Estimating the Azimuthal Mode Structure of Ultra Low Frequency Waves and Its Effects on the Radial Diffusion of Radiation Belt Electrons

Mohammad Barani
West Virginia University, mobarani@mix.wvu.edu

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Estimating the Azimuthal Mode Structure of Ultra Low Frequency Waves and Its Effects on the Radial Diffusion of Radiation Belt Electrons

Mohammad Barani

Dissertation Submitted to the Eberly College of Arts and Sciences at West Virginia University in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics

Weichao Tu, Ph.D. Chair
Mary K. Hudson, Ph.D.
Leonardo Golubovic, Ph.D.
Paul Cassak, Ph.D.
Earl Scime, Ph.D.

Department of Physics and Astronomy
Morgantown, West Virginia

2021

Keywords: ULF magnetic pulsations, azimuthal mode number, radial diffusion, electron radiation belt, spacecraft data analysis, cross spectral analysis

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ABSTRACT

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Mohammad Barani

Characterizing the azimuthal mode number $m$ of Ultra Low Frequency (ULF) waves is critical to quantifying the radial diffusion of radiation belt electrons. A Wavelet cross-spectral technique is applied to the compressional ULF waves observed by multiple pairs of GOES and MMS satellites to estimate the mode structure of ULF waves. A more realistic distribution of mode numbers is achieved by inclusion of the modes corresponding to different wave propagation directions as well as at $m$ higher than fundamental mode number. For the event study of a geomagnetic storm using GOES data, ULF wave power is found to dominate at low mode numbers during high solar wind dynamic pressure. The change of sign in $m$ around noon was observed to be consistent with anti-sunward wave propagation due to solar wind. To reduce the $2\pi$ ambiguity in resolving $m$, a cross-pair analysis is performed on GOES field measurements which is demonstrated to be effective in generating more reliable mode structure of ULF waves during high Auroral Electrojet (AE) periods.

During another event with two consecutive interplanetary shocks compressing the dayside magnetopause, contribution of low versus high modes in the power of ULF pulsations and their frequency signatures are resolved using high-fidelity multi-probe MMS magnetometer data. The analysis clearly shows that shock onset corresponds to more in-phase magnetic fluctuations in the Pc4-5 regimes than what follows, and smaller spatial scale fluctuations are implied by the dominant high mode numbers observed after both shocks hit and passed the magnetosphere. At the shock impacts, the contribution of higher frequencies (e.g., > 7 mHz, corresponding to Pc-4) to the wave power is not negligible, while after the impacts, the power distributes significantly over lower frequencies (e.g., < 7 mHz corresponding to Pc-5 compressional pulsations).

A threshold mode, $m_{th}$, is introduced to give an approximate range of the largest resolvable $m$ using ideal-MHD models. In addition, a first-principle calculation is introduced to address the long-lasting debate on the contribution of higher ULF wave azimuthal mode number (e.g., $|m|>1$) on the radial diffusion rates $D_{LL}$ of energetic electrons. We showed that the simplified assumption of $m=+1$ in ULF waves would overestimate the $D_{LL}$ by more than 300%. Therefore, contrary to the previous assumptions in earlier work, inclusion of the negative as well as higher mode values are both important and must be considered in the estimations of radial diffusion of radiation belt electrons.
Acknowledgements

I would like to thank my advisor Weichao Tu, whose full support has always been with me. I admire the kindness and support of my scientific great grandmother Mary K. Hudson whose dedication was a great lesson for me.

I should thank all other members of my PhD committee for the exceptional scientific knowledge that they generously shared with me since I came to the US and West Virginia University in 2014.

This thesis is based upon work supported by NSF grant AGS 1752736 and NASA grants NNX16AG71G, 80NSSC18K1284, and 80NSSC19M0146, as well as support by the National Center for Atmospheric Research, which is a major facility sponsored by the NSF under Cooperative Agreement NO. 1852977. Support from High Altitude Observatory Newkirk Fellowship is acknowledged. I would like to thank NOAA for providing high quality magnetic field observations from the GOES spacecraft, and Goddard Space Flight Center, Space Flight Data Facility of NASA for providing the SSC 4D Orbit Viewer 4.2.7 software for visualization of the GOES and MMS satellites orbits. Solar wind data and geomagnetic indices are provided by OmniWeb (http://omniweb.gsfc.nasa.gov/). GOES magnetic field observations are available at https://satdat.ngdc.noaa.gov/sem/goes/data/.
I dedicate this work to:
all of those whose goal is keeping the moral principles;
the real men and women of their words;
those who strive to see the beauty in Nature and God’s creatures.

A special dedication to my wife, Samira, my children AmirAli and AmirHossein, and my parents. I would hope that you all benefit from this, as my diverted attention on this work has likely cost you the most.
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Chapter 1. Introduction

1.1. The Earth’s Magnetosphere

When solar wind particles hit the Earth’s magnetic field as an obstacle, they get deflected leaving a bubble-like region inside which is dominated by the Earth’s magnetic field. This region is called the magnetosphere (Figure 1-1). Existence of the magnetosphere was predicted as a Chapman-Ferraro cavity in 1931 and almost thirty years later it was confirmed by the Explorer 12 satellite in 1959. As a result of the magnetosphere posing an obstacle to the solar wind, there is a region where the velocity of the supersonic charged particles abruptly drops to subsonic at their encountering the Earth’s magnetic field. This thin region is called bow shock and is located upstream from the magnetopause. The magnetopause is a boundary that satisfies the pressure balance between the solar wind and Earth’s magnetic field.

Figure 1-1. A 3D representation of the Earth’s magnetosphere containing some of the different plasma regions and currents (Kivelson & Russell, 1995).
The Earth’s magnetic field on the dayside magnetosphere is compressed in contrast to the elongated shape on the night side due to the continuous buffeting of the solar wind on the dayside. The magnetopause moves inward and outward as a response to the solar wind pressure variations. Magnetosheath - as a region between the magnetopause and the bow shock - is a more heated and compressed plasma layer of deflected solar wind due to this solar wind encountering the Earth’s magnetic field. The Earth’s magnetic field can be approximated as a dipole field with corrections that can be added as higher terms to this dipole field (Schulz & Lanzerotti, 1974).

Different regions inside the Earth’s magnetosphere have particles with different energy and density ranges. Figure 1-2 demonstrates different plasma populations in and around the magnetosphere (Borovsky et al., 2020). The focus of this thesis work is outer radiation belt also known as electron radiation belt.

Figure 1-2. Density-energy map of the different particle populations in and around the Earth’s magnetosphere (Borovsky et al., 2020). Dashed curves are equal pressure contours and each of the depicted contours are a part by a factor of 10. Reprinted with permission.
1.2. Basic Motions of Electrons in the Earth’s Magnetosphere

