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Theory and Techniques of Convergence of Topological Transformation Group Actions

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Theory and Techniques of Convergence
for Topological Transformation
Group Actions

Murtadha Sarray

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Abstract

Theory and Techniques of Convergence for Topological Transformation Groups

Murtadha Sarray

In this dissertation, we present new set functions called strongly limit and strongly prolongation limit sets. We show the new sets, especially strongly prolongation limit sets, characterize proper action under an arbitrary setting. That is, we characterize proper action for wider class of proper G -spaces. Also, we show the new version of the sets could be derived from strongly exceptional sets which have been used as a good technique for the characterization of a proper maps. Moreover, we review properties of well-known limit sets and prolongations and properties for the new version of limit sets under an arbitrary setting on a G -space X . Next, we give characterizations for a proper action and a locally proper action using the new set functions and some relevant areas where one could use these sets to describe some topological properties for a space X or algebraic properties for a group G acting on X . Following the technique of limit sets, we present a new version of action as weaker forms of the proper action called w -actions using convergence techniques. This approach yields new directions for the proof of some existing theorems which are then given as well as come counter-examples. We give some applications for our weaker forms and how convergence techniques could be fruitful to explain the relations among certain notions in the theory of G -spaces. Finally, according to our weaker forms, we characterize the notion of enough slice that has presented by H. Biller using w -Cartan action.

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Chapter 0

Introduction

Limit and prolongation limit sets are used in the theory of dynamical systems to describe trajectory behaviors, especially their stability in their phase spaces [7]. It is well known within that theory (R -spaces or Z -spaces), briefly, as a special case of a general theory called topological transformation groups (G -space theory), so several studies have been conducted with these kinds of set functions to describe also properties of the general topological transformation groups.

In general, two versions of proper G -spaces were discovered in the middle of 19th century. Palais in [28] introduces most of the basic concepts of G -spaces which under certain settings are taken with G as a compact (Lie) group and X a completely regular topological space. On the other hand, Bourbaki [11], introduces the concept of proper action for arbitrary topological groups and general topological spaces. Next Palais generalizes the class of a compact group action to wider one that is termed a proper action (or Palais proper) where G is an arbitrary Hausdorff locally compact topological group and X is a completely regular space. After Palais' work, many studies have arisen to investigate conditions that keep many notions **no** longer true in the sense of Palais such as the slice theorem, the conjugacy theorem, the equivariant embedding theorem and others. Some researchers have used these set functions as a technique to characterize a proper action groups on spaces under certain conditions. A fortiori one could use them also to investigate notions are related to proper action.

In chapter one, we recall some basic background of topological concepts such as the separation axioms, compactness, local compactness, nets, connectedness, and paracompact spaces. In section two, we review many concepts of the theory of topological groups. For instance, we review nets in a topological group, locally compact groups and Lie groups which form the background of our work. In the last section, we concern ourselves with some notions of G -space theory. We recall the definitions and properties of proper mapping, Bourbaki proper action, and Palais proper action.

In chapter two, especially in section one, we recall some concepts of general topology theory such as, for example, quasi-compact mappings, proper mappings, and their characterizations by convergence techniques. In section two from the same chapter, we state the definitions of the limit set and its prolongation limit set under arbitrary settings. Also, we review properties and some counter examples. In section three, we present the definitions of the strongly limit set and its strongly prolongation limit set also under arbitrary settings. Moreover, we list properties of the new version of sets and give the main result in this chapter (Theorem 2.3.3) which is given as the following:

An action of a group G on a space X is proper if and only if the strongly prolongation set is the empty.

In the last section, we first recall some interesting results for proper and locally proper (Cartan) G -spaces in the sense of Palais setting and how limit sets could be good keys to describe the orbit space. Also, we present other results under weaker conditions such as the relations of close points and their stabilizers and close orbits and their kernels. Finally, we give a new proof for conjugacy neighborhood theorem.

In chapter three, section one, we show how convergence techniques can describe well-known kinds of proper actions such as Bourbaki, Palais, and the stronger one that is presented by Buam, Connes and Higson. Moreover, we also show how the sets might be used to prove results concerning and a good technique to gain more knowledge about the local structure of G -spaces. In section two, we review the weaker conditions of the version of Cartan G -space that are introduced by Biller in [8] and show that these conditions could be replaced by a weaker condition (convergence condition) using set functions. Also, we give an example that shows that our condition is weaker than Biller's. Moreover, according to the weaker form of Cartan (we follow Biller), we introduce three kinds actions are called w -proper actions that are increasing strength. Furthermore, we investigate the relations and implications among the new version of the w - actions. Finally, in section three, an application for the weaker forms of w -actions on a topological group is given and we show some results from [5] that become immediately results of our forms and their proofs become short and understandable. By G -space X it is meant an arbitrary topological group G acts continuously on an arbitrary topological space X . We also follow Bourbaki for term "quasi-notion" is taken by a definition of the notion lacking Hausdorff axiom. Thus, by a quasi-compact, and a quasi-paracompact we mean these notions without T_2 property. Regular and completely regular spaces always possess the T_1 property.

Chapter four is divided into two sections, in the first one, we discuss the slice notion in G -space theory and give a motivation for the existence of slice theory. We rely there on the usual definition of the twisted product to describe an H -slice. In the second section, we characterize the *enough slice* notion, that has been presented by Biller [8], using the weaker form of w -Cartan action for G as a locally compact group (not necessary Lie group) and X as a completely regular

space. As a result, we give the relation between Baum-Connes-Higson proper with our notion.

Chapter 1

Fundamental Concepts

In this chapter we recall the basic background of topological spaces, topological groups, and topological transformation groups that form a base for all the notions that we shall present over next chapters.

1.1 Topological Concepts

1.1.1 Theorem. [25] Let X be a topological space. Then the following are equivalent:

1. X is T_1 , that is, every two distinct points x, y in X , each belong to an open set which does not contain the other.
2. Every singleton subset $\{P\}$ of X is closed.
3. Every singleton subset $\{P\}$ of X is the intersection of all open sets containing it.

1.1.2 Theorem. [11] Let X be a topological space. Then the following are equivalent:

1. X is T_2 , that is, every pair of distinct points $x, y \in X$ belongs respectively to disjoint open sets.

2. The intersection of the closed neighborhoods of any point of X consists of that point alone.

3. The diagonal of the product space $X \times X$ is a closed set.

1.1.3 Definition. [14] A topological space X is completely regular if whenever F is a closed set in X and $x \in X$ such that $x \notin F$, there is a continuous map $f: X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(F) = 1$.

1.1.4 Remark. In next chapter, when we mention a completely regular space X we mean in addition that it agrees with the above definition and that it is also T_1 , that is $T_{3\frac{1}{2}}$.

1.1.5 Definition. [30] A topological space X is locally compact if each point in X has a compact neighborhood.

1.1.6 Definition. [30] A topological space X is called regular provided for every closed subset F of X , and a point $x \in X$ with $x \notin F$ there are two disjoint open neighborhoods U and V with $x \in U$ and $F \subseteq V$.

1.1.7 Remark.

1. In a T_2 locally compact space X each point has a neighborhood whose closure is compact, then we often shall say that points of X have a compact closure neighborhood. Over the next chapters, locally compact space is always meant T_2 , otherwise we call it locally quasi-compact.
2. A T_2 locally compact space X has a property that every point $x \in X$ there is a collection of neighborhoods $\{U_n\}$ of x with $\bar{U}_{n+1} \subseteq U_n$ where \bar{U}_{n+1} is compact for every positive integer n . It is also important to refer here that the collection of neighborhoods above may not be neighborhood base for the point x .

1.1.8 Definition. [30] A set D is a directed set if there is a relation \leq on D satisfying:

1. $\alpha \leq \alpha$ for each $\alpha \in D$.
2. If $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_3$ then $\alpha_1 \leq \alpha_3$.
3. If $\alpha, \beta \in D$, then there is some $\gamma \in D$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

1.1.9 Definition. [30] A net in a set X is a map $x: D \rightarrow X$, where D is a directed set. The image $x(\alpha)$ is usually denoted by x_α and we often use the notation $\{x_\alpha\}_{\alpha \in D}$ or $\{x_\alpha\}$. A subnet of a net $x: D \rightarrow X$ is a composition $x \circ \varphi$, where $\varphi: M \rightarrow D$ is a map and M is a directed set such that

1. $\varphi(\alpha_1) \leq \varphi(\alpha_2)$ whenever $\alpha_1 \leq \alpha_2$.
2. For each $\alpha \in D$, there is $\gamma \in M$ such that $\alpha \leq \varphi(\gamma)$. For $\gamma \in M$ the image $x \circ \varphi(\gamma)$ is written as x_{α_γ} .

1.1.10 Definition. [30] Let $\{x_\alpha\}$ be a net in a topological space X . Then $\{x_\alpha\}$ converges to $x \in X$ (written $x_\alpha \rightarrow x$) provided it is eventually in every neighborhood of x . That is, for every neighborhood U of x there is an $\alpha_0 \in D$ such that $x_\alpha \in U$ for each $\alpha \geq \alpha_0$. We say $\{x_\alpha\}$ has a point x as a cluster point provided $\{x_\alpha\}$ is frequently in every neighborhood of x , i.e., for every neighborhood U of x and every $\alpha_0 \in D$, there is some $\alpha \in D$ with $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.

1.1.11 Theorem. [30] A net $\{x_\alpha\}$ has y as a cluster point if and only if has a subnet which converges to y .

1.1.12 Theorem. [30] If $E \subset X$, then $x \in \bar{E}$ if and only if there is a net $\{x_\alpha\}$ in E with $x_\alpha \rightarrow x$.

1.1.13 Theorem. [30] Let $f: X \rightarrow Y$ be a map. Then f is continuous at x_0 in X if and only if whenever $x_\alpha \rightarrow x_0$ in X , then $f(x_\alpha) \rightarrow f(x_0)$ in Y .

1.1.14 Theorem. [30] For a topological space X , the following are equivalent:

- a) X is called compact (\cong for every open cover of X has a finite subcover).
- b) Every net in X has a cluster point.

1.1.15 Remark.

1. A net $\{x_\alpha\}$ in X is said to have no a convergent subnet in X is written as $x_\alpha \rightarrow \infty$ if and only if every subnet of $\{x_\alpha\}$ has no a limit point [30].
2. If X is T_2 , then every convergent net in X has a unique limit [30].
3. Over the next chapters, compact space will be always T_2 . Otherwise, we call it quasi-compact.
4. From now on, a map will always mean a continuous mapping.

1.1.16 Definition. [24] Let X be a topological space. A collection \mathcal{A} of subsets of X is said to be locally finite in X if every point of X has a neighborhood that intersects only finitely many elements of \mathcal{A} .

1.1.17 Lemma. [24] Let \mathcal{A} be a locally finite collection of subsets of X . Then:

- a. Any subcollection of \mathcal{A} is locally finite.
- b. The collection of the closures of the elements of \mathcal{A} is locally finite.
- c. $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}$.

1.1.18 Definition. [24] Let \mathcal{A} be a collection of subsets of the space X . A collection \mathcal{B} of subsets of X is said to be a refinement of \mathcal{A} (or is said to refine \mathcal{A}) if for each element B of \mathcal{B} there is an element A of \mathcal{A} containing B . If the elements of \mathcal{B} are open sets, we call \mathcal{B} an open refinement of \mathcal{A} ; if they are closed sets, we call a closed refinement.

1.1.19 Definition. [24] A space X is paracompact if every open covering \mathcal{A} of X has a locally finite open refinement \mathcal{B} that covers X .

1.1.20 Example. The set of real numbers \mathbb{R} with the usual topology is paracompact. If \mathcal{A} is an open cover of \mathbb{R} . Consider a collection of open intervals $O_n = (-n, n)$, $n = 0, 1, 2, \dots$ where $O_0 = \emptyset$. Since \bar{O}_n is compact, then there is a

finite number of members of \mathcal{A} , say A_1, A_2, \dots, A_{r_n} , covers \bar{O}_n . Put $B_i = A_i \cap (\mathbb{R} \setminus \bar{O}_{n-2})$, and $\mathcal{B}_n = \{B_1, B_2, \dots, B_{r_n}\}$. Then the collection $\mathcal{B} = \cup \mathcal{B}_n$ is locally finite and a refinement of \mathcal{A} .

1.2 Topological Group Concepts

1.2.1 Definition. [18] A topological group is a set G with two structures:

1. G is a group.
2. G is a topological space.

such that the two structures are compatible; that is, the multiplication law $\mu: G \times G \rightarrow G$ and the inversion law $\nu: G \rightarrow G^{-1}$ are both continuous.

1.2.2 Remark.

1. The two conditions in definition (1.2.1) are equivalent to the fact that the map $\tau: G \times G \rightarrow G$ is defined by $\tau(g_1, g_2) = g_1 g_2^{-1}$ is continuous.
2. A continuous map $l_g: G \times G \rightarrow G$ defined by $x \rightarrow gx$ is called the left translation by g . This map has inverse $l_{g^{-1}}$ which is also continuous.

Moreover, l_g is a homeomorphism. Similarly, all right translations $r_g: G \rightarrow G$ are homeomorphisms.

3. A topological space X which satisfies the property that for every pair of points $x, y \in X$, then there is a homeomorphism on X sending x to y is called homogeneous. Therefore, every topological group is homogeneous. A map $l_{yx^{-1}}$ (or $r_{x^{-1}y}$) send a point x into y .
4. Every group admits a topological group structure with discrete topology.

1.2.3 Proposition. [18] Let G be a topological group, and A, B be subsets of G .

Then for $g \in G$ we have:

1. $Ag = Ar_g = \{ag: a \in A\}$; Ag is called the right translate of A by g .
2. $gA = Al_g = \{ga: a \in A\}$; gA is called the left translate of A by g .
3. $AB = \bigcup_{b \in B} Ab = \bigcup_{a \in A} aB$.
4. $A^{-1} = \{a^{-1}: a \in A\}$.

1.2.4 Proposition. [18] Let G be a topological group and $A, B \subset G$ and $g \in G$.

Then:

1. If A is open, then Ag and gA are open.
2. If A is closed, the Ag and gA are closed.
3. If A is open, then AB and BA are open.
4. If A is closed and B is finite, then AB and BA are closed.

1.2.5 Remark. Let G be a topological group and \mathcal{F} be a fundamental system of neighborhoods of the identity element e in G . Then for every $g \in G$ the collection $\mathcal{F}g = \{Ug: U \in \mathcal{F}\}$ is a fundamental system of open neighborhoods of g .

1.2.6 Proposition. [18] Let \mathcal{F} be a fundamental system of open neighborhoods of the identity element e . Then

1. If $U, V \in \mathcal{F}$, there is $W \in \mathcal{F}$ such that $W \subseteq U \cap V$.
2. If $g \in U \in \mathcal{F}$, then there is $V \in \mathcal{F}$ such that $gV \subseteq U$.
3. If $U \in \mathcal{F}$, $g \in G$, then there is $V \in \mathcal{F}$ such that $g^{-1}Vg \subseteq U$.
4. If $U \in \mathcal{F}$, then there is $V \in \mathcal{F}$ such that $VV^{-1} \subseteq U$.

1.2.7 Remark. Any fundamental system of open neighborhoods of the identity element e , say \mathcal{F} , also satisfies: $\forall U \in \mathcal{F}, \exists V, W \in \mathcal{F}$ such that $V^{-1} \subseteq U$ and $W^2 \subseteq U$.

1.2.8 Proposition. Let G be a topological group. Then G is regular.

Proof. Take a closed subset F of G and a point $e \in G$ such that $e \notin F$. Then there is a basic neighborhood $U \in \mathcal{F}$ such that $UU^{-1} \subseteq F^c$ or $UU^{-1} \cap F = \emptyset$. Then $U \cap$

$FU = \emptyset$. FU is a neighborhood of F . By homogenous property of G . Therefore, G is regular. ■

1.2.9 Proposition. A topological group G is T_1 if and only if the set $\{e\}$ is closed.

Proof. If $\{e\}$ is closed then for every point $g \in G$, the set $g\{e\} = \{g\}$ is closed.

Thus, G is T_1 . The other side of proof is obvious. ■

1.2.10 Proposition. Let G be a topological group and H be a subgroup of G . Then the following properties hold:

1. H is also a topological subgroup. The topology on H is just the relative topology of the topology of G .
2. The left (or right) coset set $G/H = \{gH : g \in G\}$ is endowed, in a natural way, by a quotient topology induced by the canonical quotient map $q: G \rightarrow G/H$. This topology makes the map q is open and continuous and G/H is regular.
3. If H is a normal subgroup of G , then the coset space is also a topological group.
4. The coset space G/H is T_2 if and only if H is a closed subgroup of G .
5. If H is an open subgroup, then it is also closed.

Proof. It is obvious. ■

1.2.11 Proposition. Let G be a topological group and $\{g_\alpha\}$ be a net in G . Then the following hold.

1. $\{gg_\alpha\}$ (or $\{g_\alpha g\}$) is a net in G for each $g \in G$.
2. If $g_\alpha \rightarrow e$, then $gg_\alpha \rightarrow g$ (or $g_\alpha g \rightarrow g$) for each $g \in G$.
3. If $g_\alpha \rightarrow \infty$, then $gg_\alpha \rightarrow \infty$ (or $g_\alpha g \rightarrow \infty$) for each $g \in G$.
4. If $g_\alpha \rightarrow \infty$, then $g^{-1} \rightarrow \infty$.

Proof. The proof of all these statements follows from the continuity of multiplication and inversion maps. ■

1.2.12 Definition. A topological group G is called quasi-compact provided the underlying topological space of G is a quasi-compact, that is, for every open cover of G has a finite subcover. Moreover, G is called locally quasi-compact whenever G is a locally quasi-compact as topological space and this is equivalent to that there is a quasi-compact neighborhood of the identity element e of G .

1.2.13 Definition. A topological group G is called connected provided the underlying topological space G is connected, that is, there are not two disjoint clopen subsets U and V of G separating G . Moreover, G is called locally connected if each point g in G has a connected neighborhood. If the only connected subspaces of G are the singletons, then G is called totally disconnected.

