)-path T_3 such that $V(Q) \cap V(T_3) = \{u_1\}$. Table 6 completes the proof of Case B3.

k	h	j	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e,e')$)
7,8	2	5	1 (or 5, or k)	$G[\{u_0,u_1,v_1\}] \cong K_3$ or $G[\{u_1,u_2,v_5\}] \cong K_3$ or $G[\{u_1,u_2,v_k\}] \cong K_3$
7,8	2	5	1	$v_ev_0u_1u_0P[v_l, v_{e'}]$
7,8	2	5	2,3	$v_ev_1u_0u_1P[v_l, v_{e'}]$
7,8	2	5	4	$P[v_e, v_4]u_1u_2P[v_5, v_{e'}]$
7,8	2	5	6	$P[v_e, v_6]u_1u_2v_kv_{e'}$
7	2	5	8	$P[v_e, v_7]u_2u_1v_8v_{e'}$
8	2	5	7	$P[v_e, v_7]u_1u_2v_8v_{e'}$

Table 6: Case B3 .

(C). By Claim 2(B), for any $i \in \{2, 3, 5, 6\}$, either $N_G(v_i) \subseteq V(P) \cup \{v_0, v_c\}$ or for some $u \in V(J)$,

$$v_i \in N_G(u) \subseteq V(P) \cup \{v_0, v_c\}. \tag{4.14}$$

Assume first that $G(e, e')$ does not have any graph depicted in Figure 8 as a subgraph. By Claim 2(B), for any nontrivial component L of J, we must have $A_G(L) = \{v_0, v_c\}$. But by Claim 1, we should have $|A_G(L)| \geq 3$, a contradiction. Thus (C) must hold as J does not have any nontrivial components. Hence we assume that $G(e, e')$ has a graph in Figure 8 as a subgraph.

For $2 \leq i \leq 3$, by $\kappa(G) \geq 3$, $N_G(v_i) - N_P(v_i)$ contains a vertex x_i'. Define,

$$x_i = \begin{cases} x_i' & \text{if } x_i' \notin V(J) \\ \text{a vertex in } N_G(x_i') \cap V(P) - \{v_i\} & \text{if } x_i' \in V(J) \end{cases}. \tag{4.15}$$

Case C1. $G(e, e')$ has L_1 as a subgraph.

Since $G(e, e')$ is reduced, if $x' \notin V(J)$, then $x \notin \{v_1, v_3, v_4\}$ and if $x' \in V(J)$, then as $|N_G(x') \cap (V(P) \cup \{v_0, v_c\})| \geq 3$, we can choose $x_2 \notin \{v_1, v_3, v_4\}$. Similarly, we may assume that $x_3 \notin \{v_1, v_2, v_4\}$. Table 7 shows that we must have $x_2 = x_2' = x_3 = x_3' = v_7$, and so $G(e, e')[\{v_2, v_3, v_7\}] \cong K_3$, contrary to (4.10).

x_2	x_3	$(v_e, v_{e'})$-path longer than P
v_0		$v_e v_0 v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$ or $v_e v_0 x_2' v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$
v_5, v_6		$v_e v_1 u_2 u_1 v_4 v_3 v_2 P[x_2, v_{e'}]$ or $v_e v_1 u_2 u_1 v_4 v_3 v_2 x_2' P[x_2, v_{e'}]$
v_8		$v_e v_1 u_2 u_1 u_0 v_7 v_6 v_5 v_4 v_3 v_2 v_8 v_{e'}$ or $v_e v_1 u_2 u_1 u_0 v_7 v_6 v_5 v_4 v_3 v_2 x_2' v_8 v_{e'}$
	v_0 v_0	$v_e v_0 v_3 v_2 v_1 u_1 u_2 P[v_4, v_{e'}]$ or $v_e v_0 x_3' v_3 v_2 v_1 u_1 u_2 P[v_4, v_{e'}]$
	v_5	$v_e v_1 u_0 u_1 v_4 v_3 P[v_5, v_{e'}]$ or $v_e v_1 u_0 u_1 v_4 v_3 x_3' P[v_5, v_{e'}]$
	v_6	$P[v_e, v_3] v_6 v_5 v_4 u_1 u_0 v_7 v_{e'}$ or $P[v_e, v_3] x_3' v_6 v_5 v_4 u_1 u_0 v_7 v_{e'}$
	v_8	$v_e v_1 u_0 u_1 u_2 v_7 v_6 v_5 v_4 v_3 v_8 v_{e'}$ or $v_e v_1 u_0 u_1 u_2 v_7 v_6 v_5 v_4 v_3 x_3' v_8 v_{e'}$

Table 7: Case C1 in Claim 2 .

Case C2. $G(e, e')$ has L_2 as a subgraph.

As in Case C1, since $G(e, e')$ is reduced, we may assume that $x_2 \notin \{v_1, v_3, v_4\}$. As shown in Table 8, we always obtain a contradiction to (4.9).

x_2	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_0 v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$ or $v_e v_0 x_2' v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$
v_5, v_6	$v_e v_1 u_0 u_1 v_4 v_3 v_2 P[x_2, v_{e'}]$ or $v_e v_1 u_0 u_1 v_4 v_3 v_2 x_2' P[x_2, v_{e'}]$
v_7	$P[v_e, v_2] v_7 v_6 v_5 v_4 u_1 u_2 v_8 v_{e'}$ or $P[v_e, v_2] x_2' v_7 v_6 v_5 v_4 u_1 u_2 v_8 v_{e'}$
v_8	$v_e v_1 u_2 u_1 u_0 v_8 P[v_2, v_{e'}]$ or $v_e v_1 u_2 u_1 u_0 v_8 x_2' P[v_2, v_{e'}]$

Table 8: Case C2 in Claim 2 .

Case C3. $G(e, e')$ has L_3 as a subgraph.

As in Case C1, since $G(e,e')$ is reduced, we may assume that $x_2 \notin \{v_1, v_3, v_4\}$. As shown in Table 9, we always obtain a contradiction to (4.9). This proves Case (C3) and completes the proof of Claim 2.

x_2	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_1 v_2 v_0 u_0 u_1 P[v_4, v_{e'}]$ or $v_e v_1 v_2 x_2' v_0 u_0 u_1 P[v_4, v_{e'}]$
v_5	$v_e v_0 u_0 u_1 v_4 v_3 v_2 P[x_5, v_{e'}]$ or $v_e v_0 u_0 u_1 v_4 v_3 v_2 x_2' P[x_5, v_{e'}]$
v_6, v_7	$P[v_e, v_2] P^-[x_2, v_4] u_1 u_2 v_8 v_{e'}$ or $P[v_e, v_2] x_2' P^-[x_2, v_4] u_1 u_2 v_8 v_{e'}$
v_8	$v_e v_0 u_0 u_1 u_2 v_8 P[v_2, v_{e'}]$ or $v_e v_0 u_0 u_1 u_2 v_8 x_2' P[v_2, v_{e'}]$

Table 9: Case C3 in Claim 2 .

Let $V(J) = \{u_1, u_2, \cdots\}$. By Claim 2(C), if $j \geq 1$, then $N_G(u) \subseteq V(P) \cup \{v_0, v_c\}$. Thus we may denote that $N_G(u_j) = \{v_{i_1^j}, v_{i_2^j}, v_{i_3^j}\}$ with $i_1^j < i_2^j < i_3^j$. Claim 3 below follows from the fact that P is longest.

Claim 3. Let $u_j \in V((G)(e,e')) - V(P) \cup \{v_0, v_c\}$. Then

$$i_2^j \geq \begin{cases} i_1^j + 2, & \text{if } i_1 \neq 0 \\ 3, & \text{if } i_1 = 0 \end{cases}, \text{ and } i_3^j \geq \begin{cases} i_2^j + 2, & \text{if } c > i_3^j \\ i_2^j + 3, & \text{if } c = i_3^j \end{cases}.$$

Therefore, $c \geq 6$.

Claim 4. Let $u_j \in V(J)$. If $i_2^j = i_1^j + 2$, then $N_G(v_{i_1^j+1}) \subseteq V(P) \cup \{v_0, v_c\}$. If $i_3^j = i_2^j + 2$, then $N_G(v_{i_2^j+1}) \subseteq V(P) \cup \{v_0, v_c\}$.

Proof of Claim 4. By symmetry, we only prove the case when $i_2^j = i_1^j + 2$. Let $P' = P[v_e, v_{i_1^j}] u_j P^-[v_{i_2^j}, v_{e'}]$. Then $|P| = |V(P')|$. Applying Claim 2(C) on P', we conclude that $E(G(e,e') - (V(P') \cup \{v_0, v_c\})) = \emptyset$, and so $N_G(v_{i_1^j+1}) \subseteq V(P') \cup \{v_0, v_c\}$. Since $girth(G(e,e')) \geq 4$, $u_j v_{i_1^j+1} \notin E(G)$. Thus $N_G(v_{i_1^j+1}) \subseteq V(P) \cup \{v_0, v_c\}$. This justifies Claim 4.

In Claims 5-8 below, we assume that $u_1, u_2 \in V(J)$ and define $s = \min\{i_1^1, i_2^1, i_3^1, i_1^2, i_2^2, i_3^2\}$ and $\ell = \max\{i_1^1, i_2^1, i_3^1, i_1^2, i_2^2, i_3^2\}$. For any v_i with $1 \leq i \leq 7$, by $\kappa'(G) \geq 3$, there exists $x_i' \in N_G(v_i) - N_P(v_i)$. By Claim 2(C), either $x_i' \in V(P) \cup \{v_0, v_c\}$, or $x_i' \in V(J)$ with

$N_G(x_i') \subseteq V(P) \cup \{v_0, v_c\}$. Define x_i as in (4.15). By (4.10) and (4.9), when $i-2 \geq 1$ and $i+2 \leq c-1$, we can always choose x_i so that

$$x_i \notin \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}, \text{ unless } x_i' \in V(J) \text{ and } N_G(x_i') = \{v_{i-2}, v_i, v_{i+2}\}. \tag{4.16}$$

Claim 5. Each of the following holds.
(i) $\ell - s \leq 8$. Furthermore, if $\ell - s = 8$, then $c = 8, \ell = 8$ and $s = 0$.
(ii) $i_1^2 \leq i_2^1$. (By symmetry, $i_1^1 \leq i_2^2$, $i_3^1 \geq i_2^2$ and $i_3^2 \geq i_2^1$.)

(iii) A $(v_e, v_{e'})$-path longer than P exists if for some $s \in \{1, 2\}$,

$$1 \leq i_1^1 < i_s^2 < i_2^1 < i_{s+1}^2 < i_3^1 < c \text{ or } 1 \leq i_1^2 < i_s^1 < i_2^2 < i_{s+1}^1 < i_3^2 < c.$$

(iv) We cannot have $i_1^1 = i_1^2 < i_2^1 < i_2^2 = i_3^1 < i_3^2$. (By symmetry, we cannot have $i_1^1 < i_1^2 = i_2^1 < i_2^2 < i_3^1 = i_3^2$.)
(v) We cannot have $i_1^1 = i_1^2 < i_2^1 < i_2^2 < i_3^1 = i_3^2$.
(vi) $i_2^2 = i_2^1$.

Proof of Claim 5. (i). Claim 5(i) follows immediately from $c \leq 8$.

(ii). We argue by contradiction to prove $i_1^2 \leq i_2^1$. The proof for $i_3^1 \geq i_2^2$ is omitted by symmetry. Assume that $i_1^2 > i_2^1$.

Case 5(ii).1. $i_3^2 > i_3^1$.

