Applicability of Bispectral Analysis to Unstable Plasma Waves

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Abstract

A program to implement a spectral analysis technique called the bispectrum was written and tested with computer generated time series data. The application of the algorithm to the study of nonlinear interactions was demonstrated by a comparison of computed quantities with results from model equations found in the literature. Specifically determined were: the amplitude and phase of coupling coefficients, the power transfer function, the fraction of power associated with nonlinear coupling, and the identification of waves involved in a quadratic coupling interaction. A method of distinguishing the two parent waves from the daughter wave in this three-wave interaction is proposed as a new application of the technique. These results, as well as the values computed from a Monte Carlo simulation of plasma turbulence were found to be consistent with expectations.

Two experimental systems were investigated with the bispectrum. One was the periodically pulled time series data of a driven van der Pol oscillator (unijunction transistor circuit) which contained significant bispectral features but no real evidence of quadratic coupling. The other was plasma fluctuation data from the WVU-Q Machine, where the inhomogeneous energy-density driven mode exhibited a degree of coupling to lower frequencies that was absent in the case of the current driven mode.
Acknowledgments

“No man is an island entire unto himself...”

Quoted by Hemingway in *For Whom the Bell Tolls*

It amazes me to think of the fortuitous combination of people and circumstance that led to the completion of this thesis. Some weird accident of birth left me with the perfect parents, who fooled me into thinking I could do almost anything by sheer force of will. Another quirky twist of fate (or alien influence) paired me with my wife and life partner, Cat, who is perhaps the most wonderful piece of sunbeam ever coalesced into human form. I am also indebted to my good friend and software advisor, Bruce Dean, whose contagious enthusiasm and commitment to Physics inspired and motivated me almost as much as our Arboretum runs.

No less significant are the contributions of my committee members: Dr. Koepke for his financial, intellectual, and moral support these past years, and for being patient enough to give me the luxury of struggling with these ideas, and the liberty of exploring avenues of my own choosing; Dr. Scime for valuable discussions and references; Dr. Treat, who has literally been everything to me -- recruiter, advisor, and teacher (many thanks); and Dr. Ferer who is largely responsible for sparking my interest in things chaotic.

I am also grateful to Dr.'s Pécsei and Klinger; their confirmations of my preliminary bispectrum interpretations helped water the small, sensitive sprout of my bispectrum knowledge when it was still struggling to be born, but they cannot be held responsible for any shortcomings of the present work, mutated as it is beyond their
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List of Notation

B(ωj,ωk) Bispectrum
b2(ωj,ωk) Squared bicoherence spectrum ( a normalized bispectrum)
Λjk Three wave coupling coefficient derived from Martin and Fried's mode-coupling equation in the space and time domains
Vjk Three wave coupling coefficient derived from Kim and Powers mode-coupling equation in the frequency domain
|A| Amplitude of the conversion factor between Λjk and Vjk (introduced to simplify Eq. 2.12)
α Phase of the conversion factor between Λjk and Vjk
β(ωj,ωk) Phase of the bispectrum
γ(ωj,ωk) Phase of the Martin-Fried coupling coefficient
κ Inverse scale length of Fourier amplitude variation
k(ω) Wavenumber spectrum
K(ω) Local wavenumber spectrum
S(ω) Un-normalized spectral density function
P(ω) Conventional power spectrum, or the ensemble average of the un-normalized spectral density function
S_{local}(K,ω) Local wavenumber and frequency spectral density
S_{local}(K|ω) Local conditional wavenumber spectral density
ωb, ωb, ωc, ωd 2π(10, 14, 24, 38) rad/sec respectively. ωb, ωc are the parents, and ωb, ωd are the daughters of this quadratic coupling interaction.
ωj, ωk, ωm Indices label any frequencies satisfying resonant conditions ωj+ωk=ωm.
N Number of data points per realization
M Number of realizations
R Number of different Monte Carlo wave packet types per record
S Number of shots of the same Monte Carlo wave packet type per record
Δω or Δf Elementary bandwidth, or frequency resolution
Δt Sampling interval
Ω Frequency index set (of which the ωj's are possible values)
Γ(t) Windowing function applied to each time series record
f_s Sampling frequency
f_N Nyquist frequency
η(t) Small amplitude uniformly distributed noise
T Record length in seconds
I. Introduction

A. Motivation and Scope

An understanding of the nonlinear behavior of an experimental system would be greatly enhanced by a diagnostic tool that could identify the sources and strengths of nonlinear coupling in that system. Hasselman et al.\(^1\) first applied a Fourier transform of the triple correlation function, called the bispectrum, to a study of ocean waves in 1963, and found that he was able to identify wave coupling that resulted in peaking of the crests. Since then, bispectral analysis has undergone a series of refinements, and has been used to make quantitative measurements on the strength of wave-wave interactions in plasmas as early as 1979. Three-wave coupling is considered one of the dominant nonlinear interactions in many theoretical treatments of plasma fluctuations.\(^2,3,4\) A principle thesis of this work, therefore, is that an implementation of the bispectral technique could deepen our understanding of plasma instabilities generated in the WVU-Q machine.

To this end, a computer program was written to calculate the bispectrum from a single-point measurement of a fluctuating time series and was benchmarked with results found in the literature. In a quadratic coupling scheme, two waves, referred to hereafter as “parents,” are nonlinearly coupled in such a way as to produce two new waves, or “daughters.” The wavenumber (frequency) of a daughter wave equals the sum or difference of the wavenumbers (frequencies) of the parents. Since both bispectral and wavenumber analysis are required to fully characterize a quadratic coupling interaction, a separate program was developed to compute a statistical wavenumber-frequency spectrum.
following Beall's single probe-pair technique. The bispectrum's applicability to experimental systems was demonstrated with time series measurements of a periodically driven nonlinear oscillator subjected to frequency pulling, or "periodic pulling," and of two categories of Q-machine plasma fluctuations, one driven by magnetic field-aligned current and the other driven by shear in transverse plasma flow. In the process, a method for distinguishing the parent waves of a quadratic interaction from the daughter waves was developed, and found to have some relevance to plasma experiments. Although bispectral analysis of the periodic pulling data failed to reveal anything other than what is expected from the Fourier decomposition of a repetitive pulse shape, a comparison of the bispectra associated with the two plasma instabilities proved more informational. Namely, the comparison uncovered a difference between the Current Driven Electrostatic Ion Cyclotron (CDIEC) and Inhomogeneous Energy-Density Driven (IEDD) instabilities in terms of their coupling to lower frequency drift waves.

B. The bispectrum's role in future experiments

The motivation for this work is best understood in light of the previously completed and intended future research efforts of the WVU Plasma Lab. In this context, the bispectrum is one of several tools being adopted for the analysis of the various physical systems under investigation. The use of well known analysis techniques such as auto and cross power spectra (based on Fast Fourier Transforms) is already firmly established in the WVU Plasma Lab. Min Ke was responsible for adding the next technique by implementing the Grassberger-Procaccia algorithm for determining the correlation dimension of a dynamical system as part of his master's thesis. This technique has been
used by many authors since 1983 to experimentally distinguish deterministic chaos (with a fractional number of degrees of freedom) from noise (corresponding to a large number of degrees of freedom). A future intention of the research group is to develop an algorithm for the determination of the maximum Lyapunov exponent, which can be considered a measure of a system’s chaos, or sensitivity to initial conditions (a positive exponent indicating a chaotic state).

Eventually, these tools will be used to quantify differences between turbulence generation mechanisms operating in the Q-machine. Three methods for generating low frequency turbulence will be investigated. One is the introduction of a negative ion species, SF₆⁻, into a plasma column containing Ba⁺, K⁺, or Cs⁺ ions. The state of the resulting positive-ion/negative-ion plasma can be smoothly varied from quiescent to strongly turbulent by adjusting the concentration of gaseous SF₆. Another method involves terminating the plasma column with a ceramic disc. Lashinsky used this method to show that the principle saturation mechanism of drift waves is strongly influenced by the differences in the boundary conditions associated with metal and insulating end plates. When the metal plate was replaced by a ceramic disk, the saturation occurred at a much larger fluctuation level, allowing the plasma to become quite turbulent. In a third method, large amplitude radio frequency waves (in the 5 MHz to 30 MHz range) are launched from the cold plate or via a coil surrounding the plasma column to excite ion acoustic or drift wave turbulence. The turbulent mode is selected by adjusting various experimental parameters such as plasma density, magnetic field strength, and net longitudinal current.
II. Computational Method: Bispectral Analysis

Much of the recent progress in applying bispectral analysis to plasma systems can be credited to University of Texas at Austin researchers Beall, Kim and Powers, Tsui and others. Their work forms the basis for the applications described in this chapter. The background and theory discussions are included primarily to introduce the notation and the statistical concept of cumulant spectra, while the intent of the applications section is to provide an exhaustive list of interpretations and quantities that can be determined from the bispectrum or its normalized counterpart, the bicoherence spectrum. One of these -- the method of determining causality -- is an extension of previously published work.

A. Background

Fourier's theorem states that any periodic function can be expressed as a superposition of waves such that

\[ \phi(t, x) = \sum_j \hat{\phi}(\omega_j, x) e^{i(k(\omega_j)x - \omega_j t)} = \sum_j \Phi(\omega_j, x) e^{-i\omega_j t}, \]

(2.1)

where \( \Phi(\omega_j, x) \) is the Fourier amplitude of the time series measured at a single point x

\[ \Phi(\omega_j, x) = \hat{\phi}(\omega_j, x) e^{i(k(\omega_j)x)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi(t, x) e^{i\omega_j t} dt. \]

(2.2)

Using subscripts on \( \omega \) anticipates digital (discrete) Fourier analysis. Subject to certain restrictions, the result can be extended to finite record length, non-periodic signals by applying a truncation technique such as the one described in section II.D.3. For notational simplicity, the \( \Phi(\omega_j, x) \) will be replaced by \( \Phi(\omega_j) \) when all Fourier components are
computed at the same point in space.

B. Theory

Higher order spectral techniques are introduced in the literature through the concept of moments and their corresponding cumulant spectra. For probability distributions of a single random variable, the moments of a time series are conveniently described in terms of the expectation operator \( E \) (or “estimator”):

\[
m_1 = E[\phi(t)] = \int_{\infty}^{\infty} \phi(t) p(t) dt = \bar{\phi},
\]

\[
m_r = E[(\phi(t) - \bar{\phi})^r],
\]

where \( p(t) \) is the probability distribution function. The mean value of \( \phi(t) \) is \( m_1 \); \( m_2 \) is the variance, or square of the standard deviation; and \( m_3 \) is referred to as the skewness, which is a measure of the asymmetry of the function around its mean value. The Fourier transform of the mean could be considered the first order cumulant spectrum. As a sublity of the notation, \( \phi(t) \) at each individual time point \( t \) is called the random variable (in that its value will be different for each measurement or “realization”), and the complete form of \( \phi(t) \), as \( t \) varies over all possible values is called the random process or function. When the result of an experiment requires more than one, or even a set of random variables for its full description, say \( \{\phi(t_1), \phi(t_2), \ldots, \phi(t_n)\} \), a multivariate, or “joint” moment is invoked to characterize the probability density function. For example, two variable moments are defined by

\[
m_{r} = E[(\phi(t_1) - \bar{\phi})^r (\phi(t_2) - \bar{\phi})^s].
\]

In particular, \( m_{1,1} \) is the covariance, and measures the degree of linear association between
\( \phi(t_1) \) and \( \phi(t_2) \). If \( \phi(t) \) is a stationary process, and \( t_1 \) and \( t_2 \) are separated by a time \( \tau \), it becomes the un-normalized autocorrelation function \( R(\tau) = E[\phi(t)\phi(t+\tau)] \). If there is no correlation between the two variables (i.e. if they are independent), the autocovariance vanishes. The Fourier transform of the autocorrelation function is the second order cumulant spectrum, and is identical to the conventional power spectrum

\[
P(\omega_j) = E[\Phi(\omega_j)\Phi^*(\omega_j)].
\] (2.5)

The Power spectrum can be useful in breaking down the total power of a signal into its various frequency components, but is insensitive to phase coherence among those components.

In contrast, the third order cumulant spectrum, called the bispectrum, is the Fourier transform of the triple correlation function

\[
B(\omega_j, \omega_k) = E[\Phi(\omega_j)\Phi(\omega_k)\Phi^*(\omega_j + \omega_k)],
\] (2.6)

and will be non-zero only if there is a statistical phase dependence between waves at \( \omega_j \), \( \omega_k \), and \( \omega_j + \omega_k \). For spontaneously excited independent waves, the phase of each will be statistically independent, and the resulting sum phase will be randomly distributed over the interval \(-\pi\) to \(\pi\). As a result, the average of the Fourier convolution on the right hand side of Eq. (2.6), and therefore the bispectrum, should vanish. If instead some oscillations are excited due to nonlinear interactions with other oscillations, their sum phase will be the same for each realization and therefore the average will not go to zero. Thus the bispectrum measures the degree of the statistical dependence between three waves and

\[ \text{\textsuperscript{\dag}} \] A weak form of stationarity is assumed (stationary to second order) -- only the mean and the variance need be independent of time.
provides a means of verifying and quantifying the coupling between the observed waves.