1.2.14 Components and identity components

Let G be a topological group and let R be a relation on G is defined by for every $x, y \in G$: xRy if and only if there is a connected subspace of G containing both x , and y . R is an equivalence relation on G . Therefore, R decomposes G into maximal disjoint connected subspaces are called components. The component that contains the identity element e is called the identity component is denoted by G_0 . In general, the components are closed but not open.

1.2.15 Remark.

1. The identity component G_0 is a closed normal subgroup of G .
2. The quotient G/G_0 is always a totally disconnected topological group.

1.2.16 Some properties of quasi-compact and locally quasi-compact group.

If G is a topological group and H is a subgroup of G , then the following hold:

1. If G is locally quasi-compact, then every quasi-compact subset K of G has a quasi-compact neighborhood.
2. If G is quasi-compact (or locally quasi-compact) group, and H is closed, then H and G/H are quasi-compact (locally quasi-compact).
3. If G is a quasi-compact group and H is closed. Then for $g \in G$, $gHg^{-1} \subseteq H$ if and only if $gHg^{-1} = H$. [12]
4. The closure \bar{H} is also a subgroup of G . Moreover, if H is normal then so is \bar{H} .
5. If C is a quasi-compact subset of G , then \bar{C} is quasi-compact since \bar{C} is the image of C quotient map $G \rightarrow G/\{e\}$. [17]

1.3 Lie Groups

In this section, we recall the concept of Lie group and some its relative properties, and we will not go deeply in this concept. However, it forms background for what we study in the chapter four.

1.3.1 Definition. [22] Let M be a topological space and \mathbb{R}^n be n -dimensional Euclidean space. A pair (U, φ) consisting of an open set U in M and a homeomorphism φ of U in M onto an open set in \mathbb{R}^n , is called a chart (or coordinate neighborhood) of M . When we write

$$\varphi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)), \quad x \in U,$$

$\varphi_i(x)$ is called *ith* coordinate function. If a point x of M lies in U , then we say that (U, φ) is a chart at x .

A collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ is called an atlas (or system of coordinate neighborhoods) on M if $\{U_\alpha\}$ forms a cover of M . The atlas is smooth if each pair of indices α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\varphi_\beta \cdot \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth (that is, of the class \mathbb{C}^∞).

1.3.2 Definition. [22] Two smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\alpha, \varphi_\alpha)\}$ are said to be equivalent if the union $\{(U_\alpha, \varphi_\alpha)\} \cup \{(V_\alpha, \varphi_\alpha)\}$ is smooth, i.e., if the map

$$\varphi_\beta \cdot \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap V_\beta) \rightarrow \varphi_\beta(U_\alpha \cap V_\beta)$$

is a diffeomorphism for each pair (α, β) with $U_\alpha \cap V_\beta \neq \emptyset$. An equivalence class of atlases on M is called a differential or smooth structure on M and we denote the equivalence class of an atlas $\{(U_\alpha, \varphi_\alpha)\}$ by $[(U_\alpha, \varphi_\alpha)]$.

1.3.3 Definition. [20] A smooth manifold is a pair $(M, [(U_\alpha, \varphi_\alpha)])$ where M is a Hausdorff space satisfying the second axiom of countability and $[(U_\alpha, \varphi_\alpha)]$ is a smooth structure on M . We also say that $(M, [(U_\alpha, \varphi_\alpha)])$ is a smooth n -manifold when we want to emphasize the dimension of \mathbb{R}^n .

1.3.4 Definition. [22] G is a Lie group provided that:

- 1) G is a Manifold.
- 2) G is a group.
- 3) The group operation is smooth, that is, the maps $\alpha: G \times G \rightarrow G$ and $\beta: G \rightarrow G$ defined by $\alpha(g, h) = gh$ and $\beta(g) = g^{-1}$ both smooth. Here $G \times G$ is endowed with the structure of the product manifold.

1.3.5 Example. $Gl(n, \mathbb{C})$ is one of the classic Lie groups.

Properties of Lie groups

1.3.6 Theorem. [20] A closed subgroup H of a Lie group G is an embedded Lie subgroup.

1.3.7 Theorem. [20] If H is a closed subgroup of a Lie group G , then there exists a unique smooth structure on the space G/H such that:

- 1) The projection $q: G \rightarrow G/H$ is smooth.
- 2) The projection $q: G \rightarrow G/H$ has a smooth local cross-section.

1.3.8 Corollary. [20] If H is a closed normal subgroup of a Lie group G , then G/H is a Lie group and the natural projection $q: G \rightarrow G/H$ is a Lie group homomorphism.

1.3.9 Proposition. [20] Every Lie group G is locally compact.

1.3.10 Proposition. [20] Every Lie group G is locally connected.

1.3.11 Definition. [19] A topological group has no small subgroups (*NSS*) if there exists a neighborhood U of the identity element e such that the only subgroup in U is $\{e\}$.

1.3.12 Proposition. [19] Every Lie group G is *NSS*.

1.3.13 Proposition. [19] Locally compact G of *NSS* property is a Lie group.

1.4 Topological Transformation Groups

1.4.1 Definition. [12] A topological transformation group is a triple (G, X, φ) where G is a topological group and X is a topological space and $\varphi: G \times X \rightarrow X$ is a continuous map such that:

- a. $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ for every $g_1, g_2 \in G, x \in X$.
- b. $\varphi(e, x) = x$ for every $x \in X$, where e the identity element of G .

1.4.2 Remark.

1. The map φ is called an action of G on X and the space X together with φ is called a G - space (or more precisely left G - space).
2. Since φ is understood from the context we may use a notation gx for $\varphi(g, x)$ and $g_1(g_2x) = (g_1g_2)x$ for $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1g_2, x)$.
3. For $H \subseteq G$ and $A \subseteq X$ we write $HA = \{ga \mid g \in H, a \in A\}$ for $\varphi(H, A)$.
4. A subset $A \subseteq X$ is said to be invariant under G if $GA = A$.
5. For $g \in G$, a map $\varphi_g: X \rightarrow X$ is defined by $\varphi_g(x) = \varphi(g, x) = gx$ is a continuous. Also, $\varphi_{g_1} \varphi_{g_2} = \varphi_{g_1g_2}$ and $\varphi_e = I_X$ is the identity map of X . A map φ_g has an inverse continuous map $\varphi_{g^{-1}}$. Therefore, for every $g \in G$, the map φ_g is a homeomorphism of X .
6. The set $\Gamma(X) = \{\varphi_g: g \in G\}$ forms a group under maps composition, and there is a natural group homomorphism $\Psi: G \rightarrow \Gamma(X)$ is given by $g \rightarrow \varphi_g$. Therefore, an action of a topological group G on a space X is determined by a fixed homomorphism Θ from G into the group of all homeomorphisms of X $Homeo(X)$ is given by $gx = f(x)$, where $f \in Homeo(X)$. The kernel of the action is exactly the kernel of the map Θ .

1.4.3 Definitions and notations.

For a G -space X , and $x \in X$ a subset $Gx = \{gx: g \in G\}$ is called the G -orbit through a point x . Let R be a relation on X is defined such that $(x, y) \in R$ if and only if there is $g \in G$ such that $gx = y$, that is, $(x, y) \in R$ if and only if both x , and y belong to the same orbit. Thus, G -orbits form a partition on X . The set of all G -orbits is denoted by X/G and it is called orbit space. One may endow the orbit space with a quotient topology and a natural open quotient map $\pi: X \rightarrow X/G$ which sends every point x into its orbit Gx . For a point $x \in X$, a subset $G_x = \{g \in G: gx = x\}$ is a subgroup of G is called the stabilizer subgroup over x . A kernel of

an action Θ on a space X can be defined as $\ker\Theta = \bigcap_{x \in X} G_x$. An action of G on a space X is called free if $G_x = \{e\}$ for every $x \in X$. It is called effective if $\ker\Theta = \{e\}$ while it called transitive if the orbit space X/G is just one element.

1.4.4 Definition. [12] An equivariant map (continuous) (or a G -map) of left (right) G -spaces X and Y is a map $f: X \rightarrow Y$ that is preserved the action operation of G , that is, for every $g \in G$, and $x \in X$, then $f(gx) = gf(x)$. An equivariant map which is also homeomorphism is called an equivalence.

1.4.5 Some properties of equivariant maps

1. If Y is a G -space and X is a topological space such that there is a map $f: X \rightarrow Y$ with the property $f(gx) = gf(x)$, for every $g \in G$, and $x \in X$, then the map f admits an action on X by G is given by $gx = gf(x)$.
2. If $f: X \rightarrow Y$ is an equivariant map and $x \in X$, then $f(Gx) \subseteq Gf(x)$, that is, f maps an orbit in X into an orbit in Y . Moreover, $f(G_x) \subseteq G_{f(x)}$.
3. A G -map f is called isovariant if it is one to one on each orbit, that is, $f(G_x) = G_{f(x)}$. [28]

Proper mapping and proper action

1.4.6 Definition. [30] A map $f: X \rightarrow Y$ of topological space X and Y is called closed (open) if the directed image of f for every closed subset of X is closed (open) in Y .

1.4.7 Definition. [11] Let f be a mapping of a topological space X into a topological space Y . f is said to be proper if f is a continuous and the mapping $f \times i_Z: X \times Z \rightarrow Y \times Z$ is closed, for every topological space Z .

1.4.8 Proposition: [11] Let $f: X \rightarrow Y$ be a continuous injection. Then the following three statements are equivalent:

1. f is proper.
2. f is closed.
3. f is a homeomorphism of X onto a closed subset of Y .

1.4.9 Theorem. [11] Let $f: X \rightarrow Y$ be a mapping. Then the following four statements are equivalent:

1. f is proper.
2. f is closed and $f^{-1}(y)$ is quasi-compact for every $y \in Y$.
3. If \mathcal{F} is a filter on X and if $y \in Y$ is a limit point of $f(\mathcal{F})$, then there is a cluster point x of \mathcal{F} such that $f(x) = y$.
4. If \mathcal{U} is an ultrafilter on X and if $y \in Y$ is a limit point of the ultrafilter base $f(\mathcal{U})$, then there is a limit point x of \mathcal{U} such that $f(x) = y$.

1.4.10 Definition. [29] Let X be a G -space. A subset A of X is said to be thin relative to a subset B of X if the closure of the set $((A, B)) = \{g \in G \mid gA \cap B \neq \varnothing\}$ is quasi-compact in G , and we will say that A and B are relatively thin. If A is thin relative to itself then it is called thin. X is called Cartan if every point $x \in X$ has a thin neighborhood.

1.4.11 Remark. If X is a G -space and A, B are subsets of X . Then

1. Since $(gA \cap B) = g(A \cap g^{-1}B)$ it follows that if A is thin relative to B , then B is thin relative to A .
2. Since $(gg_1A \cap g_2B) = g_2(g_2^{-1}gg_1A \cap B)$ it follows that if A and B are relatively thin, then so are any their translates g_1A and g_2B .
3. If A and B are relatively thin and $K \subset A, M \subset B$, then K and M are relatively thin.
4. If K_1, K_2 are quasi-compact subsets of X . Then $((K_1, K_2))$ is closed in G .

5. If K_1 and K_2 are quasi-compact subsets of X such that K_1 and K_2 are relatively thin, then $((K_1, K_2))$ is quasi-compact.
6. If U is a neighborhood of x in X , then $((U, U))$ is a neighborhood of the identity element e in G . So, if X is Cartan G -space, then it essential that G is locally quasi-compact.
7. If A , and B are subsets of X , then $((A, B))^{-1} = ((B, A))$.
8. If A , and B are subsets of X and $g \in G$, then $((A, gB)) = g((A, B))$ and $((A, gB)) = ((A, B))g^{-1}$.

1.4.12 Definition. (Bourbaki Proper) Let G be a topological group operating (acting) continuously on a topological space X . G is said to operate (act) properly on X if the mapping $\theta: (s, x) \rightarrow (x, sx)$ of $G \times X$ into $X \times X$ is proper.

1.4.13 Proposition. [11] If a topological group G operates properly on a topological space X , then the orbit space X/G is T_2 .

1.4.14 Definition. (Palais Proper) A subset S of a G -space X is a small subset of X if each point of X has a neighborhood which is thin relative to S . If G is a locally compact (T_2 is included) and X is a completely regular, then G acts properly on X provided that every point $x \in X$ has small neighborhood.

1.4.15 Remark. If X is a G -space, in the sense of Palais, then the following hold:

1. For each $x \in X$, then Gx , orbit of x , is closed in X .
2. The stabilizer subgroup G_x is compact for every $x \in X$.
3. Palais proper is Bourbaki proper and Cartan.

Chapter Two

Limit and Strongly Limit Sets

2.1 Basic Concepts and Convergence Techniques.

2.1.1 Definition. [11] A map $f: X \rightarrow Y$ of topological spaces X and Y is called quasi-compact if a preimage of every quasi-compact subset of Y is a quasi-compact subset of X .

2.1.2 Definition. Let x_α be a net in a topological space X . A subset $clu(x_\alpha)$ of X is the set of all limit points of subnets of the net x_α . If the net x_α has no convergent subnet, that is, $x_\alpha \rightarrow \infty$, then is clear that $clu(x_\alpha) = \emptyset$.

2.1.3 Proposition. If x_α is a net in space X , then the set $clu(x_\alpha)$ is a closed subset of X .

Proof. Take $z \in \overline{clu(x_\alpha)}$, there is a net $z_\alpha \in clu(x_\alpha)$ and $z_\alpha \rightarrow z$. For each α , there is a net $z_\gamma^\alpha \rightarrow z_\alpha$. Then there is a diagonal subnet z_γ^γ of the net z_α which converges to z . therefore, $z \in clu(x_\alpha)$ and $clu(x_\alpha)$ is closed. ■

2.1.4 Theorem. Let $f: X \rightarrow Y$ be a continuous mapping. Then the following are equivalent:

- (i) f is proper.
- (ii) If $\{x_\alpha\}$ is a net in X and y is a cluster point of a net $f(x_\alpha)$, then there is a point x in X such that x is a cluster point of $\{x_\alpha\}$ with $f(x) = y$.

Proof. If f is proper. Take $\{x_\alpha\}_{\alpha \in D}$ a net in X and y is a point in Y such that $f(x_\alpha) \propto y$, where the notation \propto meant that the net $f(x_\alpha)$ has y as cluster point.

Put for every $\gamma \in D$ a subset $E_\gamma = \{x_\alpha \mid \alpha \geq \gamma\}$. It is easy to check that the family $\{E_\alpha\}_{\alpha \in D}$ satisfies the property of finite intersection. Since f is closed, we have $f(\bar{E}_\alpha) = \overline{f(E_\alpha)}$ for all $\alpha \in D$. To show that for all $\alpha \in D$, $y \in \overline{f(E_\alpha)}$. Let U be a neighborhood of y and $\alpha \in D$. Then there is $\gamma \in D$ such that $\gamma \geq \alpha$ and $f(x_\gamma) \in U$. Since $f(x_\gamma) \in f(E_\alpha)$, then $f(E_\alpha) \cap U \neq \emptyset$ and hence $y \in \overline{f(E_\alpha)}$ for all $\alpha \in D$. For every $\alpha \in D$, $\bar{E}_\alpha \cap f^{-1}(y)$ is non-empty closed subset of $f^{-1}(y)$ which is a quasi-compact. The family $\{\bar{E}_\alpha \cap f^{-1}(y)\}_{\alpha \in D}$ satisfies the finite intersection property, that is, $\bigcap \{\bar{E}_\alpha \cap f^{-1}(y)\} = \bigcap \bar{E}_\alpha \cap f^{-1}(y) \neq \emptyset$. Then there is a point $x \in X$ such that $x \in \bigcap \bar{E}_\alpha$ and $x \in f^{-1}(y)$. We claim that $x_\alpha \rightarrow x$. Take U to be a neighborhood of x in X and $\delta \in D$. Then there is $\alpha \in D$ and $\alpha \geq \delta$ such that $E_\alpha \cap U \neq \emptyset$. Thus, there is $\tau \in D$ with $\tau \geq \alpha$ and $x_\tau \in E_\alpha \cap U$. Therefore $x_\alpha \rightarrow x$ and $f(x) = y$. Conversely, if the condition (ii) is satisfied, then $f^{-1}(y)$ is quasi-compact for all $y \in Y$, otherwise there is a net $\{x_\alpha\}$ in $f^{-1}(y)$ with $x_\alpha \rightarrow \infty$ and $f(x_\alpha) \rightarrow y$ which is a contradiction. To show that f is closed. Take a closed subset F of X and $y \in \overline{f(F)}$, then there is a net $\{x_\alpha\}$ in F such that $f(x_\alpha) \rightarrow y$. By hypotheses $\{x_\alpha\}$ has a cluster point, say $x \in X$ such that $f(x) = y$. Since F is closed then $x \in F$. Therefore, $f(x) = y \in f(F)$ and f is proper. ■

Al Sarray M.J. in [1] has presented a characterization for a proper mapping called an exceptional set for a map under the Hausdorff assumption on a codomain space Y .

2.1.5 Definition. [1] (**Exceptional Set**): Let $f: X \rightarrow Y$ be a map. We define the set:

$$E_f = \{y \in Y: \exists \text{ a net } \{x_\alpha\} \text{ in } X \text{ with } x_\alpha \rightarrow \infty \text{ and } f(x_\alpha) \rightarrow y\}.$$

We call E_f the exceptional set of f .

2.1.6 Definition. [2] (Strongly Exceptional Set): Let $f: X \rightarrow Y$ be a map. We define the set:

$$SE_f = \{y \in Y: \exists \text{ a net } \{x_\alpha\} \text{ in } X \text{ with } f(x_\alpha) \rightarrow y \text{ and } cls(x_\alpha) \cap f^{-1}(y) = \emptyset\}$$

We call SE_f the strongly exceptional set of f .