There are three basic motions for electrons or any other charged particles in the Earth’s magnetosphere: gyration, bounce, and drift. Gyration is a circular motion of electrons around the present ambient magnetic field. The time scale of this motion is millisecond corresponding to kHz frequency range for MeV energy electrons in the heart of the outer radiation belt dominated by electrons in contrast to the inner zone dominated by multi-MeV protons (Schulz & Lanzerotti, 1974). Bounce is a back-and-forth motion of electrons along the magnetic field lines between the two mirror points at northern and southern hemispheres. A typical time scale of this motion is 0.1 second corresponding to frequencies 100 times smaller than gyration for radiation belt electrons. Due to the gradient in the magnetic field strength from the nature of the approximate dipole magnetic field of Earth, electrons undergo a drift which is a translational motion around earth crossing magnetic field lines present at different longitudes. Drift has a time scale depending on particle energy, pitch angle, and distance from the Earth center. For 1 MeV electrons the drift time is around 30 minutes at \( L = 2 \) and goes down to 7 minutes (corresponding to a frequency \( f = 2.4 \) mHz) at equatorial radial distances \( L = 8 \). \( L \) is equatorial radial distance from Earth’s center normalized to the Earth radius. The drift frequency increases with an increase in the electron energy. The corresponding frequency of the drift motion of radiation belt electrons is generally in the mHz range. It is always useful to define conserved quantities while studying the dynamics of a physical system such as the motion charged particles. Corresponding to each of the three mentioned basic motions, there is an adiabatic invariant. Although the actual motion of a charged particle in Earth’s magnetosphere is always a combination of the three, the use of defined invariants is specifically practical and beneficial since the three corresponding timescales are well separated with around two orders of magnitude difference. For gyration, \( \mu \) is the first adiabatic invariant corresponding to the magnetic flux inside a particle’s gyro orbit. For bounce motion, \( K \) is the second adiabatic invariant which can be interpreted as proportional to the distance between the mirror points that charged particles are bouncing between with proportionality of square root of magnetic field strength at the mirror point in a dipole field approximation for Earth. In this study an equatorially mirroring particle is considered which corresponds to \( K = 0 \). For drift, the third
invariant is the magnetic flux through a closed surface which intercepts the particle’s drift orbit. This invariant is inversely related to the Roederer L (Roederer, 1970), which for a dipole field is equal to the equatorial radius of a drift orbit in Earth radius $Re$.

### 1.3. ULF Waves in Earth’s Magnetosphere

Ultra Low Frequency (ULF) waves in the Pc4-5 bands (Jacobs et al., 1964), corresponding to the frequency range of ~1.6-22.2 mHz, are comparable to the drift frequency of energetic electrons (100s keV to MeV) in the outer radiation belt region at equatorial radial distances from ~3 to 7 $Re$ (Kivelson & Russell, 1995). The proximity of these two frequency ranges (i.e., fluctuating fields versus particles’ drift) will bring new phenomena to the Earth’s magnetosphere such as resonance and particles’ transport, some of which are still challenging for physicists to completely understand and model since the start of the space era. ULF wave fluctuations in the magnetosphere typically have dominant power compared to other fluctuations and waves at other frequency ranges (Zong et al., 2017). This wave power dominance and field variations on the timescale of electron drift periods leads to the importance of ULF wave studies in their potential effect on radiation belt electron dynamics.

#### 1.3.1. Generation Mechanisms of ULF Waves

Although these geomagnetic pulsations had been observed in ground-based magnetometers since 1859 (Glassmeier, 1995), it took a long time for space scientists to first understand the mechanism
of MagnetoHydroDynamic (MHD) waves (Alfvén, 1942) and thereafter perceive the connection between these waves and geomagnetic pulsations (Dungey, 1954). Previous reports demonstrate well the correlation between radiation belt electron flux enhancement and the concurrent ULF wave power level (Rostoker et al., 1998).

Drivers of the ULF waves can be categorized into four main mechanisms (Glassmeier, 1995): (1) Instabilities at the boundaries of the magnetosphere; (2) Externally generated instabilities and their resulting wave propagation into the magnetosphere; (3) Macroscopic changes of the magnetospheric configuration; and (4) Instabilities generated by internal magnetospheric processes. Although it is not generally easy to distinguish between the different drivers of the ULF waves, sometimes the correlation between the ULF wave power and the solar wind characteristics and geomagnetic indices and the local time at which the waves are generated/detected can shed light, to some extent, on the possible mechanism leading to the generation of the waves. For example, if the ULF wave activity is observed to be generated at (and propagate from) the flank regions of the magnetosphere, as indicated in Figure 1-4(a), Kelvin-Helmholtz Instability (KHI) can be the mechanism of the waves’ excitation as suggested first by Dungey (1954). KHI at the magnetopause can happen when there is a shear velocity difference near the boundary between the

![Figure 1-4. Schematic picture of Ultra Low Frequency (ULF) wave excitation through different external drivers in the equatorial plane (Adapted from Elkington and Sarris (2016)). The shaded regions depict the local extent of the different wave activities, the small circles at the middle of each panel represent Earth with dark side to the left representing nightside, and the arrows show the direction of solar wind velocity that is continuously impinging upon the dayside magnetosphere. Scenario (a) shows ULF waves driven by solar wind dynamic pressure fluctuations dominated at dayside magnetic local times; Scenario (b) illustrates ULF waves driven by Kelvin-Helmholtz Instability that generally happens around the dawn and dusk flank regions.](image-url)
magnetosphere and magnetosheath which is solar wind plasma that has passed through the Earth’s bow shock.

1.3.2. Propagation of ULF Waves Through the Magnetosphere

The generated ULF waves are MHD waves which can be generally categorized into two main classes: Alfven waves and fast mode waves. These waves can be considered as the plane wave solutions to the wave equation satisfied by the magnetospheric parameters. The isotropic mode solution of the wave equation is the fast mode and the anisotropic mode is the Alfven wave. The Alfven wave is a purely transverse wave with respect to the ambient magnetic field while the fast wave can carry field-aligned perturbations.

The isotropic (fast mode) solution of the wave is \( \omega^2 = v_A^2 k^2 \) while the anisotropic (Alfven mode) is \( \omega^2 = v_A^2 k^2 \cos^2 \theta \) where \( \omega \) is the frequency, \( k \) is wave number, \( v_A \) is the Alfven velocity and \( \theta \) is the angle between \( k \) and the ambient magnetic field \( B_0 \). The Alfven velocity is \( \frac{B_0}{\sqrt{\mu_0 \rho_0}} \) where \( \mu_0 \) is the permeability of the vacuum and \( \rho_0 \) is the total mass density of the background charged plasma particles. Perturbations in velocity \( v \), charge density \( \rho \), and magnetic field \( b \) are related to each other in the same fashion as the Alfven velocity and ambient magnetic field: \( v = \pm \frac{b}{\sqrt{\mu_0 \rho}} \)

where + and − signs denote anti-parallel and parallel propagation with respect to the ambient magnetic field respectively (Glassmeier, 1995). In ideal MHD, the Alfven wave phase velocity direction and propagation direction are the same. The fast mode waves do not carry field-aligned currents while the Alfven mode can carry field-aligned current. After generation of ULF waves, they can propagate deep inside the magnetosphere and can be detected by ground magnetometers if their amplitude is large enough to survive the propagation and damping through the ionosphere.

1.4. Resolving ULF Wave Characteristics in the Magnetosphere

We only look at the magnetic pulsations in this thesis leaving the electric field study for future work. To resolve the characteristics of ULF wave electric and magnetic pulsations, one has to know at least three quantities about the fluctuating fields associated with waves: how the field is changing in time, how it changes in space, and what is the envelope in time and space in which the wave is excited. The time feature of a wave is in its frequency content, the spatial feature is
represented by its wave vector, and the envelope of the waves can be the most difficult part to resolve by sampling the wave field at different times and positions. Lack of information on each of the three quantities would limit realistic estimation of the wave behavior in the magnetosphere. Due to the measurement limitations such as difficulties in accessing the waves over the entirety of their temporal and spatial behavior, simplifications are always made. For example, if using ground magnetometers, magnetic field models must be considered to estimate the place in the magnetosphere where the detected waves come from. If using in-situ measurements, due to the limited number of probes, it is often assumed that the behavior of the wave at the positions between the two probes does not change.