As written, the bispectrum is dependent not only on the degree of coupling, but also on the amplitudes of the involved spectral components. A normalized bispectrum, the so-called bicoherence spectrum,

\[ b^2(\omega_j, \omega_k) = \frac{|B(\omega_j, \omega_k)|^2}{E[|\Phi(\omega_j)\Phi(\omega_k)|^2]E[|\Phi(\omega_j+\omega_k)|^2]}, \quad (2.7) \]

is bounded by zero and one, and is a measure of the fraction of power at a given frequency that is due to quadratic coupling interactions. The applications of Eqs. (2.6) and (2.7) to the study of non-linear coupling and turbulence is summarized below.

C. Applications

Several quantities related to the bispectrum provide information about the wave-wave interaction. In section IV.A, each of the quantities described in this section are derived from computer generated test signals.

1. Identity of the parent waves

The amplitude of the bispectrum provides a qualitative measure of the degree of coupling between three waves at \( \omega_j, \omega_k, \) and \( \omega_j+\omega_k, \) and can sometimes indicate the parent waves associated with a given interaction. A non-zero value of \( B(\omega_j, \omega_k) \) indicates a quadratic coupling of two modes, but the parent modes are not necessarily \( \omega_j \) and \( \omega_k. \) Since quadratic coupling produces both a sum and a difference frequency, other candidate pairs (\( \omega_j \) with \( \omega_j+\omega_k \) and \( \omega_k \) with \( \omega_j+\omega_k \)) can be investigated by looking for bispectrum peaks at the corresponding sum and difference frequencies. The generally accepted convention of plotting the bispectrum over a triangular region where \( \omega_j \geq 0, \omega_j \geq \omega_k, \) and
\(\omega_j \leq \omega_{k(max)} - \omega_k\), exploits certain symmetry properties discussed later in section II.D.2. In this condensed representation, any observed bispectral feature represents a three-wave interaction between the modes on the axes, say \(\omega_j\) and \(\omega_k\), and a third mode at the sum, \(\omega_j + \omega_k\). The three-wave interaction involving a daughter at the difference frequency would in this case be represented by a bispectral feature at \((\omega_k, \omega_j - \omega_k)\) where again, the sum of the mode locations gives the third wave involved in the interaction, i.e., \(\omega_j\) (see first row of Table 1). To summarize the possible combinations:

<table>
<thead>
<tr>
<th>For parent waves</th>
<th>The sum and difference interactions are represented by features at</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_j, \omega_k)</td>
<td>(\omega_j, \omega_k) (\omega_k, \omega_j - \omega_k)</td>
</tr>
<tr>
<td>(\omega_j + \omega_k) (\omega_k)</td>
<td>(\omega_j + \omega_k, \omega_k) (\omega_j, \omega_k)</td>
</tr>
<tr>
<td>(\omega_j + \omega_k) (\omega_j)</td>
<td>(\omega_j + \omega_k, \omega_j) (\omega_j, \omega_k)</td>
</tr>
</tbody>
</table>

**Table 1: Bispectral features of a quadratic coupling interaction.**

Or, rearranging for a given bispectral feature at \((\omega_j, \omega_k)\),

<table>
<thead>
<tr>
<th>If there is also a feature at:</th>
<th>The parents are:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega_k, \omega_j - \omega_k) ((\text{if } \omega_j &lt; 2\omega_k))</td>
<td>(\omega_j, \omega_k)</td>
</tr>
<tr>
<td>(\omega_j - \omega_k, \omega_k) ((\text{if } \omega_j &gt; 2\omega_k))</td>
<td>(\omega_j + \omega_k, \omega_k)</td>
</tr>
<tr>
<td>(\omega_j + \omega_k, \omega_k)</td>
<td>(\omega_j + \omega_k, \omega_k)</td>
</tr>
<tr>
<td>(\omega_j + \omega_k, \omega_j)</td>
<td>(\omega_j + \omega_k, \omega_j)</td>
</tr>
</tbody>
</table>

**Table 2: Aid in causality determination.**

So, for a given spectral feature in a Fast Fourier Transform (FFT), it should be possible to determine if it either corresponds to an independent mode, or is the result of nonlinear coupling between two waves. In practice, the identification of coupled modes is complicated by the broadness of spectral peaks typical of a physical process, and the occasional absence of one of the daughters (there is no guarantee that both daughters are resonant, or natural, modes of the system), but it is often possible to identify the parents of the interaction from the bispectrum with the help of Table 2. See, for example, the bispectrum of a magneto-hydrodynamic (MHD) and radio-frequency wave-wave...
interaction described by Intrator et al.\textsuperscript{22} Even with no knowledge of the system, the peaks at $(\omega_f, \omega_{\text{MHD}})$ and $(\omega_f - \omega_{\text{MHD}}, \omega_{\text{MHD}})$ confirm parents at $\omega_f$ and $\omega_{\text{MHD}}$. The drift wave turbulence example of Ref. 18 is another example. The concept of identifying parent modes will also be illustrated with a computer generated test signal in section IV.A.

2. Skewness of the time series

For signals with zero mean, the third order moment (skewness) is given by summing the real part of the bispectrum over all frequencies.\textsuperscript{16}

$$E[\phi^3(x,t)] = \sum_{j,k} \text{Re}[B(\omega_j, \omega_k)]$$  \hspace{1cm} (2.8)

The sign of the skewness may be determined by the physical nature of the nonlinear coupling, as seen below. This equation provides a condition with which to verify the accuracy of the computed bispectrum. In practice, a correction factor must be applied to the right hand side of Eq. (2.8), since the function used as the window for the time series (see II.D.3) influences the value of the skewness.

3. Power transfer

The growth or damping of a wave at $\omega_m$ due to the nonlinear coupling of waves at $\omega_j$ and $\omega_k$ is determined by the sign of the real part of $B$ in Eq.(2.6) and the phase of the coupling coefficient. Two different coupling coefficients are discussed in this chapter. One coefficient has its source in the time and space domain coupling equations of Hasagawa and Mima\textsuperscript{23} and Martin and Fried,\textsuperscript{24} and the other coefficient has its source in the frequency domain coupling equation of Kim and Powers.\textsuperscript{17}

For the case of a single three-wave interaction (in one dimension), the Martin and
Fried mode-coupling equation reduces to

\[ \frac{\partial \hat{\phi}(\omega_m)}{\partial x} = V_{jk} \hat{\phi}(\omega_j) \hat{\phi}(\omega_k) e^{i\Delta k x}. \]  

(2.9)

Here, \( V_{jk} \) is the coupling coefficient, \( \omega_m = \omega_j + \omega_k \), and \( \Delta k = k(\omega_j) + k(\omega_k) - k(\omega_m) \), where \( \Delta k \) is the possible mismatch in the wavenumber selection rule. Similarly, from Eq. (2.2),

\[ \frac{\partial \Phi(\omega_m)}{\partial x} - ik(\omega_m) \Phi(\omega_m) = V_{jk} \Phi(\omega_j) \Phi(\omega_k). \]  

(2.10)

Multiplying both sides of Eq.(2.10) with \( \Phi^*(\omega_j + \omega_k) \), adding it to its complex conjugate equation, and taking an ensemble average (i.e. applying the operator \( \mathbb{E} \)) gives

\[ 2 \left( \frac{\partial \Phi(\omega_m)}{\partial x} \right) = V_{jk} B(\omega_j, \omega_k) + V_{jk}^* B^*(\omega_j, \omega_k), \]  

(2.11)

an expression found, with the factor of two missing, in Ref. 17. When \( V_{jk} \) is real, Eq. (2.11) can be written

\[ \frac{\partial \Phi(\omega_m)}{\partial x} = V_{jk} \text{Re}[B(\omega_j, \omega_k)]. \]  

(2.12)

The expression on the left hand side of Eq.(2.12) is sometimes called the bispectral power transfer function, and can be interpreted as the addition of power into the spectrum at \( \omega_m \) due to the coupling of waves at \( \omega_j \) and \( \omega_k \). In general, the oscillations at a given frequency consist of phase-coherent daughter waves and some incoherent or self-excited modes. A positive value of the transfer function indicates that the portion of power due to nonlinear interactions at \( \omega_m \) is increasing in the direction of propagation. For example, if the coupling coefficient is known to be real and positive, and the wave vector of the daughter is aligned with the positive x axis, the sign of the real part of the bispectrum...
would determine if the amplitude of the daughter wave were increasing or decreasing as it traveled in the x direction. This only applies to the portion of oscillations at $\omega_m$ that are due to the interaction of waves at $\omega_j$ and $\omega_k$.

The phase of the bispectrum, or "biphase" $\beta(\omega_j, \omega_k)$, is equal to $\theta(\omega_j) + \theta(\omega_k) - \theta(\omega_j + \omega_k)$, so the sign of the real part of the bispectrum gives information about the phase relation of the daughter with respect to the beating of the two parent waves, even with no knowledge of coupling coefficients or wavenumbers. Intrator et al. 22 used the sign of the real part of the same bispectrum discussed in section II.C.1 to show that the daughters of the $(\omega_j, \omega_{\text{MHD}})$ interaction tend to cancel (negative Re$[B]$) or reinforce (positive Re$[B]$) ponderomotive forces induced by the radio frequency parent waves. If the sign of the real part of the bispectrum is positive (corresponding to a biphase of less than $\pm \pi/2$), the daughter wave will constructively interfere with the beating of the two parent waves.

4. Amplitude and phase of coupling coefficients

Substituting $\Phi(\omega_m) = |\Phi(\omega_m)| e^{i\phi}$ into Eq. (2.10) yields an expression for the amplitude of the Martin-Fried coupling coefficient.

$$\Phi(\omega_m) \left[ \frac{\partial \ln |\Phi(\omega_m)|}{\partial x} + i \left( \frac{\partial \theta(\omega_m)}{\partial x} - k(\omega_m) \right) \right] = V_{jk} \Phi(\omega_j) \Phi(\omega_k). \quad (2.13)$$

Setting amplitudes equal, multiplying both sides by $|\Phi^*(\omega_j)| |\Phi^*(\omega_k)|$ and taking an ensemble average gives,

$$|V_{jk}| = \frac{\left| A \right| \left| B^*(\omega_j, \omega_k) \right|}{E \left[ |\Phi(\omega_j) \Phi(\omega_{\alpha})|^2 \right]^2}. \quad (2.14)$$
where $|A|$ is the magnitude of the bracketed term in Eq.(2.13). The phase angle of the same term is

$$\alpha(\omega) = \tan^{-1} \left[ \frac{\partial \theta(\omega) - k(\omega)}{\partial \ln |\Phi|} \right] \approx \tan^{-1} \left[ \frac{-k(\omega)}{\kappa(\omega)} \right]. \tag{2.15}$$

where $\kappa = \partial \ln |\Phi| / \partial x$ can be interpreted as the inverse scale length of the Fourier amplitude variation in space. Setting phases equal in Eq.(2.13) gives the phase of the Martin-Fried coupling coefficient $\gamma$, in terms of quantities obtainable from the bispectrum and the two point statistical wavenumber algorithm of Beall et al.\textsuperscript{5}

$$\gamma(\omega_j, \omega_k) = \alpha(\omega_j + \omega_k) - \beta(\omega_j, \omega_k). \tag{2.16}$$

This result is verified in section V for the trivial case of sinusoidal waves with real coupling coefficients, and for the more complicated wave packet used in the Monte Carlo simulation. Note for later discussion, that the first term in the numerator of Eq.(2.15) is neglected by assumption in Beall’s method for determining the local wavenumber spectrum, and $\kappa$ is estimated by $(|\Phi_2| - |\Phi_1|)/(\Phi_{AVE}\Delta x)$.

Kim and Powers,\textsuperscript{17} using a simpler approach, assumed a quadratic nonlinearity of the form

$$\Phi(\omega_m) = \sum_{j+k=m} \Lambda_{jk} \Phi(\omega_j) \Phi(\omega_k) + \Phi'(\omega_m), \tag{2.17}$$

where $\Phi'(\omega_m)$ is a linear function independent of the product interaction. In the case where only three waves satisfy the resonance conditions for frequency and wave numbers, $\Lambda_{jk}$ is computed directly by multiplying each side of Eq.(2.17) by $\Phi^*(\omega_j)\Phi^*(\omega_k)$ and taking an expectation value. Recall that the average of $\Phi'(\omega_m)\Phi^*(\omega_j)\Phi^*(\omega_k)$ will vanish.
according to the previous bispectrum theory discussion. The coupling coefficient is then given by

\[ \Lambda_{jk} = \frac{B'(\omega_j, \omega_k)}{E[|\Phi(\omega_j)\Phi(\omega_k)|^2]} \]  

(2.18)

The coupling coefficients associated with the harmonics of ion acoustic waves in a RF glow discharge plasma have been measured using this technique and found to be in excellent agreement with theoretical predictions.\textsuperscript{17} Comparing Eqs. (2.14) and (2.18), the coupling coefficient, \( V_{jk} \) of Ref. 24 is seen to be related to the \( \Lambda_{jk} \) of Ref. 17 by

\[ V_{jk} = \Lambda_{jk} |A| e^{im} \]  

(2.19)

5. Fraction of power due to three-wave coupling

The magnitude of the bicoherence spectrum provides a quantitative measure of the degree of coupling between three waves at \( \omega_j, \omega_k, \) and \( \omega_j + \omega_k \), and can be used to measure the ratio of the spectral power of correlated daughter waves to the total power at a given frequency. For the single three-wave coupling case discussed previously in Eq.(2.17), the total power at \( \omega_m \) can be expressed as

\[ P(\omega_m) = E[|\Phi'(\omega_m)|^2] + |\Lambda_{jk}|^2 E[|\Phi(\omega_j)\Phi(\omega_k)|^2] \]  

(2.20)

As before, cross terms vanish since \( \Phi'(\omega_m) \) is not phase coherent with \( \Phi(\omega_j)\Phi(\omega_k) \).