2.1.7 Remark. In general, $E_f \subseteq SE_f$, with equality demonstrated in the following proposition.

2.1.8 Proposition. Let $f: X \rightarrow Y$ be a continuous map and Y is T_2 . Then $E_f = SE_f$.

Proof. It suffices to show that $SE_f \subseteq E_f$. Take $z \in SE_f$, then there is a net x_α in X such that $f(x_\alpha) \rightarrow z$ and $clu(x_\alpha) \cap f^{-1}(z) = \emptyset$. Suppose that there is $x \in clu(x_\alpha)$. Then there is a subnet x_{α_β} of x_α such that $x_{\alpha_\beta} \rightarrow x$. By continuity of f , we have $f(x_{\alpha_\beta}) \rightarrow f(x)$ and by T_2 property, we have $f(x) = z$. Thus, we have $x \in f^{-1}(z)$ which is a contradiction. Therefore, $clu(x_\alpha) = \emptyset$ and $x_\alpha \rightarrow \infty$. ■

2.1.9 Proposition. Let $f: X \rightarrow Y$ be a map of topological spaces. Then the set E_f is a closed subset of Y .

Proof. Let $y \in \bar{E}_f$. Then there is a net $\{y_\alpha\}_{\alpha \in M} \in E_f$ such that $y_\alpha \rightarrow y$. For every $\alpha \in M$, there is a net $\{x_\beta^\alpha\}_{\beta \in M_\alpha}$ in X with $x_\beta^\alpha \rightarrow \infty$ and $f(x_\beta^\alpha) \rightarrow y_\alpha$. Thus, there is a diagonal net $x_{\beta_\alpha}^\alpha$ in X with $x_{\beta_\alpha}^\alpha \rightarrow \infty$ and $f(x_{\beta_\alpha}^\alpha) \rightarrow y$. Therefore $y \in E_f$ and E_f is closed. ■

2.1.10 Remark. In general, SE_f is not closed. For example, let $X = \{a\}$ and $Y = \{1,2\}$ with the indiscrete topology and let $f: X \rightarrow Y$ be defined by $a \rightarrow 1$. Then, $E_f = \emptyset$ and $SE_f = \{2\}$ which is not closed in Y .

2.1.11 Theorem. [1] Let Y be Hausdorff. Then a continuous mapping $f: X \rightarrow Y$ of X and Y is proper if and only if $E_f = \emptyset$.

Proof. If f is proper, by the Bourbaki Theorem, it is easy to show that $E_f = \emptyset$. Conversely, assume that Y is Hausdorff and the set $E_f = \emptyset$. Take $y \in Y$. Since Y is T_2 , $f^{-1}(y)$ is closed subset of X . Suppose that $f^{-1}(y)$ is not quasi-compact. Then there is a net $x_\alpha \rightarrow \infty$ in $f^{-1}(y)$ and hence in X . So, we have $f(x_\alpha) = y \rightarrow y$ which implies that $y \in E_f$, which is a contradiction. To show that the map f is closed. Take F is a closed subset of X , and $y \in \overline{f(F)}$. Then there is a net x_α in F such that $f(x_\alpha) \in f(F)$ and $f(x_\alpha) \rightarrow y$. Since $E_f = \emptyset$, the net x_α must have a subnet x_{α_β} converges to a point in X , say x . Since F is closed, x is also in F . So, by continuity of f , $f(x_{\alpha_\beta}) \rightarrow f(x)$. Since Y is T_2 , $f(x) = y \in f(F)$. Therefore, $f(F)$ is closed and f is a closed map. ■

2.1.12 Remark. The T_2 condition in first direction of the above theorem is not necessarily, however if Y is not T_2 , the second direction is not true. See the example in the remark (21.10), it is obvious that every mapping from X into Y is not proper, since it is not closed. Since X is compact, then $E_f = \emptyset$.

2.1.13 Proposition. Every proper map is a quasi-compact map.

Proof. Let $f: X \rightarrow Y$ be a continuous map. Assume that f is proper. Take a quasi-compact subset K of Y . Suppose that $f^{-1}(K)$ is not quasi-compact, then there is a net x_α in $f^{-1}(K)$ such that $x_\alpha \rightarrow \infty$. Then a net $f(x_\alpha) \in K$ has convergent subnet, say itself. So, $f(x_\alpha) \rightarrow k \in K$. Since $E_f = \emptyset$, this implies that the net x_α must have a convergent subnet which is a contradiction. ■

2.1.14 Theorem. [2] A continuous map $f: X \rightarrow Y$ is proper if and only if $SE_f = \emptyset$.

Proof. Assume that $SE_f = \emptyset$. To prove f is closed and quasi compact, if F is a closed subset of X and $y \in \overline{f(F)}$, then there is a net $f(x_\alpha)$ in $f(F)$ converging to y . $SE_f = \emptyset$ implies $f^{-1}(y) \cap clu(x_\alpha) \neq \emptyset$. Then there is a point $x \in f^{-1}(y) \cap clu(x_\alpha)$ and hence there is a subnet $x_{\alpha\beta}$ of x_α such that $x_{\alpha\beta} \rightarrow x \in F$. Since F is closed, and $f(x) = y \in f(F)$. Therefore, $f(F)$ is closed. Now, if K is a quasi-compact subset of Y and x_α is a net in $f^{-1}(K)$, then there is a point $y \in K$ such that $f(x_\alpha) \rightarrow y$. $SE_f = \emptyset$ implies that there is a point $x \in f^{-1}(y) \cap clu(x_\alpha) \neq \emptyset$. Thus, there is a subnet $x_{\alpha\beta}$ of x_α such that $x_{\alpha\beta} \rightarrow x \in f^{-1}(y) \subseteq f^{-1}(K)$. Therefore, $f^{-1}(K)$ is quasi compact. Conversely, assume that f is closed and $f^{-1}(y)$ is quasi compact for every $y \in Y$. Suppose that there is a point $y \in SE_f$. Then, there is a net $x_\alpha \in X$ such that $f(x_\alpha) \rightarrow y$ and $f^{-1}(y)$ does not meet $clu(x_\alpha)$. Since f is closed, then $f^{-1}(y) \neq \emptyset$ and a quasi-compact then for every point $z \in f^{-1}(y)$ there is an open set U_z containing z and there is α_z such that $x_\alpha \notin U_z$ for each $\alpha > \alpha_z$. The collection $\{U_z\}_{z \in f^{-1}(y)}$ forms an open cover to $f^{-1}(y)$, which is quasi compact, there is a finite subcover $\{U_{z_1}, U_{z_2}, U_{z_3}, \dots, U_{z_n}\}$ and an index δ such that $x_\alpha \notin U = \bigcup_{i=1}^n U_{z_i}$ for each $\alpha > \delta$. Put $V = Y \setminus f(X \setminus U)$. f is closed, and V is open containing y . However, $f(x_\alpha) \notin V$ for each $\alpha > \delta$ which contradicts that $f(x_\alpha)$ converges to y . Therefore $SE_f = \emptyset$. ■

2.2 Exceptional Sets in G -space Theory

Recall that if X is G -space, we say G acts properly (Bourbaki proper) on X if the map $\theta: G \times X \rightarrow X \times X$ is given by $(g, x) \rightarrow (x, gx)$ is a proper mapping. So, if θ is proper, then the set $E_\theta = \emptyset$. Let us verify the definition of E_θ in G -space

notations in more explicitly. By the definition of exceptional set E_θ , it equals the set:

$$\{(x, y) \in X \times X: \exists \text{ a net } (x_\alpha, g_\alpha) \rightarrow \infty \text{ in } G \times X \text{ such that } (x_\alpha, g_\alpha x_\alpha) \rightarrow (x, y)\}.$$

Since the net $(x_\alpha, g_\alpha x_\alpha) \rightarrow (x, y)$, then $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow y$. Also, $(x_\alpha, g_\alpha) \rightarrow \infty$ implies to $g_\alpha \rightarrow \infty$. We can rewrite the above set as following:

$$\{(x, y) \in X \times X: \exists \text{ a net } x_\alpha \rightarrow x \text{ in } X \wedge \text{ a net } g_\alpha \rightarrow \infty \text{ in } G \text{ such that } g_\alpha x_\alpha \rightarrow y\}.$$

Since the point x , the first coordinate of the above set, is determined by a net converges to it. We may put $E_\theta = \Lambda(x)$ if $x_\alpha = x$ or, in general, $E_\theta = J(x)$ if $x_\alpha \neq x$. That means the sets $\Lambda(x)$ and $J(x)$ reflect the properties of the set E_θ .

2.2.1 Definition. Let X be a G -space and $x \in X$. We define the limit set and its prolongation associated with the point x as following:

$$J(x) = \{y \in X: \text{there is a net } \{x_\alpha\} \text{ in } X \text{ and a net } \{g_\alpha\} \text{ in } G \text{ with } x_\alpha \rightarrow x \text{ and } g_\alpha \rightarrow \infty \text{ such that } g_\alpha x_\alpha \rightarrow y\}.$$

$$\Lambda(x) = \{y \in X: \text{there is a net } \{g_\alpha\} \text{ in } G \text{ with } g_\alpha \rightarrow \infty \text{ and } g_\alpha x \rightarrow y\}.$$

2.2.2 Proposition. Let X be a G -space and $x \in X$. Then,

- (i) For every x in X , $\Lambda(x)$, $J(x)$ are invariant sets under G .
- (ii) If an orbit Gx is closed, then $\Lambda(x) \subseteq Gx$.
- (iii) For every x in X , $\Lambda(x)$, $J(x)$ are closed.
- (iv) $\overline{Gx} \supseteq Gx \cup \Lambda(x)$.
- (v) $y \in J(x)$ if and only if $x \in J(y)$.
- (vi) If X is the discrete G - space, then $\Lambda(x) = J(x)$ for all x in X .
- (vii) If $x \in J(x)$, then for each $y \in Gx$, $y \in J(x)$.
- (viii) If $y \in J(x)$, then for each $z \in Gx$, $y \in J(z)$.

(ix) For every $g \in G$ and $x \in X$, $g\Lambda(x) = \Lambda(gx) = \Lambda(x)$ and

$$gJ(x) = J(gx) = J(x).$$

Proof. For i , let $y \in \Lambda(x)$ and $g \in G$. Then there is a net $\{g_\alpha\}$ in G with $g_\alpha \rightarrow \infty$ and $g_\alpha x \rightarrow y$. It is clear that $\{gg_\alpha\}$ is a net in G with $gg_\alpha \rightarrow \infty$, and by continuity of the action $(gg_\alpha)x \rightarrow gy$ which implies that $gy \in \Lambda(x)$ and hence $\Lambda(x)$ is invariant. The proof for $J(x)$ is similar. For ii , let Gx is a closed subset of X and $y \in \Lambda(x)$. There is a net $g_\alpha \rightarrow \infty$ in G and $g_\alpha x \rightarrow y$. Since Gx is closed and $g_\alpha x \in Gx$, then $y \in Gx$. For iii . See proposition (2.1.11). For iv , the proof is obvious. For v , let $y \in J(x)$. Then there is a net $\{g_\alpha\}$ in G with $g_\alpha \rightarrow \infty$ and there is a net $\{x_\alpha\}$ in X with $x_\alpha \rightarrow x$ such that $g_\alpha x_\alpha \rightarrow y$. Put $y_\alpha = g_\alpha x_\alpha \rightarrow y$, then it is clear that $g_\alpha^{-1} \rightarrow \infty$. We have $g_\alpha^{-1} y_\alpha = g_\alpha^{-1} (g_\alpha x_\alpha) = x_\alpha \rightarrow x$. Thus $x \in J(y)$. For $vi, vii, viii$ proofs are obvious. For ix , let $y \in g\Lambda(x)$, and let $y = gz$ for some $z \in \Lambda(x)$. Then there is a net $g_\alpha \rightarrow \infty$ such that $g_\alpha x \rightarrow z$. By continuity of the action, we have $gg_\alpha x \rightarrow gz$ and since $gg_\alpha \rightarrow \infty$, we have $y = gz \in \Lambda(x)$. If $y \in \Lambda(x)$, then $g^{-1}y \in \Lambda(x)$ for every $g \in G$ and thus $g(g^{-1}y) \in g\Lambda(x)$. Therefore, $g\Lambda(x) = \Lambda(x)$ for every $g \in G$. Now, take $y \in \Lambda(x)$, then there is a net $g_\alpha \rightarrow \infty$ such that $g_\alpha x \rightarrow y$. Since $g_\alpha (g^{-1}g)x \rightarrow y$ and $g_\alpha g^{-1} \rightarrow \infty$. Thus, $y \in \Lambda(gx)$. If $y \in \Lambda(gx)$ for some $g \in G$, then there is a net $g_\alpha \rightarrow \infty$ such that $g_\alpha (gx) \rightarrow y$. $g_\alpha g \rightarrow \infty$ implies to $y \in \Lambda(x)$. Therefore, $\Lambda(x) = \Lambda(gx)$ for every $g \in G$. Proofs for part for $J(x)$ proofs are similar. For ix , the proof is obvious. ■

2.2.3 Remark. The second side in ii , and the quality in iv is not necessary. For example, if $G = \{-1, 1\}$ with the discrete topology acts on the plane with the indiscrete topology. It is obvious that for every x in X , $\Lambda(x) = \emptyset \subset Gx = \{x, -x\}$ and Gx is not closed subset of X . Moreover, $\bar{G}_x = X$. However, if X is T_2 the second side in ii and the quality in iv hold.

In this stage, we present some results that could be characterized by the above sets.

2.2.4 Theorem. [3] Let X be a T_2 space, and G acts continuously on X . The action is Bourbaki proper if and only if the $J(x) = \emptyset$ for every $x \in X$.

2.2.5 Remark. It is suitable in this place to be referred that for $x \in X$, the set $\theta^{-1}(x, x) = \{(g, y) \in G \times X: \theta(g, y) = (y, gy) = (x, x)\} = \{g \in G: gx = x\} = G_x \times \{x\}$. That is an action G on a space X is Bourbaki proper if and only if all stabilizers G_x are quasi-compact and the map $\theta: G \times X \rightarrow X \times X, (g, x) \rightarrow (x, gx)$ is closed.

2.2.6 Corollary. The condition $x \notin J(x)$ (without T_2) implies to $\theta^{-1}(x, x)$ be quasi-compact.

2.2.7 Remark. The converse of the above corollary is not true. For example, an action on $\mathbb{R}^2 \setminus \{0\}$ by the discrete topological group $G = \mathbb{R}$, the group of the real numbers, is given by $t(x, y) = (xe^t, ye^t)$. In such G -space we have for every point $z = (x, y)$ in the plane $\theta^{-1}(z, z) = \{0\} \times \{z\}$ which is quasi-compact.

However, $(x, y) \in J((x, y))$. Take $t_n = \frac{1}{n}$, then for every (x, y) in the plane we have $t_n(x, y) \rightarrow (x, y)$.

2.2.8 Corollary. If $y = gx$ for some $g \in G$, The condition $x \notin J(x)$ (without T_2) implies to $\theta^{-1}(x, y)$ is quasi-compact. ■

Proof. It is obvious and comes from property *vii*. ■

2.2.9 Proposition. If the orbit space X/G is T_2 , then for every pair of points $x, y \in X$ such that $\tilde{x} \neq \tilde{y}$, and $y \notin J(x)$. Moreover, if X is T_2 , the converse is also true.

Proof. Assume that the orbit space X/G is T_2 and $x, y \in X$ such that they satisfy as above. To show that $y \notin J(x)$. Suppose that $y \in J(x)$, then there is a net $g_\alpha \rightarrow \infty$ in G and a net $x_\alpha \rightarrow x$ in X such that $g_\alpha x_\alpha \rightarrow y$. Take U_x a neighborhood of x and V_y a neighborhood of y . There is a $\delta \in D$, such that $x_\alpha \in U_x$ and $g_\alpha x_\alpha \in V_y$

V_y for every $\alpha \geq \delta$. That is $g_\alpha \in ((U_x, V_y))$ for every $\alpha \geq \delta$, this implies to $\tilde{U}_x \cap \tilde{V}_y \neq \emptyset$. Since U_x and V_y are arbitrary neighborhoods of x and y , we have that the orbit space is not T_2 which is a contradiction. Therefore, $y \notin J(x)$. Now assume that X is T_2 , and every pair of points $x, y \in X$ with $y \neq gx$ for every $g \in G$. Suppose that the orbit space is not T_2 . Then is a pair of points $\tilde{x} \neq \tilde{y}$ of X/G such that there are not disjoint open sets in X/G separate them. So, every pair of neighborhoods N_x and N_y of $x \in \tilde{x}$ and $y \in \tilde{y}$, then $\bar{N}_x \cap \bar{N}_y \neq \emptyset$. We can construct a net \tilde{x}_α in X/G such that for every $\alpha \in D$, $\tilde{x}_\alpha \in \tilde{N}_x \cap \tilde{N}_y$. Moreover, $\tilde{x}_\alpha \rightarrow \tilde{x}$ and $\tilde{x}_\alpha \rightarrow \tilde{y}$. Then we find nets $z_\alpha \in \tilde{x}_\alpha$ and $g_\alpha z_\alpha \in \tilde{x}_\alpha$ such that $z_\alpha \rightarrow z \in \tilde{x}$ and $g_\alpha z_\alpha \rightarrow w \in \tilde{y}$. Since $y \notin J(x)$, then $w \notin J(z)$. So $g_\alpha \rightarrow g$ for some $g \in G$. The continuity of the action implies to $g_\alpha z_\alpha \rightarrow gz$ and w . By T_2 of X then we have $gz = w$ which contradicts $\tilde{z} = \tilde{x} \neq \tilde{y} = \tilde{w}$. Therefore X/G is T_2 . ■

2.2.10 Remark. If X is not T_2 , the converse is not true in general (see the example in the remark (2.2.3)).

2.2.11 Proposition. [3] For a G -space X and a point $x \in X$, $x \notin \Lambda(x)$ implies to the stabilizer subgroup G_x of G is quasi-compact.

2.2.12 Remark. In general, if the stabilizer of a point x is quasi-compact, then it is not necessarily that $x \notin \Lambda(x)$, see the example in the remark (2.2.7).

2.2.13 Proposition. [3] If G is non-compact, then $x \notin \Lambda(x)$ implies to that x is not periodic point [$\cong G_x K \neq G$ for every compact subset K of G].