Many of the parameters as well as the geometry in the magnetosphere enjoy an azimuthal symmetry. Among these parameters is the Earth’s magnetic field which is not dependent on local time to the zeroth order approximation (i.e., a dipole field), neglecting features which are important at low altitudes such as the South Atlantic Anomaly where Earth’s magnetic field is weaker than at other longitudes, and high altitude distortions due to magnetopause, tail and ring currents. Due to the assumed azimuthal symmetries in the magnetosphere, it is practical to talk about the rotational description of the wave number $k$ -which is azimuthal wave number- when describing the spatial content of the ULF waves. Mode number $m$ represents the total number of azimuthal wavelengths that would shape a complete circumference around Earth in the azimuthal direction. It can be analytically shown that $m$ is one of the inputs in the radial diffusion coefficient (section 1.5). Determining $m$ from in-situ magnetometer measurements is the focus of this thesis work. In the following sections the importance and significance of $m$, and the different methods to estimate it will be addressed.

1.5. Estimating the azimuthal mode number $m$ of ULF waves
1.5.1. Different Approaches in Resolving $m$ and Challenges

In the following, four general methods for resolving the mode number are outlined:

1. Cross-spectral density and phase analyses of the real-time ULF wave fields from ground-based and in-situ measurements (multi-probe) (Barani et al., 2019; Le et al., 2017; Loto’aniu et al., 2006;
Murphy et al., 2018; Olson & Rostoker, 1978; Sarris & Li, 2017; Sarris et al., 2009; Takahashi et al., 2018; Takahashi et al., 2013).

2. In-resonance flux measurement analysis of particles detected simultaneously with measured ULF waves which follow the drift-bounce resonance condition (single probe) (Mann et al., 1998; Takahashi et al., 1990; Zhang et al., 2019; Zong et al., 2007).

3. Extraction of the global ULF wave magnetic fields (compressional component) and electric fields (azimuthal component) from MHD simulations and their inclusion in the spectral analysis methods to resolve the mode behavior in time and space (Fei et al., 2006; Li et al., 2020; Li et al., 2017; Tu et al., 2012). Tu et al. (2012) and Li et al. (2016) used MHD simulations of the magnetospheric response to measured upstream solar wind parameters with double spectral analysis (frequency and azimuthal mode number) methods developed by Elkington et al. (2012) to determine azimuthal electric field and compressional magnetic field ULF wave power as a function of low $m$ numbers.

4. Finite Larmor radius effect estimation by measuring phase shift in the flux oscillations detected by oppositely oriented particle detectors (single probe) (Kivelson & Southwood, 1983; Lin et al., 1988; Su et al., 1977).

Recent studies in resolving $m$ with these methods have been briefly reviewed in Barani et al. (2019) as well as a comprehensive review by Zong et al. (2017). Shi et al. (2018) also provided a table of comparisons among some of the recent poloidal ULF wave case studies of $m$ estimation. In this work, we follow the first method and use in-situ magnetometer data to focus on the magnetic contribution of the ULF wave mode number, leaving investigation of the ULF wave the electric field for future studies. The first method will be discussed in more detail in the following sections.

### 1.5.2. The Cross-Wavelet Transform (XWT) Method

In this work, a Cross-Wavelet Transform (XWT) method (Eriksson, 1998; Grinsted et al., 2004; Torrence & Compo, 1998) is used in resolving mode structure (Sarris, 2014; Sarris et al., 2013), similar to the one used in Barani et al. (2019). Wavelet analysis is proven to be more effective in resolving the spectral signatures of nonstationary signals and signals with abrupt changes than
Fourier Transforms commonly used in previous studies (Fei et al., 2006; Li et al., 2016; Li et al., 2017; Tu et al., 2012). The XWT is specifically effective in low frequency regimes, and in offering precise correlation estimations between two signals (Eriksson, 1998; Rioul & Vetterli, 1991). The Morlet basis functions are used in this work which are sinusoidal waves with Gaussian envelope (Eriksson, 1998; Grinsted et al., 2004; Torrence & Compo, 1998). The azimuthal wavenumber (mode number) or \( m \) at time \( t \) and frequency \( f \) can be calculated as

\[
m(t, f) = \frac{XPhase(t, f)}{\Delta\lambda(t)}
\]

1-1

Here \( XPhase \) values come from our XWT analysis, which gives the phase differences between the two signals measured at positions of the two-spacecraft pair of signals, azimuthally separated by \( \Delta\lambda \). The \( XPhase \) values are the trigonometric measures of the phase difference between the two signals, which is the inverse tangent of quadrature over coincident spectral densities. Quadrature and coincident spectral densities are respectively the imaginary and real portions of the Cross Power Spectral Density (XPSD) values in Fourier space (Eriksson, 1998). The above

Figure 1-5. Different time-frequency representations of measured signals. (a) Time-domain representation lacking any frequency information. (b) Fourier representation with the best frequency resolution but no time information. (c) Windowed FFT with equal tradeoff between the signal information in time and frequency domains. (d) Octave band analysis or Wavelet Transform with frequency resolution decreases at higher frequencies such that the relative frequency resolution \( \frac{\Delta f}{f} \) is kept constant. The surface area of the 16 time-frequency rectangles (blocks) are all the same for each of these different representations. Adapted from Eriksson (1998).
equation gives the \( m \) values at every time-frequency bin while the \( XPSD \) values, as another outcome of the XWT analysis, gives the power content in the same corresponding time-frequency bins. In summary, a set of \( (XPSD, XPhase) \) values as the outcome of the XWT spectral analysis, combined with calculation of \( m \) from equation 1-1 at each time and frequency, is the core of what is needed for the mode structure analysis. Figure 1-5 shows the different time-frequency representations of different methods of spectral analysis. The width of each block in time and frequency direction denotes the error or uncertainty in that direction. The area of each single block regardless of the four depicted methods of signal analysis is constant and is governed by the uncertainty principle in signal analysis which prohibits resolving any signal in time and frequency simultaneously with no uncertainty. For example, in Panel (a), time domain representation of a signal resolves temporal changes very well but suffers from any information in frequency (very wide width in the frequency direction) and it is opposite to Panel (b), which only resolves the frequency content of a signal as a Fourier representation. Spectrograms in Panels (c) and (d) show the difference between the windowed Fast Fourier Transform (WFFT) and Wavelet Transform (WT). WFFT works well for stable signals, and smooth changes. However, WT gives more accurate results for abrupt changes and non-stationary signals (signals with time-dependent frequency content). In WT, the frequency resolution decreases at higher frequencies such that the relative resolution \( \frac{\Delta f}{f} \) is kept constant as can be seen in the blue shaded regions in Panel (d). One basic difference between WT and WFFT that gives WT a superior capability is that the basis functions that construct the signal are not infinite in time as opposed to the basis functions of WFFT that are sin and cos functions. For instance, the Morlet basis functions used in the current work have a Gaussian shape in time and therefore in the frequency domain as well. The WT scheme multiplies the given signal by different Morlet functions with different widths to resolve the given frequency range that is subject to study. The inputs (given signals) are the in-situ level-2 magnetic field values from space magnetometers.

In the spectral analysis method, a pair of measurements that are azimuthally separately by \( \Delta \lambda \) which are measuring the same wave signal (i.e., with high coherence) is needed. Performing the XWT on the pair of measurements provides the relative phase or \( XPhase \) values between the two signals. To derive equation 1-1, we start with two signals located at longitudes \( \lambda_1 \) and \( \lambda_2 \), respectively, which are sampling the same wave. Consider Signal 1 to be the measured field values.
from the trailing probe while Signal 2 is the field values from the leading probe of a pair of spacecraft. They can be written in the forms of:

\[
\begin{align*}
\text{Signal 1: } b_1 &= B \sin(m\lambda_1 + \omega t) \\
\text{Signal 2: } b_2 &= B \sin(m\lambda_2 + \omega t)
\end{align*}
\]

The use of the same notation for amplitude \(B\), frequency \(\omega\), and mode \(m\) in relations of both signals 1 and 2 mathematically assures that the same wave is detected by both spacecraft.