If $i_2^2 > i_3^1$, then $i_3^2 \geq i_2^2 + 2$. As $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_2^2}, v_{e'}]$ is not longer than P, we have $(i_1^2 - i_2^1) + (i_2^2 - i_3^1) \geq 4$ and so $i_3^2 - i_1^1 \geq 9$, contrary to Claim 5(i). If $i_2^2 = i_3^1$, then as $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_3^2}, v_{e'}]$ is not longer than P, we have $(i_1^2 - i_2^1) + (i_3^2 - i_2^2) \geq 4$, and so $i_3^2 - i_1^1 \geq 8$. By Claim 5(i), $c = 8, i_1^1 = 0$ and $i_3^2 = 8$. Thus $i_2^1 = 3, i_3^1 = 6$, and so the path $P[v_e, v_6]u_2v_8v_{e'}$ is longer than P, contrary to (4.9). If $i_2^2 < i_3^1$, then as $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_3^2}, v_{e'}]$ is not longer than P, and so $(i_3^2 - i_3^1) + (i_1^2 - i_2^1) \geq 4$. Since $i_2^1 - i_1^1 \geq 2$ and $i_2^2 - i_1^2 \geq 2$, we have $i_3^2 - i_1^1 \geq 9$, contrary to Claim 5(i).

Case 5(ii).2. $i_3^2 = i_3^1$.

By (4.9), $i_3^1 \geq i_1^1 + 7$. As $i_1^1 = 0$ implies $i_2^1 \geq 3$, we conclude that we always have

$c = v_3^1 = 8$, $i_2^2 = 6$, $i_1^2 = 3$, and $i_1^1 \in \{0, 1\}$. Hence $P[v_e, v_6]u_2v_8v_{e'}]$ is longer than P, contrary to (4.9).

Case 5(ii).3. $i_3^2 < i_3^1$.

By (4.9), $i_3^1 \geq i_1^1 + 8$, and so we must have $i_1^1 = 0, i_3^1 = 8$. That $i_1^1 = 0$ forces $i_2^1 \geq 3$, and so $i_3^1 \geq 9$, contrary to Claim 5(i).

(iii). If $1 \leq i_1^1 < i_s^2 < i_2^1 < i_{s+1}^2 < i_3^1 < c$, then either $P[v_e, v_{i_1^1}]u_1P^-[v_{i_2^1}, v_{i_s^2}]\ u_2P[v_{i_{s+1}^2}, v_{e'}]$ or $P[v_e, v_{i_s^2}]u_2\ P^-[v_{i_{s+1}^2}, v_{i_2^1}]u_1P[v_{i_3^1}, v_{e'}]$ is longer than P. The proof for the other case is similar. This proves Claim 5(iii).

(iv). Assume that $i_1^1 = i_1^2 < i_2^1 < i_2^2 = i_3^1 < i_3^2$.

By (4.10), $i_3^2 - i_1^1 \geq 6$. If $i_1^1 = 0$, then by (4.9), we must have $i_2^1 \geq 3$, $i_3^1 \geq 5$ and $i_3^2 \in \{7, 8\}$. Then $P[v_e, v_{i_3^1}]u_1v_0u_2v_{i_3^2}v_{e'}$ is longer than P, and so we assume that $2 \geq i_1^1 \geq 1$. If $i_1^1 = 2$, then we must have $i_3^1 = 6$, $c = 8$ and $v_8u_2 \in E(G)$, and so $P[v_e, v_6]u_2v_8v_{e'}$ is longer than P. Hence we have $(i_1^1, i_2^1, i_3^1) = (1, 3, 5)$ with $i_3^2 \in \{7, 8\}$. If $i_3^2 = 8$, then Table 10A shows that a contradiction can always be found. By (4.16), $x_4 \notin \{v_3, v_4, v_5\}$, and $x_4 \in \{v_2, v_6\}$ only if $x_4' \neq x_4$.

x_4	$(v_e, v_{e'})$-path longer than P	Explanation
v_0	$v_ev_0v_4v_3v_2v_1u_1P[v_5, v_{e'}]$	
v_1		$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 1.2.3(i).
v_2	$v_ev_1v_2x_4'v_4v_3u_1P[v_5, v_{e'}]$	
v_6, v_7	$P[v_e, v_4]v_6v_5u_2v_8v_{e'}$ or $P[v_e, v_4]v_7v_6v_5u_2v_8v_{e'}$	
v_8	$P[v_e, v_4]v_8u_2v_5v_6v_7v_{e'}$	

Table 10A: Claim 5(iv) with $i_3^2 = 8$.

If $i_3^2 = 7$, then Table 10B shows that a contradiction can always be found. By (4.16), $x_2 \notin \{v_1, v_2, v_3\}$ and $x_2 = v_4$ only if $x_2' \neq x_2$.

x_2	$(v_e, v_{e'})$-path longer than P	Explanation
v_0	$v_e v_0 v_2 v_1 u_1 P[v_3, v_{e'}]$	
v_4	$v_e v_1 v_2 x'_2 v_4 v_3 u_1 P[v_5, v_{e'}]$	
v_5		$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 1.2.3 (i).
v_6	$v_e v_1 u_1 v_5 v_4 v_3 v_2 v_6 v_7 v_{e'}$	
v_7	$(x'_2 \neq v_7)$, $v_e v_1 u_1 v_5 v_4 v_3 v_2 x'_2 v_7 v_{e'}$	$(x'_2 = v_7)$ and $G[\{v_1, v_2, v_3, v_4, v_5, v_7, u_1, u_2\}]$ is not reduced, by Lemma 1.2.3(i).
v_8	$v_e v_1 u_2 P^-[v_7, v_2] v_8 v_{e'}$	

Table 10B: Claim 5(iv) with $i_3^2 = 7$.

(v). Assume that $i_1^1 = i_1^2 < i_2^1 < i_2^2 < i_3^1 = i_3^2$.

By (4.10), $i_3^2 - i_1^1 \geq 5$. If $i_3^2 - i_1^1 = 5$, then $G[V(P[v_{i_1^1}, v_{i_3^2}]) \cup \{u_1, u_2\}]$ is not reduced, by Lemma 1.2.3(i). Hence we assume that $i_3^2 - i_1^1 \geq 6$. If $i_1^1 = 0$, then by (4.9), $i_2^1 \geq 3$ and $i_2^2 \geq 4$, and so $P[v_e, v_{i_2^2}]u_2 v_0 u_1 P[v_{i_3^1}, v_{e'}]$ is longer than P. Thus $i_1^1 > 0$. By symmetry, $i_3^2 < c$. As $c \leq 7$ implies $i_3^2 - i_1^1 = 5$, we must have $c = 8$ and $(i_1^1, i_2^1, i_2^2, i_3^2) \in \{(1, 3, 4, 6), (1, 3, 5, 7), (1, 3, 4, 7)\}$. By (4.16), $x_4 \notin \{v_3, v_4, v_5\}$ and $x_4 \in \{v_2, v_6\}$ only if $x'_4 \neq x_4$; $x_2 \notin \{v_1, v_2, v_3\}$ and $x_2 = v_4$ only if $x'_2 \neq x_2$.

x_4	x_2	$(v_e, v_{e'})$-path longer than P	Explanation or Conclusion
v_0		$v_e v_0 v_4 v_3 v_2 v_1 u_2 v_5 v_6 v_7 v_{e'}$	
v_1			$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 1.2.3 (i).
v_2		$v_e v_1 u_1 v_3 v_2 x'_4 P[v_4, v_{e'}]$	(same for $x_2 = v_4$).
v_6		$P[v_e, v_4] x'_4 v_6 v_5 u_2 v_7 v_{e'}$	
v_7			either $v_4 v_7 \in E(G)$ and $G[\{v_1, v_2, v_3, v_4, v_5, v_7, u_1, u_2\}]$ is not reduced, by Lemma 1.2.3(i); or $v_4, v_7 \in N_G(x'_4)$, for $\pi = \langle \{u_2, v_6\}, \{v_5, v_7\} \rangle$, $G[(V(P) - \{v_e, v_{e'}\}) \cup \{u_1, u_2, x'_4\}]/\pi$ is collapsible by Theorem **??**, G is not reduced.
v_8		$P[v_e, v_3] u_1 v_5 x'_4 v_8 v_{e'}$	Hence $v_4 v_8 \in E(G)$.
	v_0	$v_e v_0 P[v_2, v_5] u_1 v_1 u_2 v_7 v_{e'}$	
	v_5		$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 1.2.3 (i).
	v_6	$v_e v_1 u_1 P^-[v_5, v_2] v_6 v_7 v_{e'}$	
v_8	v_7	$v_e v_1 u_1 v_3 v_2 v_7 v_6 v_5 v_4 v_8 v_{e'}$	$v_4 v_8 \in E(G)$.
v_8	v_8	$v_e v_1 u_1 v_3 v_2 v_8 P[v_4, v_{e'}]$	$v_4 v_8 \in E(G)$.

Table 11A: Claim 5(v) with $(i_1^1, i_2^1, i_2^2, i_3^2) = (1, 3, 5, 7)$.

If $(i_1^1, i_2^1, i_2^2, i_3^2) = (1, 3, 4, 6)$, by Theorem 1.2.3(i), $G(e, e')[V(P[v_1, v_6]) \cup \{u_1, u_2\}]$ is collapsible, contrary to (4.10). If $(i_1^1, i_2^1, i_2^2, i_3^2) = (1, 3, 5, 7)$, then Table 11A first indicates that $v_4 v_8 \in E(G)$ and then shows that a contradiction can always be found.

Thus we assume $(i_1^1, i_2^1, i_2^2, i_3^2) = (1, 3, 4, 7)$. Table 11B shows that a contradiction can always be found. By (4.15), if $v_2 v_7 \notin E(G)$, then $x_2 \neq v_7$.

x_2	$(v_e, v_{e'})$-path longer than P	Explanation or Conclusion
v_0	$v_e v_0 v_2 v_3 u_1 v_1 u_2 P[v_4, v_{e'}]$	
v_4	$v_e v_1 u_1 v_3 v_2 x'_2 P[v_4, v_{e'}]$	
v_5, v_6	$v_e v_1 v_2 x'_2 P^-[x_2, v_3] u_1 v_7 v_{e'}$	$G[V(P[v_1, v_7]) \cup \{u_1, u_2\}]$ is not reduced, by Lemma 1.2.3 (iv).
v_7		$G[\{v_1, v_2, v_3, v_4, v_7, u_1, u_2\}]$ is not reduced, by Lemma 1.2.3(i).
v_8	$v_e v_1 u_2 P^-[v_7, v_2] v_8 v_{e'}$	

Table 11B: Claim 5(v) with $(i_1^1, i_2^1, i_2^2, i_3^2) = (1, 3, 4, 7)$.

(vi). We shall prove that assuming $i_2^2 > i_2^1$ will lead to contradictions, and so $i_2^2 \leq i_2^1$. By symmetric arguments, we also have $i_2^1 \leq i_2^2$, which proves (vi). In the rest of the proof of Claim 5(vi), we assume that $i_2^2 > i_2^1$. By Claim 5(ii), (iv) and (v), $i_1^2 \leq i_2^1$, $i_3^1 \geq i_2^2$ and $i_1^1 \neq i_1^2$ when $i_1^3 \in \{i_2^2, i_3^2\}$.

Case 5(vi).1. $i_3^1 \notin \{i_2^2, i_3^2\}$.

Case 5(vi).1A. $i_3^1 > i_3^2$.

If $i_1^2 < i_1^1$, then by (4.9), either $(i_1^2, i_1^1, i_3^2, i_3^1) = (0, 1, 6, 7)$, whence $v_e v_0 u_2 P^-[v_6, v_1] u_1 v_7 v_{e'}$ is longer than P; or $(i_1^2, i_1^1, i_3^2, i_3^1) = (1, 2, 7, 8)$, whence $v_e v_1 u_2 P^-[v_7, v_2] u_1 v_8 v_{e'}$ is longer than P. If $i_1^1 = i_1^2 \in \{0, 1\}$, then $i_2^1 \geq 3$, $i_3^2 - i_2^1 \geq 3$ and $i_3^1 \in \{7, 8\}$. Thus $v_e v_{i_1^1} u_2 P^-[v_{i_3^2}, v_{i_2^1}] u_1 v_{i_3^1} v_{e'}$ is longer than P. Hence we assume $i_1^2 > i_1^1$.