Multiplying Eq. (2.17) by \( \Phi^*(\omega_j)\Phi^*(\omega_k) \), the statistical average of its magnitude squared gives

\[ b^2(\omega_j, \omega_k)P(\omega_m) = |\Lambda_{jk}|^2 E[|\Phi(\omega_j)\Phi(\omega_k)|^2] \]  

(2.21)

which is the power at \( \omega_m \) that is due to quadratic wave coupling. Therefore \( b^2 \) can be
interpreted as
\[ b^2(\omega_j, \omega_k) = \frac{P_{\text{coupled}}}{P_{\text{coupled}} + P_{\text{not coupled}}} \]  

(2.22)

It should be noted that the presence of many modes will also reduce the typical value of \( b^2 \), and herein lies the principal difficulty of using the bicoherence to analyze turbulence. Though a large value of \( b^2 \) at \((\omega_j, \omega_k)\) is indicative of coherent nonlinear coupling (which may lead to turbulence), a fully turbulent state would actually be characterized by small values for \( b^2 \). Tsui et al. use a nonlinear wave coupling equation similar to Eq.(2.9) to derive the relation
\[ \sum_{j,k} b^2(\omega_j, \omega_j) = 1 - \delta^2, \]  

(2.23)

where \( \delta \) is related to turbulent spectral broadening, and \( b \) is calculated over the reduced triangular region described in section II.D.2. Because of the unity upper limit on this sum, \( b^2 \) will closely approximate the fraction of power due to non-linear coupling only if there are few modes involved in the interaction and as long as the spectral peaks are not too broad. In practice, a more meaningful value of bicoherence is obtained by summing over all \( j \) and \( k \) that satisfy the resonant conditions (of frequency and wavenumber) for a given \( \omega_m \) \((=\omega_j + \omega_k)\). A value near unity for that sum would indicate coherent wave coupling, and anything less would imply the presence of turbulent frequency broadening. The application of the bicoherence spectrum is best illustrated by an example with a generated test signal, such as the one given by Eq. (3.2).

**D. Computerization of the method**

It is difficult to imagine time series analysis without computers and computer
programming. Bispectral analysis requires the successive calculation of a sufficient number of realizations to achieve an acceptable level of statistical uncertainty, and the careful selection of certain parameters associated with discrete sampling of the signal. Priestley’s book Spectral Analysis and Time Series is a good reference, and provides the basis for many of the concepts discussed in this section.

1. Equations

The expectation operator $E$ is defined for functions of a discrete (random) variable by

$$E[\Phi(\Omega)] = \sum_{i=1}^{M} \Phi(\Omega_i)p[\Omega = \Omega_i], \quad (2.24)$$

where the probability, $p$, is $1/M$ for this process (one value of $\omega_j$ per realization, for $M$ realizations). Following Priestley’s notation, the lower-case Greek letter $\omega_j$ is used to denote one of the possible values of frequency set $\Omega$. Eqs. (2.6) and (2.7), can then be rewritten as

$$B(\omega_j, \omega_k) = \frac{1}{M} \sum_{i=1}^{M} \Phi^{(i)}(\omega_j)\Phi^{(i)}(\omega_k)\Phi^{*(i)}(\omega_j + \omega_k), \quad (2.25)$$

where the superscript labels the $i$th record or realization, and

$$b^2(\omega_j, \omega_k) = \frac{\left| \frac{1}{M} \sum_{i=1}^{M} \Phi^{(i)}(\omega_j)\Phi^{(i)}(\omega_k)\Phi^{*(i)}(\omega_j + \omega_k) \right|^2}{\left[ \frac{1}{M} \sum_{i=1}^{M} \left| \Phi^{(i)}(\omega_j)\Phi^{(i)}(\omega_k) \right|^2 \right] \left[ \frac{1}{M} \sum_{i=1}^{M} \left| \Phi^{(i)}(\omega_j + \omega_k) \right|^2 \right] + \epsilon} \quad (2.26)$$

These are equivalent to the equations published by Kim and Powers except for the $\epsilon$ used.
for division-by-zero protection. Each $\Phi(\omega_j)$ is computed using the discrete form of Eq. (2.2), i.e.

$$
\Phi(\omega_j) = \Phi(j\Delta\omega) = \frac{1}{N/2} \sum_{n=1}^{N/2} \phi(x, n\Delta t) e^{i(j\Delta\omega)(n\Delta t)}. \tag{2.27}
$$

where $\Delta\omega$ is the elementary bandwidth, $\Delta t$ is the sampling interval, and $N$ is the number of data points per realization.

2. Symmetries

![Diagram of bispectrum calculation region and lines representing all possible interactions with one of the triplets in $\omega_j + \omega_k = \omega_m$. (b)](image)

Referring to Eq. (2.27), there exists a maximum measurable frequency, $j_{\text{max}}\Delta\omega$, where $j_{\text{max}}$ is $N/2$ for reasons discussed later. The bispectrum, since it involves Fourier components at $j\Delta\omega$, $k\Delta\omega$, and $j\Delta\omega + k\Delta\omega$, is defined only in the interval $j\leq N/2$, $k\leq N/2$, and $j+k\leq N/2$, which define the boundaries of the hexagon of Fig. 1. Using the symmetry relations $B(\omega_j, \omega_k) = B(\omega_k, \omega_j) = B^*(-\omega_j, \omega_k)$, Kim and Powers show that it is sufficient to

‡ Alternatively, one could avoid division by zero (actually 0/0) by calculating only those terms with a sufficiently large unnormalized value of $B$ (and setting the rest equal to zero). This possibility was explored with small data sets and yielded encouraging results.
calculate the bispectrum over regions A and B of the hexagon. Changing variables \((j'\rightarrow-k,\) and \(k'\rightarrow j+k)\) and using the relations \(B(\omega_j,\omega_k) = B^*(-\omega_k,\omega_j+\omega_k) = B^*(-\omega_j, \omega_j+\omega_k),\) eliminates all but the triangular region A, which is \(1/12\)-th of the total computation region. The ASYST program of Appendix A was written to exploit these symmetry relations, and resulted in a substantial reduction in memory requirements and computation time. This also simplifies interpretation, since redundant information is removed. Some authors prefer to compute the bispectrum over areas A and B in Fig. 1a. In this representation, region A contains the sum interactions and B contains the difference interactions, the advantage being that all the interactions involving a given frequency are represented on a single line \((\omega_j+\omega_k=\omega_m=\text{constant})\). In the compact representation, these interactions are represented on three separate lines: \(\omega_j+\omega_k=\omega_m, \omega_j=\omega_m, \text{ and } \omega_k=\omega_m\) (see Fig. 1b).

### 3. Windowing

The limiting form of the Fourier Transform for non-periodic functions (Eq. 2.2) is only valid if the integrand tends to zero at the limits of integration.\(^{27}\) For finite record lengths, it is therefore necessary to multiply the time series by a function that vanishes at the beginning and end of the record. Many of these functions, called “windows,” have been suggested, and I have chosen the Tukey-Hanning window for its effectiveness at reducing spurious side-lobes in the spectrum due to finite length record lengths (and for computational convenience). Of the eleven functions surveyed by Priestley in Ref.28, the Tukey-Hanning window has the lowest “relative mean square error” and highest efficiency (based on leakage and variance). Every record of length \(T\) is multiplied by the windowing function, given by
\[ \Gamma(t) = \frac{1}{2} \left[ 1 + \cos \left( \frac{t - T/2}{T/2\pi} \right) \right], \quad \text{for } 0 \leq t \leq T, \]
\[ \Gamma(t) = 0, \quad \text{for } t > T \text{ and } t < 0, \]  

before any Fourier transforms are computed.

4. Selection of \( \Delta t, \Delta x, N, M \)

A minimum of two points per cycle is required to determine the frequency of an oscillation. The highest distinguishable frequency, termed the Nyquist frequency, is then given by \( f_N = \frac{f_s}{2} \), where \( f_s \) is the sampling frequency. The highest frequency of interest determines the minimum acceptable sampling rate or Nyquist frequency, and choosing a frequency much above this lower limit involves compromises in computation time, memory, or frequency resolution. Just as \( \Delta t \) must be small enough so that any frequency of interest is less than \( f_N (= \frac{1}{2}\Delta t) \) to avoid aliasing, the spatial separation \( \Delta x \) between simultaneous measurements must be small enough to ensure that any wavelength of interest is less than \( \frac{1}{2}\Delta x \). This is a concern for the wavenumber spectrum calculation discussed in section V.A.3. The sampling period \( N\Delta t \) and frequency resolution \( \Delta f \) are related by \( \Delta f = \frac{1}{N\Delta t} \). Due to the limitations of the Fast Fourier Transform (FFT) algorithm, \( N \) must be a power of two. In practice, \( N \) is limited to 512, since the next power of two requires \((1024)^2 \) complex numbers \( \times 32 \) bytes/complex number = 33.6MB of disk space for a single bispectrum array, which overwhelms available graphics and data analysis software. The number of realizations \( M \) is selected to give a sufficiently small variance in the bicoherence spectrum. Kim & Powers\(^{17} \) showed that the variance in \( b \) is less than \( 1/M \) for all combinations of frequency. Typical values of \( M \) range from 16
(Ref. 16) to 240 (Ref. 19). Many data acquisition systems (including the digitizer we used) impose additional restrictions on sampling frequency, so one must optimize between available sampling intervals, memory requirements, and frequency resolution.
5. ASYST program

ASYST (a pseudo acronym for “A Scientific System”) is a programming language developed specifically to facilitate data acquisition, instrument control, analysis and graphing. As such, it includes efficient key words (subroutines) for numerous mathematical and statistical functions such as the Fast Fourier Transform. Appendix A contains the ASYST source code for the calculation of bicoherence spectra of variable-length time series records. The length and apparent complexity of the program belies its underlying simplicity -- the calculation itself involves perhaps a dozen lines of code and can be summarized as follows:

1. Retrieve one record of a stationary time series.
2. Subtract the mean value.
3. Apply a Hanning window and compute the FFT.
4. For the reduced triangular region described in section II.D.2, compute the triple product inside the estimator of Eq. 2.6, and the normalization factors in the denominator of Eq. 2.7 (EST1 and EST2).
5. Repeat steps one through four for M realizations and calculate the expectation values.
6. Divide the modulus squared bispectrum by the normalization factors to get the bicoherence spectrum.

The remainder of the program serves mostly to bypass the 640K conventional memory
limitation of the personal computer by writing successive cross-sections of the array to a
"virtual" Random Access Memory drive. The resulting 131,000 data point arrays are then
converted to ASCII for export to a graphics/data analysis program. Note that the spectra
of smaller record lengths (up to 128 points) can easily be accommodated (and graphed) by
ASYST, so a much simpler version was written to quickly evaluate smaller length,
computer generated test signals.

6. Interpretation pitfalls

The bispectrum must be interpreted with caution, since there are instances where
phase coherence between waves will not be indicated by a bispectral feature, and others
where the bispectrum may give a false indication of coupling. The first situation occurs
when the phase difference between the daughter and the beating of the parent waves
averages to ±π/2. In this case, only the imaginary part of the bispectrum (which is not
typically considered) will be non-zero. Or, if the dispersion relation prohibits three-wave
coupling, higher order wave interactions may become more important, requiring a
"trispectral" analysis. The second situation occurs much more frequently, and can be
attributed to a variety of mechanisms. As previously mentioned, both the frequency and
wavenumber selection criteria must be satisfied for true quadratic coupling. The
bispectrum will still be non-zero for phase coherent waves even when km ≠ kj + kk, provided the frequencies sum to zero. Conversely, because of the limited spectral
resolution associated with finite record lengths, it is possible to obtain a significant peak in
the bispectrum even when the frequency selection rule is not exactly satisfied.

Filtering effects also need to be considered. When filters are used to eliminate
aliased Fourier spectral components or amplifier noise, they introduce frequency-dependent phase shifts. Since the bispectrum will be non-zero for any constant phase relation between three waves, the magnitude of $B$ is expected to remain unaffected by filtering, but not necessarily the biphase and the other phase-dependent quantities of section II.C.3 and II.C.4.
III. Experimental method

A significant portion of this chapter is devoted to the generation and digital processing of test signals. To maximize the effectiveness of the bispectrum, it must be used in conjunction with a wavenumber-frequency spectrum. In the WVU-Q machine, obtaining a measurement simultaneously at more than several spatially separated points is not practical, so a method for obtaining a local statistical wavenumber-frequency spectrum with a single probe pair is presented here, and tested with a Monte Carlo signal designed to simulate plasma turbulence. Though no wavenumber analysis suitable for this wavenumber-frequency spectrum was conducted on the plasma process described at the end of this chapter, the examples in this section should facilitate future experimental bispectral interpretations -- particularly those involving the more challenging power transport and coupling relations of section II.C.