As the arguments of \(\sin\) for signal 1 and signal 2 are \(\varphi_1\) and \(\varphi_2\), respectively, the phase difference \(\text{XPhase}\) is

\[
\text{XPhase} = \varphi_2 - \varphi_1 = m(\lambda_2 - \lambda_1) = m\Delta\lambda
\]

Thus, mode number is derived as \(m = \text{XPhase}/\Delta\lambda\) which is equation 1-1. Since the mode number is completely determined by the ratio between \(\text{XPhase}\) and \(\Delta\lambda\) values, it does not need to comply with the Nyquist sampling theorem. The important assumption made in this method is that the pair of satellites are measuring the same wave signal, that is why we require strong coherence/correlation between the two signals.

The nearest integer is used in the \(m\) estimation, which means in general the positive mode number of \(+|m|\) corresponds to \(\text{XPhase}\) in the range of \(\left(\frac{-2|m|-1}{2}\Delta\lambda, \frac{2|m|+1}{2}\Delta\lambda\right)\), and the negative mode \(-|m|\) corresponds to \(\text{XPhase}\) in the range of \(\left(-\frac{2m+1}{2}\Delta\lambda, -\frac{2|m|-1}{2}\Delta\lambda\right)\). This representation assumes the uncertainty in \(m\) estimation to be less than 0.5, which is consistent with previous works (e.g., Sarris et al. (2013) and Sarris (2014)). A more detailed discussion of error analysis will be addressed in the future sections.

The sign of mode number indicates the propagation direction of the wave: considering rotational phase velocity of ULF waves, \(\Omega_{ph} = \frac{\omega}{m}\), if the mode number changes from \(m\) to \(-m\), the phase velocity would go from \(V_{ph}\) to \(-V_{ph}\). As the relative phase between the two signals (which is
\( XPhase \) in a pair is defined to be \( \varphi_2 - \varphi_1 \), then \( m > 0 \) corresponds to eastward-propagating waves, which is trailing satellite to the leading satellite direction while \( m < 0 \) means westward-propagating waves, leading to trailing direction. The change of the wave propagation direction slightly before noon is consistent with the picture of compressional ULF waves driven by solar wind dynamic pressure variations [e.g., Olson and Rostoker (1978); Hughes (1994)] and will be addressed in the mode analysis using GOES data (Chapter 2).

1.5.3. Mode Number Estimation Challenges and \( 2n\pi \) Ambiguity

As in-situ or ground-based measurements can be applied in the first (Cross-Spectral analysis) and third (double spectral analysis of MHD fields) methods, the estimation of \( m \) depends on the number and spatial separation of measurements. For example, to resolve mode numbers up to \( m_{\text{max}} \) in a full circumference around Earth at constant radial distance \( r \), for a given frequency, at least \( 2m_{\text{max}} \) equally spaced azimuthal coherent field measurements are needed. Therefore, the maximum mode number that can be resolved is crucially dependent upon the number and spacing of measurements undertaken by the spacecraft or the number and spacing of ground measurements that are mapped through a magnetic field line model to the regions of interest in space. If ground-based magnetic field measurements are used, the effect of ionospheric screening on transmission of the pulsations to the ground, as well as possible errors in magnetic field mapping, limits us to resolving mode number values less than \( \sim 40 \) due to ionospheric shielding (Chisham & Mann, 1999). In the third method, estimating \( m \) from MHD simulations, simplifying assumptions include the lack of a plasmasphere (Tu et al., 2012) or a plasmaspheric model based on a fixed initial Kp value (Li et al., 2016; Li et al., 2017) included in the MHD simulations on which ULF wave analysis is performed. Claudepierre et al. (2016) have shown that including a cold corotating plasmasphere of ionospheric origin in global MHD simulations modifies the ULF wave mode structure and power distribution. In addition, the use of MHD simulations inherently imposes limitations to the maximum resolvable \( m \). Another precaution of using MHD driven fields in estimating \( m \) is that if the corresponding spatial scale of the changes in fields are approaching the ion gyro radius in the plasma system, there would be a range of \( m \) values which extends beyond what MHD simulations are capable of resolving. In that case, resorting to the fields data from in-situ measurements or more advanced simulations would be required.
One fundamental difference between the first and the third approaches in resolving \( m \) is that the third approach directly performs Fourier Transforms on the discrete signals located at different azimuthal angles covering the full globe (\( 2\pi \)), which obeys the Nyquist sampling theorem, while the first approach is applied to a pair of time signals that are azimuthally separated and observing the same ULF wave, which does not necessarily obey the Nyquist theorem. In other words, if the azimuthal separation between the pair of field measurements is \( \Delta \lambda \) in degrees, the \( m \) resolved using the first approach is determined by the ratio between the phase difference between the two signals and \( \Delta \lambda \), while in the third approach the resolved \( m \) cannot go beyond the Nyquist limit which is half of \( 360^\circ / \Delta \lambda \).

Since the XWT method only resolves trigonometric values of the phase difference between the two signals, the resulting \( XPhase \) has a so-called \( 2n\pi \) ambiguity, which leads to possible uncertainty in the mode number estimation.

The real \( XPhase \) values are:

\[
XPhase_{\text{real}} = XPhase_{\text{calc}} \pm n 360^\circ
\]

Where \( XPhase_{\text{real}} \) and \( XPhase_{\text{calc}} \) are real and calculated phase differences respectively. Then based on equation 1-1:

\[
m_{\text{real}} = m_{\text{cal}} \pm n \frac{360^\circ}{\Delta \lambda}
\]

Where \( m_{\text{real}} \) and \( m_{\text{cal}} \) are realistic and calculated \( m \) values respectively, and \( n \) (which can be 0,1,2,...) is interpreted as a measure for the level of ambiguity.

First, the \( 2n\pi \) ambiguity can affect the sign of the mode number. The \( XPhase \) values from the XWT analysis can be either constrained to be positive only, i.e., in the range of \([0^\circ, 360^\circ]\), or to include both positive and negative values, i.e., in the range of \([-180^\circ, 180^\circ]\). Many previous applications of the XWT and cross-spectrogram Fourier analyses [e.g., Sarris et al., 2013; Sarris, 2014; Li et al., 2016; Sarris and Li, 2017] assumed only positive \( XPhase \) (or mode number), which corresponds to eastward propagating ULF waves that can drift-resonate with radiation belt electrons. However, waves can propagate in both directions and therefore in this work negative
mode numbers are also considered (i.e., $XPhase$ in the range of $[-180^\circ, 180^\circ]$) to incorporate both eastward and westward propagating waves. This is similar to the studies in Chisham and Mann (1999) and Murphy et al. (2018) using ground-based and in-situ magnetometer data respectively and the SuperDARN studies in Yeoman et al. (2010). Second, the $2n\pi$ ambiguity suggests that the $XPhase$ cannot be uniquely determined since the number of the full wave periods in between the two measurements is unknown. However, this ambiguity can be largely reduced by validating the mode number results between two pairs of signals that overlap azimuthally as will be discussed later.