2.3 Strongly Exceptional Sets in G -space

Let X be a G -space and let G act on X properly. Then, the map $\theta: G \times X \rightarrow X \times X$ such that $(g, x) \rightarrow (x, gx)$ is proper in the sense of Bourbaki. So, θ is proper if and only if the set $SE_\theta = \emptyset$. Again, let us verify the definition of SE_θ in G -spaces notations. By the definition of the strongly exceptional set SE_θ , it equals to the set: $\{(x, y) \in X \times X: \exists \text{ a net } (g_\alpha, x_\alpha) \text{ in } G \times X \text{ such that } (x_\alpha, g_\alpha x_\alpha) \rightarrow (x, y) \text{ and } clu((g_\alpha, x_\alpha)) \cap \theta^{-1}(x, y) = \emptyset\}$. Since the net $(x_\alpha, g_\alpha x_\alpha) \rightarrow (x, y)$, then $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow y$. Also, $clu((g_\alpha, x_\alpha)) \cap \theta^{-1}(x, y) = \emptyset$ implies to either $clu(g_\alpha) \cap (\{x\}, \{y\}) = \emptyset$ or $clu(x_\alpha) \cap \{x\} = \emptyset$. Since $x_\alpha \rightarrow x$, we can rewrite the above set as following:

$\{(x, y) \in X \times X: \exists \text{ a net } x_\alpha \rightarrow x \text{ in } X \wedge \text{ a net } g_\alpha \text{ in } G \text{ such that } g_\alpha x_\alpha \rightarrow y \wedge clu(g_\alpha) \cap (\{x\}, \{y\}) = \emptyset\}$. Since the point x , the first coordinate of the above set, is determined by a net converges to it. We may put $SE_\theta = \Lambda_S(x)$ if $x_\alpha = x$ or, in general, $SE_\theta = J_S(x)$ if $x_\alpha \neq x$. That means the sets $\Lambda_S(x)$ and $J_S(x)$ reflect the properties the set SE_θ .

2.3.1 Remark. It is obvious that for every $x \in X$ then $\Lambda(x) \subseteq \Lambda_S(x)$ and $J(x) \subseteq J_S(x)$ and if X is T_2 , then we have $\Lambda(x) = \Lambda_S(x)$ and $J(x) = J_S(x)$.

We give some properties for these sets in an arbitrary G -space X .

2.3.2 Proposition. Let X be a G -space and $x \in X$. Then

1. $J_S(x), \Lambda_S(x)$ are invariant.
2. $\overline{Gx} = Gx \cup \Lambda_S(x)$.
3. Gx is closed if and only if $\Lambda_S(x) \subseteq Gx$.
4. If $y \in J_S(x)$ then $x \in J_S(y)$.
5. For every $g \in G$, then $g\Lambda_S(x) = \Lambda_S(gx) = \Lambda_S(x)$.
6. For every $g \in G$, then $gJ_S(x) = J_S(gx) = J_S(x)$.

7. If $y \in J_S(x)$ then $y \in J_S(z)$ for every $z \in Gx$.

8. If $y \in J_S(x)$ then $z \in J_S(x)$ for every $z \in Gy$.

Proof. For 1, we shall show proof for the set $J_S(x)$ and the proof for $\Lambda_S(x)$ is similar. For invariance of $J_S(x)$, take $g \in G$ and $y \in J_S(x)$. Then there is a net $x_\alpha \rightarrow x$ in X and a net g_α in G such that $g_\alpha x_\alpha \rightarrow y$ and $clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$. Since $gg_\alpha x_\alpha \rightarrow gy$ and gg_α is a net in G , then it suffices to show that $clu(gg_\alpha) \cap ((\{x\}, \{gy\})) = \emptyset$. Suppose that there is a point $g_1 \in clu(gg_\alpha) \cap ((\{x\}, \{gy\}))$, then there is a subnet gg_{α_β} of the net gg_α in G such that $gg_{\alpha_\beta} \rightarrow g_1$ and $g_1 x = gy$. Then we have $g_{\alpha_\beta} \rightarrow g^{-1}g_1$ and $g^{-1}g_1 x = y$. Thus $g^{-1}g_1 \in clu(g_{\alpha_\beta}) \cap ((\{x\}, \{y\})) \subseteq clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$ which is a contradiction. Therefore, $clu(gg_\alpha) \cap ((\{x\}, \{gy\})) = \emptyset$ and $gy \in J_S(x)$. For 2, it is obvious that $Gx \cup \Lambda_S(x) \subseteq \overline{Gx}$. Conversely, take $y \in \overline{Gx}$. Then there is a net $y_\alpha = g_\alpha x \in Gx$ and $g_\alpha x \rightarrow y$. Either $g_\alpha \rightarrow \infty$, then $y \in \Lambda(x) \subseteq \Lambda_S(x)$ or g_α has a cluster point g_1 , then if $g_1 x = y \in Gx$ and we are done. Otherwise, $clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$. Thus, $y \in \Lambda_S(x)$. For 3, if Gx is closed, then $\overline{Gx} = Gx$. By 2, we have $\Lambda_S(x) \subseteq Gx$. On the other hand, if $\Lambda_S(x) \subseteq Gx$, then also by 2, $\overline{Gx} = Gx \cup \Lambda_S(x) = Gx$. Therefore, Gx is closed. For 4, if $y \in J_S(x)$, there is a net $x_\alpha \rightarrow x$ in X and a net g_α in G such that $g_\alpha x_\alpha \rightarrow y$ and $clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$. Put $y_\alpha = g_\alpha x_\alpha$. Then $y_\alpha \rightarrow y$ is a net in X and g_α^{-1} is a net in G such that $g_\alpha^{-1} y_\alpha \rightarrow x$. It suffices to show that $clu(g_\alpha^{-1}) \cap ((\{y\}, \{x\})) = \emptyset$. Suppose that there is $g \in clu(g_\alpha^{-1}) \cap ((\{y\}, \{x\}))$. Then there is a subnet $g_{\alpha_\beta}^{-1}$ of g_α^{-1} with $g_{\alpha_\beta}^{-1} \rightarrow g$ and $gy = x$. Thus, $g_{\alpha_\beta} \rightarrow g^{-1}$ and $g^{-1}x = y$. By continuity of the action, we have $g_{\alpha_\beta} x_\alpha \rightarrow g^{-1}x = y$. Thus, $g^{-1} \in clu(g_{\alpha_\beta}) \cap ((\{x\}, \{y\})) \subseteq clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$ which is a contradiction. Therefore, $clu(g_\alpha^{-1}) \cap ((\{y\}, \{x\})) = \emptyset$ and $x \in J_S(y)$. The proofs for 7, and 8 come immediately from the property in 1, and the proof in 5 is a

special case in 6. For 6, let $z \in gJ_S(x)$, and let $z = gy$ for some $y \in J_S(x)$. Then there is a net $x_\alpha \rightarrow x$ in X and a net g_α in G such that $g_\alpha x_\alpha \rightarrow y$ and $clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$. Consider a net $g_\alpha g^{-1}$ in G . It is obvious that $(g_\alpha g^{-1})g x_\alpha \rightarrow y$ and $g x_\alpha \rightarrow gx$. We want to show that $clu(g_\alpha g^{-1}) \cap ((\{gx\}, \{y\})) = \emptyset$. Suppose there is $g_1 \in clu(g_\alpha g^{-1}) \cap ((\{gx\}, \{y\}))$. Then there is a subnet $g_{\alpha_\beta} g^{-1}$ of $g_\alpha g^{-1}$ with $g_{\alpha_\beta} g^{-1} \rightarrow g_1$ and $g_1 gx = y$. We have $g_{\alpha_\beta} \rightarrow g_1 g$, and then $g_{\alpha_\beta} x_\alpha \rightarrow g_1 gx = y$. Thus, $g_1 g \in clu(g_{\alpha_\beta}) \cap ((\{x\}, \{y\})) \subseteq clu(g_\alpha) \cap ((\{x\}, \{y\})) = \emptyset$. Which is a contradiction. Therefore, $y \in J_S(gx)$ and by 1, $gy = z \in J_S(gx)$. By 1, it is obvious that $J_S(gx) = J_S(x)$. We shall show that $J_S(x) \subseteq gJ_S(x)$. Take $z \in J_S(x)$. Then $g^{-1}z \in J_S(x)$ and again $z = gg^{-1}z \in gJ_S(x)$. ■

2.3.3 Theorem. [Main Theorem] Let X be a G -space. Then G acts properly on X if and only if for every point $x \in X$, the set $J_S(x) = \emptyset$.

Proof. The proof comes immediately from the theorem (2.1.14) and from the fact that $J_S(x) = SE_\theta$. ■

2.3.4 Remark. As result that Bourbaki proper implies that the orbit spaces are T_2 .

2.3.5 Corollary. Let X be a G -space. If $J_S(x) = \emptyset$ for every $x \in X$, then the orbit space X/G is Hausdorff.

Proof. Let \tilde{x}, \tilde{y} be two distinct points in X/G . Suppose that every pair of open neighborhoods $U_{\tilde{x}}, V_{\tilde{y}}$ of \tilde{x}, \tilde{y} respectively do not separate them. Then we may construct a net from those intersections for every pair. That is, there is a net $\tilde{x}_\alpha \in U_\alpha \cap V_\alpha \neq \emptyset$ such that \tilde{x}_α converges to both \tilde{x}, \tilde{y} . Put $x_\alpha^1 \in \pi^{-1}(\tilde{x}_\alpha)$ and $x_\alpha^2 \in \pi^{-1}(\tilde{x}_\alpha)$ such that $x_\alpha^1 \rightarrow x \in \pi^{-1}(\tilde{x})$ and $x_\alpha^2 \rightarrow y \in \pi^{-1}(\tilde{y})$ and $g_\alpha x_\alpha^1 = x_\alpha^2$. Since $J_S(x) = \emptyset$, then $clu(g_\alpha) \cap ((\{x\}, \{y\})) \neq \emptyset$ which contradicts that $\tilde{x} \neq \tilde{y}$.

Therefore X/G is Hausdorff. ■

2.4 Limit Sets and Proper actions

In this section, we shall present some interesting results using the limit set and its prolongation set in their proofs as convergence technique and also, we show how such these sets could be good keys to describe the behaviors of orbits in orbit spaces. First, we recall some well-known results have proved by limit sets under certain conditions. Second, we present more results on G -space notions under weaker conditions using our keys the sets $\Lambda(x)$ and $\Lambda_S(x)$, $J(x)$ and $J_S(x)$. In a compact (T_2 axiom included) topological group acts on a Hausdorff space, there is concert relations between points and their stabilizers [9]. We shall review these relations and then we give our proofs under weaker conditions. Finally, we give a new proof for a well-known theorem “Conjugacy Neighborhoods Theorem”.

2.4.1 Remark. From now on, a setting where G is a locally compact (T_2 is included) acting on a completely regular space X is called a Palais setting (or setting in the sense of Palais). In the rest, whenever we mention Cartan or Palais proper G -space that we mean certainly G and X endowed with these properties in this setting.

2.4.2 Proposition. [13] Let X be a G -space. X is Cartan G -space (in the sense of Palais) iff $x \notin J(x)$ for each $x \in X$.

2.4.3 Proposition. [26] Let X be a G -space, where X and G are merely Hausdorff. Then X is Bourbaki proper G -space if and only if $J(x) = \phi$ for each $x \in X$.

Also, the author, in [3], has given several properties of limits sets when G is a locally compact and X is a completely regular topological space and their relations with small and thin notions. The following results in [3] could be proved in certain weaker conditions.

2.4.4 Proposition. Let X be a T_2 G -space and $x \in X$. Then the orbit space X/G is T_2 iff for each $x \in X$, $J(x)$ is a subset of Gx .

Proof. If X/G is T_2 , then for every pair \bar{x} and \bar{y} in X/G with $\bar{x} \neq \bar{y}$ we have $y \notin J(x)$. For an arbitrary point $x \in X$, if $x \notin J(x)$, then $J(x) = \emptyset \subseteq Gx$. Otherwise, that is $x \in J(x)$, then by inveracity of $J(x)$ we have $Gx \subseteq J(x)$. From the first step of the proof we have shown that for every $y \neq gx$ for some $g \in G$, $y \notin J(x)$. Thus $J(x)$ is one component, that is $J(x) = Gx$. Conversely, let $J(x) \subseteq Gx$ and suppose that the orbit space is not T_2 . Then there is a pair of two distinct points \bar{x} and \bar{y} in X/G such that they could not be separated by two disjoint open neighborhoods in X/G . Then we can construct a net \bar{x}_α in X/G such that $\bar{x}_\alpha \rightarrow \bar{x}$ and $\bar{x}_\alpha \rightarrow \bar{y}$. Then there are nets x_α^1, x_α^2 in \bar{x}_α and a net g_α in G such that $x_\alpha^1 \rightarrow x$, $x_\alpha^2 \rightarrow y$ and $g_\alpha x_\alpha^1 = x_\alpha^2$. Either $g_\alpha \rightarrow \infty$ or $g_\alpha \rightarrow g$ for some $g \in G$. For the first case we have $y \in J(x) \subseteq Gx$ and in the other we have $g_\alpha x_\alpha^1 = x_\alpha^2 \rightarrow gx = y$ where the quality is hold if X is T_2 . Thus, both cases implies to a contradiction. Therefore, X/G is T_2 . ■

2.4.5 Proposition. [3] Let X be a Cartan G -space. Then for each $x \in X$, $\Lambda(x) = \emptyset$.

2.4.6 Proposition. Let G be a locally quasi-compact group acting on a T_2 space X and let $x \in X$. Then $x \notin J(x)$ if and only if there is a neighborhood U of x and a compact neighborhood V of e , e is the identity element of G , such that $gU \cap U = \emptyset$ for each $g \notin V$.

Proof. If G is quasi-compact, the proof is trivial since then $J(x) = \emptyset$ for every $x \in X$ and so one could take $V = G$ for the condition in the proposition. Let G be an arbitrary locally quasi-compact. First, we assume that $x \notin J(x)$ for some $x \in X$. Suppose that if U is a neighborhood of x and V is a quasi-compact of the identity,

then there is a $g \notin V$ and $gU \cap U \neq \emptyset$. Let $\{U_\alpha\}$ be the neighborhood system of the point x . Then for every α there a point $g_\alpha \notin V$ and a point $x_\alpha \in U_\alpha$ such that $g_\alpha x_\alpha \in U_\alpha$. Since $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow x$ and $x \notin J(x)$, there is a $g \in G$ such that $g_\alpha \rightarrow g$. By the continuity of the action of G and X is T_2 , we have $g_\alpha x_\alpha \rightarrow gx = x$. There is no loss of generality if we assume that V be a neighborhood of the stabilizer G_x of x (actually, it is because G_x is quasi-compact when $x \notin J(x)$). Thus, the net g_α is eventually in V which contradicts that $g_\alpha \notin V$ for every α . Therefore, the condition of the proposition is true. For the other direction, suppose that $x \in J(x)$ for some point $x \in X$. There is a net $x_\alpha \rightarrow x$ in X and a net $g_\alpha \rightarrow \infty$ in G such that $g_\alpha x_\alpha \rightarrow x$. By the condition, there is a neighborhood U of x and a quasi-compact V of the identity such that $((U, U)) \subseteq V$. Since x_α and $g_\alpha x_\alpha$ are eventually in U . We have $g_\alpha \in ((U, U)) \subseteq V$. V is a quasi-compact, then g_α must have a convergent subnet which is a contradiction. Therefore, $x \notin J(x)$. ■

Note. We could also prove the proposition (2.4.6) without T_2 condition on the space X using only the set $J_s(x)$.

2.4.7 Corollary. Let X be a G -space as in the above proposition and $x \in X$. Then $x \in J(x)$ if and only if x has no thin neighborhood.

Proof. Assume that $x \in J(x)$ for some $x \in X$. Suppose there is a thin neighborhood U of x . Then the closure of the set $((U, U))$ is quasi-compact. $x \in J(x)$ implies that there is a net $x_\alpha \rightarrow x$ in X and a net $g_\alpha \rightarrow \infty$ in G such that $g_\alpha x_\alpha \rightarrow x$. Thus, $g_\alpha \in ((U, U))$ and g_α must have convergent subnet which contradiction. Conversely, assume that a point $x \in X$ has no thin neighborhood. Suppose that $x \notin J(x)$. Then by proposition (2.4.6), there is a neighborhood U of x and a quasi-compact V of the identity $((U, U)) \subseteq V$. Since V is a quasi-compact, the its closure \bar{V} is also a quasi-

compact. Thus, $\overline{((U, U))} \subseteq \bar{V}$. Therefore, U is a thin neighborhood of x which contradicts our assumption. ■

2.4.8 Remark. By corollary (2.2.8), the condition $y \notin J(x)$ implies to that a set $((\{x\}, \{y\}))$ is a quasi-compact. Then we can put the following criterion.

2.4.9 Proposition. Let G be a locally quasi-compact group acting on T_2 space X and let x, y be a pair of points in X . $y \notin J(x)$ if and only if there is a neighborhood U of x and a neighborhood W of y and a quasi-compact neighborhood V of the $((\{x\}, \{y\}))$ such that $gU \cap W = \emptyset$ if $g \notin V$.