A. Generated test signals

1. Sinusoidal waves

The first three test signals analyzed are simple cosine functions suggested by Kim and Powers. The inclusion of these test signals in this work is motivated by a desire to check the validity of our code against previously established results and to demonstrate the method of determining causality (see Table 2), the phase of the coupling coefficient (trivial case of Eq. 2.16), and the fraction of power at a certain frequency due to non-linear coupling (Eq. 2.22). As an additional check, the sum of the real part of the computed
The bispectrum was compared to the mean cube value (Eq. 2.8) in all three cases. The Kim and Powers procedure was duplicated explicitly: 64 records of 128-point time series were generated with small amplitude noise added to each record (η in Eqs. 3.1 and 3.2). Since only the frequency ratios were given (normalized with the Nyquist frequency), a record length of one second was chosen arbitrarily, and the sampling rate was determined by the method of the previous section. Thus, in the test signal,

$$\phi(t) = \cos(\omega_b t + \theta_b) + \cos(\omega_c t + \theta_c) + \cos(\omega_d t + \theta_d) + \eta(t), \quad (3.1)$$

the Nyquist frequency was 64 Hz with a sampling interval of (1/128) seconds. Here

$$\omega_b = 2\pi(14.08) \text{ s}^{-1}, \quad \omega_c = 2\pi(24.00) \text{ s}^{-1}, \quad \text{and} \quad \omega_d = 2\pi(38.08) \text{ s}^{-1} = \omega_b + \omega_c,$$

satisfy the frequency ratios $$f_b/f_N = 0.220, \quad f_c/f_N = 0.375,$$ and $$f_d = f_b + f_c.$$ The initial phases $$\theta_b, \theta_c$$ and $$\theta_d$$ were randomly distributed between $$\pm \pi$$ for each of 64 realizations except in those signals used to show the effects of phase coherence, where $$\theta_d$$ was determined by the sum of $$\theta_b$$ and $$\theta_c.$$

The next signal was designed by Kim and Powers specifically to demonstrate the use of Eq. (2.22).

$$\phi(t) = \cos(\omega_b t + \theta_b) + \cos(\omega_c t + \theta_c) + \frac{5}{2} \cos(\omega_d t + \theta_d) + \cos(\omega_b t + \theta_b) \cos(\omega_c t + \theta_c) + \eta(t) \quad (3.2)$$

The same frequency ratios, sampling rates, etc. of Eq. (3.1) apply here, and the sum of the random phases of the first two waves has no relation to the phase of the third ($$\theta_d$$ is an independent random number). The fourth term produces the daughter modes from quadratic coupling (at the sum and difference frequencies), whereas the third represents an

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8 We used a -20dB (one tenth of the maximum signal amplitude) uniform distribution of white noise, as opposed to the Gaussian noise of the Kim & Powers example for computational simplicity.
independent mode at the same frequency as one of the daughters.

Since the phases in the Kim and Powers examples evolve randomly with time, there is no way to recover information about the signal's evolution in space, i.e., there is no information in Eqs. (3.1) or (3.2) related to wave numbers, or spatial coordinates. Some method of signal generation is needed that preserves the temporal nature of nonlinear coupling in plasmas, while still containing spatial information so that a determination can be made of the quantities described in section II.C.3 and II.C.4. The so-called Monte Carlo method, used to simulate turbulence in plasmas since the 1970's, is well suited to this application.

2. Monte Carlo method

A "Monte Carlo" method uses a probabilistic model of a system to estimate quantities of interest. An example commonly used to introduce this method is that of determining the area enclosed by an irregularly shaped boundary, but the principle has been applied to quite complex problems, including fluid (and magnetofluid) turbulence. Some of the statistical turbulence theories discussed in Ref. 29 model such fluctuations with a series of superposed wave packets. Pécseli and Trulsen used a Monte Carlo simulation based on a superposition of pulses to demonstrate, among other things, interpretation pitfalls of the bispectrum. This idea inspired the following model, based on a different pulse shape described by

$$\psi_r(t, x) = \exp \left\{ -\frac{1}{\tau^2} \left( \frac{k_x + \delta_x}{\omega_r} - t \right)^2 \right\} \cos(k_x - \omega_r t + \delta_x),$$
where the δ's are randomly distributed phases. The signal for each realization is constructed using a random superposition of these pulse shapes such that

\[ \phi(x,t) = \sum_{r=1}^{R} \sum_{s=1}^{S} \psi_s(x - x_s, t - t_s) \]  

(3.3)

where \( R \) is the number of different pulse types (with different frequency and wavenumber characteristics) and \( S \) is the number of shots of each type. The number of points in each realization is more than one order of magnitude larger than \( R \), following Pecseli's example, and \( \tau \) was adjusted to give a sufficiently stationary power spectrum. If \( \tau \) is too small, the frequency spectrum becomes dominated by the frequencies corresponding to the envelopes of the pulses, rather than the oscillations within each pulse, and changes too quickly to satisfy the conditions required for local wavenumber determination (refer to the discussion related to Eq. 2.15). The frequency ratios of Eq.(3.1) and Eq.(3.2) were preserved for this test, as well as the sampling rate and record lengths.

3. Wavenumber spectrum

One of the principle motivations of using the Monte Carlo model is to assess the possibility of retrieving wavenumber information using the fixed-probe-pair technique described in Ref. 5. The frequency spectrum of a “time series” measured at a single spatial point is determined from Eq. (2.27). The corresponding transformation for the wavenumber spectrum would require the simultaneous measurement of a signal at \( N \) points separated by a small distance \( \Delta x \) -- a “space series.” Since it is impractical to
achieve levels of resolution in wavenumber spectra that compare with the resolution in frequency spectra, Beall et al.\textsuperscript{5} introduced the concept of a local wavenumber and frequency spectrum, $S_{\text{local}}(k, \omega)$, which can be estimated using a single probe pair. They conclude that under certain conditions (one of which is the requirement that the Fourier amplitudes don’t vary significantly within a sampling interval, $\Delta t$), the statistical average of $M$ realizations of the local wavenumber may be a good approximation of the conventional spectral density.\textsuperscript{5} The local wavenumber, $K$, between two fixed probes is approximately

$$K(x, \omega) \approx \frac{\theta(x_2, \omega) - \theta(x_1, \omega)}{\Delta x}. \quad (3.4)$$

This is another way of expressing the phase of the cross-correlation function $\Phi^*(x_1, \omega)\Phi(x_2, \omega)$ divided by $\Delta x$, and requires that the phases not vary significantly within the space of $\Delta x$. In a turbulent system, there may not exist a deterministic dispersion relation, so the power in a given frequency band $\Delta \omega$ may be broadly distributed in $k$ space. Assadi\textsuperscript{32} succeeded in using Beall’s weighted-average “statistical dispersion relation” to estimate the wavenumber spectrum in a turbulent plasma, and provided some useful information on computerizing the method. The weighting coefficients, referred to as the “local conditional spectrum estimates”, $S_{\text{local}}(K|\omega)$, are simply the fraction of power at a given frequency with wavenumber $K$. To add a final bit of notational complexity, the $i^{th}$ sample spectral density function given by $\Phi^{(i)}(x, \omega_i)\Phi^{(i)}(x, \omega_i)$ is labeled $S^{(i)}(\omega_i)$ to distinguish it from the power spectrum, which is an ensemble average of this quantity (see Eq. 2.5). Beall’s digital estimation technique can then be summarized as follows:
1. Determine the Fourier components $\Phi_1(\omega)$ and $\Phi_2(\omega)$ of two time series measured simultaneously at locations separated by $\Delta x$.

2. Calculate the local wavenumber spectrum from Eq. (3.4).

3. Find the average of the spectral densities at the two points.

$$\bar{S}(\omega) = \frac{S(\omega_1) + S(\omega_2)}{2}$$

4. For each wavenumber bin, $m$, multiply the $\bar{S}(\omega)$ by the wavenumber spectrum, provided the value of $K^0$ falls within that wavenumber bin ($m\Delta K$), then repeat steps one through four for each realization and take an ensemble average to get $S_{local}(K, \omega)$. In the concise language of mathematics this reads;

$$S_{local}(m\Delta K, j\Delta \omega) = \frac{1}{M} \sum_{m=1}^{M} \bar{S}^{(m)}(j\Delta \omega) I_{m\Delta K}[K^{(m)}(j\Delta \omega)]$$

(3.5)

where the indicator function, $I_{m\Delta K}$, equals one if its argument falls within the $m$th wavenumber bin, and is zero otherwise. $\Delta K$ is given by $2\pi/N\Delta x$, and the wavenumbers to be estimated must be less than $\pm \pi/\Delta x$, as previously discussed.

5. Divide $S_{local}$ by the ensemble average of $\bar{S}$ to find the weighting coefficients, or the local conditional spectral estimates, $s_{local}(K|\omega)$.

6. Finally, compute the statistical dispersion relation by taking the weighted average of each wavenumber bin.

$$\hat{K}(\omega_j) = \sum_{m} m\Delta K \times s_{local}(m\Delta k|\omega)$$

(3.6)
For the test signals described previously, Δx was chosen as 1 cm, so the maximum measurable k is π cm⁻¹ and the wavenumbers of each generated pulse were $k_b = -1.3$ cm⁻¹, $k_c = 0.5$ cm⁻¹, and $k_d = 1.8$ cm⁻¹.

B. Unijunction Transistor circuit

![Circuit diagram for the UJT relaxation oscillator. L=19.5 mH, C=0.03 mF, $V_S = 11.2$, and $V_a = 12.8$ VDC.]

For a first attempt at bispectral analysis of a real system, we chose a driven unijunction transistor (UJT) oscillator circuit that displays the nonlinear phenomenon of periodic pulling. The system is well modeled by the forced van der Pol equation. The strength of the nonlinear interaction between the self-oscillating circuit and the driving force can be continuously varied from weak, similar in appearance to conventional amplitude modulation, to strong, with pulse-like amplitude modulation and frequency modulation. When the driving force amplitude becomes sufficiently large and the driving frequency gets close enough to the self-oscillating frequency, the system becomes entrained. The experimental setup is similar to the one used by Koepke and Hartley, but due to individual differences in UJT’s, the parameters differed slightly from Ref. 6. Here $V_S = 11.2$ Vdc, $V_r = 12.8$ Vdc, $L = 19.5$ mH, $C = 0.03$ mF. In order to modify the dynamical state of the system, the driving frequency $f_0 = \omega_0 / 2\pi$ was varied between 5 kHz and 9 kHz while keeping the amplitude of the applied driving force constant at 3.12 Vpp. The ratio of the applied driving amplitude $V_D$ (on the primary side of the transformer), to the driving amplitude in the RLC part of the circuit (on the secondary side) is
approximately $10^3$. Fourteen sets of 100 realizations were recorded with the LeCroy 6810 digitizer for several different strengths of periodic pulling. Each realization contained 1024 points (yielding 512 point FFT's), sampled at intervals of 20 µsec. The fluctuations were monitored on an oscilloscope with FFT processing capability and no significant spectral features above the Nyquist frequency of 25 kHz were observed. A microcomputer transferred the 1024 samples to the hard drive as the digitizer's buffers became full. The various bispectral quantities were computed following the procedure outlined in Chapter III.

C. IEDD and CDEIC plasma instabilities

As an application of bispectral techniques to actual plasma fluctuations, we analyzed data from the Ph.D. dissertation of W. Amatucci, and published in recent papers. Two categories of waveforms were used. One category is associated with the Inhomogeneous Energy-Density Driven (IEDD) instability and the other is associated with the Current-Driven Electrostatic Ion-Cyclotron (CDEIC) instability. The setup for this experiment is documented in Refs. 7 and 8. The sampling frequency for each 1024-point realization was one mega-sample per second, and a band-pass filter (10 kHz to 240 kHz) was used to suppress drift waves at 3 kHz and eliminate interference at 240 kHz. The type of instability is selectable by adjusting the bias voltage on two concentric rings of a segmented disk electrode ($V_s$ and $V_b$). The data corresponding to IEDD fluctuations and CDEIC fluctuations were taken from a large data set which included combinations of $V_s$ and $V_b$ over a wide range. For each combination, eight realizations were collected. A small fraction of this data set was chosen to arrive at an 88k data subset identified with
IEDD waves. A slightly larger, but different portion of this data set was chosen to arrive at a 112k data subset identified with CDEIC waves. Each subset was made small enough to ensure that the variations of $V_a$ and $V_b$ associated with the different combinations did not noticeably effect the spectral characteristics of the mode. Only a small time interval (fractions of a second) elapsed between each group of eight realizations while bias voltages were incremented.
IV. Results and Discussion

The bispectrum algorithm was applied to three different situations and interpreted in light of the relations of section II.2 (applications).