To better address the issue of $2n\pi$ ambiguity quantitatively, here we give an example of resolving mode number with azimuthal separation of $15^\circ$ considering two cases: 1- For the “Positive and Negative” case, corresponding to $XPhase$ values in the range of $[-180^\circ, 180^\circ]$, the resolved $m$ range is $[-12, 12]$ based on equation (1-1) with $\Delta \lambda = 15^\circ$. Therefore, $m = -12$ corresponds to $XPhase$ in the range of $(-12\Delta \lambda, -11.5\Delta \lambda)$, $m = 0$ corresponds to $XPhase$ in the range of $(-0.5\Delta \lambda, 0.5\Delta \lambda)$, and $m = 12$ corresponds to $XPhase$ in the range of $(11.5\Delta \lambda, 12\Delta \lambda)$. 2- If we define $XPhase$ to be in the range of $[0^\circ, 360^\circ]$ for the “All Positive” case, considering the $2n\pi$ ambiguity all the $XPhase$ values in the range of $[-180^\circ, 0^\circ]$ in the “Positive and Negative” case are added by $2\pi$ or $24\Delta \lambda$. Then $m = -1$ flips to $m = 23$, $m = -2$ flips to $m = 22$, etc. The corresponding power or $XPSD$ in $m = 0$ in the “Positive and Negative” case splits into the $XPSD$ of $m = 0$ and $m = 24$ in the “All Positive” case. Similarly, the $XPSD$ of $m = 12$ in the “All Positive” case is the sum of the $XPSD$ values in $m = 12$ and $m = -12$ in the “Positive and Negative” case. Therefore, wave power in the low negative modes is basically mirrored to the power in high positive modes due to the different assumptions of the $XPhase$ range (basically adding $2\pi$ to $XPhase$ or adding 24 to $m$ based on the $15^\circ$ azimuthal separation between satellites, thus $m = -1$ flipping to $m = 23$, $m = -2$ flipping to $m = 22$, etc.) A detailed discussion with visualization of the $2n\pi$ ambiguity phenomenon will be given in Chapter 2 using GOES data.

In this work we take a step forward to reduce the $2n\pi$ ambiguity in the resolved $XPhase$ by performing a new cross-pair analysis to compare and reconcile the mode number results from two pairs of signals that overlap azimuthally. Specifically, the suggested scheme to reduce the ambiguity is that for each pair we allow the phase difference to be three possible values, $XPhase$ —
360°, XPhase, and XPhase + 360°, where XPhase is the phase difference directly from the XWT analysis in the range of [-180°, 180°]. That means we allowed a level of ambiguity for \( n \) to be 0 and 1 and we consider both signs in the ambiguity relation to take both wave directions into account. Correspondingly, three possible values of mode number \( m1 \) are calculated using Eq. (1-6) for pair 1, and the same for \( m2 \) of pair 2. Then we cross compare the possible mode values from each pair to find the \( m1 \) from pair 1 that is the closest to the \( m2 \) from pair 2, i.e., \(|\Delta m| = |m1 - m2| \) reaching a minimum, and those values are used as the final and more realistic mode numbers for the two pairs. Using GOES magnetometer data, we will show that this scheme is very effective and successful in reducing the ambiguity in mode number.

1.5.4. Previous Works in \( m \) Estimation

Sarris and Li (2017) performed the same Wavelet analysis to obtain the mode structure of ULF waves and found that lower geomagnetic activity (weaker Dst) is generally accompanied by low \( m \)-numbers, whereas intense geomagnetic activity favors a more even distribution of power across all modes. In a CLUSTER satellite data study, Zong et al. (2007) estimated \( m \leq 10 \) for toroidal ULF waves with an azimuthal magnetic and radial electric field perturbation based on the drift resonance condition. Tu et al. (2012) performed FFT analysis on the global MHD fields (third method) resolving the full mode structure and connection of solar wind dynamic pressure to the fundamental mode of compressional ULF waves which have a radial and parallel magnetic and azimuthal electric field perturbation, while power in \( m > 1 \) is related primarily to the night-side substorm activity.

Despite the significant progress in quantifying the mode number of ULF waves, two aspects of the mode structure are usually over-simplified, specifically concerning the estimation of the radial diffusion. First, some of the methods (e.g., Sarris and Li (2017); Tu et al. (2012)) assumed only positive \( m \) values for the ULF waves, whereas in reality the mode number can be either positive or negative corresponding to the waves propagating in either eastward or westward direction (Chisham & Mann, 1999; Le et al., 2017; Murphy et al., 2018; Yeoman et al., 2010). This assumption will lead to uncertainty in \( D_{LL} \) quantification since only positive-mode ULF waves would resonate with radiation belts electrons which drift eastward. Second, the resolved mode structure of ULF waves was usually assumed to be global around the Earth, which may not always
be an accurate assumption. As evidence, through ionospheric radar observations, Fenrich et al. (1995) found that \( m > 17 \) ULF waves are concentrated in the midnight and local afternoon, while \( m < 17 \) waves are concentrated near the flank regions. Similarly, Sarris and Li (2017) found a larger mode number contribution of Pc-5 waves to the nightside than the dayside. Therefore, assuming a uniform mode structure around the Earth, rather than using a realistic local time distribution, can lead to significant uncertainty in \( D_{LL} \) estimation where the local time dependent mode structure needs to be properly drift-averaged. In summary, a reliable estimate of the diffusion coefficients requires including both the positive and negative mode numbers of ULF waves, as well as specifying the local time coverage of the specific mode structure, the calculation of which is the subject of the work presented herein. More discussion on this important point will be given in the following Chapters.

### 1.6. Radial Diffusion and Importance of \( m \)

Among different mechanisms responsible for the transport of energetic charged particles in the magnetosphere, radial diffusion has long standing as an important mechanism. Radial diffusion is a phenomenon through which charged particles undergo a radial displacement with respect to their initial positions in the presence of a radial gradient by interacting with fluctuating electric and magnetic fields which oscillate on a time scale comparable to their drift period (Fälthammar, 1965; Kellogg, 1959; Schulz & Lanzerotti, 1974). Despite its established importance (see review by Shprits et al. (2008)), many uncertainties and simplifying assumptions remain, some of which are addressed in (Barani et al., 2019). These include a \( 2n\pi \) ambiguity in azimuthal mode number \( m \) and the effect of both positive and negative \( m \). Radial diffusion is defined as a random walk on a radial gradient at a rate determined by the coefficient

\[
D_{LL} = \frac{\langle (\Delta L)^2 \rangle}{2\tau}
\]

wherein the magnetic flux through a closed surface which intercepts the particle’s drift orbit is not conserved (Schulz & Lanzerotti, 1974). This invariant is inversely related to the Roederer L (Roederer, 1970), which for a dipole is the equatorial radius of a drift orbit in Earth radii \( Re \). If the frequency range of pulsations in the magnetosphere coincides with the drift frequencies of the
charged particles drifting around Earth (mHz range), particles are transported radially as described by a Fokker-Planck equation,

\[ \frac{\partial}{\partial t} f(\mu, L) = L^2 \frac{\partial}{\partial L} \left( D_{LL} \frac{1}{L^2} \frac{\partial}{\partial L} f \right) \]

causing non-zero mean squared displacements from their initial radial distance from the Earth, \( \langle (\Delta L)^2 \rangle \), averaged over many drift periods. \( f \) is the phase space density (distribution function) of particles. The interaction time \( \tau \) is the time it takes for charged particles to remain coherent in phase with the driving Fourier component of the wave spectrum at the drift resonant frequency (Schulz & Lanzerotti, 1974). \( \mu \) is the first adiabatic invariant, and equatorially mirroring particles are considered here corresponding to the second adiabatic invariant of \( K = 0 \).

In some references, the time \( \tau \) in the \( D_{LL} \) relation is introduced as the drift period (Roederer & Zhang, 2014). However, based on first principles calculations, \( \tau \) must be larger than the particles’ drift period, large enough to allow the particles to have enough interactions with fluctuating ULF waves (Fälthammar, 1965) to be considered similar to a random walk over many drift periods. The presumed discrepancy comes from the way we perceive the phenomenon of diffusion. If a process is diffusive, the result must remain the same regardless of the ways \( \tau \) is defined since diffusion is a time-independent process: The rate of change in \( L \) with respect to time is linear in diffusion. Interaction time should satisfy the relation \( \tau >> 1/\omega_d \) where \( \omega_d \) is azimuthal drift frequency and

![Figure 1-6. Schematic representation of global ULF wave activity with azimuthal wave (mode) numbers \( m = 1 \) and \( m = 2 \) in panels (a) and (b) respectively. Shaded regions can be interpreted as positive magnetic field amplitudes with maximum at the middle of each sector. Adapted from Elkington and Sarris (2016).](image)
\( \tau \), interaction time, is the integration time of the derivative of \( L \) with respect to time (Schulz & Lanzerotti, 1974). Therefore, to keep the generality of the discussion, \( \tau \) should be larger than the drift period although if the drift period is considered as the interaction time, it means that the system is already in the diffusion mode.