If $i_1^2 = i_2^1$, then by $girth(G(e, e')) \geq 4$, $i_3^1 - i_1^1 \geq 8$, and so $i_1^1 \in \{0, 1\}$ and $(i_1^2, i_2^2, i_3^2, i_3^1) = (3, 5, 7, 8)$. By (4.16), $x_4 \notin \{v_3, v_4, v_5\}$, and $x_4 \in \{v_2, v_6\}$ only if $x'_4 \neq x_4$. Table 11C shows that a contradiction always exists.

x_4	i_1^1	$(v_e, v_{e'})$-path longer than P	Explanation or Conclusion
v_0	0, 1	$v_e v_0 v_4 v_5 v_6 v_7 u_2 v_3 u_1 v_8 v_{e'}$	
v_1	0	$v_e v_0 u_1 v_3 v_2 v_1 P[v_4, v_{e'}]$	
v_1	1	$v_e v_1 P[v_4, v_7] u_2 v_3 u_1 v_8 v_{e'}$	
v_2	0, 1	$v_e v_1 v_2 x'_4 P[v_4, v_7] u_2 v_3 u_1 v_8 v_{e'}$	
v_6	0,1	$P[v_e, v_4] x'_6 v_6 v_5 u_2 v_7 v_{e'}$	
v_7	0,1	$(x'_4 \neq v_7)$ $P[v_e, v_3] u_2 v_5 v_4 x'_4 v_7 v_{e'}$	$(x'_4 = v_7)$ $G[\{v_3, v_4, v_5, v_6, v_7, u_2\}]$ is not reduced, by Lemma 1.2.3(i).
v_8	0,1	$P[v_e, v_3] u_2 P^-[v_7, v_4] v_8 v_{e'}$	

Table 11C: Claim 5(vi).1A with $i_1^1 \in \{0, 1\}$ and $(i_1^2, i_2^2, i_3^2, i_3^1) = (3, 5, 7, 8)$.

Thus we have $i_1^1 < i_1^2 < i_2^1 < i_2^2 < i_3^2 < i_3^1$. By Claim 5(iii), a $(v_e, v_{e'})$-path longer than P exists. This excludes this subcase.

Case 5(vi).1B. $i_2^2 < i_3^1 < i_3^2$.

Now we have $i_1^1 < i_2^1 < i_2^2 < i_3^1 < i_3^2$, As $P[v_e, v_{i_2^1}] u_1 P^-[v_{i_3^1}, v_{i_2^2}] u_2 \; P[v_{i_3^2}, v_{e'}]$ is not longer than P, $(i_2^2 - i_2^1) + (i_3^2 - i_3^1) \geq 4$. By Claim 5(iii), we cannot have $i_1^1 < i_1^2 < i_2^1$, and so $i_1^2 < i_1^1$ or $i_1^2 \in \{i_1^1, i_2^1\}$.

If $i_1^2 < i_1^1$, then $i_2^1 - i_1^1 \geq 2$. Since $(i_2^2 - i_2^1) + (i_3^2 - i_3^1) \geq 4$, it follows that $l - s \geq i_3^2 - i_1^2 \geq 8$. By Claim 5(i), $i_1^2 = 0$, $i_1^1 = 1$, $i_2^2 = 5$, $i_3^1 = 6$ and $i_3^2 = 8$. Hence $v_e v_0 u_2 P^-[v_5, v_1] u_1 v_6 v_7 v_{e'}$ is longer than P.

If $i_1^2 = i_1^1$, then by $girth(G(e, e')) \geq 4$ and by Claim 3, we must have $i_2^1 \geq 3$. It follows by $l - s \leq 8$ and by $(i_2^2 - i_2^1) + (i_3^2 - i_3^1) \geq 4$, we must have $i_1^1 \in \{0, 1\}$, and $(i_2^1, i_2^2, i_3^1, i_3^2) \in \{(3, 5, 6, 8), (3, 6, 7, 8)\}$. If $i_1^1 = 0$, then $P[v_e, v_{i_3^1}] u_1 v_0 u_2 v_{i_3^2} v_{e'}$ is longer than P. If $i_1^1 = 1$, then $P[v_e, v_{i_2^1}] u_1 P^-[v_{i_3^1}, v_{i_2^2}] u_2 v_8 v_{e'}$ is longer than P. This excludes this subcase.

Case 5(vi).1C. $i_3^1 < i_2^2$.

Then $i_1^1 < i_2^1 < i_3^1 < i_2^2 < i_3^2$. By Claim 5(ii), $i_2^1 \geq i_1^2$. If $i_2^1 > i_1^2 > i_1^1$, then As $P[v_e, v_{i_1^1}] u_1 P^-[v_{i_3^1}, v_{i_1^2}] u_2 P[v_{i_2^2}, v_{e'}]$ is not longer than P, we have $(i_1^2 - i_1^1) + (i_2^2 - i_3^1) \geq 4$.

By $girth(G(e,e')) \geq 4$, $i_3^2 - i_2^2 \geq 2$ and $i_3^1 - i_2^1 \geq 2$, it follows $i_3^2 - i_1^1 \geq 9$, contrary to Claim 5(i). If $i_1^2 < i_1^1$, then by $girth(G(e,e')) \geq 4$, we must have $i_3^2 - i_1^2 \geq 8$, and so by Claim 5(i), we have $i_3^2 = 8$. By Claim 3, $i_2^2 \leq 5$, which forces $i_3^2 - i_1^2 \geq 9$, contrary to Claim 5(i).

Hence $i_1^2 \in \{i_1^1, i_2^1\}$. If $i_2^1 = i_1^1 = 1$, then by $girth(G(e,e')) \geq 4$, we have $i_3^2 = 8$. By Claim 3, $i_2^2 \leq 5$, which forces $i_3^2 - i_1^2 \geq 9$, contrary to Claim 5(i). If $i_2^1 = i_1^1 = 0$, then by Claim 3 and by $girth(G(e,e')) \geq 4$, we have $i_3^1 \in \{5,6\}$ and $i_2^2 \in \{6,7\}$. Thus $P[v_e, v_{i_3^1}]u_1 v_0 u_2 P[v_{i_2^2}, v_{e'}]$ is longer than P. If $i_1^2 = i_2^1$, then by $girth(G(e,e')) \geq 4$, we have $i_3^2 - i_1^1 \geq 7$. As $P[v_e, v_{i_1^1}]u_1 P^-[v_{i_3^1}, v_{i_2^1}]u_2 P[v_{i_2^2}, v_{e'}]$ is not longer than P, we have $(i_2^1 - i_1^1) + (i_2^2 - i_3^1) \geq 4$, forcing $i_3^2 - i_1^1 \geq 8$. By Claim 5(i), we have $(i_2^2, i_3^2) = (7,8)$, and so $P[v_e, v_7]u_2 v_8 v_{e'}$ is longer than P. This excludes this subcase and completes the proof of Case 5(vi).1.

Case 5(vi).2. $i_3^1 = i_3^2$. Thus $i_3^1 = i_3^2 > i_2^2 > i_2^1 > i_1^1$.

By Claim 3, we cannot have $i_1^1 < i_1^2 < i_2^1$. By Claim 5(iv) and (v), $i_1^2 \notin \{i_1^1, i_2^1\}$. Hence we have $i_1^2 < i_1^1$. Then either $v_e v_0 u_2 P^-[v_{i_2^2}, v_{i_1^1}]u_1 P[v_{i_3^2}, v_{e'}]$ (if $i_1^2 = 0$) or $P[v_e, v_{i_1^2}]u_2 P^-[v_{i_2^2}, v_{i_1^1}]u_1 P[v_{i_3^2}, v_{e'}]$ (if $i_1^2 > 0$) is longer than P. This proves this case.

Case 5(vi).3. $i_3^1 = i_2^2$. Thus $i_3^2 > i_3^1 = i_2^2 > i_2^1 > i_1^1$.

By $girth(G(e,e')) \geq 4$, $i_3^2 - i_1^1 \geq 6$. By Claim 5(iv), $i_1^1 \neq i_1^2$. Assume first that $i_1^2 < i_1^1$, and so $i_3^2 - i_1^2 \geq 7$, implying $i_3^2 \in \{7,8\}$. Note that when $i_3^2 \in \{7,8\}$, we must have $i_3^1 - i_1^1 = 4$, and so by Theorem 6.2.1(v) and (vi), adding any edge to

$$G(e,e')[V(P[v_1^1, v_3^1])] \text{ will result in a non reduced subgraph.} \tag{4.17}$$

As $i_3^2 \in \{7,8\}$, by Claim 3 and by $girth(G(e,e')) \geq 4$, we must have $(i_1^2, i_1^1, i_2^1, i_2^2) = (0,1,3,5)$. By (4.17) and (4.16), $x_4 \notin \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Table 11D shows that for any other values of x_4, a contradiction to (4.9) can always be found, which completes the proof of this case.

x_4	i_3^2	$(v_e, v_{e'})$-path longer than P
v_0		$v_e v_0 P^-[v_4, v_1]u_1 P[v_5, v_{e'}]$
v_7 or v_8	7, 8	$v_e v_0 u_2 v_5 u_1 P[v_1, v_4] v_{i_3^2} v_{e'}$

Table 11D: Claim 5(vi).3 with $i_3^2 \in \{7,8\}$ and $(i_1^2, i_1^1, i_2^1, i_2^2) = (0,1,3,5)$.

Claim 6. $s = |\{v_{i_1^1}, v_{i_2^1}, v_{i_3^1}\} \cap \{v_{i_1^2}, v_{i_2^2}, v_{i_3^2}\}| \geq 2$.

If $s < 2$ then by Claim 5(i), $s = 1$. Without loss of generality, we assume that $i_1^1 \leq i_1^2$. Hence by Claim 5, we have $i_1^1 < i_1^2 < i_2^2 = i_2^1 \leq i_3^1$. By Claim 5(iii), we have $i_1^1 < i_1^2$ and $i_3^1 < i_3^2$. By (4.10), $i_3^1 - i_1^2 \geq 4$. It follows that $P[v_e, v_{i_1^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_3^2}, v_{e'}]$ is longer than P, contrary to (4.9). Claim 6 is justified.

Claim 7. $s = |\{v_{i_1^1}, v_{i_2^1}, v_{i_3^1}\} \cap \{v_{i_1^2}, v_{i_2^2}, v_{i_3^2}\}| = 3$.

By contradiction and Claim 6, assume that $s = 2$. Without loss of generality, we assume that $i_1^1 \leq i_1^2$. Hence by Claim 5, we have $i_1^1 \leq i_1^2 \leq i_2^2 = i_2^1 \leq i_3^1$, and so $s = 2$ implies, in addition to $i_2^1 = i_2^2$, either $i_1^1 = i_1^2$ or $i_3^1 = i_3^2$. By symmetry, it suffices to assume that both $i_1^1 = i_1^2$ and $i_2^1 = i_2^2$ to find contradictions.

Hence we have $i_1^1 = i_1^2 < i_2^2 = i_2^1 < i_3^1$. By symmetry, we may assume that $i_3^1 < i_3^2$. As $P[v_e, v_{i_1^1}]u_1\ P^-[v_{i_3^1}, v_{i_2^1}]\ u_2P[v_{i_3^2}, v_{e'}]$ is not longer than P, $(i_3^2 - i_3^1) + (i_2^1 - i_1^1) \geq 4$, and so by (4.9) and (4.10), $i_3^2 - i_1^1 \geq 6$. Hence $i_2^1 \geq 3$, $i_1^1 \leq 1$ or $i_3^2 \geq 7$. If $v_1^1 = 0$, then by (4.9), $i_3^1 \geq 5$, and so $P[v_e, v_{i_3^1}]u_1v_0u_2v_{i_3^2}v_{e'}$ is longer than P. This forces that $i_1^1 = 1$ and $i_3^1 - i_2^1 = 2$, and so $(i_2^1, i_3^1, i_3^2) \in \{(4,6,7), (4,6,8), (3,5,7), (3,5,8)\}$.