Proof. The proof is obvious if G is a quasi-compact. We assume that G be an arbitrary locally quasi-compact. Take a pair of points x, y in X such that $y \notin J(x)$. Suppose that if U, W and V are neighborhood of x, y and $((\{x\}, \{y\}))$ respectively, then there is a $g \notin V$ and $gU \cap W \neq \emptyset$. Let $\{U_\alpha\}$ and $\{W_\alpha\}$ be the neighborhood systems for x and y and let V be a quasi-compact of the identity. Then for every α there is a $g_\alpha \notin V$ and $g_\alpha U_\alpha \cap W_\alpha \neq \emptyset$. Thus, for every α there is $x_\alpha \in U_\alpha$ such that $g_\alpha x_\alpha \in W_\alpha$. Since $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow y$ and $y \notin J(x)$, there is a $g \in G$ such that $g_\alpha \rightarrow g$. By the continuity of the action of G and T_2 property of X , we have $g_\alpha x_\alpha \rightarrow gx = y$ and $g \in ((\{x\}, \{y\})) \subseteq V$. V is a neighborhood of g , then the net g_α is eventually in V which contradicts $g_\alpha \notin V$ for every α . Conversely, Suppose that $y \in J(x)$. Then there is a net $x_\alpha \rightarrow x$ in X and a net $g_\alpha \rightarrow \infty$ in G such that $g_\alpha x_\alpha \rightarrow y$. By hypothesis, there is a neighborhood U_x of x and a neighborhood W_y of y and V is a quasi-compact neighborhood of $((\{x\}, \{y\}))$ such that $((U_x, W_y)) \subseteq V$. Then, the nets x_α and $g_\alpha x_\alpha$ are eventually in U_x and W_y respectively and $g_\alpha \in ((U_x, W_y)) \subseteq V$. That is, g_α must have a convergent subnet which is impossible. Therefore, $y \notin J(x)$. ■

Note. In the above proposition we may drop the T_2 condition on the space X , however, we must use the set $J_S(x)$ instead of the set $J(x)$.

2.4.10 Corollary. Let X be a G -space. For each pair of points x, y in X there is a neighborhood U of x and a neighborhood W of y and there is a quasi-compact neighborhood V of $((\{x\}, \{y\}))$ such that $gU \cap W = \emptyset$ for each $g \notin V$ if and only if $J(x) = \emptyset$ for each $x \in X$.

Proof. The proof immediately comes from proposition (2.4.9). ■

2.4.11 Proposition. Let X be a G -space and y be a point in X . Then y has no small neighborhood whenever $y \in J(x)$ for some point $x \in X$.

Proof. Let $y \in J(x)$ for some $x, y \in X$. If S is a small neighborhood of y , there is a neighborhood U of x such that a set $((U, S))$ has a quasi-compact closure. Since $y \in J(x)$, there is a net $x_\alpha \rightarrow x$ in X and a net $g_\alpha \rightarrow \infty$ in G such that $g_\alpha x_\alpha \rightarrow y$. Thus, $g_\alpha \in ((U, S))$ which is a quasi-compact which implies to a contradiction since $g_\alpha \rightarrow \infty$. ■

2.4.12 Proposition. Let G be a locally quasi-compact group acting on T_2 space X . Then $J(x) = \emptyset$ for each $x \in X$ if and only if every pair of points of X has relatively thin neighborhoods.

Proof. For the first side, for every pair of points $x, y \in X$. Since $y \notin J(x) = \emptyset$, then by proposition (2.4.9), there are neighborhoods U_x, W_y , and a quasi-compact neighborhood V satisfy the condition in the proposition. That is, $((U_x, W_y)) \subseteq V$. Since \bar{V} is still quasi-compact, then $\overline{((U_x, W_y))}$ is quasi-compact. Therefore, x, y have relatively thin neighborhoods. Conversely, for a pair of points $x, y \in X$, Take $V = \overline{((U, W))}$ where U and W relatively thin neighborhoods of x and y respectively. We have $((U, W)) \subseteq V$ and V is a quasi-compact neighborhood of a

set $(\{x\}, \{y\})$). By proposition (2.2.10), then $y \notin J(x)$. Since the points x and y are an arbitrary. Therefore, $J(x) = \emptyset$ for every $x \in X$. ■

2.4.13 Remark. In general, if for every pair of points of X , there are relatively thin neighborhoods then for each $x \in X$, $J(x) = \emptyset$. The converse is not in general. For example, the natural action of the group of rational numbers on itself. Also, if for every pair of points of X , there are relatively thin neighborhoods, it does not imply to that X is Palais proper. For instance, see the dynamical system in [7, pp. 46]. However, if X is a locally compact, the notions of Bourbaki and Palais proper are equivalent.

2.4.14 Theorem. [3] Let X be a locally compact G -space. Under Palais setting, Palais proper and Bourbaki proper are equivalent.

If G is a compact Lie group acting on T_2 space, there is a relation between close points and their stabilizers and there is also a relation between close orbits and their kernels. One can show those interesting results in the version of compact Lie group in Montgomery and Zippin [23] and [12]. Palais also has given analogous results for more wider classes of G -spaces are called locally proper G -spaces (or Cartan G -spaces) when G is an arbitrary T_2 -locally compact (Lie) group acting continuously on completely regular space. In the next, we will review those results in both versions and then we shall give our proofs for the same propositions to get the same results using this convergence technique. We merely need the following definition.

2.4.15 Definition. [10] Let G be a topological group acting continuously on a space X . and let x be a point in X . The kernel of the action G on X is the subgroup $\bigcap_{x \in X} G_x$ and the kernel of action of G on the orbit Gx is the subgroup $\bigcap_{y \in Gx} G_y$.

2.4.16 Lemma. [9] (Close points have close stabilizers) Let G acts properly on a space X , choose $x \in X$ and let U be a neighborhood of the stabilizers G_x of x in G . Then x has a neighborhood V such that for every $y \in V$, then $G_y \subseteq U$.

2.4.17 Proposition. Let G be an arbitrary topological group acting on T_2 space X . If $x \notin J(x)$ for some point $x \in X$. Then for every neighborhood U of the stabilizer G_x of x in G there is a neighborhood V of x such that for every $y \in V$, then $G_y \subseteq U$.

Proof. Suppose that the result is not true. That is, there is a neighborhood U of the stabilizer G_x of x in G such that for every neighborhood V of x , there is a point $y \in V$ such that $G_y \not\subseteq U$. Let $\{V_\alpha\}$ be a neighborhoods system of the point x . Then for every V_α there is a point $x_\alpha \in V_\alpha$ such that there is a $g_\alpha \in G_{x_\alpha}$ and $g_\alpha \notin U$. Let x_α be the net constructed by our choice of the family $\{V_\alpha\}$, then the net $x_\alpha \rightarrow x$. Since $g_\alpha x_\alpha = x_\alpha \rightarrow x$ and $x \notin J(x)$. We have $g_\alpha \rightarrow g$ for some $g \in G$. The continuity of the action gives $g_\alpha x_\alpha \rightarrow gx$ and the T_2 property on X gives $gx = x$. Thus, $g \in G_x \subseteq U$. Then $g_\alpha \in U$ for every $\alpha > \alpha_0$ for some α_0 which contradicts our assumption. Therefore, the result is true. ■

2.4.18 Lemma. [9] (Close orbits have close kernels). Let G be a T_2 group acting properly on a space X , choose a point $x \in X$, and let U be a neighborhood of the kernel of the action G on the orbit Gx . Then x has a neighborhood V such that U contains all kernels which correspond to orbits of points in V .

2.4.19 Proposition. Let G be an arbitrary topological group acting on T_2 space X , let x in X and $x \notin J(x)$. Then for every neighborhood U of the kernel of the action G on the orbit Gx , x has a neighborhood V such that the kernels of the action of G on all orbits through points in V contained in U .

Proof. Suppose that the result is not true, and there is a neighborhood U of the kernel of the action G on Gx such that every neighborhood V of x there a point

$y \in V$ with $\bigcap_{g \in G} G_{gy} \not\subseteq U$. Let $\{V_\alpha\}$ be a neighborhoods system of the point x . Then for every V_α there is a point $x_\alpha \in V_\alpha$ such that there is a $g_\alpha \in \bigcap_{g \in G} G_{gx_\alpha}$ and $g_\alpha \notin U$. Let x_α be the net constructed by our choice of the family $\{V_\alpha\}$, then net $x_\alpha \rightarrow x$ and let $g \in G$. Then $gx_\alpha \rightarrow gx$. Since $g_\alpha(gx_\alpha) = gx_\alpha \rightarrow gx$ and $gx \notin J(gx)$. We have $g_\alpha \rightarrow g_1$ for some $g_1 \in G$. The continuity of the action implies to $g_\alpha(gx_\alpha) \rightarrow g_1(gx) = gx$ and $g_1 \in G_{gx}$ for every $g \in G$. Thus $g_1 \in \bigcap_{g \in G} G_{gx_\alpha} \subseteq U$ and $g_\alpha \in U$ eventually which is a contradiction. Therefore, the result is true. ■

Note. Propositions (2.4.17) and (2.4.19), we may drop the T_2 condition on the space X using the condition $x \notin J_s(x)$ instead of $x \notin J(x)$.

According to the proposition (2.4.17), and the conjugacy theorem via Zippin and Montgomery [23], if G is a Lie group, then all stabilizer subgroups of points close enough to the point x are conjugate to subgroups of G_x . Palais has given a direct proof for this theorem using the notion of slices [29]. We will apply convergence techniques to obtain the same result but before taking up the proof, we need some of the following background.

2.4.20 Definition. [16] A locally compact topological group is called having a property **(S)** provided for every $x \neq e$ in a sufficiently small neighborhood U there exists an integer n so that $x^{2^n} \notin U$.

2.4.21 Theorem. [16] A locally compact group has a property **(S)** if it has no small subgroup.

2.4.22 Remark. Every Lie group has a property **(S)**. [16]

2.4.23 Theorem. Conjugacy neighborhood theorem using convergence

techniques. Let X be a G -space where G is Hausdorff locally compact with property (\mathcal{S}) and let $x \notin J(x)$ for every $x \in X$. Then for a point $x \in X$, there is a neighborhood U of x in X such that the stabilizers of points in U are conjugate to a subgroup of G_x .

Proof. Suppose that there is a point $x \in X$ such that for every neighborhood U of x there is $y \in U$ such that $G_y \not\subset G_x$. We can choose the neighborhoods system U_α of the point x in X . We can find for every α there is a point $x_\alpha \in U_\alpha$ and $g_\alpha \in G_{x_\alpha}$ such that $g_\alpha \notin G_x$. We have $g_\alpha x_\alpha = x_\alpha$ but $g_\alpha x \neq x$. By our choice of the neighborhoods of x , we have $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha = x_\alpha \rightarrow x$. $x \notin J(x)$ implies that $g_\alpha \rightarrow g$ for some $g \in G$. By the continuity of the action, we have $g_\alpha x_\alpha \rightarrow gx$ and $gx = x$. Thus, $g \in G_x$. Now $g_\alpha \rightarrow g$ so that $k_\alpha = g^{-1}g_\alpha \rightarrow e$ where e is the identity element in G . Let V be a symmetric neighborhood of e in G with \bar{V} is compact. Since G has the property (\mathcal{S}) , we can construct a sequence of neighborhoods $\{U_n\}$ of e with $U_n \supseteq U_{n+1}$ and $\bigcap U_n = \{e\}$. On other the hand, since $k_\alpha \rightarrow e$, it allows us to construct a sequence $y_n = k_{\alpha_n} \neq e$ with $y_n \in U_n$ for every n . Let W be a neighborhood of e such that $W^2 \subset V$. By our assumption, for every n there is an integer m_n such that $y_n^{2^{m_n}} \notin W$ so we can consider m_n being the smallest integer satisfies this property for every n . Now, the inequality $2^{m_n-1} < 2^{m_n}$ implies $y_n^{2^{m_n-1}} \in W$ and $y_n^{2^{m_n-1}} y_n^{2^{m_n-1}} = y_n^{2^{m_n}} \in W^2 \subset V$. Since the closure of V is compact then there is a subsequence $y_{n'}$ of y_n such that $y_{n'} = y_{n'}^{2^{m_{n'}}} \rightarrow a$ in \bar{V} . The condition $y_n^{2^{m_n}} \notin W$ implies that $a \notin W$ and $a \neq e$. Let K be the set of the existence limits of such subsequences. We claim that K is subgroup of G . Pick $a, b \in K$, then $a = \lim y_{n'}^{2^{m_{n'}}}$ and $b = \lim y_{n''}^{2^{m_{n''}}}$. Put $s_n = 2^{m_{n'}} - 2^{m_{n''}} \pmod{2^{m_{n'}}}$. Then we have $s_n \leq 2^{m_{n'}}$ and $\lim y_{n'}^{s_n} = e = ba^{-1} \in \bar{V}$. We have $ba^{-1} \in K$ and then K is a subgroup of G containing in \bar{V} which contradicts

that G has property **(S)**. Therefore, there is a neighborhood of x such that points in it whose stabilizers are subgroups of G_x ■.

Chapter Three

Weaker Forms of Proper Actions

3.1 Three Kinds of Proper Actions

In the chapter two, we have stated two kinds of proper actions, Palais, and Bourbaki proper. We have mentioned that Palais proper implies Bourbaki proper. In general, the two actions are not equivalent. On the other hand, they have been put in different settings. There is another action that has been presented by Baum, Connes and Higson (the stronger one). The latter has been put in a setting different than Palais'. Biller in [8] has defined three weaker kinds of proper that are increasingly strong. In [8], the author makes three kinds of proper actions in unambiguous directions. There the author also establishes a weaker form termed Cartan depending on some of the properties of Palais' Cartan. In our view, the convergence technique makes the relation between three kinds of proper clearer and technique of proofs shorter and more understandable. Moreover, we shall show that our weaker forms are weaker than Biller's. In this way, we propose a new insight into the study of wider classes of proper action that use only convergence techniques. Before taking up those forms, we give Baum-Connes-Higson proper definition.

3.1.1 Definition. [6] Let G be a locally compact and second countable (T_2 included) and let G be act continuously on a metrizable space X . The action of X is called proper if for every $x \in X$ there is a triple (U, H, ρ) such that:

1. U is an open neighborhood of x , with $gy \in U$ for every $(g, y) \in G \times U$,
2. H is a compact subgroup of G ,
3. $\rho: U \rightarrow G/H$ is a G -map from U to the homogeneous space G/H .

3.1.2 Lemma. Let G be a topological group and g is an element of G and \mathcal{N} be a fundamental open neighborhoods system of the identity. Then for every $V \in \mathcal{N}$, there is an open neighborhood U of g such that $((U, U)) = \{g \in G: gU \cap U \neq \emptyset\} \subseteq V$.

Proof. From topological group properties, especially for a fundamental open neighborhood system \mathcal{N} of the identity element in G , we have that every element $V \in \mathcal{N}$, and any element $g \in G$ there is an element $N \in \mathcal{N}$ such that $N \subseteq gVg^{-1}$. Then we have $((N, N)) \subseteq gVg^{-1}$ or $g^{-1}((N, N))g \subseteq V$, and hence $((gN, gN)) \subseteq V$. Take $U = gN$. ■

3.1.3 Definition. [3] Let X be a G -space. A subset S of X is called star if for every $x \in X$ there is a $g \in G$ such that the set $gx \in S$ and if, additionally, S is also small then it is called strong.

3.1.4 Remark. We have shown that the extended sets $J(x)$ and $J_S(x)$, from the sets E_f and SE_f respectively, are good keys as to whether an action is Bourbaki proper or not. It is useful in this stage that we give explanations that one may gain from the set $J(x)$. First, let us prove the following theorem that we have stated it in the chapter two.

3.1.5 Theorem. Let X be a T_2 space and let G be a topological group acting continuously on X . The action is Bourbaki proper if and only if the set $J(x) = \emptyset$ for every $x \in X$.

Proof. If the G acts properly on X , suppose that there are points $y, x \in X$ and $y \in J(x)$. There is a net $g_\alpha \rightarrow \infty$ in G and a net $x_\alpha \rightarrow x$ in X such that $g_\alpha x_\alpha \rightarrow y$. That is $(x_\alpha, g_\alpha x_\alpha) \rightarrow (x, y)$. Since G acts properly on X , then there is a point $(g, x) \in G \times X$ such that $(g_\alpha, x_\alpha) \rightarrow (g, x)$. This implies that $g_\alpha \rightarrow g$ which is a

contradiction. Therefore, since x, y are arbitrary, $J(x) = \emptyset$ for every $x \in X$.

Conversely, to show that the condition $J(x) = \emptyset$ for every $x \in X$ implies that the map $\theta: (g, x) \rightarrow (x, gx)$ is proper, take a point $(x, y) \in X \times X$, then if $y \neq gx$ for every $g \in G$, then $\theta^{-1}(x, y) = \emptyset$ and hence it is quasi compact. Suppose that $y = gx$ for some g in G and $\theta^{-1}(x, y)$ is not quasi compact. There is a net $(g_\alpha, x_\alpha) \rightarrow \infty$ in $\theta^{-1}(x, y)$ and hence in $G \times X$ since $\theta^{-1}(x, y)$ is closed subset of $G \times X$. Since $\theta(g_\alpha, x_\alpha) = (x_\alpha, g_\alpha x_\alpha) = (x, y) \rightarrow (x, y)$, then we have $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow y$. $(g_\alpha, x_\alpha) \rightarrow \infty$ implies that $g_\alpha \rightarrow \infty$ and $y \in J(x)$ which is a contradiction. Take F as a closed subset of $G \times X$ and $(x, y) \in \overline{\theta(F)}$. There is a net $(g_\alpha, x_\alpha) \in F$ and $\theta(g_\alpha, x_\alpha) = (x_\alpha, g_\alpha x_\alpha) \rightarrow (x, y)$. So $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow y$. Since $J(x) = \emptyset$, the net g_α must have a convergent point in G , say g . That is $g_\alpha \rightarrow g$ and $x_\alpha \rightarrow x$ or $(g_\alpha, x_\alpha) \rightarrow (g, x) \in F$, since F is closed. The continuity of θ implies to $\theta(g_\alpha, x_\alpha) = (x_\alpha, g_\alpha x_\alpha) \rightarrow \theta(g, x) = (x, gx) = (x, y)$ by T_2 condition of X . Therefore $\theta(F)$ is closed in $X \times X$ and θ is closed map. ■

3.1.6 Remark. If X is not T_2 in the theorem (3.1.4), then the second direction may not be true. For example, $G = \{\pm 1\}$ with the discrete topology, and $X = \mathbb{R}^2$ with the indiscrete topology. Then G acts continuously on X by $t(x, y) = (x, ty)$ for every $t \in G$ and $(x, y) \in X$. Obviously, then $J((x, y)) = \emptyset$ for every $(x, y) \in X$. However, X/G is not T_2 (not Bourbaki proper).