When the resonance condition is met, radial diffusion can occur. The drift-resonance criterion is described as \( \omega = m \omega_d \) where \( \omega \) and \( m \) are characteristics of waves: frequency and azimuthal mode (or wave) number of the driving ULF waves respectively, and \( \omega_d \) is a feature of charged particle motion in an inhomogeneous magnetic field such as a dipole, the electron drift frequency. The dominant relation that connects the spatial content of ULF wave magnetic field with radial diffusion of electrons in the magnetosphere is

\[
D_{LL}^B = \alpha(\mu, L) \sum_{m=1}^{m_{max}} m^2 PSD(m, m\omega_d)
\]

which is inserted in the previous equation (1-8) to calculate the time evolution of the electron phase space density. \( D_{LL}^B \) is the radial diffusion coefficient of electrons due to magnetic pulsations and \( PSD(m, m\omega_d) \) are compressional magnetic power spectral densities embedded at different mode numbers and resonant frequencies. \( m_{max} \) is the maximum existing mode that contains power of ULF pulsations. The proportionality factor \( \alpha \) is a function of first adiabatic invariant and \( L \) value and for dipole field is \( \frac{1}{8} \frac{\mu^2 L^4}{e^2 R_E^4 B_E^2} \) (Fei et al., 2006). The above relation further emphasizes the importance of \( m \) in resolving the radial diffusion of the electrons in Earth’s magnetosphere (Fälthammar, 1965; Fei et al., 2006; Schulz & Lanzerotti, 1974). A related expression is obtained for electric field power which includes both a potential and inductive contribution as discussed below.

The way ULF wave electric and magnetic pulsations are understood to contribute to the diffusion rate of electrons depends on the models and how the driver fields are distinguished:

In the Fälthammar (1965) scheme (discussed in more detail in the next section), \( D_{LL} \) is a summation of the electrostatic radial diffusion coefficient \( (D_{LL,e}) \) and electromagnetic radial diffusion coefficient \( (D_{LL,m}) \) which gives \( D_{LL} = D_{LL,e} + D_{LL,m} \). While in the Fei scheme (Fei et al., 2006),
\( D_{LL} \) is a summation of the electric radial diffusion coefficient (\( D_{LL,E} \)) and magnetic radial diffusion coefficient (\( D_{LL,b} \)) that gives \( D_{LL} = D_{LL,E} + D_{LL,b} \) without separating electric field contributions into potential and inductive sources.

In general, the drivers of the particles are electric (convective, \( E^{con} \), and inductive, \( E^{ind} \)) fields and time varying magnetic field (\( b \)) which is considered to have a stationary stochastic nature. To the first order approximation, we obtain

\[
\frac{dr}{dt} = \frac{E}{B_d} - \frac{\mu}{eB_d r_0} \frac{\partial b}{\partial \varphi} \tag{1-10}
\]

Regardless of the methods used to determine field variations, integration of this relation determines the radial diffusion coefficient in equation (1-9). For the derivation of equation (1-9) from relation (1-10), refer to Fälthammar (1965), Fei et al. (2006), as well as Tu and Li (2011). The resulting diffusion coefficients can be divided between electric and magnetic contributions with two different approaches described next (Lejosne, 2019).

**1.6.1. Fälthammar Scheme**

In the Fälthammar (1965) description, change of the radial distance of electrons from Earth (\( r \)) due to the drivers (fields) is:

\[
\frac{dr}{dt} = \frac{E^{con}}{B_d} + \frac{E^{ind}}{B_d} - \frac{\mu}{eB_d r_0} \frac{\partial b}{\partial \varphi} \tag{1-11}
\]

where \( e \) is electron charge, \( B_d \) is the dipole magnetic field and \( \varphi \) is the azimuthal component in the standard spherical coordinate system (which has the opposite direction compared to the coordinate considered by Fälthammar). \( r_0 \) is the radial distance of the electrons prior to the perturbation by fields which satisfies \( r_0 = L R_E \) and \( R_E \) is Earth’s radius. In this relation, the electric field is decomposed into two components, convective and inductive. The convective electric field \( E^{con} \) term which is the first on the right hand side (RHS) of the above expression, is responsible for the generation of the electrostatic radial diffusion coefficient (\( D_{LL,e} \)) and comes from only the electric potential (curl-free) field. The other two terms on the RHS produce the
electromagnetic radial diffusion coefficient \( D_{LL,m} \). As can be seen, Fälthammar separates the convective electric field from the inductive one which is produced by temporal fluctuations of magnetic field following Faraday’s law of \( \nabla \times E^{ind} = -\frac{\partial b}{\partial t} \) and both terms of \( \frac{\partial b}{\partial t} \) and \( E^{ind} \) are considered in the same diffusion term \( D_{LL,m} \) while satisfying Faraday’s law. The total magnetic field is considered to be \( B = B_d + b(t) \) with \( b(t) = [S(t) + A(t) \times \hat{z}] \) where \( b(t) \) has the symmetric contribution \( S(t) \) and the day-night asymmetric time-varying contribution of \( A(t) \). \( \times \) is distance in the Earth-sun pointing direction.

1.6.2. Fei et al. Scheme

In the Fei et al. (2006) description, the two electric fields (inductive and convective) are not separated and the first term on the RHS of equation (1-10) is considered to be the electric radial diffusion coefficient \( D_{LL,E} \) and the second term in (1-10) is the magnetic radial diffusion coefficient \( D_{LL,b} \) which only has explicit magnetic variations in it. As Fei et al. did not separate the inductive electric field and the time varying magnetic field, Faraday’s law might not always hold since the two terms \( \frac{\partial b}{\partial t} \) and \( E^{ind} \) are being treated differently in two separate terms to derive the total radial diffusion.

The total magnetic field is considered to follow \( B = B_d + \Delta B \cos \varphi \hat{z} + b(t) \) with fluctuating \( b(t) = \delta B(t) \hat{z} \) which is considered to have a stochastic nature of fluctuations. The second term is due to the background day-night asymmetry caused by the solar wind continuously buffeting the Earth and \( \Delta B \) is the measure of this effect and depends on the geomagnetic activity. As can be seen here, the Fei et al. model does not separate time-varying symmetric versus asymmetric contributions of the fluctuations. For further discussion on the possible advantages of Fälthammar’s scheme over Fei’s scheme, refer to Lejosne (2019). On the other hand, using the Fei et al. scheme has practical advantages since there is no need to calculate which portion of the detected electric field measurements is the induced one, which in general requires global knowledge of the fields, even though it may have uncertainties.
1.7. MFA Coordinate