Assume first that $(i_2^1, i_3^1, i_3^2) \in \{(3,5,7), (3,5,8)\}$. By (4.16), $x_4 \notin \{v_2, v_3, v_4, v_5\}$. If $v_1v_4 \in E(G)$, then by Lemma 1.2.3(i), $G[V(P[v_1, v_5]) \cup \{u_2\}]$ is not reduced. Hence when $x_4 = v_1$, $x_4' \in V(J)$. By $\kappa'(G) \geq 3$, we can choose $x_4 \in N_G(x_4') - \{v_1, v_4\}$. Thus we may assume that $x_4 \neq v_1$ as well. If $u_2v_7, x_4v_7 \in E(G)$, then by Lemma 1.2.3(i), $G[V(P[v_3, v_7]) \cup \{v_1, u_1, u_2\}]$ is not reduced. Hence when $i_3^2 = 7$ and $x_4 = v_1$, $x_4' \in V(J)$. Table 12A indicates that a contradiction can always be found.

x_4	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_0 P^-[v_1, v_4]u_1 P[v_5, v_{e'}]$ or $v_e v_0 x'_4 P^-[v_1, v_4]u_1 P[v_5, v_{e'}]$
v_6	$P[v_e, v_3]u_1 v_5 v_4 P[v_6, v_{e'}]$ or $P[v_e, v_3]u_1 v_5 v_4 x'_4 P[v_6, v_{e'}]$
v_7	$P[v_e, v_3]u_1 v_5 v_4 x'_4 P[v_5, v_{e'}]$, if $i^2_3 = 7$.
v_7	$v_e v_1 u_1 v_5 v_6 v_7 v_4 v_3 u_2 v_8 v_{e'}$ or $v_e v_1 u_1 v_5 v_6 v_7 x'_4 v_4 v_3 u_2 v_8 v_{e'}$, if $i^2_3 = 8$.
v_8	$P[v_e, v_3]u_2 P^-[v_4, v_7]v_8 v_{e'}$ or $P[v_e, v_3]u_2 P^-[v_4, v_7]x'_4 v_8 v_{e'}$, if $i^2_3 = 7$.
v_8	$P[v_e, v_3]u_2 v_8 P[v_4, v_{e'}]$ or $P[v_e, v_3]u_2 v_8 x'_4 P[v_4, v_{e'}]$, if $i^2_3 = 8$.

Table 12A: Claim 7, with $i^1_1 = i^2_1$, $i^1_2 = i^2_2$ and $(i^1_2, i^1_3, i^2_3) \in \{(3,5,7), (3,5,8)\}$.

Hence $(i^1_2, i^1_3, i^2_3) \in \{(4,6,7), (4,6,8)\}$. By (4.16), $x_5 \notin \{v_3, v_4, v_5, v_6\}$. If $v_1 v_v \in E(G)$, then by Lemma 1.2.3(i), $G[\{v_1, v_4, v_5, v_6, u_1, u_2\}]$ is not reduced. Hence when $x_5 = v_1$, $x'_5 \in V(J)$. By $\kappa'(G) \geq 3$, we can choose $x_5 \in N_G(x'_5) - \{v_1, v_5\}$. Thus we may assume that $x_5 \neq v_1$ as well. Table 12B indicates that a contradiction can always be found. This completes the proof for Claim 7.

x_5	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_0 P^-[v_5, v_1]u_1 P[v_6, v_{e'}]$ or $v_e v_0 x'_5 P^-[v_5, v_1]u_1 P[v_6, v_{e'}]$
v_2	$v_e v_1 u_1 v_4 v_3 v_2 P[v_5, v_{e'}]$ or $v_e v_1 u_1 v_4 v_3 v_2 x'_5 P[v_5, v_{e'}]$
v_7 or v_8	$P[v_e, v_4]u_1 v_6 v_5 x_5 v_{e'}$ or $P[v_e, v_4]u_1 v_6 v_5 x'_5 x_5 v_{e'}$

Table 12B: Claim 7, with $i^1_1 = i^2_1$, $i^1_2 = i^2_2$ and $(i^1_2, i^1_3, i^2_3) \in \{(4,6,7), (4,6,8)\}$.

Claim 8. $|V(G(e,e')) - V(P) \cup \{v_0, v_c\}| = 1$.

By contradiction, we assume that $u_1, u_2 \in V(G(e,e')) - V(P) \cup \{v_0, v_c\}$. By Claim 7, $|N_G(u_1) \cap N_G(u_2)| = 3$, and so $i^1_j = i^2_j$ for $1 \leq j \leq 3$. If $i^2_1 = i^1_1 + 2$ or $i^1_3 = i^1_2 + 2$, then $G(e,e')$ has a collapsible subgraph $K^-_{3,3}$, contrary to (4.10). Therefore, we must have both $i^1_2 \geq i^1_1 + 3$ and $i^1_3 \geq i^1_2 + 3$. Since $G(e,e')$ does not contain a collapsible subgraph $K_{3,3}$, by Claim 7, $V(G) = V(P) \cup \{v_0, v_c, u_1, u_2\}$.

Since $V(G) = V(P) \cup \{v_0, v_c, u_1, u_2\}$, $x_3 = x'_3$. By (4.16), $x_3 \notin \{v_1, v_2, v_3, v_4, v_5\}$. If $x_3 = v_7$, then $v_3 v_7 \in E(G)$, and so by Lemma 1.2.3(i), $G[\{v_1, v_2, v_3, v_4, v_7, u_1, u_2\}]$ is not reduced. Thus $x_3 \neq v_7$ and so $x_3 \in \{v_0, v_6, v_8\}$. It follows that $v_e v_0 v_3 v_2 v_1 u_1 P[v_4, v_{e'}]$ (if $x_3 = v_0$) or $P[v_e, v_3]v_6 v_5 v_4 u_1 v_7 v_{e'}$ (if $x_3 = v_6$), or $v_e v_1 u_1 P^-[v_7, v_3]v_8 v_{e'}$ (if $x_3 = v_8$) is longer than P.

Thus either $i_1^1 = 0$ or $i_3^1 = c$. Without loss of generality, we assume that $i_1^1 = 0$. As neither $v_e v_0 u_1 P[v_{i_2^1}, v_{e'}]$ nor $P[v_e, v_{i_2^1}]u_1 v_0 u_2 P[v_{i_3^1}, v_{e'}]$ is longer than P, we must have $i_2^1 = 3$. By symmetry, that $i_3^1 = c = 8$ implies $i_2^1 = 5 > 3$, and so $i_1^1 = 1$ implies $i_3^1 = 7$. Hence $(i_1^1, i_2^1, i_3^1) \in \{(0,3,6),(0,3,7)\}$. If $(i_1^1, i_2^1, i_3^1) = (0,3,6)$, then $v_e v_1 v_2 v_3 u_2 v_0 u_1 P[v_6, v_{e'}]$ is longer than P. Therefore $(i_1^1, i_2^1, i_3^1) = (0,3,7)$.

By (4.16), $x_4 \notin \{v_2, v_3, v_4, v_5, v_6\}$. Since $G(e,e')$ cannot have a $K_{3,3}^-$, $x_4 \notin \{v_0, v_7\}$, and so $x_4 \in \{v_1, v_8\}$. Hence $v_e v_0 u_1 v_3 v_2 v_1 P[v_4, v_{e'}]$ (if $x_4 = v_1$) or $P[v_e, v_3]u_1 P^-[v_7, v_4]v_8 v_{e'}$ (if $x_4 = v_8$) is longer than P, contrary to (4.9). This proves Claim 8.

Define a new graph L_z from $G(e,e')$ by adding a new vertex z and new edges zv_e and $zv_{e'}$. Then $G(e,e')$ has a spanning $(v_e, v_{e'})$-trail if and only if L_z is supereulerian. If $c \leq 7$, or if $c = 8$ and $v_0 = v_c$, then by Claim 8, $|V(L_z)| \leq 12$. As L_z has exactly one edge cut of size 2, it follows by Lemma 1.2.3(iv) that either L is supereulerian, whence $G(e,e')$ has a spanning $(v_e, v_{e'})$-trail; or $L_z = P(10)(e)$, whence $G = H_8$. In either case, a contradiction to the assumptions of Theorem 4.3.1 is found. Hence we will assume that $c = 8$ and that $v_0 \neq v_c$.

By Claim 8, we denote $V(J) = \{u\}$. By Claim 7, $N_G(u) \subseteq V(P) \cup \{v_0, v_c\}$. Let $N_G(u) = \{v_{i_1}, v_{i_2}, v_{i_3} ...\}$. By Claim 4, we may assume $i_1 < i_2 < i_3$. By (4.10), both $i_2 > i_1 + 2$ and $i_3 \geq i_2 + 2$, and if $i_1 = 0$, then $i_2 > 2$, and if $i_3 = c$, then $i_2 < c - 2$. Therefore, the possibilities of (i_1, i_2, i_3) can be listed below:

i_1	i_2	(i_1, i_2, i_3)	Symmetric case	i_1	i_2	(i_1, i_2, i_3)	Symmetric case
0	3	(0, 3, 5)	(3,5,8)	1	3	(1, 3, 5)	(3,5,7)
0	3	(0, 3, 6)	(2,5,8)	1	3	(1, 3, 6)	(2,5,7)
0	3	(0, 3, 7)	(1,5,8)	1	3	(1, 3, 7)	(1,5,7)
0	3	(0, 3, 8)	(0,5,8)	1	4	(1, 4, 6)	(2,4,7)
0	4	(0, 4, 6)	(2,4,8)	1	4	(1, 4, 7)	(1,4,7)
0	4	(0, 4, 7)	(1,4,8)	2	4	(2, 4, 6)	(2,4,6)
0	4	(0, 4, 8)	(0,4,8)				
0	5	(0, 5, 7)	(1,3,8)				

Table 13: Possibilities of (i_1, i_2, i_3).

We shall show that in each of these cases of (i_1, i_2, i_3), either a longer $(v_e, v_{e'})$-path is found or a nontrivial collapsible subgraph of $G(e, e')$ is found, leading to contractions to (4.10). For each i with $1 \le i \le 7$, denote $x_i (= x_i') = v_{s_i}$. By (4.16),

$$s_1 \in \{4, 5, 6, 7, 8\}, s_2 \in \{0, 5, 6, 7, 8\}, s_3 \in \{0, 6, 7, 8\}$$
$$s_4 \in \{0, 1, 7, 8\}, s_5 \in \{0, 1, 2, 8\}, s_6 \in \{0, 1, 2, 3, 8\}, s_7 \in \{0, 1, 2, 3, 4\} \quad (4.18)$$

Claim 9. If $i_1 = 0$, then each the following statements holds.
(i) Let $t \in \{i_2, i_3\}$. Then $v_0v_{t-1}, v_0v_{t-2} \notin E(G)$, and $v_0v_{t+1}, v_0v_{t+2}, v_1v_{t+1}, v_1v_{t+2} \notin E(G)$ if $t + 1, t + 2 \le 8$.
(ii) If $i_3 \ne 8$, then $v_1v_8 \notin E(G)$.

As $i_3 \ge 5$, Table 14 proves Claim 9.