A. Generated test signals

The bispectrum measures the extent of joint statistical phase coherence between three waves satisfying $\omega_m = \omega_j + \omega_k$. A constant phase relation between these waves will give a non-vanishing bispectrum. Figures 4 and 5 demonstrate the bispectrum’s sensitivity to phase coherence. In Figs. 4a and 5a are the power spectra of test signals consisting of three cosine functions with angular frequencies $\omega_a = 2\pi \cdot 14s^{-1}$, $\omega_b = 2\pi \cdot 24s^{-1}$, and $\omega_c = 2\pi \cdot 38s^{-1}$. The initial phases of the oscillators at $\omega_a$, $\omega_b$, and $\omega_c$ are different for each realization (they are randomly distributed between $\pm \pi$), but both signals obey the frequency summation rule $\omega_c = \omega_a + \omega_b$, and their power spectra are indistinguishable. The only difference between the test signals used to generate Figs. 4 and 5 is the phase consistency, or coherence, of the third oscillator at $\omega_c$. In the test signal represented in Fig. 4, the phase relation of the three oscillators is constant for each realization, and the statistical average of 64 realizations results in the pronounced bicoherence feature at $(\omega_a, \omega_b)$ seen in Fig. 4b. Conversely, in the signal of Fig. 5, the phase of the third mode is allowed to vary randomly and independently of the parent waves, resulting in the very low value of bicoherence (less than 1/64).
Figs. 4c and 4d, illustrate the effect of the phase relation between the three oscillators on the real part of the bispectrum. The bicoherence spectra of Figs. 4a and 4b are not affected by the value of $\beta$ ($= \theta_b + \theta_c - \theta_d$), as long as it is the same for each realization, but the sign of the real part of the bispectrum depends on whether the daughter wave is interfering constructively or destructively with the beating of the parent waves (as discussed in section II.C.3). Note that the bispectrum peak is positive in Fig. 4c, corresponding to the reinforcing phase relation $|\theta_j + \theta_k - \theta_m| = 0$, and negative in Fig. 4d, where $\pi$ has been added the phase of the daughter mode.

The computer generated test signal used to arrive at the bicoherence spectrum of Fig. 6 includes a quadratically coupled term, $\cos(\omega_b t + \theta_b)\cos(\omega_c t + \theta_c)$, where $\omega_b + \omega_c = \omega_d$, and an independent term, $0.5\cos(\omega_d t + \theta_d)$, which both contribute to the power of the fluctuations at $\omega_d$. The product term can be rewritten as

$$\frac{1}{2}\cos[(\omega_b + \omega_c)t + (\theta_b + \theta_c)] + \frac{1}{2}\cos[(\omega_c - \omega_b)t + (\theta_c - \theta_b)],$$

(4.1)

so one half of the power at $\omega_d$ is due to the product interaction term, and the other half is from an independent mode term. We could not unambiguously make this determination from the power spectrum, shown in Fig. 6a, since it contains no phase information. In
contrast, the bicoherence spectrum of Figs. 6b and 6c can be used to deduce whether or not a given spectral feature is the result of quadratic coupling, and to measure the fraction of power at a given frequency that is due to such coupling. Here, $b^2(\omega_c, \omega_b) \approx .5$ and $b^2(\omega_b, \omega_c-\omega_b) \approx 1.0$, confirming that only half the power at $\omega_d$ and all the power at the difference frequency, $\omega_d = \omega_c - \omega_b$, is due to the nonlinear interaction of waves at $\omega_b$ and $\omega_c$ (see Eq. 2.22). The amplitude of the coupling coefficient is calculated using the real part of the bispectrum and Eq. (2.18), and also agrees well with values used in the generated signal (see Table 3). Fig. 6c is the contour plot of the same bicoherence spectrum shown in Fig. 6b, and allows one to more easily locate the peaks in the $(\omega_j, \omega_k)$ plane. By referring to Table 2 on page 8 and Fig. 6c, one can correctly answer the question, “Which two of the three waves involved in a given non-linear interaction are the parent modes?” In this simple example, for the bispectral feature at $(\omega_b, \omega_a)$, the corresponding peak at $(\omega_b + \omega_a, \omega_b)$, and the third row of Table 2, point to parents at $\omega_c = \omega_b + \omega_a$ and $\omega_b$. Similarly, the corresponding feature of $(\omega_a, \omega_b)$ is $(\omega_b, \omega_c - \omega_b)$, on the first row of Table 2, which also indicates parents at $(\omega_a, \omega_b)$. These results are summarized in Table 3.

<table>
<thead>
<tr>
<th>$b^2$ feature</th>
<th>Parents</th>
<th>Daughter</th>
<th>Fraction of power at $\omega_{dipole}$ due to coupling</th>
<th>Coupling coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Calculated</td>
<td>Model</td>
</tr>
<tr>
<td>$\omega_c, \omega_b$</td>
<td>$\omega_c, \omega_b$</td>
<td>$\omega_d$</td>
<td>$0.49 \pm 0.01$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\omega_b, \omega_a$</td>
<td>$\omega_c, \omega_b$</td>
<td>$\omega_a$</td>
<td>$1.00 \pm 0.01$</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 3: Comparison of modeled and calculated parameters for quadratically coupled cosines

The differences can be attributed to the noise power and the normalized standard error of $1/M$. Unfortunately, it is not always possible to unambiguously determine causality in
this manner, since more than one row of Table 2 could apply in a more complicated bispectrum (i.e. a bispectrum containing more than two peaks).

As an additional quantitative check on the computed bispectrum, the sum of the real part of the bispectrum, which equals the volume under the surface plot of Re(B), was compared to the mean cube value of each test signal. Before processing, each realization was multiplied by the window function \( \Gamma \) of Eq. 2.28, so the bispectrum must first be corrected for the attenuation this windowing produces. The multiplicative factor applied to the bispectrum,

\[
\left( \frac{1}{N} \sum_{n=1}^{N} \Gamma^3(n\Delta t) \right)^{-1} = \left( \frac{40}{128} \right)^{-1},
\]

where \( N \) is the number of data points per realization, is analogous to Stoneking's correction to the power spectrum.\(^{35}\) The sum of Re(B) as computed over the reduced triangular region must additionally be multiplied by 12 before being compared to the mean cube value on the left hand side of Eq. (2.8) (c.f. section II.D on symmetries).

Shown in Table 4 are the averages of five trials conducted on each of the test signals, with the corresponding mean cube value.

<table>
<thead>
<tr>
<th>Test signal</th>
<th>( \sum \text{Re (B)} )</th>
<th>Mean cube</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase coherent cosines</td>
<td>1.478</td>
<td>1.478</td>
</tr>
<tr>
<td>Phase incoherent cosines</td>
<td>-0.087</td>
<td>-0.086</td>
</tr>
<tr>
<td>Quadratically coupled cosines</td>
<td>1.449</td>
<td>1.449</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the sum of the real part of the bispectrum to the mean cube value

Note that so far, all the results depend only on time series data collected at a single point in space. The power transfer and Martin-Fried coupling coefficient relations of sections

35
II.C.3 and II.C.4 remain to be tested, both of which require wavenumber information.

Such a test requires a signal that contains spatial quantities.

The code of Appendix B was used to recover the wavenumber-frequency spectrum of the Monte-Carlo signal

\[
\phi(t, x) = \sum_{s=1}^{N} \psi_b + \sum_{s=1}^{N} \psi_e + \frac{1}{2} \sum_{s=1}^{N} \psi_d + \sum_{s=1}^{N} \psi_b \sum_{s=1}^{N} \psi_e
\]

with pulse shape given by \(\psi_j(t, x) = \exp\left(-\frac{1}{\tau^2}\left(\frac{k_j x + \delta_x}{\omega_j} - t\right)^2\right) \cos(k_j x - \omega_j t + \delta_x)\)

where \((\omega_b, k_b) = (2\pi \cdot 14 s^{-1}, -1.3 cm^{-1}), (\omega_e, k_e) = (2\pi \cdot 24 s^{-1}, 0.5 cm^{-1}), (\omega_c, k_c) = (2\pi \cdot 38 s^{-1}, 1.8 cm^{-1})\). Fig. 7 shows the principal spectra involved in the wavenumber determination. The local wavenumber spectrum \(K(\omega)\) for a typical realization in Fig. 7a is essentially useless, by itself, in estimating the wavenumbers at the frequencies of interest. The local wavenumber and frequency spectral density plotted in Fig. 7b produces the local conditional wavenumber spectrum of Fig. 7c when normalized by the power at each frequency. Taking the weighted average of the wavenumber peaks along each line of constant \(\omega\) in \(s_{local}\) yields the statistical wavenumber spectrum shown in Fig. 7d. The local wavenumbers of frequencies that have little or no power in the frequency spectrum are attenuated by Beall’s algorithm (they appear as \(k=0\) in Fig. 7d). When the power at a certain frequency reaches the noise level, the wavenumbers are randomly distributed between \(\pm \pi\), and average to the center value (zero) for large \(M\). In the appendix B implementation of Beall’s algorithm, \(s_{local}\) is set to zero for all points where \(S(k, \omega)\) is below the level of statistical uncertainty (about 0.5% of the maximum \(P(\omega)\)).
amplitude), to eliminate spurious 0/0 peaks in the conditional wavenumber-frequency spectrum. Because of this, small, random fluctuations about k=0 are not observed in Fig. 7d. The squared bicoherence value at (ωn, ωk) was measured to be 0.57 for a time series collected halfway between the two probe tips at x1 and x2 (see top line of Fig. 8). Therefore, the weighted average of the independent mode’s wavenumber (1.8 cm⁻¹) and the wavenumber of the ωd daughter (-0.8 cm⁻¹) is 0.7 cm⁻¹, in agreement with the measured wavenumber indicated at 38 Hz in Fig. 7e. A more accurate determination, with better frequency and wavenumber resolution would require a modification of the code to overcome the conventional memory limitations of the operating system (as was done with the bispectrum program). The estimate is still sufficient for the purposes of this work, since it demonstrates the feasibility of wavenumber determinations with only two spatial point measurements for future experiments, and can be used to extract the spatially dependent quantities of II.C.3 and 4.

Recall from sections II.C.3 and II.C.4 that the power transfer function depends on the magnitude and phase of the Martin-Fried coupling coefficient and the bispectrum. If the coupling coefficient Vjk is not known in advance, it may be determined experimentally by measuring α with the wavenumber-frequency spectrum program, and subtracting the biphase at ωj, ωk (see Eqs. 2.15 and 2.16). In applying this concept, the first obstacle to overcome is the calculation of the modeled or “theoretical” value of γ. For the previous test signals, which can be considered waves with k=0 (and therefore α=0), the Fourier transform of cosωt is proportional to a delta sequence in ωT with no x dependence. The phases of the Fourier components are equal to the phases of the corresponding cosine
arguments, which are randomized in our example. The left-hand side of Eq. (2.10) is zero, and the phases of $\Phi(\omega_3)\Phi(\omega_k)$ on the right average to zero for large $M$. Equating phases gives $\gamma=0$ for these waves. As a trivial example of the application of Eq. (2.16), the average $\beta(\omega_c, \omega_b)$ of 8 bispectrum calculations yielded $-0.01 \pm 0.06$ radians $\approx \gamma$, confirming that the phase of the bispectrum is equal to the negative phase of the coupling coefficient when $\alpha = 0$.

As a more complicated example, each pulse in the Monte Carlo simulation transforms to

$$\Phi(\omega, x) = \frac{T \sqrt{\pi}}{2T} e^{-ik(\omega)x} \left(1 + \frac{\omega - \omega_0}{\omega} \right) e^{-\frac{x^2}{2}(\omega - \omega_0)^2},$$

(4.4)

as derived in Appendix D. Substituting this result into Eq. (2.10) gives

$$\gamma = -\bar{x} \left[ h(\omega_j) + h(\omega_k) \right]$$

(4.5)

where $\bar{x} = (x_1 + x_2)/2$ is the centered of the two probe tips. For this simulation, $x_1 = 0.375 \text{cm}$, $x_2 = 1.375 \text{cm}$, $k(14 \text{Hz}) = 0.5 \text{cm}^{-1}$, $k(24 \text{Hz}) = 1.35 \text{cm}^{-1}$, resulting in a phase of $-1.23 \text{rad}$ for $V_{jk}$. The program of Appendix C was used to measure statistical wavenumber and inverse scale length (for $N=1$, $\tau=1$), yielding the averaged $\alpha$'s and $\beta$'s of five trials given in Table 5.

<table>
<thead>
<tr>
<th>$\alpha(\omega_c+\omega_b)$</th>
<th>$\beta(\omega_c,\omega_b)$</th>
<th>$\alpha(\omega_c+\omega_b) - \beta(\omega_c,\omega_b)$</th>
<th>$\gamma_{\text{obs}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.18 rad</td>
<td>+1.12 rad</td>
<td>-1.30 rad</td>
<td>-1.23 rad</td>
</tr>
</tbody>
</table>

Table 5: Phase dependent quantities of the Monte Carlo simulation

When the number of structures was increased to eight or more per record, the variability in the calculated values of $\kappa$, $k$, and $\alpha$ was too large to make a meaningful measurement.
Even in the \( N=1 \) simulation, the standard deviation was quite large (15%).