Depending on the application, ease of use, and the symmetries in physical systems subject to study, different coordinate systems can be chosen to study the magnetic and electric field disturbance in the ULF wave frequency range. As the ambient magnetic field is the dominant component of a typical magnetic field vector in the electron radiation belt region due to the strength of the Earth’s dipole, it is desirable to express the magnetic and electric fields either from measurements or simulations in a coordinate system that has an easily noticeable direction as the background field. Therefore, all components are defined based on the background direction. The proper coordinate system that fits this desire is the Mean Field Aligned (MFA) coordinate system. There are different ways to define the direction aligned with the mean field as the background or ambient direction. For a review on this coordinate system, the reader is suggested to refer to Mauro et al. (2017). Caution must be paid to distinguish between the MFA coordinate and field aligned coordinate. In the field aligned coordinate, at every time/place the base or background component based on which other components are defined is always in the direction of the instantaneous measured field. That gives a vector field with only non-zero values at the base direction which is typically the third component. So, values in a field aligned coordinate description have only the third component nonzero. A field aligned coordinate system is not desired for ULF wave study since it is important to express fluctuating fields based on the ambient field which is not an instantaneously variable and fluctuating value. That means a coordinate system is needed to consider a semi-constant direction as the ambient and then define fluctuations on top of this base direction. An MFA coordinate system well serves this purpose. Fields in MFA would be the same as the fields in a field aligned coordinate system in case of no fluctuations. To get the base direction, in our work we implement the moving average technique. The time window for the moving average depends on the time scale of the physics intended to study and other parameters such as the speed of the spacecraft and the location in space that the spacecraft is probing. For the outer radiation belt studies, the typical time window is 30 minutes for data from geostationary spacecraft (such as GOES) and 20 minutes for data from the NASA Magnetospheric Multiscale Mission (MMS) for MFA coordinate analysis, shorter for MMS because the spacecraft traverse L shells rapidly in the inner magnetosphere while GOES is geostationary. Figure 1-7 shows a schematic representation of the MFA coordinate system and the geometry of the Geocentric Solar Ecliptic (GSE) coordinate.
system. On the top left, different components of the ULF magnetic field fluctuations in MFA coordinates are depicted on top of the spacecraft orbit and the ambient magnetic field $<B>$ is a running averaged field close to the dipole field during quiet geomagnetic times. On the bottom right, a pair of spacecraft are shown with names Probe1 and Probe2 of azimuthal separation $\Delta \lambda$. The red fluctuating curve that passes through the probes is schematically the compressional component of ULF pulsations in space.

As can be seen in Figure 1-7, after subtracting the window-averaged baseline fields, the remaining fluctuating magnetic fields are decomposed into poloidal, toroidal, and compressional components, denoted as $B_\nu$, $B_\phi$, and $B_\parallel$ respectively:

$$B_\nu = B \cdot (<\hat{\nu}>)$$  \hspace{1cm} 1-12
$$B_\phi = B \cdot (<\hat{\phi}>)$$  \hspace{1cm} 1-13
$$B_\parallel = B \cdot (<\hat{\parallel}>) - < |B| >$$  \hspace{1cm} 1-14

Figure 1-7. Mean Field Aligned coordinate and the geometry of GSE coordinate system. The red fluctuating curve is schematically the compressional component of ULF pulsations in space measure by passing probes.
The unit vectors in the poloidal, toroidal, and compressional directions are $\langle \hat{\nu} \rangle$, $\langle \hat{\phi} \rangle$, and $\langle \hat{b} \rangle$ respectively:

\begin{align*}
\langle \hat{\nu} \rangle &= \langle \hat{\phi} \rangle \times \langle \hat{b} \rangle & \text{1-15} \\
\langle \hat{\phi} \rangle &= \frac{\mathbf{r} \times \langle \mathbf{B} \rangle}{|\mathbf{r} \times \langle \mathbf{B} \rangle|} & \text{1-16} \\
\langle \hat{b} \rangle &= \frac{\langle \mathbf{B} \rangle}{|\langle \mathbf{B} \rangle|} & \text{1-17}
\end{align*}

Where $\langle c \rangle$ of a general quantity $c$ represents moving average of the quantity, $|c|$ denotes norm of $c$, and $\hat{c}$ represents unit vector for a vector $c$. Vector $\mathbf{r}$ here is position of Earth with respect to the spacecraft. All of the quantities can be time dependent.

In looking at the radial diffusion of radiation belt electrons it is mainly the compressional component that plays the dominant role since considering the symmetric geometry of the magnetosphere and approximately dipole geomagnetic field, time variations of compressional magnetic field (which is in the $z$ direction on the equatorial plane) produces an induction electric field in the same direction as electrons’ drift, which is east-west on the equatorial plane following Faraday’s law. Therefore $E_{\parallel}$ accompanies $B_{\parallel}$ as the dominant components responsible for particle diffusion. Because of this fact, the focus of this thesis work is on the contribution of the compressional fluctuations of ULF magnetic pulsations.

The thesis following chapters are organized as follows: Chapter 2 is on mode number estimation of ULF waves using multi-point GOES spacecraft magnetometer data, published as Barani et al. (2019). Chapter 3 is on high-fidelity analysis of mode structure using multi-point MMS observation. Chapter 4 summarizes what has been accomplished in this thesis with plans for future study.
Chapter 2. Mode Number Estimation of PC-5 Compressional ULF Waves Using Multi-Point GOES Spacecraft Magnetometer Data

2.1. Introduction

Radial diffusion plays an important role in the transport, acceleration, and loss of energetic electrons in the Earth’s radiation belts (Fälthammar, 1965; Kellogg, 1959; Schulz & Lanzerotti, 1974). Therefore, understanding and physically quantifying the radial diffusion process is critical in modeling the complex dynamics of radiation belt electrons. The third adiabatic invariant of electrons, which is inversely proportional to the Roederer L (Roederer, 1970), is violated in radial diffusion. The effects of radial diffusion can be quantified by the coefficient \( D_{LL} \), defined as \( D_{LL} = \langle (\Delta L)^2 \rangle / 2 \tau \), which represents the mean-square-displacement of L value for a large number of particles over a time scale \( \tau \) much longer than the particles’ drift period. Enhanced radial diffusion is a result of the drift-resonant interactions between radiation belt electrons and large-scale fluctuations of the magnetosphere’s magnetic and electric fields with frequencies comparable to the electrons’ drift frequencies (Schulz & Lanzerotti, 1974). Since the drift frequencies of MeV electrons in the outer radiation belt are on the order of mHz, Ultra Low Frequency (ULF) waves in the Pc5 range (1.6–7 mHz (Kivelson & Russell, 1995)) are most effective in driving radial diffusion. Specifically, the drift resonance condition follows \( \omega = m \Omega_D \) where \( \omega \) is the wave frequency, \( \Omega_D \) is the electron drift frequency, and \( m \) is the azimuthal mode number (or wave number) of ULF waves, which represents (for simple harmonic behavior) the number of wavelengths in the azimuthal direction around the Earth. Characterizing and estimating the azimuthal mode structure of ULF waves are required for calculating the \( D_{LL} \) of radiation belt electrons.

Despite its crucial importance, physical quantification of the azimuthal mode number of ULF waves is difficult and has been a missing part in the calculation of the radial diffusion coefficients (Sarris, 2014; Sarris & Li, 2017; Tu et al., 2012). In principle, to determine the mode structure of a simple harmonic wave at a given frequency up to mode number \( m_{max} \), we need at least \( 2m_{max} \) equally-spaced azimuthal coherent field measurements. This is hard to achieve from in-situ
measurements in space due to insufficient satellite coverage. Arrays of longitudinally separated
ground magnetometers offer much better coverage and have also been used for mode number
estimation (Chisham & Mann, 1999). However, for high m values, significant amplitude
attenuation is expected for ULF waves propagating down to the ground due to the ionospheric
screening effect (James et al., 2013; Nishida, 1978), which imposes an extra limitation in resolving
physical m values of ULF waves from ground measurements. Due to the difficulties in resolving
the actual mode structure of ULF waves, simplified assumptions have been made in previous
estimations of the radial diffusion coefficients. For example, in Brautigam et al. (2005), Fei et al.
(2006), and Ali et al. (2015) and many previous analyses of $D_{LL}$, the power of ULF waves was
assumed to originate from only $m = 1$. Perry (2005) simulated the effect of ULF waves on the
guiding centers of particles through a 3D model by assuming the waves possessing only $m = 2$.
These simplified assumptions of ULF wave mode structure can lead to significant uncertainties in
the $D_{LL}$ estimation (Li et al., 2017; Sarris et al., 2006; Tu et al., 2012). For example, Tu et al. (2012)
calculated the $D_{LL}$ of MeV electrons based on ULF waves from global MHD simulation and found
that assuming all the power comes from $m = 1$ generally underestimates the total $D_{LL}$, sometimes
by more than an order of magnitude due to the non-power-law spectral shape of the electric field.
Therefore, a better quantification of the ULF mode structure is critical.