	Edge in $E(G)$	$(v_e, v_{e'})$-path longer than P	
(i)	$v_0v_{t-1} \in E(G)$	$P[v_e, v_{t-1}]v_0uP[v_t, v_{e'}]$	
	$v_0v_{t-2} \in E(G)$	$P[v_e, v_{t-2}]v_0uP[v_t, v_{e'}]$	
	$v_0v_{t+1} \in E(G)$	$P[v_e, v_t]uv_0P[v_{t+1}, v_{e'}]$	$t + 1 \le 8$
	$v_0v_{t+2} \in E(G)$	$P[v_e, v_t]uv_0P[v_{t+2}, v_{e'}]$	$t + 2 \le 8$
	$v_1v_{t+1} \in E(G)$	$v_ev_0uP^-[v_t, v_1]P[v_{t+1}, v_{e'}]$	$t + 1 \le 8$
	$v_1v_{t+2} \in E(G)$	$v_ev_0uP^-[v_t, v_1]P[v_{t+2}, v_{e'}]$	$t + 2 \le 8$
(ii)	$v_1v_8 \in E(G)$	$v_ev_0u\ P^-[v_{i_3}, v_1]v_8v_{e'}$	

Table 14: Proof of Claim 9.

Claim 10. If $i_1 = 0$, then $i_2 = 3$.

If $i_2 \geq 4$, then $(i_1, i_2, i_3) \in \{(0,4,6),(0,4,7),(0,4,8),(0,5,7)\}$. By (4.18), $s_5 \in \{0,1,2,8\}$. Hence $P[v_e, v_4]uv_0P[v_5, v_{e'}]$ (if $v_0v_5 \in E(G)$), or $v_ev_0uP^-[v_4, v_1]P[v_5, v_{e'}]$ (if $v_1v_5 \in E(G)$), or $v_ev_0uv_4v_3v_2P[v_5, v_{e'}]$ (if $v_2v_5 \in E(G)$) is longer than P. Thus $v_5v_8 \in E(G)$. If $i_3 \in \{6,7\}$, then $P[v_e, v_4]uv_{i_3}P^-[v_{i_3}, v_5]v_8v_{e'}$ is longer than P. Hence $(i_1, i_2, i_3) = (0,4,8)$. In this case, $P[v_e, v_4]uv_8P[v_5, v_{e'}]$ is longer than P, contrary to (4.9), and so Claim 10 holds.

Claim 11. Both $i_1 \geq 1$ and $i_3 \leq c-1$. Furthermore, when $i_1 = 1$, each of the following holds.
(i) Let $t \in \{i_2, i_3\}$. Then $x_0v_{t-1} \notin E(G)$.
(ii) If $i_2 \geq 4$, then $v_0v_{i_2-2} \notin E(G)$. If $i_3 \geq i_2 + 3$, then $v_0v_{i_3-2} \notin E(G)$.

By symmetry, we assume that $i_1 = 0$. By Claim 10, $(i_1, i_2, i_3) \in \{(0,3,5),(0,3,6),(0,3,7),(0,3,8)\}$. By (4.18), $s_1 \in \{4,5,6,7,8\}$. By Claim 9, $(i_1, i_2, i_3) \neq (0,3,5)$.

If $(i_1, i_2, i_3) = (0,3,6)$, then by Claim 9, $v_1v_6 \in E(G)$, and so $v_2v_6 \notin E(G)$. By (4.18), $s_2 \in \{5,7,8\}$.

If $(i_1, i_2, i_3) = (0,3,8)$, then by (4.18) and Claim 9, $s_1 \in \{6,7,8\}$. If $s_1 = 6$, then $s_2 \neq 6$ and so by (4.18), and Claim 9, $s_2 \in \{5,7,8\}$.

If $(i_1, i_2, i_3) = (0,3,7)$, then by (4.18) and Claim 9, $s_4 \in \{7,8\}$; and if $s_4 = 7$, $s_5 \in \{2,8\}$. Similarly, When $v_4v_7, v_2v_5 \in E(G)$, $s_1 \in \{6,7\}$.

With these analysis, Table 15 proves Claim 11.

Cases	(i_1, i_2, i_3)	Edge in $E(G)$	$(v_e, v_{e'})$-path longer than P	Conclusions
$i_1 \geq 1$ $i_3 \leq c-1$	(0,3,6)	$v_2v_5 \in E(G)$	$v_ev_1v_2v_5v_4v_3$ $uP[v_6, v_{e'}]$	
		$v_2v_{s_2} \in E(G)$	$v_ev_0uP^-[v_6,$ $v_2]v_{s_2}v_{e'}$	$s_2 \in \{7, 8\}$ $s_2 \in \{7, 8\}$
	(0,3,8)	$v_1v_7 \in E(G)$	$v_ev_1P^-[v_7,$ $v_3]uv_8v_{e'}$	
		$v_1v_8 \in E(G)$	$v_ev_0uv_8P[v_1, v_{e'}]$	$s_1 = 6$, and so $s_2 \in \{5, 7, 8\}$
		$v_2v_5 \in E(G)$	$v_ev_0uv_3v_4v_5$ $v_2v_1v_6v_7v_{e'}$	
		$v_2v_{s_2} \in E(G)$	$v_ev_0uP[v_3,$ $v_6]v_1v_2v_{s_2}v_{e'}$	$s_2 \in \{7, 8\}$
	(0,3,7)	$v_4v_8 \in E(G)$	$P[v_e, v_3]uP^-[v_7,$ $v_4]v_8v_{e'}$	$v_4v_7 \in E(G)$, and so $s_5 \in \{2, 8\}$
		v_4v_7, $v_5v_8 \in E(G)$	$P[v_e, v_4]v_7v_6v_5$ $v_8v_{e'}$	$v_2v_5 \in E(G)$, and so $s_1 \in \{6, 7\}$
		v_2v_5, $v_1v_6 \in E(G)$	$v_ev_0uv_3v_4v_5v_2$ $v_1P[v_6, v_{e'}]$	
		v_2v_5, $v_1v_7 \in E(G)$	$v_ev_0uv_3v_4v_5v_2$ $v_1v_7v_{e'}$	
$i_1 = 1$ $t \in \{i_2, i_3\}$		x_0v_{t-1} $\in E(G)$	$v_ev_0P^-[v_{t-1},$ $v_1]uP[v_t, v_{e'}]$	
$i_1 = 1$ $i_2 \geq 4$		$v_0v_{i_2-2}$ $\in E(G)$	$v_ev_0P^-[v_{i_2-2},$ $v_1]uP[v_{i_2}, v_{e'}]$	
$i_1 = 1$ $i_3 \geq i_2 + 3$		$v_0v_{i_3-2}$ $\in E(G)$	$v_ev_0P^-[v_{i_3-2},$ $v_1]uP[v_{i_3}, v_{e'}]$	

Table 15: Proof of Claim 11.

Claim 12. $i_1 \geq 2$ and $i_3 \leq c-2$.

By contradiction, by symmetry and Claim 11, we assume that $i_1 = 1$ and so $(i_1, i_2, i_3) \in \{(1, 3, 5), (1, 3, 6), (1, 3, 7), (1, 4, 6), (1, 4, 7)\}$.

Case 12.1. $(i_1, i_2, i_3) = (1, 3, 5)$.

By (4.10), $G(e, e')[\{v_1, v_2, \cdots, v_5, u\}] \not\cong K_{3,3}^-$, and so $s_1 \neq 4$ and $s_2 \neq 5, 6 \notin E(G)$. By (4.18) and Claim 11, $s_2 \in \{0, 7, 8\}$. If $v_2v_8 \in E(G)$, then $P' = v_ev_1uv_3 \cdots v_7v_{e'}$ satisfies $|V(P')| = |V(P)|$ and $N_G(v_2) \cap V(P') = \{v_8, v_3, v_1\}$, and so P' violates applying Claim 11 (when P is replaced by P'). Hence $v_2v_7 \in E(G)$.

Note that $v_4v_8, v_6v_8, v_0v_6 \notin E(G)$, as otherwise, $v_ev_1uv_3v_2v_7v_6v_5v_4v_8v_{e'}$ or $v_ev_1v_5v_4v_3v_2v_7v_6v_8v_{e'}$ or $v_ev_0v_6v_5v_4v_3uv_1v_2v_7v_{e'}$ would be longer than P. Hence by (4.18) and Claim 11, $s_4 \in \{1, 7\}$ and $s_6 \in \{1, 2, 3\}$. Let $H_1 = G[V(P[v_1, v_7] \cup \{u\}]$. As $s_4 \in \{1, 7\}$ and $s_6 \in \{1, 2, 3\}$, by Theorem 1.2.2(vi) and (v), $F(H_1) \leq 2(8) - 12 - 2 = 2$, and so H_1 is not reduced, contrary to (4.10). This excludes Case 12.1.

Case 12.2. $(i_1, i_2, i_3) = (1, 3, 6)$.

By (4.18) and Claim 11, $s_2 \in \{5, 6, 7, 8\}$. Note that $v_2v_5, v_2v_7, v_2v_8 \notin E(G)$, as otherwise, $v_ev_1v_2v_5v_4v_3uv_6v_7v_{e'}$, or $v_ev_1uP^-[v_6, v_2]v_7v_{e'}$, or $v_ev_1u\ P^-[v_6, v_2]v_8v_{e'}$ would be longer than P. This implies $v_2v_6 \in E(G)$. By (4.18) and Claim 11, $s_5 \in \{1, 8\}$. If $v_5v_8 \in E(G)$, then $v_ev_1uv_6P[v_2, v_5]v_8v_{e'}$ would be longer than P, and so $v_5v_1 \in E(G)$. By Lemma 1.2.3(i), with $v_2v_6, v_5v_1 \in E(G)$, $G[(e, e')[\{v_1, v_3, v_6, u, v_2, v_5\}]$ is not reduced, contrary to (4.10). This excludes Case 12.2.

Case 12.3. $(i_1, i_2, i_3) = (1, 3, 7)$.

First note that $v_2v_6 \notin E(G)$, as otherwise $v_ev_1v_2v_6v_5v_4v_3uv_7v_{e'}$ is longer than P. Hence by (4.18), Claim 11, and symmetry, $s_2 \in \{5, 7\}$. But then $v_6v_8 \notin E(G)$, as otherwise, either $v_ev_1v_2v_5v_4v_3uv_7v_6v_8v_{e'}$ (if $s_2 = 5$) or $v_ev_1uv_7P[v_2, v_6]v_8v_{e'}$ (if $s_2 = 7$) is longer than P. By Claim 11 and as $s_6 \neq 8$, $s_6 \in \{1, 3\}$. As $G(e, e')[\{v_1, v_3, v_7, u, v_6, v_2\} \not\cong K_{3,3}^-$, we have $s_2 \neq 7$, and so $s_2 = 5$.

As $s_2 = 5$, $v_0v_4 \notin E(G)$, as otherwise, $v_ev_0v_4v_3uv_1v_2P[v_5, v_{e'}]$ is longer than P. By Claim 11, $s_4 \in \{1, 7\}$. As $s_2 = 5$ and $s_4 \in \{1, 7\}$, by Lemma 1.2.3(i), $G(e, e')[V(P[v_1, v_7]) \cup \{u\}]$ is not reduced, contrary to (4.10). This proves Case 12.3.

Case 12.4. $(i_1, i_2, i_3) = (1, 4, 6)$.

By (4.18) and Claim 11, $s_2 \in \{5, 6, 7, 8\}$. Note that $v_2v_5, v_2v_7, v_2v_8 \notin E(G)$ as otherwise, $v_ev_1uv_4v_3v_2P[v_5, v_{e'}]$, or $v_ev_1uP^-[v_2, v_6]v_7v_{e'}$, $v_ev_1uP^-$ $[v_6, v_2]v_8v_{e'}$ is longer than P. Thus $s_2 = 6$. By (4.18), Claim 11, and symmetry, $s_5 = 1$. Thus by Lemma 1.2.3(i), $G(e, e')[\{v_1, v_4, v_6, u, v_5, v_2\}]$ is not reduced, contrary to (4.10). This proves Case 12.4.

Case 12.5. $(i_1, i_2, i_3) = (1, 4, 7)$.