3.1.7 Proposition. [27] Let X be a Cartan G -space (in the sense of Palais) and x be a point in X . If U is a neighborhood of G_x , then there is a neighborhood V of x such that $((V, V)) \subseteq U$.

3.2 Biller's Cartan Action and Weaker Cartan Actions

3.2.1 Definition. [8] Let X be a topological space and G be a topological group. An action of G on X is said to be Cartan if the following two conditions are satisfied:

1. All stabilizers are quasi compact.
2. If $x \in X$, and V_{G_x} is a neighborhood of G_x , there is a neighborhood U_x of x such that $((U_x, U_x)) \subseteq V_{G_x}$.

3.2.2 Proposition. Cartan action in the sense of Biller implies to the condition that $x \notin J(x)$ for every $x \in X$.

Proof. Assume that the two conditions of Biller be satisfied. Suppose that there is a point $x \in X$ such that $x \in J(x)$. Then there is a net $g_\alpha \rightarrow \infty$ in G and a net $x_\alpha \rightarrow x$ in X such that $g_\alpha x_\alpha \rightarrow x$. For each $g \in G_x$ there is, since $g_\alpha \rightarrow \infty$, $\delta_g \in D$ and V_g is a neighborhood of g such that $g_\alpha \notin V_g$ for every $\alpha \geq \delta_g$. The collection $\{V_g\}$ forms an open cover of G_x which is a quasi-compact, then there is a finite sub-collection $\{V_{g_i}\}_{i=1}^n$ covering G_x . Then we find $\delta \in D$ such that $g_\alpha \notin V_{g_i}$ for every $\alpha \geq \delta$, and $i \in \{1, 2, 3, \dots, n\}$. Put $V = \bigcup_{i=1}^n V_{g_i}$ then V is a neighborhood of G_x and $g_\alpha \notin V$ for every $\alpha \geq \delta$. By the second condition of Biller, there is a neighborhood U_x such that $((U_x, U_x)) \subseteq V$. Since $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow x$, then there is $\beta \in D$ such that $x_\alpha \in U_x$ and $g_\alpha x_\alpha \in U_x$ or $g_\alpha \in ((U_x, U_x))$ for every $\alpha \geq \beta$. Choose $\gamma \geq \beta$ and $\gamma \geq \delta$, then we have $g_\alpha \in ((U_x, U_x)) \subseteq V$ and $g_\alpha \notin V$ for every $\alpha \geq \gamma$ which is a contradiction, x is an arbitrary point in X and hence $x \notin J(x)$ for every $x \in X$. ■

3.2.3 Remark.

- The converse of the above proposition is not true in general. For example, consider the action $t(x, y) = (x, ty)$ by the discrete topological group $G = \{+1, -1\}$ acting on the plane with the indiscrete topology. Take a point $x = (2, 3)$ in the plane, then $G_x = \{1\}$. The subset $V = \{1\}$ of G is an open neighborhood of G_x . However, there is a unique neighborhood $U = \mathbb{R}^2$ of x and $((U, U)) = \{-1, 1\} \not\subseteq V$. It is obvious that $x \notin J(x)$.
- Under the second condition of definition of Biller Cartan action, the first condition is equivalent to the condition $x \notin \Lambda(x)$ for every $x \in X$.

In this stage, we have shown that there is a convergence condition that is weaker than Biller conditions. A fortiori one could define a weaker form of Cartan action by using such convergence techniques. We will call our version weak Cartan or just w -Cartan.

3.2.4 Definition. (New version of Cartan action) Let X be topological space and G be a topological group acts continuously on X . The action of G on X is said to be w -Cartan if the following condition is satisfied: $\forall x$ in X , $x \notin J(x)$.

3.2.5 Proposition. In the sense of Palais, Cartan and w -Cartan coincide [see, proposition (2.4.2)].

3.2.6 Proposition. Let X be a T_2 w -Cartan G -space and x a point in X . Then the map $g \rightarrow gx$ is an open map from G onto Gx .

Proof. Let V be an open neighborhood of e in G . To show Vx is an open in Gx . Suppose it is not. There is a net g_α in G with $g_\alpha x$ is not eventually in Vx but it converges to x . If $g_\alpha \rightarrow \infty$, then $x \in \Lambda(x) \subseteq J(x)$ which contradicts that X is w -Cartan. So, $g_\alpha \rightarrow g$ for some $g \in G$. Then $g_\alpha x \rightarrow gx = x$ and hence $g \in G_x$.

However, $g_x x \notin Vx$ is equivalent to $g_\alpha \notin VG_x$ which is a neighborhood of g . This contradiction proves the proposition. ■

3.2.7 Proposition. Let X be a w -Cartan G -space and x a point in X . Then the map $G/G_x \rightarrow Gx$ is a homeomorphism.

Proof. It is obvious that the map is a bijection map, and it is also equivariant. Let U be an open subset of G/G_x , then $\pi^{-1}(U) = UG_x$ is open in G where π is the canonical map from G onto G/G_x . By the above lemma, Ux is open in Gx . ■

3.2.8 Example. Let $G=\mathbb{Q}$ be the group of rational numbers with usual addition operation and the relative topology of the usual topology on \mathbb{R} , and let G act on itself by the usual translations. That is, the action $(g, x) \rightarrow g + x$ for each $g, x \in \mathbb{Q}$. Then G acts continuously on itself. Also, the orbit space is just one point. Now for every net $g_\alpha \rightarrow \infty$ then either g_α goes to $\pm\infty$ or g_α converges to an irrational number r , and in the both cases if there is a net $x_\alpha \rightarrow x \in \mathbb{Q}$, then $g_\alpha + x_\alpha \rightarrow \infty$. So $x \notin J(x)$ for each $x \in \mathbb{Q}$. This action is not Cartan in the sense of Palais because $0 \in \mathbb{Q}$ and for any neighborhood V of 0, the set $((V, V))$ has not compact closure in \mathbb{Q} .

3.2.9 Definition. (New versions of w -proper actions) Let X be a topological space and G be a topological group and let G acts continuously on X . Then the action G on X is said to be:

- i-** w -proper action if is w -Cartan and the orbit space X/G is Hausdorff.
- ii-** w -Palais proper if it is w -Cartan and the orbit space X/G is regular (T_2 is included).
- iii-** w -strongly proper if it is w -Cartan and the orbit space X/G is paracompact (T_2 is included).

3.2.10 Remark. It is obvious that every w -strongly proper action is a w -Palais proper action and the latter is a w -proper action. A good universal example for converse sides is given by Biller in [8] as following; Let G be a topological group and X is a topological space. Let G acts on the space $G \times X$ by $g(t, x) = (gt, x)$ for every $g, t \in G$ and $x \in X$. To show this action is w -Cartan. Take a point $\bar{z} = (t, x) \in G \times X$. Suppose that $\bar{z} \in J(\bar{z})$, there is a net $(t_\alpha, x_\alpha) \rightarrow (t, x)$ in $G \times X$ and a net $g_\alpha \rightarrow \infty$ such that $g_\alpha(t_\alpha, x_\alpha) = (g_\alpha t_\alpha, x_\alpha) \rightarrow (t, x)$. Then we have that $g_\alpha t_\alpha \rightarrow t$ and $x_\alpha \rightarrow x$. By the previous lemma (3.1.2), if V is a neighborhood of e , there is a neighborhood U of t such that $((U, U)) \subseteq V$. Since $g_\alpha \rightarrow \infty$, there is $\delta \in D$ such that $g_\alpha \notin V$ for every $\alpha \geq \delta$. Since $g_\alpha t_\alpha \rightarrow t$, and $t_\alpha \rightarrow t$ there is $\beta \in D$ such that $g_\alpha t_\alpha \in U$, and $t_\alpha \in U$ for every $\alpha \geq \beta$. So, for $\alpha \geq \delta$ and $\alpha \geq \beta$, we have $g_\alpha \in ((U, U))$ and $g_\alpha \notin V$ is impossible. Therefore, $\bar{z} \notin J(\bar{z})$ for every point $\bar{z} = (t, x) \in G \times X$ and the action is w -Cartan. Moreover, the orbit space of this action is homeomorphic to X , as a result there is w -proper action which is not w -Palais proper and w -Palais proper which is not w -strongly proper.

3.2.11 Proposition. w -proper action is equivalent to the Bourbaki proper if X , in addition, is T_2 .

Proof. Bourbaki proper implies that $J(x) = \emptyset$ for every $x \in X$. That is, $x \notin J(x)$ and hence w -Cartan. Also, the condition implies that for every pair of points $x, y \in X$, we have $y \notin J(x)$ which implies to X/G is T_2 whenever X is T_2 . Conversely, if X is a w -proper, then $x \notin J(x)$ and since the orbit space is T_2 . The condition $y \notin J(x)$ for every pair of points $x, y \in X$ with $Gx \neq Gy$ implies that $J(x) = \emptyset$ for every point $x \in X$, and the latter implies that X is Bourbaki proper whenever X is T_2 . ■

3.2.12 Proposition. In the Palais setting, that is, G is a locally compact (T_2 included) and X is a completely regular ($T_{3\frac{1}{2}}$), Palais proper and w -Palais proper coincide.

Proof. If X is a w -Palais proper, then it is a w -Cartan which is under Palais setting X is also Palais Cartan action from the previous proposition. Moreover, X/G is regular by the definition of w -Palais proper. Thus, X is Palais proper (since Cartan with regularity of the orbit space is equivalent to Palais proper). The proof of the other side is obvious. ■

3.2.13 Proposition. If X is a T_2 G -space, A is an open thin subset of X , then GA is w -proper (also Bourbaki) G -space ($\cong J(x) = \emptyset$, for every $x \in GA$).

Proof. To show that the action $G \times GA \rightarrow GA$ is w -proper. It must be w -Cartan G -space and the orbit space GA/G is Hausdorff. To show that for every point $x \in GA$, $x \notin J(x)$. Suppose that is not true. That is, there is a point $g \cdot x \in GA$, but $g \cdot x \in J(g \cdot x)$. Then $\exists x_\alpha \rightarrow g \cdot x$ in GA and $\exists g_\alpha \rightarrow \infty$ in G such that $g_\alpha x_\alpha \rightarrow g \cdot x$ in GA . Since $x \in A$ and $g_\alpha^{-1} g_\alpha x \rightarrow x$, then there is $\delta \in D$ such that $g_\alpha^{-1} g_\alpha x \in A$ for every $\alpha \geq \delta$, so $g_\alpha^{-1} g_\alpha \in ((A, A))$ which has a quasi-compact closure. Then there is a $g \in G$ and $g_\alpha^{-1} g_\alpha \rightarrow g$ or $g_\alpha \rightarrow g \cdot g$ which is a contradiction. Therefore, for every $x \in GA$, $x \notin J(x)$. To show that GA/G is T_2 . Suppose that there are $\bar{x}, \bar{y} \in GA/G$ such that $\bar{x} \neq \bar{y}$ and $y \in J(x)$ where $x \in \bar{x}$ and $y \in \bar{y}$, then there are $g_0, g_1 \in G$ such that $a_0 = g_0 \cdot x \in A$ and $a_1 = g_1 \cdot y \in A$. Since $y \in J(x)$, then $a_1 \in J(a_0)$, then there are a net $g_\alpha \rightarrow \infty$ and $x_\alpha \rightarrow a_0$ such that $g_\alpha x_\alpha \rightarrow a_1$. A is open, then there is $\delta \in D$ such that $x_\alpha \in A$ and $g_\alpha x_\alpha \in A$ for each $\alpha \geq \delta$. This means that $g_\alpha \in \overline{((A, A))}$ which is quasi compact. So $g_\alpha \rightarrow g$ for some $g \in G$ which is a contradiction. Therefore, GA/G is T_2 , and the action is w -proper. ■

3.2.14 Corollary. In the Palais setting, for a w -Cartan G -space X , every point has a neighborhood N such that GN is Palais proper.

Proof. It follows from the fact that Cartan and w -Cartan coincide in the Palais setting. ■

3.2.15 Remark. If X is a G -space and A is a closed thin subset of X , then GA is not necessarily to be closed in X . For example, a subset

$A = \{(x, y) \in \mathbb{R}^2 : (x - 16)^2 + y^2 \leq 1\}$ is a closed thin in the G -space \mathbb{R}^2 , where G is the set of real numbers with the usual topology acts on \mathbb{R}^2 by the formula $t(x, y) = (xe^{-t}, ye^t)$. However, GA is not closed. That is, $\pi(A)$ is not closed in the orbit space X/G . If A is a small subset of a T_2 G -space, then we can obtain the following results.

3.2.16 Proposition. Let A be a small subset of a T_2 G -space X , then G acts on GA w -properly.

Proof. First, we show $J(x) = \emptyset$ for every $x \in GA$. Suppose there are points $x = g_1a_1, y = g_2a_2 \in GA$ such that $y \in J(x)$. Then there is a net $g_\alpha a_\alpha \rightarrow g_1a_1$ in GA and a net $t_\alpha \rightarrow \infty$ such that $t_\alpha g_\alpha a_\alpha \rightarrow g_2a_2$. Take U be a neighborhood of a_1 in X such that a set $((A, U))$ has a quasi-compact closure. Since $g_1^{-1}g_\alpha a_\alpha \rightarrow a_1$ and $a_\alpha \in A$, thus $g_1^{-1}g_\alpha \in ((A, U))$ which implies to there is a $g \in G$ such that $g_1^{-1}g_\alpha \rightarrow g$ or $g_\alpha \rightarrow g_1g$. Similarly, there is a neighborhood V of a_2 in X such that $((A, V))$ has a quasi-compact closure. Then, we have $g_2^{-1}t_\alpha g_\alpha \in ((A, V))$ which implies to there is a $t \in G$ such that $g_2^{-1}t_\alpha g_\alpha \rightarrow t$ or $t_\alpha \rightarrow g_2t g_1g$ which is a contradiction. Since x, y are arbitrary points in GA , therefore, $J(x) = \emptyset$ for every $x \in GA$. It is obvious that GA is a subspace of T_2 space. Therefore, GA is a T_2 and w -proper G -space. ■

3.2.17 Proposition. [4] If A is a small subset of a G -space X , then \bar{A} is also small. Moreover, if $f: X \rightarrow Y$ is a morphism (G -map) of G -spaces Y and X , then $f^{-1}(A)$ is a small subset of Y .

3.2.18 Proposition. Let X be a T_2 G -space and let S be a strong subset of X , then X is w -proper.

Proof. From the proposition (3.2.16), $GS = X$ is a w -proper. ■

3.2.19 Proposition. Let X be a T_2 G -space and A be a closed small subset of X , then:

- a) The restriction map ρ of the action on $G \times A$ into X is a proper map.
- b) The restriction map φ of the orbit map on A into X/G is a proper map.

Proof. For (a), to show ρ is a proper map, it suffices to show that its exceptional set E_ρ is empty. Suppose that there is a point $x \in E_\rho$, then there is a net $(g_\alpha, a_\alpha) \rightarrow \infty$ in $G \times A$ such that $g_\alpha a_\alpha \rightarrow x$. Since A is small, there is a neighborhood U_x of x such that the set $((A, U_x))$ has quasi-compact closure. $g_\alpha a_\alpha \rightarrow x \in U_x$, then there is $\delta \in D$ such that $g_\alpha a_\alpha \in U_x$ for every $\alpha \geq \delta$ that implies, since $a_\alpha \in A$, to $g_\alpha \in ((A, U_x))$ which has quasi compact closure, so there is $g \in G$ such that $g_\alpha \rightarrow g$ and then $g_\alpha^{-1} \rightarrow g^{-1}$ and $g_\alpha^{-1}(g_\alpha a_\alpha) = a_\alpha \rightarrow g^{-1}x \in A$, since A is closed. We have that $(g_\alpha, a_\alpha) \rightarrow (g, g^{-1}x)$ which is a contradiction and $E_\rho = \emptyset$. Since X is T_2 , then ρ is proper. To prove (b), first we show that GA is closed in X . Take $x \in \overline{GA}$. Then there is a net $g_\alpha a_\alpha \rightarrow x$ and $g_\alpha a_\alpha \in GA$. Since A is small, there is a neighborhood U of x in X such that the set $((U, A))$ has quasi-compact closure. Since $g_\alpha a_\alpha \in U$ for some $\alpha \geq \delta$ and $g_\alpha^{-1}(g_\alpha a_\alpha) = a_\alpha$. Thus, $g_\alpha^{-1} \in ((U, A))$ and there is a $g \in G$ such that $g_\alpha^{-1} \rightarrow g$. Also, we get $a_\alpha \rightarrow g^{-1}x \in A$ since A is closed. Therefore, $g(g^{-1}x) = x \in GA$ and GA is closed in X . GA is G -invariant

and hence GA/G is closed in X/G . Now, suppose that $SE_\varphi \neq \emptyset$, that is, there is a point \tilde{x} in SE_φ . Thus, there is a net a_α in A such that $a_\alpha \rightarrow \tilde{x}$ and $clu(a) \cap \varphi^{-1}(\tilde{x}) = \emptyset$ in A . Since $\varphi^{-1}(\tilde{x}) = ((A, \{x\})) x$ which is closed since A is closed and quasi-compact since A is small. X is T_2 , there is a neighborhood U of $\varphi^{-1}(\tilde{x})$ which does not meet $clu(a_\alpha)$ this implies to $a_\alpha \not\rightarrow \tilde{x}$ and a contradiction. Therefore, $SE_\varphi = \emptyset$. ■

w-Action by locally quasi-compact groups

Our aim is to put the three kinds of proper actions in clear implications under suitable conditions. Palais setting is reasonable, so we shall not go away from those conditions. The following shows the analogous properties that are still true for our forms of w -actions and we need G is a locally quasi-compact and X is T_2 .

3.2.20 Lemma. Let G be a topological group. Then

- a) For every quasi-compact subset of G its closure is still a quasi-compact,
- b) if G is a locally quasi compact and K is a subset of G , then \bar{K} is a quasi-compact if and only if K contained in a quasi-compact closed neighborhood.