Lastly, Eq. (2.11) in polar form (given with a factor of two on the right hand side in Ref. 16),

\[
\frac{\partial P}{\partial x} = \left| v_{jh} \right| |B(\omega_j, \omega_k)| \cos[\alpha(\omega_m)]
\]  

implies that the direction of power transfer is determined by \( \alpha \). Since \( |\alpha| \) is less than \( \pi/2 \) in the present example, the spectral power of the primary waves at \( \omega_0 \) and \( \omega_n \) is being transferred to \( \omega_d \) as it travels in space. From Fig. 7e, the wavenumber at \( \omega_d \) is positive, so as the daughter wave propagates in the +x direction, its amplitude is being increased by the coupling. In light of this new information, we compare the bicoherence cross-sections at 24 Hz in Fig. 8. The value \( b^2(24 \text{ Hz}, 14 \text{ Hz}) \), which is the fraction of the total power at 38 Hz that is due to the quadratic coupling of waves at 24 Hz and 14 Hz (c.f. Eq. 2.22), is increasing with \( x \).

\section*{B. Periodic pulling in a relaxation oscillator circuit}

Figures 9 and 10 are the power spectrum and bispectrum plots of UJT oscillations in two different cases, representative of weak and strong periodic pulling regimes. Figure 9 is the case of conventional-like amplitude modulation, where the driving frequency and
amplitude are too far from the entrainment boundary for \( f_0 \) to deviate significantly from \( f_0 \), the natural, undriven frequency of 7.14 kHz (c.f. Fig. 6, p. 44 of Ref. 6). Here \( f_0 = 5.9 \) kHz, \( f_{10} = 7.10 \) kHz (so the deviation is detectable), and the amplitude of the driver, \( V_D \), is 3.12 \( V_{\text{rms}} \) on the primary side of coil T1. The resulting driving amplitude on the secondary side is 3.1 m\( V_{\text{rms}} \), which is much smaller than the UJT’s 27 m\( V_{\text{rms}} \) oscillation amplitude.

Only the slightest asymmetry is noticeable in the sidebands of \( f_{10} \). The notation introduced here allows for a convenient way of tracking harmonics. In \( f_{jk} \), \( j \) represents the harmonic number (starting with 1 for the fundamental), the separator ‘s’ stands for “sideband,” and \( k \) is 0 for the natural frequency, positive for side-bands opposite the driving frequency, and negative for those on the same side as the driver (starting with -1 for the driver itself).

The side-bands are spaced at intervals equal to the modulation frequency, \( f_{10} - f_{1+1} \). The frequency selection rule \( f_{jk} + f_{mn} = f_{j+m,k+n} \) is evident in Fig. 9a. More than one combination of \( k \) and \( n \) may give rise to a given higher order harmonic, any of which would satisfy the frequency selection criteria necessary (but not sufficient) for a non-zero bispectrum. The four bicoherence peaks of Fig. 9b all have an amplitude of 1.0 (fully coherent), and are separated by the modulation frequency of 1.13 kHz.

The asymmetric frequency spectrum characteristic of strong periodic pulling is shown in Figure 10a. All experimental parameters are identical to the weak pulling case except for the driving frequency, which was raised to 6.57 kHz -- just outside of the entrainment regime. This bicoherence spectrum is not different in any meaningful way from the spectrum corresponding to weak coupling; the peaks all have unity magnitude, and are separated by the 200 Hz modulation frequency. More features are observed in
Fig. 10b, due to the increase in number and amplitude of harmonics present in the power spectrum. Conspicuous by its absence is the \((f_m, f_0)\) or \((f_0, f_0)\) bispectral feature that could explain the origin of the first-order side-bands. There are two contributing factors. One is the extremely low power of the modulation frequency. Any triple product of Fourier amplitudes involving this frequency will be several orders of magnitude smaller than the fully coherent peaks, and thus practically indistinguishable from background noise. Part of the reason for the low power at \(f_m\) is that it is not constant for each record. In the periodic pulling regime, the system response oscillates between \(f_0\) and \(f_0\), a fact that is hidden by the time-averaged FFT spectrum. Since every realization contains a different frequency and phase relation between the driving and natural frequencies, the statistical averaging of \(M\) records eliminates any indication of coupling. The combined wavelet and bispectral analysis of Ref. 37 is perhaps better suited to data records containing such pulses or short lived events, and could be a vehicle for a continued study of the subject.

In fact, both bicoherence plots are what would be expected from the Fourier decomposition of a simple (but non-sinusoidal) periodic pulse shape, and are not to be interpreted in terms of the three wave coupling theory of section II. As an example, Fig.

![Figure 11: (a) Power spectrum and (b) contour plot of \(b^2\) for sawtooth wave](image-url)
11a is the Fourier decomposition (power spectra) of a computer generated, slightly noisy, 8 Hz sawtooth wave. Every feature in the power spectrum is fully phase coherent with every other, as seen in the bicoherence plot of Fig. 11b. The similarity with Fig. 10b is unmistakable, and both call attention to some interesting general features of bispectral contour plots. First, any features off of the main diagonal ($\omega_j = \omega_k$) imply a phase coherence among different frequency components (i.e. not just harmonics, which will always be fully coherent). Second, any features on the same off-diagonal line delineated by constant values of $\omega_j + \omega_k$ represent, in the context of section II, different parents of the same daughter -- clearly not the simple three wave coupling considered in the discussion of bispectrum theory. Inspection of these "iso-daughter" lines in Figures 10b and 11b is somewhat illuminating; in the spectra of both the sawtooth wave and the "pulled" sine wave, every possible combination that satisfies the frequency selection rule is represented, provided the triple product of Fourier amplitudes is above the 0.5% maximum amplitude threshold.

There are some subtle differences in the bispectrum plots that were not fully investigated here. For instance, in case of the pulled time series, the sign of \( \text{Re}(B) \) was
found to be positive for only one peak in each iso-daughter line, and only for those “interactions” of the driving frequency with the positive side-bands (see Fig. 10c). In contrast, the real part of the sawtooth’s bispectrum was inconsistently positive and negative, in roughly equal portions. It is known that there is a canceling phase relationship between side-bands produced by AM and FM, which produces the asymmetrical frequency spectrum of Fig. 10a. At first sight, this effect offers some hope of eventually interpreting the reinforcing phase relation implied by the biphase at \((f_{1+n}, f_{1+m})\), for \(n = 1, 2, 3, \ldots\), but a more complete wave analysis of periodic pulling is not the intent of this work.

C. IEDD and CDEIC plasma instabilities

![Figure 12a: Average CDEIC power spectrum](image1)
![Figure 13a: Average IEDD power spectrum](image2)

The average power spectra of the two plasma instabilities as shown in Figures 12a/b and 13a/b are quite distinct. The current driven mode, seen at 71 kHz, is almost monochromatic, and the spectrum displays only small amounts of energy in the lower frequencies. Conversely, the frequency spectrum associated with the energy density-driven mode, seen at approximately 65 kHz, is broadband with significantly more power in the 2 kHz - 12 kHz range. The low frequency fluctuations have been identified by their
three-dimensional mode characteristics and magnetic field dependent frequency as drift waves. Drift wave type oscillations are expected to satisfy the Hasegawa-Mima three-wave equation of Ref. 23 used to derive Eq.(2.23).

The coherency of the CDEIC mode is particularly evident in the contour plot of the bispectrum and bicoherence spectrum (Figure 12c and 12d), where the most significant feature indicates "coupling" of $f_{\text{CDEIC}}$ with its first harmonic. Also note the 71 kHz iso-daughter lines reminiscent of the sawtooth or pulled sine wave bispectra of Figures 10b and 11b. Every possible combination of frequencies satisfying the selection rule for $\omega_i + \omega_k = 71$ kHz exhibits some, albeit much smaller degree of coherence with the CDEIC mode, implying that 71 kHz is just one of the many components required in the Fourier decomposition of the longer wavelength pulse shapes. Such a bispectrum would be expected from any time series containing the sum of a periodic, non-sinusoidal low frequency waveform and a sinusoidal high frequency waveform and should not be interpreted as an indication of active nonlinear coupling.

The cluster in the lower left hand corner of the bispectrum plot, however, is indicative of some non-linear interactions present among the low frequency fluctuations, a result consistent with previous experiments on drift wave type turbulence in Tokamaks and RF glow discharges. Tsui, in particular, used the Hasegawa-Mima drift wave turbulence equation to derive (Eq. 2.23)

$$\sum_{\omega_j+\omega_k=\omega_m} b^2(\omega_j,\omega_k) = 1 - \delta^2$$

where the bicoherence spectrum is computed over the reduced triangular region $A$ of Figure 1, and $\delta^2$ increases with turbulence broadening of spectral components. Clearly this
result does not apply to the harmonically
generated spectra of Figures 9 through 11, but
it should apply to the sum of $b^2$ for all
interactions involving the turbulent drift wave
oscillations of this experiment.

As an interesting confirmation of Tsui's
result in the present experiment, the sum of the
squared bicoherence of the interactions involving each frequency was calculated using the
program of Appendix C. Recalling the discussion on symmetries of section II.D, the
bicoherence of all interactions at a certain frequency lies along three separate lines in the
reduced triangular region. Figure 12e(i) is the bicoherence of all interactions involving the
mode at 71 kHz, where the x axis is the frequency of one of the triplets satisfying

$$\omega_m = \omega_1 + \omega_k,$$

for $\omega_m$ fixed at 71 kHz. The amplitude of the cluster at 35 to 70 kHz, though
smaller than the self-coherent 71 kHz peak, is still much larger than the statistical
uncertainty of $1/M = 0.009$.

Plotting the area under each such curve as a function of the interaction frequency
produces the graph shown in Figure 12e. For the range of frequencies associated with drift wave turbulence, the sum is indeed less than one, as predicted by Eq. (2.23). The difference of .4 between unity and the sum of $b^2$ for $f_{\text{shift}}=3\text{kHz}$ is attributed to the spectral broadening term $\delta^2$.

The IEDDI bispectrum indicates a coupling of the high frequency mode to drift waves that is absent in the CDEIC fluctuations. Inspection of the 3-D surface plots of the low frequency group (Figure 13f) and the high frequency group (Figure 13e) reveals the four most significant spectral features: $(f_1, f_2) = (3,3), (6,3), (62,6)$, and $(65,3)$ all in kHz. The third feature, characterized in Table 2, unequivocally points to parents at $(68,62)$, and the last probably indicates parent waves at $(68,65)$. The low frequency peaks are more ambiguous, since none of the rows of Table 2 can be eliminated with any degree of certainty, except for the $(9,6)$ parent of the second feature. The power spectrum suggests the formation of harmonics of the 3 kHz wave. The presence of the $(3,3)$ peak on the bispectrum confirms that the mode at 6 kHz peak is indeed the second harmonic of 3 kHz, and not a self-excited mode. In summary, bispectral analysis suggests a “mixing down” of the IEDDI waves to the lower frequency drift waves, and a coupling among various spectral components of the drift wave oscillations.

A few words of caution are perhaps in order: First, resonant wave-wave interactions cannot be verified without a corresponding k-space analysis, since there is no way of knowing if the wavenumbers of the three modes satisfy the wavenumber selection criteria required for quadratic coupling. Second, no correction has been made to offset the frequency dependent phase shifts introduced by the band-pass filtering of the time
series data, so even an a priori knowledge of wavenumbers and coupling coefficients would complicate a meaningful interpretation of the biphase. A filter similar to the one used to collect the plasma data was used to measure the phase shifts of a generated sine wave at the frequencies of interest. The result was -72 degrees at 6 kHz, +22 degrees at 62 kHz, and +24 degrees at 68 kHz, where the plus sign indicates that the output lags the input. Using this convention, and letting $\varphi$ be the phase shift introduced by the filter,

$$\beta_{\text{corrected}} = (\theta_j + \varphi_j) - (\theta_m + \varphi_m) = \beta_{\text{corrected}} + \varphi_j + \varphi_k - \varphi_m$$

(4.7)

gives a $\beta_{\text{corrected}}(68 \text{ kHz}, 62 \text{ kHz})$ value of $-4 \pm 12^\circ$. This indicates a reinforcing phase relation of the parent waves with respect to the low frequency daughter wave. A more comprehensive power flow analysis awaits the implementation of Beall’s two point correlation technique (with the results of section II) for $\partial P/\partial x$.

The plot of summed bicoherence vs. interaction frequency for the IEDD data provides an additional indication that something other than the coupling of drift waves among themselves is responsible for the growth of the low frequency turbulence (see Fig. 13e). Compared to drift waves in the CDEIC case, the drift waves present in the IEDD spectrum seem to exhibit a much stronger degree of coherence -- in fact more than is

![Figure 13e: Sum of squared bicoherence of the interactions involving a frequency given by the x axis for IEDD fluctuations.](image)
allowed by Eq. (2.23) for drift wave nonlinear interactions. One could infer that the
difference is the result of coupling with the higher frequency, IEDD instability (harmonic
generation of drift waves is not excluded in Tsui's derivation of Eq. 2.23). It may be
plausible that this coupling is facilitated by the more broadband spectrum associated with
IEDD fluctuations and is inhibited by the narrow CDEIC wave spectrum. Also note that
at the center frequency of the IEDD mode, the sum of the squared bicoherence is close to
one. In the context of three-wave coupling models, a value of one indicates coherent
wave coupling. The low value of the sum at intermediate (20 kHz -55 kHz) and high
frequencies (greater than 75 kHz) is consistent with random, non-coherent interactions
among the background turbulence (a somewhat surprising result given the relatively high
power of the intermediate frequencies).
V. Conclusion

A working computer implementation of Hasselman’s bispectrum algorithm is now available as a diagnostic tool for any future WVU Plasma Lab experiments involving nonlinear coupling. The bispectrum plots obtained from analyzing the test signals suggested by Kim and Powers are identical to published results, and the numerical solutions of $\Sigma Re(B)$, fraction of power due to nonlinear interactions, and the coupling coefficient, $A_{jk}$, are consistent with the theoretical predictions of section II. A method of determining causality is proposed and subjected to empirical tests with these signals. The proposed method is shown to be valid (with some limitations) in experimental systems.