Note that $v_2v_5, v_3v_6 \notin E(G)$, as otherwise $v_ev_1uv_4v_3v_2P[v_5, v_{e'}]$ or $P[v_e, v_3]v_6v_5v_4uv_7v_{e'}$ is longer than P. By (4.18), Claim 11, and by symmetry, we may assume that $s_2 \in \{6, 7\}$, $s_3 = 7$ and $s_5 = 1$. By Lemma 1.2.3(i), $G(e, e')[V(P[v_1, v_7]) \cup \{u\}]$ is not reduced, contrary to (4.10). This precludes Case 12.5, and so Claim 12 holds.

By Claim 12, $(i_1, i_2, i_3) = (2, 4, 6)$. By (4.18) and Claim 11, $s_3 \in \{0, 6, 7, 8\}$. Note that $v_3v_0, v_3v_7, v_3v_8 \notin E(G)$, as otherwise $v_ev_0v_3v_2uP[v_4, v_{e'}]$, or $v_ev_1v_2uP^-[v_6, v_3]v_7v_{e'}$, or $v_ev_1v_2uP^-[v_6, v_3]v_8v_{e'}$ is longer than P. Hence $s_3 = 6$. By Lemma 1.2.3(i), $G(e, e')[V(P[v_2, v_6]) \cup \{u\}]$ is not reduced, contrary to (4.10). The proof for Theorem 4.3.1 is now complete.

Chapter 5

The Discharging Method and 3-Connected Essentially 10-Connected Line Graphs

5.1 Introduction

A subgraph of G isomorphic to a $K_{1,2}$ or a 2-cycle is called a P_2-subgraph of G, and a 2-cycle is a graph consisting of two non loop edges sharing two end-vertices. An edge cut X of G is a P_2-edge cut of G if at least two components of $G - X$ contain P_2-subgraphs. By the definition of a line graph, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2-edge cut with size less than k. In 2006, Lai et al. considered the hamiltonicity of 3-connected line graphs and showed that the high essential connectivity of a 3-connected line graph can guarantee the existence of a hamiltonian cycle as follows.

Theorem 5.1.1 *(Lai, Shao, Wu and Zhou [26]) Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Recently, Li and Yang improved Theorem 5.1.1 by directly counting the number of edges between partitioned vertex subsets in applying spanning trees packing theorem of Nash-Williams [38] and Tutte [47].

Theorem 5.1.2 *(Li and Yang [32]) Every 3-connected, essentially 10-connected line graph is hamiltonian connected.*

In this note we shall use discharging to give a short proof of Theorem 5.1.2. Using the cl^M-closure introduced by Ryjáček and Vrána on claw-free graphs (Theorem 9, [44]), we have the following.

Corollary 5.1.3 *Every 3-connected, essentially 10-connected claw-free graph is hamiltonian connected.*

5.2 Proof of Theorem 5.1.2

Recall that subset D of the vertex set $V(G)$ is a dominating set if every edge has at least one end-vertex in D. Let $e_1, e_2 \in E(G)$. We use "(e_1, e_2)-trail" to denote a trail having the end-edges e_1 and e_2. An (e_1, e_2)-trail is a dominating trail if each edge of G is incident with at least one internal vertex of the trail. An (e_1, e_2)-trail is a spanning trail if it is a dominating trail which contains all the vertices of G. A graph is dominating trailable if for each pair of e_1 and e_2 of edges of G there exists a dominating trail with end-edges e_1 and e_2. Similarly, one can define the spanning trailable graphs.

Theorem 5.2.1 *(Nash-Williams [39]) Let G be a graph. If $|E(G)| \geq k(|V(G)|-1)$, then G has a nontrivial subgraph H such that $\tau(H) \geq k$.*

Theorem 5.2.2 *Let G be a graph, and let H be a subgraph of G.*
(i) (Catlin and Lai, Theorem 4 of [10]) Suppose that $\tau(G) \geq 2$. For any $e_1, e_2 \in E(G)$, G has a spanning (e_1, e_2)-trail if and only if $\{e_1, e_2\}$ is not an essential edge cut of G.
(ii) (Liu et al, Lemma 2.1 of [34]) If $\tau(H) \geq 2$ and $\tau(G/H) \geq 2$, then $\tau(G) \geq 2$.

Let G be a connected, essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The core of the graph G, denoted by G_0, is obtained from G by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 5.2.3 *(Shao [45]) Let G be a connected, essentially 3-edge-connected graph. Then the core G_0 of G satisfies the following.*
(i) G_0 is uniquely defined and $\kappa'(G_0) \geq 3$.
(ii) If G_0 is spanning trailable, then $L(G)$ is hamiltonian connected.

Proof of Theorem 5.1.2. To prove $L(G)$ is hamiltonian connected, by Theorem 5.2.2(i) and Lemma 5.2.3, it suffices to prove that $\tau(G_0) \geq 2$, where G_0 is the core of G. By contradiction, assume that $\tau(G_0) < 2$. We choose $L(G)$ such that $|V(G_0)|$ is minimized. By Theorem 5.2.2(ii),

$$G_0 \text{ does not have a nontrivial subgraph } T \text{ such that } \tau(T) \geq 2. \qquad (5.1)$$

Thus, if H is a P_2-subgraph in G_0, then $H = K_{1,2}$.

Claim 1. Let H be a P_2-subgraph in G_0. Then $e(V(H), V(G) - V(H)) \geq 10$.

Assume that $e(V(H), V(G) - V(H)) \leq 9$. Let $H = K_{1,2}$ with $\{xy, yz\} \subseteq E(H)$. Then $X = E(V(H), V(G) - V(H))$ is an edge cut in G_0. Let $H, H_1, H_2, \cdots, H_k$ be components of $G_0 - X$. Then each $H_i (i = 1, \cdots, k)$ does not contain a P_2-subgraph. Thus each H_i is either a single vertex or a single edge.

Assume that $E(H_1) = \{uw\}$. As $d_{G_0}(v) \geq 3$ and $d_{G_0}(w) \geq 3$, we have $|N_{G_0}(v) \cap \{x, y, z\}| \geq 2$ and $|N_{G_0}(w) \cap \{x, y, z\}| \geq 2$. As G_0 has no subgraph T such that $\tau(T) \geq 2$, $d_{G_0}(v) = 3$ and $d_{G_0}(w) = 3$, and the subgraph induced by $\{x, y, z\} \cup \{u, w\}$ is one of graphs in Figure 1. Thus $k \geq 2$. Without loss of generality, we assume that $V(H_2) \cap (N_{G_0}(y) \cup N_{G_0}(z)) \neq \emptyset$. Consider the new edge cut $X' = E(\{x, u, w\}, V(G_0) - \{x, u, w\})$. Then X' is a P_2-edge-cut, and so $|X'| \geq 10$. As $e(\{x, u, w\}, \{y, z\}) = 3, 4$, we have $|N_{G_0}(x) \cap (V(H_2) \cup \cdots \cup V(H_k))| \geq 6$. Therefore, $|X| \geq 6 + 4 = 10$, a contradiction. So each H_i is a single vertex.

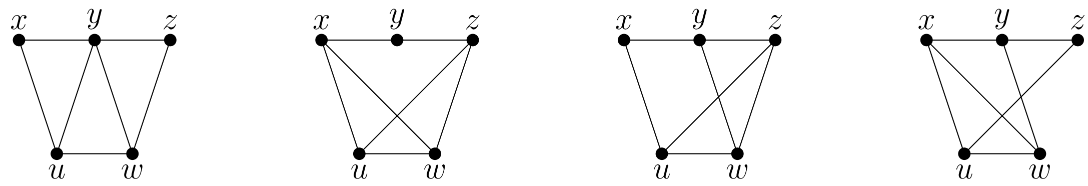


Figure 1.

Let $V(H_i) = \{a_i\}$. Since G_0 is 3-edge-connected, we have $a_ix, a_iy, a_iz \in E(G)$ and $k \geq 2$. Thus the subgraph induced by $V(H) \cup \{a_1, a_2\}$ in G_0 contains 2 edge-disjoint trees, contrary to (1). Therefore, Claim 1 holds.

Next we will get a contradiction by using the discharging method to find a nontrivial subgraph H in G_0 with $\tau(H) \geq 2$. We define the initial charge at v as $charge(v) = d_{G_0}(v)$. Recharging rule is defined as follows.

(R1) Let $v \in D_i(G_0)(i = 5, 6, 7)$. By Claim 1, $|N_{G_0}(v) \cap D_3(G_0)| \leq 1$. We define the charge of v as

$$charge(v) := \begin{cases} charge(v) - 1, & \text{if } |N_{G_0}(v) \cap D_3(G_0)| = 1 \\ charge(v), & \text{if } |N_{G_0}(v) \cap D_3(G_0)| = 0 \end{cases}.$$

If $w \in D_3(G_0) \cap N_{G_0}(v)$, we define the discharge of w as

$$charge(w) := charge(w) + 1.$$

(R2) Let $v \in D_i(G_0)(i \geq 8)$, we define the charge of v as

$$charge(v) := 4.$$

For every $w \in N_{G_0}(v)$, we define the discharge of w as

$$charge(w) := charge(w) + \frac{d_G(v) - 4}{d_G(v)}.$$

As $d_{G_0}(v) \geq 8$, we have $\frac{d_{G_0}(v)-4}{d_{G_0}(v)} \geq \frac{1}{2}$.

Claim 2. After all vertices are recharged, for each vertex $v \in V(G_0)$, $charge(v) \geq 4$.

Let $v \in D_i(G_0)(i \geq 4)$. By (R1) and (R2), we have $charge(v) \geq 4$. Next we consider $v \in D_3(G_0)$.

Case 1. $N_{G_0}(v) \cap D_3(G_0) = \emptyset$.

If $N_{G_0}(v) \cap (D_5(G_0) \cup D_6(G_0) \cup D_7(G_0)) \neq \emptyset$, by (R1), we have $charge(v) = 4$. Otherwise, $N_{G_0}(v) \cap (D_5(G_0) \cup D_6(G_0) \cup D_7(G_0)) = \emptyset$. By Claim 1, $|N_{G_0}(v) \cap D_4(G_0)| \leq 1$, Thus $|N_{G_0}(v) \cap D_{\geq 8}(G_0)| \geq 2$. By (R2), we have

$$charge(v) \geq 3 + \frac{1}{2} + \frac{1}{2} = 4.$$

Case 2. $N_{G_0}(v) \cap D_3(G_0) \neq \emptyset$.

By Claim 1, $|N_{G_0}(v) \cap D_3(G_0)| = 1$. Let $N_{G_0}(v) \cap D_3(G_0) = \{u\}$. By Claim 1, for any $w \in (N_{G_0}(v) \cup N_{G_0}(u) - \{u, v\})$, $d_{G_0}(w) \geq 8$. By (R2), we have

$$charge(v) \geq 3 + \frac{1}{2} + \frac{1}{2} = 4, \text{ and } charge(u) \geq 3 + \frac{1}{2} + \frac{1}{2} = 4.$$

Claim 2 hods.

By Claim 2, we have

$$2|E(G_0)| = \sum_{v \in V(G_0)} d_{G_0}(v) = \sum_{v \in V(G_0)} \text{charge } (v) \geq 4|V(G_0)|,$$

and so $|E(G_0)| \geq 2|V(G_0)|$. By Theorem 5.2.1, G_0 has a nontrivial subgraph T such that $\tau(T) \geq 2$, contrary to (1). □

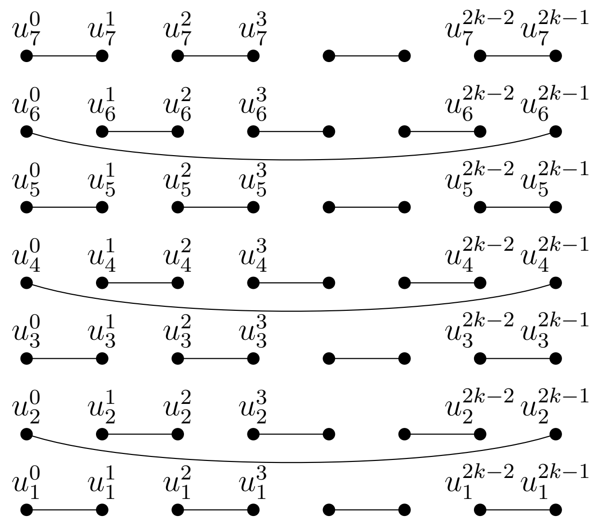


Figure 2.