Proof. The proof for (a), it is follows that if K is a quasi-compact, then $\bar{K} = \{\bar{e}\}K$ [19] which is a quasi-compact, e is the identity element in G . For (b), it is obvious that if $K \subseteq V$, where V is closed quasi-compact neighborhood of K , then $\bar{K} \subseteq V$, and then \bar{K} is quasi-compact. Conversely, if K has quasi-compact closure, then for every $g \in \bar{K}$ there is quasi-compact neighborhood V_g and hence by (a), there is a quasi-compact \bar{V}_g . So $\bar{K} \subseteq \bigcup_{g \in \bar{K}} \bar{V}_g$, then there is a finite set g_1, \dots, g_n such that $\bar{K} \subseteq \bigcup_{i=1}^n \bar{V}_{g_i}$. Choose $V = \bigcup_{i=1}^n \bar{V}_{g_i}$. Then V is a quasi-compact closed neighborhood of K . ■

3.2.21 Proposition. Let G be a locally quasi-compact topological group acts continuously on X , and for every $x \in X$ the subspace Gx is T_2 . Then the action of G on X is w -Cartan if and only if for every point $x \in X$ has a thin neighborhood.

Proof. Assume that the action is w -Cartan. Since $x \notin J(x)$, then by proposition. (2.4.6) there is a neighborhood U of x and a quasi-compact neighborhood V of the identity such that $\overline{(U, U)} \subseteq \bar{V}$ which is a quasi-compact, that is x has a thin neighborhood. The converse is obvious. ■

3.2.22 Example. Let $G = \{1, -1\}$ be a group with discrete topology and let X be the plane with a topology structure is identified by open sets \emptyset, X, U_1, U_2 where U_1 is open upper half of the plane and U_2 is the open lower half of the plane. Then a map $G \times X \rightarrow X$ is given by the formula $\pm 1 \cdot (x, y) = (x, \pm y)$ is an action. The continuity of the map come from that the inverse image of U_i , $i = 1$ or 2 is the set $\{\{1\} \times U_i\} \cup \{-1\} \times U_j\}$ which is open in $G \times X$, where $j = 1$ if $i = 2$ and vice versa. X is not T_2 but every orbit is T_2 space of X . Also, every point has a thin neighborhood since G is compact. X is not Cartan G -space in the sense of Palais since U_1 is thin but $GU_1 = X \setminus L$ is not Palais proper, where L is the set of points on x -axis. By proposition (3.2.21), X is w -Cartan.

3.2.23 Theorem. (w -proper actions by locally quasi-compact groups) Let G be a locally quasi-compact group acts continuously on a T_2 topological space X . Then the action is:

- i-** w -proper if and only if every pair of points of X has relatively compact neighborhoods.
- ii-** w -Palais proper if and only if X is regular and each point has small neighborhood.

iii- w -strongly proper if and only if X is paracompact and there is an open strong subset of X .

Proof. For (i), assume that the second side in (i) is satisfied. Suppose that there a pair of points $x, y \in X$ such that $y \in J(x)$. Then there is a net $x_\alpha \rightarrow x$ in X and a net $g_\alpha \rightarrow \infty$ in G such that $g_\alpha x_\alpha \rightarrow y$. By the assumption, there are neighborhoods U, V of x, y respectively, such that $((U, V))$ has a quasi-compact closure. Also, since $x_\alpha \rightarrow x$ and $g_\alpha x_\alpha \rightarrow y$ there is $\delta \in D$ such that $x_\alpha \in U$ and $g_\alpha x_\alpha \in V$ or $g_\alpha \in ((U, V))$ for every $\alpha \geq \delta$. The set $((U, V))$ has a quasi-compact closure, so there is $g \in G$ and $g_\alpha \rightarrow g$ which is a contradiction. Thus, $J(x) = \emptyset$ for every $x \in X$. The last result with our assumption that X is T_2 implies to that the action is Bourbaki proper. By proposition (3.2.11), X is w -proper. Conversely, assume the action is w -proper and again by proposition (3.2.11), that means $J(x) = \emptyset$ for each $x \in X$. Suppose that there is a pair of points $x, y \in X$ such that every pair of neighborhoods U_x and U_y of x, y respectively, the set $((U_x, U_y))$ has no a quasi-compact closure. Since $J(x) = \emptyset$, then the set $((\{x\}, \{y\}))$ is a quasi-compact. Let V be a quasi-compact neighborhood of the set $((\{x\}, \{y\}))$ and then \bar{V} is still quasi-compact by lemma (3.2.20). Then, by our hypothesis, $((U_x, U_y)) \not\subseteq \bar{V}$ for every pair of neighborhoods of x , and y . We can construct a net $x_\alpha \rightarrow x$ in X and a net g_α in G such that $g_\alpha x_\alpha \rightarrow y$ and $g_\alpha \notin \bar{V}$. Since $J(x) = \emptyset$, then $g_\alpha \rightarrow g$ for some $g \in G$. By the continuity of the action and X is T_2 , then we have $g_\alpha x_\alpha \rightarrow gx = y$ and $g \in ((\{x\}, \{y\})) \subseteq V$ and hence $g_\alpha \in V \subseteq \bar{V}$ which is a contradiction. Therefore, the statement is true. For (ii), assume that the action is w -Palais proper. To show that every point in X has small neighborhood, take $x \in X$. Since $J(x) = \emptyset$, x has thin neighborhood, say U_x , and then GU_x is a neighborhood of Gx . By regularity of the orbit space, there is an invariant closed neighborhood W of Gx in X/G such that $x \in W \subseteq GU_x$. Put $S = U_x \cap W$. We

claim that the subset V is a small neighborhood of x in X . If y is an arbitrary point in X , then either $y \in GU_x$ or $y \in X \setminus GU_x$. If $y \in GU_x$, say $y \in gU_x$ for some $g \in G$, then gU_x is a neighborhood of y and $((gU_x, S)) \subseteq ((gU_x, U_x))$ where the latter has a quasi-compact closure. If $y \in X \setminus GU_x$, then $X \setminus W$ is a neighborhood of y and $((X \setminus W, S)) = \emptyset$. So, in either case y has a neighborhood which is a relatively thin to S . Thus S is a small neighborhood of x . Since x is an arbitrary point in X , every point in X has small neighborhood. Now, pick a point $x \in X$ and let S be a small neighborhood of x . Then the closure \bar{S} is also small neighborhood of X . By proposition (3.2.19, *b*), the map $\bar{S} \rightarrow X/G$ is proper, so \bar{S} is regular since the inverse image of regular space is regular under proper map and the domain space is T_2 . We have that every point in X has a closed neighborhood which is regular subspace of X . Therefore, X is regular [11, I, prop.13]. Conversely, if every point has a small neighborhood then $J(x) = \emptyset$ for every $x \in X$ and hence every point has a small closed neighborhood K which is regular and admits a proper map, by proposition (3.2.19, *b*), onto the orbit space X/G . Therefore, X/G is regular. For (iii), the proof of the first part, we adapt it from Abel [4, prop.1.8]. Since X is w -Palais proper, there is a closed small subset S of X from part *ii*. Now for every $x \in X$ there is an open neighborhood U_x such that $A \cap GU_x \subseteq U_x$. The open cover $\pi(U_x)$, $x \in X$ has a locally finite open refinement $\{V_{x_\alpha}\}$ such that $V_{x_\alpha} \subseteq \pi(U_{x_\alpha})$. Set $W_\alpha = U_{x_\alpha} \cap \pi^{-1}(V_{x_\alpha})$. Then the family $\{W_\alpha\}$ is a locally finite of small subsets of X . Then $F = \cup W_\alpha$ is an open small star subset of X . Moreover, the product $G \times F$ is a quasi-regular and a quasi-paracompact. By proposition (3.2.19, *a*), the map $G \times F \rightarrow X$ is surjective proper, then X is a quasi-paracompact. X is T_2 and hence X is also paracompact. For the other side, assume that X is paracompact and F is an open strong subset of X , then a map $\bar{F} \rightarrow X/G$ is proper, proposition (3.2.19, *b*). Since \bar{F} is paracompact, then X/G is also paracompact. Being there is a

strong subset of X implies to every point in X has a thin neighborhood proposition (3.2.19), and hence $x \notin J(x)$. Therefore, the action is strongly proper. ■

The following facts in [4] then become results following from theorem (3.2.23). We sum them up in the following corollary.

3.2.24 Corollary. Let G be a locally quasi-compact group acts continuously on T_2 space X and the action of G is w -Cartan proper. Then

- a) If X/G is a paracompact, X has a strong set.
- b) X has a paracompact (closed) strong set if and only if X/G is paracompact.
- c) If X/G is paracompact, then X is paracompact.

3.2.25 Proposition. Let G be a locally compact acting w -strongly properly on a metrizable space X . Then then X/G is metrizable.

Proof. It follows from proposition (3.2.19) and theorem (3.2.23, *iii*) that there is a strong (also closed) subset S of X and there is a surjective proper mapping from S onto the orbit space X/G . Metrizable is a proper map invariant. ■

3.3 Application

In this section, we show an application for limit sets, especially the weaker forms of w -actions in another topic that are related to the theory of G -spaces. It is well-known that if X is a topological group and G is a subgroup of X , there is a natural action of G on X is given by $g * x = x * g^{-1}$ for every $g \in G$, and $x \in X$. In [5], the author has answered the following open problem:

Let G be a locally compact group. Then the orbit space X/G of any paracompact proper G -space X is paracompact.

The author in that work has solved the above problem in a specific case, when a space G is a topological subgroup of X . We shall show how our technique could be a good way to describe the above problem. Before we take up the main proof of the open problem, the following results can be established for an arbitrary case.

3.3.1 Proposition. Let G be a topological subgroup of a group X , and let G act on X naturally. For $x \in X$, $x \notin J(x)$.

Proof. Suppose there is a point $x \in X$ and $x \in J(x)$. Then there are nets $g_\alpha \rightarrow \infty$ in G and $x_\alpha \rightarrow x$ in X such that $g_\alpha * x_\alpha = x_\alpha \cdot g_\alpha^{-1} \rightarrow x$ in X . By continuity of multiplication law of the group X , then $x_\alpha^{-1} \rightarrow x^{-1}$ in X and $g_\alpha^{-1} = x_\alpha^{-1}(x_\alpha \cdot g_\alpha^{-1}) \rightarrow x^{-1} \cdot x = e$ in X or $g_\alpha \rightarrow e$ in X and then in G which is a contradiction. Since the point x is an arbitrary point in G , we have $x \notin J(x)$ for every point $x \in X$. ■

3.3.2 Remark. One can deduce that every natural action is w -Cartan.

3.3.3 Proposition. The natural action of G on X is w -Palais proper if and only if G is closed subgroup of X .

Proof. If the action is w -Palais proper, then the orbit space X/G is T_2 and this equivalent to G is closed. Conversely, if G is a closed subgroup of X , then X/G is T_2 and hence it is also regular. By the proposition (3.3.1), we have $x \notin J(x)$ for every point $x \in X$. Therefore, the action is w -Palais proper. ■

3.3.4 Remark. One can deduce that the natural action is also w -proper when G is closed subgroup of X .

3.3.5 Proposition. Let X be a Hausdorff topological group and G be a closed subgroup of X , the natural action of G on X is Bourbaki proper.

Proof. This follows from remark (3.3.4) and proposition (3.2.11). ■

3.3.6 Proposition. Let H be a subgroup of G . Then the natural action, $t(gH) = tg(H)$, of G on G/H is a w -Cartan if and only if H is a quasi-compact subgroup of G .

Proof. Since the action of G on G/H is transitive, then $\Lambda(x) = J(x)$ for every $x \in G/H$. Thus, $H \notin \Lambda(H)$ implies that the stabilizer subgroup of $H \in G/H$ under G , that is $G_H = \{g \in G: gH = H\} = H$ is quasi-compact. Conversely, if H is quasi-compact, suppose that the action is not w -Cartan. Then there is a point gH in G/H such that $gH \notin J(gH)$, then by proposition (2.4.6) there is a neighborhood U_{gH} of gH in G/H and a quasi-compact neighborhood V of the identity element in G such that $((U_{gH}, U_{gH})) \subseteq V$. Then $gH \in J(gH)$ implies that there is a net $g_\alpha \rightarrow \infty$ in G and a net $t_\alpha H \rightarrow gH$ in G/H such that $g_\alpha t_\alpha H \rightarrow gH$. Thus, $g_\alpha \in ((U_{gH}, U_{gH})) \subseteq V$ which means the net g_α must have convergent subnet which is a contradiction. ■

3.3.7 Remark.

1. If X is a Hausdorff topological group and G is a locally compact subgroup of X , then the natural action of G on X has Palais setting because X is completely regular (T_2 included) and G is a locally compact group.
2. Corollary (3.3.8) provides an argument for theorem (1.1) in [5].

3.3.8 Corollary. If X is a T_2 topological group and G is a locally compact subgroup of X , the natural action of G on X is Palais proper.

Proof. It follows from remark (3.3.7, 1), proposition (3.3.3), and proposition (3.2.12). ■

Now let us give a sketch for the proof of the suggested open problem using our notions. Under assumptions of theorem (3.2.23, *iii*), we have that if X is paracompact and it has an open strong subset, then it is equivalent to the action being w -strongly proper. That is, X is w -Cartan and the orbit space is paracompact. Therefore, under the open problem assumption, the orbit space is paracompact if and only if X has closed strong subset. The last claim always holds for a locally compact subgroup G of T_2 group X acting naturally on X [5, Th., 1.2]. We can summarize the above discussion by the following proposition.

3.3.9 Proposition. Let G be a locally compact subgroup of T_2 group X , and let G act naturally on X . Then X is paracompact if and only if the action is w -strongly proper.

Proof. It follows from theorem (3.2.23, *iii*) and the above proof sketch. ■

Finally, it is also interesting to refer in this stage to the advantage of the existence of a closed strong subset A of G -space X . Proposition (3.2.19, *b*) admits a surjective proper map $A \rightarrow X/G$. Moreover, it is well-known that proper mapping and its inverse could be preserve many topological properties such as, for example, compactness, para-compactness, metrizability, and so on. In the following, we state more general result that it holds whenever a G -space has a strong subset.

3.3.10 Proposition. Let P be a topological property preserved under proper maps and inherited by closed subsets. Assume that X is a topological group with the property P and let G be a locally compact subgroup of X . Then the orbit space X/G of the natural action also has the property P .

Proof. It follows from the fact that the natural action has closed strong set and proposition (3.2.19, *b*). ■

Chapter Four

Convergence Techniques for Slices

The existence of slices in G -spaces has the advantage that it supports additional theorems in transformation group action theory. For example, the Conjugacy Theorem due to Montgomery, the Equivariant Embedding Theorem [27, 28, 29], and some other notions. In this Chapter, we discuss a slice theorem in the theory of G -spaces, especially the *enough* slice notion that has been presented by Biller in [8]. The convergence technique is used to verify the notion of *enough* slice with weaker forms of w -actions. We show that if G is an arbitrary locally compact group acting on a completely regular G -space X , then the existence of *enough* slice at each point is equivalent to X being w -Cartan. Moreover, Biller in [9] has defined an H -slice where H is a closed subgroup of an arbitrary locally compact group acting on a completely regular space. Also, the author gives a characterization for the existence H -slice without assumption of Lie group on G . In the first section, we give helpful motivation for slice in an arbitrary G -space.

Motivation for slices

Let G be a topological group and H be a subgroup of G and A be a H -space. Then a twisted product of a fiber A is the orbit space $G \times_H A$ of an H -action θ on the space $G \times A$ given by $h(g, a) = (gh^{-1}, ha)$. The H -orbits are denoted by $H(g, a)$ or $[g, a]$, and the canonical quotient map is denoted by $q: G \times A \rightarrow$

$G \times_H A$ is given by $(g, a) \mapsto [g, a]$. It is clear that q is an open map. The orbit space may be considered as a G -space and takes the form $g[g_1, a] = [gg_1, a]$. Then q may be considered as H -map. Take $S = q(e, A)$. Then S satisfies the following natural three properties [21].

1. $((S, S)) = H$.
2. S is an H -invariant.
3. The map $G \times S \rightarrow G \times_H A$, given by $(g, s) \rightarrow gs$, is open.

Let us now call S an H -slice in $G \times_H A$. On the other hand, a natural question may arise concerning the conditions under which an arbitrary G -space X can be written as $G \times_H A$ where A is an arbitrary H -space. In this case, one may construct, in a natural way, an H -slice S in X . Let X be a G -space and A is a subset of X such that $((A, A)) A = A$, then $((A, A))$ is a subgroup of G , say H . Also, we can easily construct the twisted product $G \times_H A$ with fiber A and a quotient map $q: G \times A \rightarrow G \times_H A$. Also, one can consider a map $G \times A \rightarrow X$ is given by $(g, a) \mapsto ga$ which is a restriction of the G -action on $G \times A$ and it is always considered as an H -map. Moreover, we can construct a map from the two previous maps, say $\tau: G \times_H A \rightarrow X$ given as $[g, a] \rightarrow ga$, which is always injection, that makes the following diagram is commutative.

$$\begin{array}{ccc}
 G \times A & \rightarrow & X \\
 \downarrow q & \nearrow \tau & \\
 G \times_H A & &
 \end{array}$$

According to our aim, if the map $G \times A \rightarrow X$ is an open (or closed), then the map τ is an embedding onto an open subspace of X . Since the diagram commutes, we

have that $G \times_H A \cong GA \subseteq X$. In this case the subset $S = \tau(q(e, A)) \subseteq GA$ is an H -slice in GA which is open in X , and so S is also an H -slice in X . On the other hand, suppose there is a G -map $f: X \rightarrow G/H$ of G -spaces X and G/H . Take $S = f^{-1}(H)$. Then S is an H -space and the twisted product $G \times_H S$ with fiber S exists. By the same procedure, and because the map τ , in this case, is even onto (since f is a G -map), then it is a homeomorphism onto X whenever the map $G \times S \rightarrow X$ is open or closed and so we have S is an H -slice in X and $X \cong GS \cong G \times_H S$.

4.1 Slices

There are several definitions for the slice notion in the G -space theory, and here we recall the most well-known ones. In general, they are not equivalent, but if G is a locally compact group acting on T_2 space, then they coincide.