The ASYST code to compute Beall’s two point statistical wavenumber spectrum is developed and tested with a Monte-Carlo model using computer generated signals designed to simulate turbulent plasma fluctuations. Though the variability of the spatial quantities is rather large, the program is successful (for a low density of structures) in recovering the wavenumber-frequency spectrum and other parameters used in conjunction with the bispectrum to measure the coupling coefficient $V_{jk}$, and the power transfer function, $\partial P / \partial x$.

Applying Hasselman’s method to time series data from a periodically pulled UJT oscillator, the bispectrum is found to be consistent with the Fourier decomposition of a periodic, arbitrary pulse shape (by comparison with the bispectrum of a sawtooth waveform). The time-averaged nature of the bispectrum calculation cloaks the temporal variations in instantaneous frequency characteristic of the pulse-like periodic pulling phenomenon, so the detection of coupling is suppressed.
Finally, a bispectral analysis of two distinct plasma instabilities reveals a degree of coupling of the IEDD mode to drift waves that is absent in the CDEIC fluctuations. It is concluded, with some ambiguity, that the parents of the nonlinear interaction are the high frequency IEDD modes. The quadratic coupling of these waves produce daughters at their difference frequency that tend to increase the power of the drift waves.
References


12 D.P. Sheehan, R. McWilliams and N. Rynn, Adjustable levels of strong turbulence in a positive/negative ion plasma, Phys. Fluids B 5, 1523 (1993).


19 H.Y.W. Tsui, K. Rypdal, Ch. P. Ritz and A.J. Wootton, Coherent Nonlinear Coupling


36 Arfken, p. 481.

Appendices

Appendix A: Bispectrum code

******************************************************************************
\ BICOHERENCE & BISPECTRUM for ASYST data
\ files loaded in an 8MB ram drive, outputs a
\ (m X m/2) file in ascii where m is the length of the
\ record and M ("records") = the number of subfiles.
\ Adjust default values for "records" and "m" (100 & 512)
& type "hi"
******************************************************************************

forget it : it ;
32 string filename \ input filename
32 string bic.asc.file \ bA2 output file
32 string bis.asc.file \ bispectrum (real part) n
32 string m.asc.file \ average fit output file
integer scalar n
scalar m 512 m := \ # of columns(# elements/row)
scalar 2m 2 m * 2m := \ m must be a power of 2
scalar start 1 start := \ starting subfile #
scalar records 100 records := \ # of records (M)
scalar epsilon \ division by zero protection
scalar meancube \ stores ave cube of timeseries
scalar biggest \ stores max value of each B row
token banning exp.mem> banning
  token estl exp.mem> est1
  token est2 exp.mem> est2
  token bicoexp.exp.mem> bicoexp
  token aveeff.exp.mem> aveeff
  token dat.exp.mem> dat
  token bctrans.exp.mem> bctrans
  token brtrans.exp.mem> brtrans
  trap.underflow.on
  : makearrays
  m real ramp becomes> est1 0. est1 :=
  m real ramp becomes> est2 0. est2 :=
  m real ramp becomes> bicoexp 0. bicoexp :=
  m real ramp becomes> aveeff 0. aveeff :=
  m / 2 real ramp becomes> bctrans 0. bctrans :=
  m / 2 real ramp becomes> brtrans 0. brtrans :=
  2m complex ramp becomes> dat 0. dat :=
  m complex ramp becomes> bspec 0. bspec :=
  2m real ramp m 1 + abs m / pi * cas 1 + 2 / becomes>
  : hamming

  : makefiles
  " f: defer> data.file
  file.template
  complex dim[ m ] subfile m 2 / times end
  file.create bicoexp.dat
  file.template
  real dim[ m ] subfile m 2 / times end
  file.create est1.dat
  file.create est2.dat
  file.create bicoexp.dat
  file.template

real dim[ m 2 / ] subfile m times end
file.create bctrans.dat
file.create brtrans.dat

  : writetofiles
  " f: defer> data.file
  file.open bicoexp.dat
  n subfile bicoexp array>file
  file.close
  file.open est1.dat
  n subfile est1 array>file
  file.close
  file.open est2.dat
  n subfile est2 array>file
  file.close

  : readfromfiles
  " f: defer> data.file
  file.open bicoexp.dat
  n subfile bicoexp file>array
  file.close
  file.open est1.dat
  n subfile est1 file>array
  file.close
  file.open est2.dat
  n subfile est2 file>array
  file.close

  : bico  
  \ *** Calculate bicoherence spectrum ***
  \ cr." Calculating bicoherence (squared)"
  0. biggest :=
  m 2 / 1 + 1 do i n := \ cr." Row " n .
  readfromfiles
  bicoexp zmag dup * [i]max dup biggest >
  if biggest := else drop then
  loop cr." Max = " biggest .
  biggest 500. / epsilon := \ min normalization
  m 2 / 1 + 1 do i n := \ cr." Row " n.
  readfromfiles
  m 1 + n - n do
  bicoexp [ i ] zmag dup * epsilon >
  if
  bicoexp [ i ] zmag dup *
  est1 [ i ] est2 [ i ] * /
  bicoexp [ i ] :=
  else
  0.0 bicoexp [ i ] :=
  then
  loop
  file.open bicoexp.dat
  n subfile bicoexp array>file
  file.close
  0. bicoexp :=
  loop

  : transit \ prereqs: makefiles and makearrays
  cr." Transposing bicoherence (squared)."

53
m 1 + 1 do \ transpose rows to columns for b\^2
file.open f:\bicohmc.dat
m 2 / 1 + 1 do
  i subfile bicohmc file>array
  bicohmc [ j ] bctrans [ i ] :=
loop file.close
file.open f:\bctrans.dat
i subfile bctrans array>file file.close
loop

file.open f:lbicohmc.dat
m 2 / 1 + 1 do
  i subfile bicohmc file>array
  bicohmc [ j ] bctrans [ i ] :=
loop file.close
file.open f:\bctrans.dat
i subfile bctrans array>file file.close
loop

\ Transposing Re(bispectrum).

m 1 + 1 do \ transpose rows to columns for Re(B)
file.open f:lbispect.dat
m 2 / 1 + 1 do
  i subfile bispect file>array
  bispect [ j ] brtrans [ i ] :=
loop file.close
file.open f:lbrtrans.dat
i subfile brtrans array>file file.close
loop

**bicoherence to ascii**

filename 11 :=
then
filename "len 2 - "right file name
"cat bic.asc.file" :=
" f:\ bic.asc.file" cat bic.asc.file " :=
cr. "Converting bicoherence (squared) to ascii file "
bic.asc.file "type
-1 3 fix.format
m 1 + 1 do in :=
file.open f:brspect.dat
n subfile brspect file>array
file.close
bic.asc.file defer> out>file
console.off
m 2 / 1 + 1 do
  brspect [ i ] . \ matrix form
loop out>file.close
console
loop

**Re (bispectrum) to ascii**

bic.asc.file "len 5 - "right file name
"cat br.asc.file" :=
" f:\" bic.asc.file "cat br.asc.file " :=
cr. "Converting real part of bispectrum to ascii file "
bic.asc.file "type
m 1 + 1 do in :=
file.open f:brtrans.dat
n subfile brtrans file>array
file.close
bic.asc.file defer> out>file
console.off
m 2 / 1 + 1 do
  brtrans [ i ] . \ matrix form
loop out>file.close
console
loop

**Ave FFT to ascii**

bic.asc.file "len 5 - "right file name
"cat fft.asc.file" :=
cr. "Converting the average fft to ascii file " fft.asc.file
"type
fft.asc.file defer> out>file
console.off
m 1 + 1 do avefft [ i ] . cr loop
out>file.close console

**Type avefft yp to see average of" records .. " fft's." ;

: bi
cr. " Initializing...

2 m * 2m := \ freq res. = sampling freq / 2m
m . .. by " m 2 / ." arrays
makarrays

cr. " Making files...

makfiles

0. avefft := 0. meancube :=
cr. " Input filename? " "input filename ":=

cr. " Calculating bispectrum for " records .. " records"
cr. " starting with subfile # " start . cr. " Set 

records start + start do i .
  filename defer> file open
  i subfile file>unnamed.array sub [ 1 , 2m ]
  file.close
dup mean - header . fft

2m / \ Normalize each fft
dat :=
dat sub [ i , m ] zmag avefft + avefft :=

\*** Calculate bispectrum and normalization factors
m 2 / 1 + 1 do in :=
readfromfiles
m 1 + n - n do
  dat [ i + ] dat [ n 1 + ] * dat [ i n 1 + ] conj *
  bispect [ i ] + bispect [ i ] :=
  dat [ i + ] dat [ n 1 + ] * zmag dup *
est1 [ i ] + est1 [ i ] := \ calculate the nth
dat [ i n 1 + ] zmag dup * \ row of bispect
est2 [ i ] + est2 [ i ] := \ and keep total
loop
  writetofiles \ store nth row
loop \ for each of m rows
loop

\ records
\ bico

ascit
cr. " Mean cube = " meancube records /

;
Appendix B: Wavenumber spectrum code

```plaintext
forget it; it;
trap.unflow.on

integer scalar 2m 128 2m := \ Number of data points / record
scalar m 2m 2/m := \ Number of freq and k bins
scalar records 30 records := \ Number of records
scalar l 1 2m := \ Loop variable
scalar n 1 n := \ Number of pulses / record
scalar w 14 w := \ Index of coupled wave
real scalar k 0.5 k1 :=
scalar k2 1.3 k2 :=
scalar k3 1.8 k3 :=
scalar w1 2 pi• 14.08 •
w1 :=
scalar W2 2 pi• 24.00 • w2 :=
scalar W3 2 pi• 38.08 • w3 :=
scalar kappa 1 Inverse scale length
scalar alpha 1 Phase of A in eqn 2.16
scalar deltak \ wavenumber resolution
scalar t0.4 t := \ Time of x series
scalar deltax 1.0 deltax := \ Separation of probe tips
scalar x0.6875 x := \ probe tip location for b
scalar tau 1.0 tau := \ Width of wavelet
scalar noise 0.2 noise :=

real dim[n] array randphasel
dim[n] array randphase2
dim[n] array randphase3
dim[2m] array ta
dim[2m] array f1
dim[2m] array f2
dim[2m] array f3
dim[2m] array z

dim[m] array theta1
dim[m] array theta2
dim[m] array power1
dim[m] array power2
dim[m] array powerave1
dim[m] array powerave2
dim[m] array sbar
dim[m] array s_ave
dim[m] array klocal
dim[m] array ksppectrum

\ The following are f(K, freq):
dim[m, m] array slocal

dim[m, m] array second \ Conditional spectral est.
dim[64, 32] array est1
dim[64, 32] array est2
dim[64, 32] array bicoherence
complex dim[64, 32] array bispact
dim[128] array dat

2m hanning.window.drop \ Create hanning window
2m real ramp 1 - 2m / ta := \ Create time array

: ascit
out>file gen.asc
65 1 do
33 1 do
```

bicoherence [j, i] .","
loop er
loop
out>file.close

: make_phase
n 1 + 1 do i1 :=
    rand.unif.5 - pi • randphasel [1] :=
    rand.unif.5 - pi • randphase2 [1] :=
    rand.unif.5 - pi • randphase3 [1] :=
    \ randphasel [1] + pi 2 / +
    \ randphase3 [1] :=
loop

: timeser
0.0 fl := 0.0 f2 := 0.0 f3 :=
n 1 + 1 do i1 :=
    k1 x * w1 ta * -
    randphasel [1] + cos
    0.0 k1 x * randphase1 [1] + w1 / ta -
    dup * - tau / exp *
    fl + fl :=
    k2 x * w2 ta * -
    randphase2 [1] + cos
    0.0 k2 x * randphase2 [1] + w2 / ta -
    dup * - tau / exp *
    f2 + f2 :=
    k3 x * w3 ta * -
    randphase3 [1] + cos
    0.0 k3 x * randphase3 [1] + w3 / ta -
    dup * - tau / exp *
    f3 + f3 :=
loop
    fl f2 + .5 f3 * + fl f2 * + fl := \ Fig 4 of K&P
\ fl f2 + f3 + ft := \ Fig 2 & 3 of K&P
2m 1 + 1 do
    rand.unif.5 - noise * ft [i] + ft [j] :=
loop