Remark: If the essential connectivity of $L(G)$ in Theorem 5.1.2 is less than 10, we fail to prove $|E(G_0)| \geq 2|V(G_0)|$ using the discharging method. For example, we take $2k(k \geq 1)$ copies of $H_i = K_{2,7}(i = 0, 1, \cdots, 2k-1)$, where $V(H_i) = \{a^i, b^i, u_1^i, u_2^i, \cdots, u_7^i\}$ with $d_{H_i}(a^i) = d_{H_i}(b^i) = 7$ and $d_{H_i}(u_j^i) = 2(j = 1, 2, \cdots, 7)$. The graph G is defined as $V(G) = \bigcup_{i=0}^{2k-1} V(H_i)$ and $E(G) = \bigcup_{i=0}^{2k-1} E(H_i) \cup \bigcup_{i=0}^{2k-1} E(H_i, H_{i+1})$ (i is modulo by $2k$), where $H(2t, 2t+1) = \{u_1^{2t}u_1^{2t+1}, u_3^{2t}u_3^{2t+1}, u_5^{2t}u_5^{2t+1}, u_7^{2t}u_7^{2t+1}\}$ and $H(2t+1, 2t+2) = \{u_2^{2t+1}u_2^{2t+2}, u_4^{2t+1}u_4^{2t+2}, u_6^{2t+1}u_6^{2t+2}\}(t = 0, 1, \cdots, k-1)$ (The subgraph induced by $\bigcup_{j=1}^{7} \bigcup_{i=0}^{2k-1} \{u_j^i\}$ is shown in Figure 2). Then $|V(G)| = 18k$ and $|E(G)| = 35k$, and so $2|V(G)| - |E(G)| = 36k - 35k = k$. Therefore, the difference of $2|V(G)| - |E(G)|$ can be as large as possible. However, $L(G)$ is 3-connected, essentially 9-connected.

Chapter 6

Cycle Chains and hamiltonian 3-connected claw-free graphs

6.1 Introduction

Graphs in this chapter are finite and may have multiple edges or loops. Let H_5 (usually called the hour-glass) denote the graph obtained from K_5 by deleting two independent edges, and let T_3 be the graph with

$$V(T_3) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \text{ and } E(T_3) = \{v_i v_{i+1}, v_i v_{i+4}, v_{i+1} v_{i+4} : 1 \leq i \leq 3\}.$$

Theorem 6.1.1 *(T. Kaiser, M. Li, Ryjáček and L. Xiong [24]) A graph G is said to have the* **hourglass property** *if in every induced hourglass H_5, G has two non-adjacent vertices which have a common vertex in $V(G) - V(H_5)$. Then every 4-connected claw-free graph is hamiltonian.*

Theorem 6.1.2 *(F. Pfender [40]) Every 4-connected $\{K_{1,3}, T_3\}$-free graph is Hamiltonian.*

For a vertex $v \in V(G)$, $N_G(v)$ is the set of all vertices adjacent to v in G. For a subset $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$. A vertex v is **locally connected** in G if $G[N_G(v)]$ is connected. Let G be a claw-free graph. If $v \in V(G)$ is locally connected, the operation of adding all possible edges between vertices in $N_G(v)$ is referred as local completion at v. The **line graph closure** of G, denoted by $cl(G)$, is obtained from this claw-free graph G by repeatedly applying local completion at every locally connected vertices until none left. Ryjáček [43] showed that $cl(G)$ is well-defined and unique. Furthermore, he proved the following very useful result.

Theorem 6.1.3 *Let G be a claw-free graph. Then $cl(G)$ is the line graph of a K_3-free graph, and G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

Theorem 6.1.4 *(Fujisawa and Ota [20]) Let G be a 4-connected claw-free graph. Assume that $G[N_G(T)]$ is cyclically 3-connected if T is a maximal K_3 in G which is also maximal in $cl(G)$. Then G is Hamiltonian.*

The purpose of this paper is to find a unified approach to generalize all these results. A graph G is *supereulerian* if G has a spanning eulerian subgraph. For any edge $e = uv \in E(G)$, define

$$d_G(e) = d_G(u) + d_G(v) - 2.$$

Our main theorem states as follows.

Theorem 6.1.5 *Let G be a graph with $\kappa'(G) \geq 3$. If for any 3-bond D of G,*
(i) either D intersects a cycle of length at most 3, or
(ii) every $e \in D$ lies in at least $\min\{d_G(e) - 3, 2\}$ *cycles of length at most 4,*
then G is supereulerian.

In the last section, we shall show that Theorem 6.1.5 can be applied to obtain short proofs for each of Theorems 6.1.1 6.1.2, and 6.1.4. Moreover, Theorem 6.1.5 can also be applied to obtained new sufficient conditions for a 3-connected claw-free graph to be hamiltonian.

A claw-free graph G is said to have the **maximal K_3-property** if for every maximal K_3 subgraph K of G, every edge in K lies in a cycle of length at most 4 in G other than K itself. For a vertex cut X of G, a cycle of G that contains at least two vertices of X is called an **X-cycle**. If L is a component of $G-X$, then $G[V(L)\cup X]$ is an **X-component** of G. An X-component is a **clique X-component** if it is a maximal complete subgraph of G. A vertex cut X is **essential** if $G-X$ has at least two nontrivial components. Our main result also implies the following.

Theorem 6.1.6 *Let G be a 3-connected claw-free graph with the maximal K_3-property. Then G is Hamiltonian if for any vertex 3-cut X of G, either*
(i) G does not have a clique X-component, and either G has an X-cycle C with $|E(C)| \leq 3$, or every pair of vertices of X are in an X-cycle of length at most 4 in G, or
(ii) one of the X-component K is a complete graph, and either G has an X-cycle C not in K with $|E(C)| \leq 3$, or every pair of vertices of X are in an X-cycle not in K of length at most 4.

Theorem 6.1.6 has the following immediate corollary.

Corollary 6.1.7 *Every 4-connected claw-free graph with maximal K_3-property is hamiltonian.*

6.2 Proof of Theorem 6.1.5

In this section, we will prove a stronger result (Theorem 6.2.6 below) which implies Theorem 6.1.5. If K is a subgraph of G, then we write G/K for $G/E(K)$. If K is a connected subgraph of G, and if v_K is the vertex in G/K onto which K is contracted, then K is called the **preimage** of v_K. A vertex u in G/K is a **trivial vertex (of the contraction)** if the preimage of u in G is u itself.

Definition 6.2.1 *A graph G is* **collapsible** *if for any subset R of $V(G)$ with $|R| \equiv 0$ (mod 2), G has a spanning connected subgraph Γ_R with $O(\Gamma_R) = R$. In particular, K_1 is collapsible.*

Collapsible graphs were discovered by Catlin in [8]. Catlin in [8] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The *reduction* of G is the graph obtained from G by successively contracting $H_1, H_2, \cdots, H_k$. A graph G is *reduced* if it equals to its reduction. Thus reduced graphs do not have any nontrivial collapsible subgraphs. Note that any collapsible graph G has a spanning Eulerian subgraph, and so itself is supereulerian. Let $a_1(G)$ be the minimum number of spanning trees of G such that every edge of G is in at least one of them.

Theorem 6.2.2 *Each of the following holds for a a connected graph G.*
(i) (Catlin, Theorem 3 of [8]) Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible; and G is supereulerian if and only if G/H is supereulerian.
(ii) (Catlin, Corollary 1 of [8]) If G has a spanning tree Tsuch that every edge of T lies in a collapsible subgraph of G, then G is collapsible.
(iii). (Catlin, Theorem 7 of [8], Jaeger [23]) If $\kappa'(G) \geq 4$, then G is collapsible.
(iv). (Catlin, Theorem 8 of [8], Lemma 1 of [7]) If G is reduced, then $a_1(G) \leq 2$, and G contains no subgraph isomorphic to C_2, K_3, or $K_{3,3} - e$. When $G \notin \{K_1, K_2\}$ and $a_1(G) \leq 2$, $F(G) = 2|V(G)| - |E(G)| - 2$.

Let G be a graph. Define a relation $\sim$ on $E(G)$ as follows: $\forall e, e' \in E(G)$, $e \sim e'$ if and only if either $e = e'$ or G has a sequence of cycles $C_1, C_2, \cdots, C_k$ such that
(4C1) for each i with $1 \leq i \leq k$, $|E(C_i)| \leq 4$,
(4C2) for $i = 1, 2, ..., k-1$, $E(C_i) \cap E(C_{i+1}) \neq \emptyset$, and
(4C3) $e \in E(C_1$ and $e' \in E(C_k)$.

Such a sequence $C_1, C_2, ..., C_k$ is referred as a **4-cycle (e, e')-chain**. It is routine to verify that this is an equivalence relation. The equivalence classes on $E(G)$ induce subgraphs in G which are referred as **4-cycle-connected components**. Part (i) of Lemma 6.2.3 follows from the definition.

Lemma 6.2.3 *Let G be a connected graph.*
(i) Every 4-cycle-component H of G is either 2-connected or a K_2.
(ii) (Theorem 1 of [25]) If every edge of G lies in a cycle of length at most 4 in G, and if both $\kappa(G) \geq 2$ and $\delta(G) \geq 3$, then G is collapsible.
(iii) If H is a 4-cycle-component of G and if H has a collapsible subgraph with least one edge, then H is collapsible.
(iv) If H is a 4-cycle-component of G, then for any $e \in E(H)$, H/e is collapsible.

Proof. If (iii) holds, then (iv) follows. In fact, if H contains a 3-cycle, then by (iii), H is collapsible, and so H/e is also collapsible. Assume that H does not contain a 3-cycle. For any edge $e \in E(H)$, e must be in a 4-cycle C in H. Therefore C/e is a 3-cycle and by definition, H/e is either 4-cycle connected, or is an edge-disjoint union of 4-cycle-components with the contraction image of e being the only cut vertex. As in the latter case, each component of H/e contains a 3-cycle (contraction image of 4-cycles containing e in H) and so by (iii), H is collapsible. Hence (iv) follows from (iii).

It suffices to prove (iii). Since H has a collapsible subgraph with least one edge, H has a maximal nontrivial collapsible subgraph L. If $H = L$, then done. Otherwise assume that $H \neq L$. Since H is a 4-cycle component, and since $H \neq L$, H must have an edge $e \in E(H) - E(L)$ such that e is incident with a vertex in L. Since H is 4-cycle connected, e must be in a 4-cycle C with $E(C) \cap E(L) \neq \emptyset$, and so $C/(C \cap L)$ is a cycle of length at most 3 in H/L. By Theorem 6.2.2 (iv), $C/(C \cap L)$ is collapsible. By Theorem 6.2.2 (i), $C \cup L$ is collapsible in H, contrary to the maximality of L. Hence H must be collapsible. □

Let $\mathcal{F}$ denote the family of graphs satisfying the hypothesis of Theorem 6.1.5. By the definition of contraction,

$$\text{if } G \in \mathcal{F}, \text{ then for any } e \in E(G),\ G/e \in \mathcal{F}. \tag{6.1}$$

Lemma 6.2.4 *Let $G \in \mathcal{F}$ and let H be a 4-cycle-connected component of G with $|E(H)| > 2$. Then H is collapsible.*

Proof. Since 3-cycles are collapsible, by Lemma 6.2.3 (iii), we assume that H does not have a cycle of length at most 3. Since every edge of H lies in a cycle of length at most 4, $\delta(H) \geq 2$. By Lemma 6.2.3(i) and (ii), it suffices to show that $\delta(H) \geq 3$. Suppose, by contradiction, that $d_H(z_0) = 2$ for some vertex $z_0 \in V(H)$. Let $N_G(z_0) = \{z_1, z_2, \cdots, z_d\}$ such that $d = d_G(z_0)$ and such that $N_H(z_0) = \{z_1, z_2\}$.