4.1.1 Definition. (Slice in Palais Sense). Let G be a locally compact group acting on a completely regular space X and let H be a closed subgroup of G . A subset S of X is said to be an H -slice provided that GS is open in X and there is G -map $f: GS \rightarrow G/H$ such that $f^{-1}(H) = S$. By a slice at x we mean a G_x -slice in X which contains x .

4.1.2 Proposition. [29] Let X be a G -space and let H be a closed subgroup of G . Then the following are equivalent:

- a) H is compact and S is an H -slice in X .
- b) S has the following properties:
 1. S is closed in GS , and GS open in X ,
 2. S is H -invariant,
 3. $((S, S)) = H$,

4. S has a thin neighborhood in GS .

4.1.3 Theorem. (Palais Slice Theorem). Let G be a Lie group. If X is a Cartan G -space (in the sense of Palais), then there exists a slice at each x in X .

4.1.4 Lemma. [9] Let G be a locally compact group acting on a Hausdorff space X and let H be a closed subgroup of G . We call a nonempty subset $S \subseteq X$ an H -slice in X for the action of G if any one of the following equivalent conditions is satisfied:

1. There is a G -invariant open subset $U \subseteq X$ that admits a G -equivariant map $\varphi: U \rightarrow G/H$ such that $S = \varphi^{-1}(H)$. (Note that $U = GS$, and φ is determined uniquely by S because $\varphi(gs) = g\varphi(s)$).
2. There is an H -space A and a G -equivariant open embedding $\psi: G \times_H A \rightarrow X$ such that $\psi([e, A]) = S$.
3. The subset S is invariant under H , the subset $GS \subseteq X$ is open, and $\chi: G \times_H S \rightarrow GS$ is given by $[g, s] \rightarrow gs$ is a homeomorphism.

Before taking up the *enough* slice notion [8], we shall review some natural properties of twisted product.

4.1.5 Proposition. The action H on $G \times A$ is free and the map θ is proper whenever H is quasi-compact.

Proof. Take $h \in H$, and $(g, a) \in G \times A$ and let $h(g, a) = (g, a)$. Then $(gh^{-1}, ha) = (g, a)$ which implies to $gh^{-1} = g$ and hence $h = e$. Assume H is quasi-compact. Suppose that there is a point $(g, a) \in SE_\theta$. Then there is a net $(h_\alpha, (g_\alpha, a_\alpha))$ in $H \times (G \times A)$ such that $(g_\alpha h_\alpha^{-1}, h_\alpha a_\alpha) \rightarrow (g, a)$. Since H is a quasi-compact, then $h_\alpha \rightarrow h$ for some h , and so $g_\alpha \rightarrow gh$ and $a_\alpha \rightarrow h^{-1}a$. Since

$h(gh, h^{-1}a) = (g, a)$, then $(gh, h^{-1}a) \in \theta^{-1}(g, a) \cap \text{cls} \left((h_\alpha, (g_\alpha, a_\alpha)) \right) = \emptyset$

which is a contradiction. Therefore, $SE_\theta = \emptyset$ and θ is proper. ■

4.1.6 Corollary: The map θ is closed. ■

4.1.7 Proposition. Let G be a topological group and H be a closed subgroup of G .

Then for every H -space A the action of H on $G \times A$ is w -Cartan. Moreover, if G and A are T_2 , then the action is also w -proper ($G \times_H A$ is Hausdorff).

Proof. We shall prove that $J(g, a) = \emptyset$ for every (g, a) . Suppose that $(g_1, a_1) \in J(g, a)$ for some $(g_1, a_1) \in G \times A$. Then there are nets $h_\alpha \rightarrow \infty$ in H and $(g_\alpha, a_\alpha) \rightarrow (g, a)$ such that $(g_\alpha h_\alpha^{-1}, h_\alpha a_\alpha) \rightarrow (g_1, a_1)$. Then $g_\alpha h_\alpha^{-1} \rightarrow g_1$ and $h_\alpha^{-1} \rightarrow g^{-1}g_1$. Since H is a closed, we have a contradiction. Therefore, $J(g, a) = \emptyset$. If $G \times A$ is T_2 , then θ is w -proper. ■

4.1.8 Proposition. Let K be a quasi-compact subset of T_2 topological group G , and H be a closed subgroup of G . Then for a T_2 H -space A , the restriction of quotient map $q: K \times A \rightarrow G \times_H A$ is proper.

Proof. To show that $E_q = \emptyset$, suppose that there is a point $[g, a] \in E_q$. Then there is a net $(k_\alpha, a_\alpha) \rightarrow \infty$ in $K \times A$ such that $[k_\alpha, a_\alpha] \rightarrow [g, a]$. We have $k_\alpha \rightarrow g \wedge a_\alpha \rightarrow a$ which contradicts our assumption. Therefore, $E_q = \emptyset$. Since $K \times A$ is T_2 , q is proper. ■

4.1.9 Corollary. The map $A \rightarrow [e, A]$ is a G - map and an embedding from A into the orbit space $G \times_H A$.

Proof: It is a restriction of the quotient map q on $\{e\} \times A$ which is proper (closed) by proposition (4.1.8). ■

4.1.10 Remark.

1. Since $G \times_H S$ is considered a G -space by the formula $g[g', s] = [gg', s]$, then every G - orbit $[g, s]$ can be written as $G[e, s]$ where e is the identity element.
2. If $s \in S$ and $g \in G$, then $G_{[g,s]} = H_s$. If $g \in G_{[e,s]}$ and $g[e, s] = [e, s]$, then $[ge, s] = [g, s] = [e, s]$ this implies to that there exists an $h \in H$ such that $g = eh^{-1}$ and $hs = s$. Thus $g = h$ and $g \in H_s$. On the other hand, if $h \in H_s$, then $h[e, s] = [h, s] = [hh^{-1}, hs] = [e, s]$. Therefore, $G_{[g,s]} = H_s$. ■
3. If $g_1, g_2 \in G$ and $s_1, s_2 \in S$, then the set $(([g_1, s_1], [g_2, s_2])) = g_2((\{s_1\}, \{s_2\}))_H g_1^{-1}$. Take $g \in (([g_1, s_1], [g_2, s_2]))$, then $[gg_1, s_1] = [g_2, s_2]$. By the definition of action, there is an $h \in H$ such that $gg_1 h^{-1} = g_2$ and $hs_1 = s_2$. Thus, $h \in ((\{s_1\}, \{s_2\}))_H$ and $g = g_2 h g_1^{-1} \in g_2((\{s_1\}, \{s_2\}))_H g_1^{-1}$. Conversely; if $g \in g_2((\{s_1\}, \{s_2\}))_H g_1^{-1}$, then $g = g_2 h g_1^{-1}$ for some $h \in ((\{s_1\}, \{s_2\}))_H$. Since $hs_1 = s_2$, then we have $g(g_1 h^{-1}) = g_2$ or $(gg_1)h^{-1} = g_2$. Therefore, $[gg_1, s_1] = g[g_1, s_1] = [g_2, s_2]$ and $g \in (([g_1, s_1], [g_2, s_2]))$. ■

4.2 Slice in w -Actions

In this section, we shall follow the definition of Biller for slice. However, we will not consider separation axiom properties on G and X .

4.2.1 Definition. Let G be a topological group that acts continuously on a space X and let H be a closed subgroup of G . An H -slice in X for the action of G is an H -invariant subset $S \subseteq X$ such that the continuous G -invariant map $G \times_H S \rightarrow X$ which is given by $[g, s] \mapsto gs$ is an open embedding. Moreover, the action said to

be having *enough* slice if for each point $x \in X$, there is a closed quasi-compact subgroup K of G such that x is contained in a K -slice.

4.2.2 Remark. Taking the definition (4.2.1), it is obvious that GS is open in X . Consider a map $GS \rightarrow G/H$ which takes $gs \mapsto gH$. It is well-defined since if $g_1s_1 = g_2s_2$, then $g_2^{-1}g_1s_1 = s_2$ and because S is only invariant under H , then $g_2^{-1}g_1 \in H$ and $g_1H = g_2H$. It is clear the map is an equivariant and a surjection. Its continuity comes from the continuity of the map $G \times_H S \rightarrow G/H$ and the openness and surjection of the map $G \times_H S \rightarrow GS$. We have that the definition (4.2.1) implies the Palais' (definition 4.1.1). Therefore, in the Palais setting the two definitions are equivalent (lemma 4.1.4).

The following result comes already from our sense of w -Cartan action.

4.2.3 Proposition. If G is a topological group and S is an H -space where H is a subgroup of G , then the action of G on $G \times_H S$ is w -Cartan if and only if the action of H on S is w -Cartan.

Proof: Assume that S is a Cartan H -space. Suppose there is $s \in S$ and $g \in G$ such that $[g, s] \in J([g, s])$. Then there is a net $g'_\alpha \rightarrow \infty$ in G and $[g'_\alpha, s_\alpha] \rightarrow [g, s]$ such that $g'_\alpha[g'_\alpha, s_\alpha] = [g'_\alpha g'_\alpha, s_\alpha] \rightarrow [g, s]$. This implies that $g'_\alpha \rightarrow g$ and $g'_\alpha g'_\alpha \rightarrow g$ and $s_\alpha \rightarrow s$. By the definition of the twisted product, let $h_\alpha \in H$ be a net in H such that $g'_\alpha g'_\alpha h_\alpha^{-1} \rightarrow g$ and $h_\alpha s_\alpha \rightarrow s$. Since $s \notin J(s)$, then we have $h_\alpha \rightarrow h \in H$. Thus $g'_\alpha \rightarrow ghg^{-1} \in G$ which is a contradiction. Conversely, suppose that there is a point $s \in S$ such that $s \in J(s)$. Then there is a net $h_\alpha \rightarrow \infty$ in H and a net $s_\alpha \rightarrow s$ in S with $h_\alpha s_\alpha \rightarrow s$. Since the map $G \times S \rightarrow G \times_H S$ is continuous and $(e, h_\alpha s_\alpha)$ is a net in $G \times S$ converges to (e, s) , then $[h_\alpha, s_\alpha] = h_\alpha[e, s_\alpha] = [e, h_\alpha s_\alpha] \rightarrow [e, s]$ in $G \times_H S$ this means that $[e, s] \in J([e, s])$ which contradicts that $G \times_H S$ is w -Cartan. Therefore, the proposition is true. ■

4.2.4 Proposition. Let G be a topological group, and H be a closed subgroup of G which acts on S . Then there is a homeomorphism $\bar{\mu}$ from $S/H \rightarrow (G \times_H S)/G$.

Proof. It follows from the composition of induced maps from composition maps μ such that the following diagram is commutative,

$$\begin{array}{ccccc} \mu: H \times S & & \rightarrow & G \times S & \rightarrow & G \times_H S \\ & & & \downarrow & & \downarrow \\ \bar{\mu}: S/H \cong H \times_H S & \rightarrow & & G \times_H S & \rightarrow & (G \times_H S)/G \end{array}$$

and is determined by the maps $(h, s) \rightarrow (h, s) \rightarrow [e, s]$, and then $\bar{\mu}(Hs) = G[e, s]$ which is a continuous bijection. To show $\bar{\mu}$ is open, take US/H and open S/H , then its image under $\bar{\mu}$ is exactly the image of the subset $G \times U$ under $q \circ \mu$ which is open where q is the canonical quotient map. ■

4.2.5 Corollary. If G is a topological group and S is an H -space where H is a subgroup of G . Then the action of G on $G \times_H S$ is w -proper (w -strongly proper) if and only if the action of H on S is w -proper (w -strongly proper).

Proof. It follows immediately from the previous proposition. ■

4.2.6 Proposition. If X is a T_2 G -space and X has *enough* slice, then X is w -Cartan.

Proof. Assume that the action has *enough* slice. To show that for every point $x \in X$ the condition $x \notin J(x)$, pick a point $x \in X$. Then there is a closed quasi-compact subgroup H such that x contained in H -slice $S \subseteq X$. That is, the map $\rho: G \times_H S \rightarrow X$ is an open embedding. Since H is quasi-compact and S is an H -space, then the action of H on S is w -Cartan, that is $x \notin J(x)$ with respect to this H -space S . By proposition (4.2.3). the G -action on $G \times_H S$ is w -Cartan. Since the map $\rho: [g, s] \rightarrow gs$ is an open embedding, then $gx \notin J(gx)$ and hence $x \notin J(x)$ in the G -space GS which is an open subset of X . Therefore, $x \notin J(x)$ in the G -space X and the action of G on X is w -Cartan. ■

4.2.7 Remark. For the slice theorem in the Palais setting we know that w -notions and Palais' notions coincide. We may also then give the inverse of the other direction of the above proposition, but we need the following facts.

4.2.8 Definition. A locally quasi-compact group is called almost connected provided that the orbit space G/G_0 is quasi-compact (and hence compact since every totally disconnected topological group is T_2).

4.2.9 Lemma 1. If G is locally quasi-compact, and H is a closed subgroup of G , then H is locally quasi-compact and the orbit space G/H is locally compact.

Proof. It is obvious. ■

4.2.10 Lemma. [23] If G is locally quasi-compact, then G/G_0 contains arbitrarily small compact open subgroups and corresponding to these groups, G contains open subgroups G' such that G'/G_0 is compact.

4.2.11 Remark. According to lemma 2, every locally compact group contains an arbitrary open subgroup which is almost connected.

4.2.12 Lemma. [15] If the component G_0 of the identity element of a locally compact group has a compact factor group G/G_0 , then every neighborhood of the identity of G contains a compact normal subgroup N such that G/N is a Lie group.

4.2.13 Lemma. [9] Let G be a locally compact group acting properly on a completely regular space X . If $x \in X$ and N is a compact normal subgroup of G such that G/N is a Lie group, then x is contained in a $G_x N$ -slice for the action of G on X .

4.2.14 Theorem. An action of a locally compact group K on a completely regular space X has enough slice if and only if it is w -Cartan.

Proof. The first direction of the proof comes immediately from the proposition (4.2.6). Assume that an action of a locally compact group K on X is w -Cartan.

Since K is a locally compact, then K contains an arbitrary open almost connected

subgroup K' by lemma (4.2.10) such that $G = K'/G_o$ is compact. By lemma (4.2.12), every neighborhood of the identity element e in K' contains a compact normal subgroup N such that K'/N is a Lie group. Since X is a w -Cartan G -space, every point $x \in X$ has a thin (open) neighborhood U on which G acts w -Palais proper. Then by lemma (4.2.13), x contained in $K'_x N$ -slice S with respect to the action of G on X . ■

According to our aim, namely putting the three actions in clear arrangement, we return to Baum-Connes-Higson proper, definition (3.1.1). We shall show the latter proper is just strongly proper under its assumption. The three conditions of the definition (3.1.1) are equivalent to that X is w -Cartan.

4.2.15 Proposition. Under the assumption of the definition (3.1.1), the three conditions of the definition are equivalent to that X is w -Cartan.

Proof. Assume the three conditions of the definition (3.1.1) are satisfied. By proposition (3.2.11), it suffices to show every point has a thin neighborhood. Take a point $x \in X$, there is a triple (U, H, ρ) such that the three conditions are satisfied. Put $S = \rho^{-1}(H)$, and $\rho(g^{-1}x) = H$ for some $g \in G$. Then S is an H -invariant closed subset of U . Since G is a locally compact and H is compact, then there is a compact neighborhood \bar{V} of H in G/H . Let $W = \cup \{\bar{v}: \bar{v} \in \bar{V}\}$. Then W is a compact neighborhood H in G . $V = \rho^{-1}(\bar{V})$ is a neighborhood of S in U . It is easy to see that $g^{-1}x \in S \subseteq V$ and $((V, V)) = W$, that is $g^{-1}x$ has a thin neighborhood V in U that is equivalently x has a thin neighborhood gV in U and hence it is open in X . Therefore, X is w -Cartan. Conversely, by theorem (4.2.14) if $x \in X$, then x is contained in H -slice, say S . Put (GS, H, ρ) where $\rho(gv) = gH$ for every $g \in G$ and $v \in V$. The triple (GS, H, ρ) satisfies the three conditions of the definition (3.1.1). ■

4.2.16 Corollary. Let G be a locally compact and second countable (T_2 included) and let G act continuously on a metrizable space X . Then the following are equivalent.

1. X is a w -strongly proper G -space.
2. X is a *Ba-Co-Hi* proper G -space.

Proof. (2 \rightarrow 1) follows from proposition (4.2.15), X is paracompact and hence X/G is as well. For (1 \rightarrow 2), it is immediately implied by proposition (4.2.15). ■

4.2.17 Remark. Palais has presented many results using the slice theorem. Here we also confirm some of the results in the context of an arbitrary locally compact (non-necessary Lie) group.

4.2.18 Corollary. Let S be H -slice at a point x in X . Then there is a neighborhood V of x such that $y \in V$ implies G_y is conjugate in G to a subgroup of H . Moreover if O is any neighborhood of $e \in G$, then V can be chosen so that in fact for each $y \in V$ we have $G_y \subseteq gHg^{-1}$ for some $g \in O$.

Proof. Take $x \in X$ and S be a H -slice containing x . Then there is a G -map $f: GS \rightarrow G/H$. Take an open neighborhood of the identity element e in G , say N . Put $U = f^{-1}(NH/H)$, then U is an open neighborhood of x in GS and hence in X . For every $u = nh$ in U , then we have $G_u \subseteq G_{f(u)} = G_{nH} = nHn^{-1}$. ■

4.2.19 Corollary. Let S be an H -slice at a point x in X . If K is any closed subgroup of G which includes H , then KS is a K -slice in X containing x .

Proof. Let $f: G \times_H S \rightarrow G/H$ be the G -map that associated with the H -slice S . Since $H < G$, there is a canonical map $g: G/H \rightarrow G/K$ given by $gH \rightarrow gK$. The composition of the two is a map $G \times_H S \rightarrow G/K$. Since $G \times_K (K \times_H S) \cong G \times_H S$ and $(K \times_H S) \cong KS$ since S is H -slice under the subgroup K . Therefore, the

composition is the G -map $G \times_K KS \rightarrow G/K$ and hence KS is K -slice containing x . ■

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