: b
x=0.0, y=0.0, bispect := 0 est1 := 0 est2 := 0 ft := 0.
meancube :=
64 1 do
    make_phase
    timeser
    ft dup dup ** [sum meancube + meancube :=
    ft dup mean - hanning.window.apply ft dat :=
    \*** Calculate bispectrum and normalization
33 1 do i1 :=
    65 1 - 1 do
    dat [i 1 +] dat [j 1 +] * dat [i j + 1 +] conj *
    bispect [i, j] + bispect [i, j] :=
    dat [i 1 +] dat [j 1 +] * zmag dup *
    est1 [i, j] + est1 [i, j] :=
    dat [i j + 1 +] zmag dup *
    est2 [i, j] + est2 [i, j] :=
loop
```
loop \ Repeat for each record

\*** Calculate bicoherence spectrum ***
bispect zmag [max] [max]
500.0 / eps := \ Determine min normalization
33 1 do i 1 :=
65 1 - do
bispect [i , j ] zmag eps >
if
bispect [i , j ] zmag dup *
est1 [i , j ] est2 [i , j ] * /
bicoherence [i , j ] :=
else
0.0 bicoherence [i , j ] :=
then
loop
loop
bicoherence trans[1 , 2 ] dup axon.plot
axis.defaults [max] [max]
" Max b'/2 = " .
\ cr. " Sum ofRe(E) = " bispect zreal [sum] [sum]
128 128 * / 
\ cr. " Mean cube = " meancube records /. 
ascit ; ;
axon.plot axis.defaults
; ; : k
2.0 pi * m / deltax / deltak := \ Calculate size of k bin
0.0 s_ave := 0.0 slocal := 0.0 kspectrum :=
0.0 power1ave := 0.0 power2ave :=
\ rand.unif .5 - 2 * 2m * x :=
.375 x :=
records 1 + 1 do
make_phase
timeser
ft dup mean - hanning.window.apply ft dup
zarg sub[1 , m ] theta1 :=
zmag sub[1 , m ] dup * power1 :=
power1ave power1 + power1ave :=
x deltax + x :=
timeser
x deltax - x :=
ft dup mean - hanning.window.apply ft dup
zarg sub[1 , m ] theta2 :=
zmag sub[1 , m ] dup * power2 :=
power2ave power2 + power2ave :=
theta1 theta2 - deltak / klocal :=
power1 power2 + 2 / sbar :=
sbar s_ave + s_ave :=

\*** Calculate statistical wavenumber spectra ***
m 1 + 1 do
m 2 / 1 + 0.0 m 2 / - 1 + do
i deltak * scond [i m 2 / , j ]*
slocal [i] sbar [j] / scond [i , j ] :=
else
0.0 scond xsect[i , j ] :=
then
loop
loop
\*** Calculate conditional spectral estimate ***
m 1 + 1 do
m 1 + 1 do
slocal [i , j ] eps >
if
slocal [i , j ] sbar [j ] / scond [i , j ] :=
else
0.0 scond xsect[i , j ] :=
then
loop
loop
\*** Calculate alpha ***
power2ave [w] power1ave [w] - s_ave [w] / deltak /
kappa :=
0.0 kspectrum [39] - kappa / atan alpha :=
kspectrum yp
cr. " Kappa = " kappa .
cr. " Alpha = " alpha .
; ; : kreal
0.0 s_ave :=
2m real ramp 1 - k1 * wP + cos ft :=
ft dup mean - hanning.window.apply
fft zmag sub[1 , m ] s_ave + s_ave := \ Actually
k_ave
m real ramp 1 - m / pi * deltak /
s_ave records / xy.auto.plot
;
Appendix C: Summed bicoherence spectrum code

\TSULASY - calculates the sum of the bicoherence for a
\ constant interaction frequency -- for each frequency.
\ Outputs a sum
\ of b2 vs interaction freq via sumarray or individual
\ b2 vs w1 plots via intractn.dat (Tsui paper refers).
\ bicohc.dat should be in the ramdrive (my fdrive)
\ Can't change m after loading w/o tokens
forget it: it;

integer scalar m 512 m := \ 2 * m = # of points/record
scalar flag \ in the bispectrum
scalar col
scalar row
real dim[m] array bico \ Interaction frequency
real dim[m] array sumarray \ Total interaction b
real dim[m] array tsui \ Individual b^2 lines for w1

: ascsum
delete f\sumb.asc
out>file sumb.asc
m 1 + 1 do
sumarray [i].cr
loop
out>file.close

: asc_one
"f\" defer> data.file
delete f\intractn.asc
cr."w3 is the argument of this word. Hit any key"
dokey drop
file.open intractn.dat
w3 subfile bico file>array
file.close
out>file intractn.asc
console.off
m 1 + 1 do
bico [i].cr
loop
out>file.close console
cr."w3 index of b line converted to ascii was * w3 .

: makesfiles
delete f\intractn.dat
"f\" defer> data.file
file.template
real dim[m] subfile m times
end file.create intractn.dat

: go
makesfiles
m 1 + 1 do i w3 :=

0.0 tsui :=
w3 2 modulo 0 :=

else
w3 1 + 2 / row := \ Starting subfile # of bicohc.dat
w3 1 + 1 + col := \ Starting index #

file.open bicohc.dat

\ *** Diagonal leg ***
w3 1 + col do
flag :=
row subfile bico file>array
bico [i] tsui [i w3 1 - + ] :=
row 1 - row :=
loop

\ *** Vertical leg ***
w3 m 2 /<=
if \ 1st half triangle
w3 1 >
if \ skip if it's the first w3 (prevents 2 2 do error)
w3 1 + 2 do 2 flag :=
i subfile bico file>array
bico [w3] tsui [i w3 1 - + ] :=
loop then
else \ for the 2nd half triangle
w3 0 >
if \ skip if it's the last w3 (prevents 2 2 do error)
m w3 - 2 + do 3 flag :=
i subfile bico file>array
bico [w3] tsui [i w3 1 - + ] :=
loop then

\ *** Horizontal leg ***
w3 m 2 /< if \ skip if at or past 1/2 way point
m w3 - 2 + w3 1 + do
4 flag :=
w3 subfile bico file>array
bico [i] tsui [i w3 1 - + ] :=
loop then

file.close
tsui [sum sumarray [w3 ]] := \ Sum of b for w3
file.open intractn.dat
w3 subfile tsui array>file \ Record the results
file.close
loop \ Repeat for each freq
sumarray yp

w3 2 / row := \ Starting subfile # of bicohc.dat
w3 2 / 1 + col := \ Starting index #
else
w3 1 + 2 / row := \ row = subfile
w3 1 + 2 / 1 + col := \ col = index
then
file.open bicohc.dat

\ *** Diagonal leg ***
w3 1 + col do
flag :=
row subfile bico file>array
bico [i] tsui [i w3 1 - + ] :=
row 1 - row :=
loop

\ *** Vertical leg ***
w3 m 2 /<=
if \ 1st half triangle
w3 1 >
if \ skip if it's the first w3 (prevents 2 2 do error)
w3 1 + 2 do 2 flag :=
i subfile bico file>array
bico [w3] tsui [i w3 1 - + ] :=
loop then
else \ for the 2nd half triangle
w3 0 >
if \ skip if it's the last w3 (prevents 2 2 do error)
m w3 - 2 + do 3 flag :=
i subfile bico file>array
bico [w3] tsui [i w3 1 - + ] :=
loop then

\ *** Horizontal leg ***
w3 m 2 /< if \ skip if at or past 1/2 way point
m w3 - 2 + w3 1 + do
4 flag :=
w3 subfile bico file>array
bico [i] tsui [i w3 1 - + ] :=
loop then

file.close
tsui [sum sumarray [w3 ]] := \ Sum of b for w3
file.open intractn.dat
w3 subfile tsui array>file \ Record the results
file.close
loop \ Repeat for each freq
sumarray yp

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Appendix D: Derivations

For the Monte Carlo pulse shape described by

$$\phi(t, x) = e^{-\left(\frac{kx + it}{\omega_0}\right)^2} \cos(kx - \omega_0 t)$$

The Fourier transform is

$$\Phi(\omega, x) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \phi(t, x) e^{i\omega t} dt \approx \frac{1}{T} \int_{-T}^{T} \phi(t, x) e^{i\omega t} dt$$

for large T, where the windowing function sets $\phi$ to zero at the limits of integration.

Let $\theta = kx - \omega_0 t$.

$$\Phi(\omega, x) = \frac{1}{T} \int_{-T}^{T} e^{-\theta^2} e^{i\theta} \left[ e^{\frac{-\theta^2}{2}} \right] \exp \left[ \frac{-\theta^2}{\omega_0^2} + i\theta + i\omega t \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ \frac{-\theta^2}{\omega_0^2} + i\theta + i\omega t \right] \exp \left[ \frac{-\theta^2}{\omega_0^2} - i\theta + i\omega t \right] d\theta$$

$$I_+ = \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \exp \left[ -\frac{k^2x^2}{\omega_0^2} + i kx - \frac{1}{\tau^2} \right] \left[ t^2 - 2t \left( \frac{kx}{\omega_0} + i \left( \frac{\omega \pm \omega_0}{2} \right) \right)^2 \right] dt$$
Let \( \xi = t - \left( \frac{kx}{\omega_0} + i \frac{\omega \pm \omega_0}{2} \right) \)

Since \( \exp[-(\omega+\omega_0)\tau^2/2] \) is much less than \( \exp[-(\omega-\omega_0)\tau^2/2] \), \( I \gg I_1 \) and

\[
\Phi(\omega, x) \approx \frac{1}{2\tau^2} e^{-ikx \left( \frac{1}{2} \frac{\omega - \omega_0}{\omega} \right)} e^{-\frac{\tau^2}{2} (\omega - \omega_0)^2} \int_{-\infty}^{\infty} e^{-\xi^2/\tau^2} d\xi
\]

\[
\Phi(\omega, x) = \frac{\tau \sqrt{\pi}}{2\tau^2} e^{-ikx \left( \frac{1}{2} \frac{\omega - \omega_0}{\omega} \right)} e^{-\frac{\tau^2}{2} (\omega - \omega_0)^2}
\]

This is Eq. (4.4) in section IV.A.
Figure 4: Phase coherent cosines \( \cos(\omega_1 t + \theta_1) + \cos(\omega_2 t + \theta_2) + \cos(\omega_3 t + \theta_3) \) where \( \omega_1 = \omega_3 + \omega_3 \) and \( \theta_1 = \theta_2 + \theta_3 \). \( \omega_1 = 2\pi \cdot 14.8 \), \( \omega_2 = 2\pi \cdot 24 \), \( \omega_3 = 2\pi \cdot 5.8 \). (a) Power spectrum and (b) squared bicoherence spectrum.
Figure 5: Phase incoherent cosines \( \cos(\omega_x t + \theta_x) + \cos(\omega_y t + \theta_y) + \cos(\omega_z t + \theta_z) \) where \( \omega_x = \omega_y + \omega_z \) but \( \theta_x \neq \theta_y + \theta_z \).

(a) Power spectrum and (b) squared bicoherence spectrum.
Figure 6: Quadratically coupled cosines

\[ x(t) = \cos(\omega_1 t + \theta_1) + \cos(\omega_2 t + \theta_2) + \cos(\omega_3 t + \theta_3) \cos(\omega_4 t + \theta_4) + \eta(t) \]

where \( \omega_1 = 2\pi 4.08 \text{ s}^{-1} \), \( \omega_2 = 2\pi 24 \text{ s}^{-1} \), and \( \omega_3 = 2\pi 38.08 \text{ s}^{-1} \). The phases of each cosine argument are randomly distributed between \( \pm \pi \) for each record. (a) Power spectrum and (b) squared bicoherence spectrum.
Figure 6c: Contour plot of the bicoherence spectrum of quadratically coupled cosines.
Figure 7: Determination of statistical wavenumber-frequency spectrum: (a) is the local wavenumber spectrum obtained by taking the phase of the cross-correlation divided by the probe separation, (b) is the local wavenumber frequency spectrum, and (c) is the corresponding conditional wavenumber spectral density (normalized by the power at each frequency). The right-hand axis of (d) shows the statistical wavenumber (diamonds) at the peaks of the frequency spectral density (solid line). The expected values of $k$ are 1.8, -1.3, .5, and .7 cm$^{-1}$ at 10, 14, 24, and 38 Hz respectively.
Figure 9: Weakly pulled time series from the UJT relaxation oscillator. (a) Power spectrum and (b) contour plot of squared bicoherence. The magnitude of each peak is 1.0.
Figure 10: Strongly pulled time series from the UJT relaxation oscillator. (a) Power spectrum and (b) squared bicoherence contour plot. Each peak has magnitude of 1.0.
Figure 12: Average power spectra (b), the associated real part of the bispectrum (c), and the squared bicoherence spectrum (d) of 112 CDEIC realizations.
Figure 13: Average power spectra (b), the associated real part of the bispectrum (c), and the squared bicoherence spectrum (d) of 88 IEDD realizations.
Figure 13: Enlarged view of the high frequency group (e) and low frequency group (i) of the real part of the IEDDI bispectrum.
APPROVAL OF EXAMINING COMMITTEE

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3 July 1995

Date