We claim that $d \geq 4$. If not, as $d = d_G(v) \geq \kappa'(G) \geq 3$, we must have $d = 3$. Since G has no cycles of length at most 3, z_0z_3 must be in a 4-cycle C of G. Since $d = 3$, $|E(C) \cap E_G(z_0)| = 2$. It follows that $z_3 \in V(H)$ by the definition of 4-cycle-connectedness, contrary to the fact that $z_3 \notin V(H)$.

By this claim and since $\kappa'(G) \geq 3$, $d_G(z_0z_1) = 2$ and so by the definition of $\mathcal{F}$, z_0z_1 must be in 2 distinct 4-cycles C_1, C_2 (say) of G. By the definition of 4-cycle-connectedness, both $E(C_1) \cup E(C_2) \subseteq E(H)$, and so we must have $d_H(z_0) \geq 3$, contrary to the assumption that $d_H(z_0) = 2$. □

Lemma 6.2.5 *Let G be a 3-edge-connected graph. If for any 3-bond D, G has a collapsible subgraph H_D with $D \cap E(H_D) \neq \emptyset$, then G is collapsible.*

Proof. Let $D_1, D_2, \cdots, D_t$ denote the list of all 3-bonds of G. By assumption, G has collapsible subgraphs $H_{D_1}, \cdots, H_{D_t}$ such that $E(H_{D_i}) \cap D_i \neq \emptyset$, for each $i = 1, 2, \cdots, t$. Let $G' = G/(H_{D_1} \cup \cdots \cup H_{D_t})$. By the definition of contractions, $\kappa'(G') \geq \kappa'(G)$. Since G' does not have any 3-bonds, $\kappa'(G') \geq 4$. By Theorem 6.2.2(iii), G' is collapsible. By repeated applications of Theorem 6.2.2(i), G is collapsible. □

Theorem 6.2.6 *Let G be a graph with $\kappa'(G) \geq 3$. If for any 3-bond D, either D intersects a cycle of length at most 3, or every $e \in D$ lies in at least $\min\{d_G(e) - 3, 2\}$ cycles of length at most 4, then G is collapsible.*

Proof. By Lemma 6.2.4, every edge e in a 3-bond of G lies in a collapsible subgraph of G. By Lemma 6.2.5, G is collapsible. □

6.3 Proofs of Theorems 6.1.1, 6.1.2, 6.1.4 and 6.1.6

Let G be a graph such that $\kappa(L(G)) \geq 3$ and such that $L(G)$ is not complete. For each $v \in D_2(G)$, let $E_G(v) = \{e_1^v, e_2^v\}$ and define

$$X_1(G) = \cup_{v \in D_1(G)} E_G(v), \text{ and } X_2(G) = \{e_2^v : v \in D_2(G)\}. \tag{6.2}$$

Since $\kappa(L(G)) \geq 3$, $D_2(G)$ is an independent set of G and for any vertex $v \in D_2(G)$, $|X_2(G) \cap E_G(v)| = 1$. Define the *core* of the graph G as

$$G_0 = G/(X_1(G) \cup X_2(G)) = (G - D_1(G))/X_2(G). \tag{6.3}$$

Recall thta an eulerian subgraph H of a graph G is **dominating** if $E(G - V(H)) = \emptyset$. Edges in $\cup_{v \in D_2(G)} E_G(v) - X_2(G)$ are referred as *nontrivial edges* in G_0. Vertices of G adjacent to a vertex in $D_1(G)$ are viewed as the contraction image of edges in $\cup_{v \in D_1(G)} E_G(v)$. Utilizing the well-known theorem of Harary and Nash-Willaims ([21]) and Catlin's collapsible graphs ([8]), Shao proves the following useful theorem. A detailed justification for Theorem 6.3.1(iii) can be found in [36].

Theorem 6.3.1 *(Shao, Section 1.4 of [45]) Let G_0 be the core of graph G, then each of the following holds.*
(i) G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$.
(ii) G_0 is well defined.
(iii) $L(G)$ is hamiltonian if and only if G_0 has a dominating eulerian subgraph containing all nontrivial vertices and both end vertices of each nontrivial edges.

Theorem 6.3.2 *Let G be a claw-free graph and let $cl(G)$ be its closure. Each of the following holds.*
(i) (Kaiser, Li, Ryjáček and Xiong, Proposition 3 of [24]) If G has the hourglass property, then $cl(G)$ also has the hourglass property.
(ii) (Pfender, Theorem 14 of [40]) The closure of a $\{K_{1,3}, T_3\}$-free graph is also $\{K_{1,3}, T_3\}$-free.
(iii) (Fujisawa and Ota, Lemma 5 in [20]) Any maximal K_3 in $cl(G)$ is also a maximal

K_3 of G.
(iv) If G satisfies the hypothesis of Theorem 6.1.6, then $cl(G)$ also satisfies the hypothesis of Theorem 6.1.6.

Proof. It remains to prove (iv). By (iii), $cl(G)$ also has maximum K_3-property. Since $cl(G)$ is obtained from G by adding edges, and adding edges will not increase the length of the cycles in G. Therefore, $cl(G)$ also satisfies Theorem 6.1.6 (i) and (ii). □

By Theorem 6.1.3 and Theorem 6.3.2, it suffices to prove Theorems 6.1.1 6.1.2, 6.1.4 and 6.1.6 for line graphs.

Proof of Theorems 6.1.1 and 6.1.2 Let $L(G)$ be a 4-connected line graph with the hourglass property or without an induced T_3. Let G_0 be the core of G. Then by the definitions of a line graph and of the core, every edge in G_0 lies in a cycle of length at most 4. By Theorem 6.1.5, G_0 is collapsible, and so by Theorem 6.3.1, $L(G)$ is hamiltonian. □

Proof of Theorem 6.1.4 Let $L(G)$ be a line graph satisfying the conditions of Theorem 6.1.4. As indicated in Proposition 9 in [20], the core G_0 of G must satisfy the condition of Theorem 6.1.5, and so by Theorem 6.1.5, G_0 is collapsible. By Theorem 6.3.1, $L(G)$ is hamiltonian. □

Proof of Theorem 6.1.6 Let $L(G)$ be a line graph satisfying the conditions of Theorem 6.1.6 and denote the core of G by G_0.

We argue by contradiction and assume that $L(G)$ is a line graph satisfying the conditions of Theorem 6.1.6 but $L(G)$ is not hamiltonian. Denote the core of G by G_0. We choose G so that

$$|V(G)| \text{ is minimized.} \tag{6.4}$$

Claim 1. G does not have an essential vertex cut of size 1, and so the only cut vertices of G are those adjacent to a vertex of degree 1 in G.

By contradiction, assume that G has an essential vertex cut $\{v_0\}$. Then G has nontrivial connected subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v_0\}$, and for each $i \in \{1, 2\}$, $G_i - v_0$ is nontrivial. Let G_i' be the graph obtained from G_i by adding a new vertex z_i and a new edge z_iv_0. Thus every essential edge cut of G_i' is also an essential edge cut of G. Since G is essentially 3-edge-connected, each G_i' is also essentially 3-edge-connected, and so $L(G_i')$ is 3-connected.

Fix $i \in \{1, 2\}$. As G is essentially 3-edge-connected, $d_{G_i}(v_0) \geq 3$, and so $d_{G_i'}(v_0) \geq 4$. Hence any maximal K_3 in $L(G_i')$ must be a maximal K_3 of G. It follows from the assumption that $L(G)$ has the maximal K_3-property that $L(G_i')$ also has the maximal K_3-property. Let $X = \{e_1, e_2, e_3\}$ be a vertex 3-cut of $L(G_i')$. Then X must be an essential edge cut of G_i', and so $X \subseteq E(G_i)$. If $X \neq E_{G_i}(v_0)$, then X is also an essential edge cut of $L(G)$, and so a vertex 3-cut of $L(G)$. It follows that X must satisfy Theorem 6.1.6 (i) and (ii) in $L(G)$, and so X satisfies Theorem 6.1.6 (i) and (ii) in $L(G_i')$. By (6.4), $L(G_i')$ has a Hamilton cycle C_i.

Let $f_i = z_iv_0$. Suppose that in C_i, e_1^i, e_2^i are adjacent to f_i. Then $e_1^i, e_2^i \in E_G(v_0)$. Since $E_G(v_0)$ induces a complete graph in $L(G)$, $(C_1 - f_1) \cup (C_2 - f_2) \cup \{e_1^1e_1^2, e_2^1e_2^2\}$ is a Hamilton cycle of $L(G)$, contrary to the assumption that G is a counterexample. This proves Claim 1.

If G_0 satisfies Theorem 6.1.5 (i) and (ii), then by Theorems 6.1.5 and 6.3.1, Theorem 6.1.6 follows, and so a contradiction would be obtained. Therefore, we shall show that G_0 satisfies Theorem 6.1.5 (i) and (ii). By contradiction, we assume that G_0 has a 3-bond X that does not satisfy Theorem 6.1.5 (i) or (ii). Denote $X = \{e_1, e_2, e_3\}$. Since G_0 is a contraction of G, X can also be viewed as a subset of $E(G)$, and so X is also an edge cut of G.

Claim 2. For any $v \in V(G)$, we cannot have $X = E_G(v)$. (This implies that X must be an essential edge cut of G).

If $X = E_G(v)$, then X induces a maximal K_3 of $L(G)$. Since $L(G)$ has the maximal K_3-property, every edge of X satisfies Theorem 6.1.5 (i) and (ii), contrary to the assumption on X. Hence this proves Claim 2.

Claim 3. For any $v \in V(G)$, we cannot have $X \subset E_G(v)$.

Suppose that for some v, $X \subseteq E_G(v)$. Since X is an edge cut of G, v must be a cut vertex of G. By Claims 1 and 2, and since X is an essential edge cut of G, every edge in $E_G(v) - X$ must be incident with a vertex of degree 1 in G. It follows that $E_G(v)$ induces a clique X-component K. By Theorem 6.1.6(ii), $L(G)$ must have either a 3-cycle C containing two vertices in X, or every pair of vertices $e_i, e_j \in X$ are in a cycle C_{ij} of length at most 4 in $L(G)$. Since $X \subseteq E_G(v)$, G must have a cycle C' or cycles C'_{ij} such that $C = L(C')$ or $C_{ij} = L(C'_{ij})$, for all possible values i and j. Hence X satisfies Theorem 6.1.5 (i) and (ii), contrary to the assumption on X. This proves Claim 3.

Claim 4. $L(G)$ does not have a clique X-component.

If $L(G)$ has a clique X-component K, then $|V(K)| = d \geq |X| + 1 = 4$. It follows that there must be some $v \in V(G)$ of degree d such that $X \subseteq E_G(v)$, contrary to Claims 2 or 3. This proves Claim 4.

By Claim 2, X does not induce a maximal K_3 in $L(G)$. By Claim 4, $L(G)$ doe snot have a clique X-component. Since X is a vertex cut of $L(G)$, by Theorem 6.1.6 (i), $L(G)$ must have either a 3-cycle C containing two vertices in X, or every pair of vertices $e_i, e_j \in X$ are in a cycle C_{ij} of length at most 4 in $L(G)$. Since $L(G)$ does not have a clique X-component, G must have a cycle C' or cycles C'_{ij} such that $C = L(C')$ or $C_{ij} = L(C'_{ij})$, for all possible values i and j. Hence X satisfies Theorem 6.1.5 (i) and (ii), contrary to the assumption on X. This proves that G_0 satisfies the hypothesis of Theorem 6.1.5, and so G_0 must be collapsible. By Theorem 6.3.1, $L(G)$ is hamiltonian, contrary to the assumption that $L(G)$ is a counterexample. This completes the proof for Theorem 6.1.6.

□

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