There are three main approaches to estimating the mode number of ULF waves: 1. Performing the
cross-phase analysis on pairs of real-time ULF wave measurements (Barani et al., 2019; Le et al.,
2017; Loto'aniu et al., 2006; Murphy et al., 2018; Olson & Rostoker, 1978; Sarris & Li, 2017;
Sarris et al., 2009; Takahashi et al., 2018; Takahashi et al., 2013). 2. Analyzing flux observations
of particles that are in resonance with the concurrent ULF waves and estimating the mode number
based on the drift resonance or drift-bounce resonance condition (Mann et al., 1998; Takahashi et
al., 1990; Zong et al., 2007). 3. Directly calculating the mode spectrum using the global ULF fields
from MHD simulations of the magnetosphere (Fei et al., 2006; Li et al., 2017; Tu et al., 2012).
The fundamental difference between the first and the third approaches is that the third approach
directly performs Fourier Transforms on the discrete signals located at different azimuthal angles
covering the full globe ($2\pi$), which obeys the Nyquist sampling theorem, while the first approach
is applied to a pair of time signals that are azimuthally separated and observing the same ULF
wave, which does not obey the Nyquist theorem. In other words, if the azimuthal separation
between the pair of field measurements is $\Delta \lambda$ in degrees, the $m$ resolved using the first approach is determined by the ratio between the phase difference between the two signals and $\Delta \lambda$ which will be discussed in detail in Section 3, while in the third approach the resolved $m$ cannot go over the Nyquist limit which is half of $360^\circ/\Delta \lambda$. Zong et al. (2017) provided a comprehensive review on recent works in calculating the $m$ values using the first two data-driven approaches. Shi et al. (2018) also provided a table of comparisons among some of the recent poloidal ULF wave case studies. Sarris and Li (2017) performed cross-spectral and cross-phase calculations to three azimuthally aligned GOES magnetometer data to obtain the mode structure of ULF waves during two storm periods. They found that lower geomagnetic activity (weaker Dst) generally favors the distribution of power in primarily low $m$-numbers, whereas intense geomagnetic activity favors a more even distribution of power across all resolved $m$-numbers. In studying the ULF modulation of energetic particles observed by CLUSTER satellite, Zong et al. (2007) estimated $m \leq 10$ for toroidal ULF waves based on the drift resonance condition. Tu et al. (2012) calculated the full mode structure of ULF waves by performing FFT analysis on the global MHD fields generated by the Lyon-Fedder-Mobarry MHD code. They found that the power of the compressional ULF waves in $m = 1$ mode is related primarily to solar wind dynamic pressure variations, while power in $m > 1$ is related primarily to the night-side substorm activity.

Even though significant progress has been made in quantifying the mode number of ULF waves, when applying the mode number results to calculations of radial diffusion coefficients, two aspects of the mode structure are usually over-simplified. First, some of the methods mentioned above using the cross-phase techniques (Sarris & Li, 2017) or global MHD simulations (Tu et al., 2012) assumed only positive $m$ values for the ULF waves, whereas in reality the mode number can be either positive or negative corresponding to the waves propagating in either eastward or westward direction (Chisham & Mann, 1999; Le et al., 2017; Murphy et al., 2018; Yeoman et al., 2010). This assumption will lead to uncertainty in $D_{LL}$ quantification since only positive-mode ULF waves would resonate with radiation belts electrons which drift eastward. Therefore, when assuming $m > 0$, the resulting $D_{LL}$ could overestimate the real $D_{LL}$ and should only be considered as an upper limit. Second, in many of the previous calculations of $D_{LL}$, the resolved mode structure of ULF waves was usually assumed to be global around the Earth, which may not always be an accurate assumption. For example, through ionospheric radar observations, Fenrich et al. (1995)
found that $m > 17$ ULF waves are concentrated in the midnight and local afternoon, while $m < 17$ waves are concentrated near the flank regions. Similarly, Sarris and Li (2017) found that the mode number values of Pc5 ULF waves are generally higher at the nightside than the dayside. Therefore, assuming a uniform mode structure around the Earth in radial diffusion calculations, rather than using a realistic local time distribution of the mode number, can lead to significant uncertainty in $D_{LL}$ estimation where the local time dependent mode structure needs to be properly drift-averaged. In summary, a reliable estimate of the diffusion coefficients requires including both the positive and negative mode numbers of ULF waves, as well as specifying the local time coverage of the specific mode structure, the calculation of which is the subject of the work presented herein.

In this paper we quantify the azimuthal mode structure of the compressional Pc5 ULF waves observed by multiple pairs of GOES satellites during the 28-31 May 2010 geomagnetic storm. Even though compressional ($B_z$), poloidal ($E_\phi$), and toroidal ($E_r$) mode ULF waves can all lead to radial diffusion of energetic particles in realistic geomagnetic field (Elkington, 2003; Elkington et al., 1999; Fälthammar, 1965; Mann et al., 2013), here we only focus on the compressional mode since GOES only measures magnetic field but not electric field. We apply the cross-spectral technique used in Sarris (2014) and Sarris and Li (2017) but using up to five GOES satellites available during the event to better address the limitations in the previous mode number calculations mentioned above. The five GOES satellites cover a wide range of local times at the same time, which provides an excellent opportunity to more reliably estimate temporal and spatial variations of mode numbers, as well as to resolve the local time extent of the mode structure. Both positive and negative mode numbers are included in our calculation and the availability of multiple overlapping pairs of satellites enables a comparative analysis to reduce the $2\pi$ ambiguity in the phase difference and mode number calculation, which is performed here for the first time for ULF wave measurements in space. The paper is organized as follows: Section 2 introduces the 28-31 May 2010 storm with its corresponding solar wind and geomagnetic conditions; the cross-wavelet transform (XWT) method is described in Section 3; the results from our analysis of the 28-31 May 2010 storm regarding the mode structure are discussed in Section 4; validation work to reduce the $2\pi$ ambiguity in the mode estimation is discussed in Section 5; and discussions and conclusions are included in Section 6.
2.2. 28-31 May 2010 Storm Event

For the period of 28-31 May 2010, we analyze the compressional Pc5 ULF magnetic pulsations using in-situ GOES magnetometer measurements. The geomagnetic indices and solar wind parameters are shown in Figure 2-1. A small interplanetary shock arrived at the Earth at ~03UT on 28 May; subsequently a long storm commencement was observed, from ~03 to ~21UT on 28 May. The solar wind dynamic pressure ($P_{dyn}$) remained high following the shock and Interplanetary Magnetic Field (IMF) $B_z$ turned southward and then northward twice during the storm commencement period. During the main phase of the storm, from ~21UT on 28 May to ~12UT on 29 May, the IMF $B_z$ turned southward and stayed southward over the entire storm main phase. The Auroral Electrojet (AE) index increased during the main phase and reached its maximum value at the end of the main phase. The storm then continued with a long recovery phase.

![Figure 2-1](image-url).

Figure 2-1. The geomagnetic indices and solar wind parameters during the 27-31 May 2010 storm based on the OMNI data set, including IMF $B_z$, total bulk solar wind flow speed $V_{SW}$, solar wind dynamic pressure $P_{dyn}$, AE index, and SYM-H index. The flux of $>0.6$ MeV electrons measured by the GOES 13 satellite is shown in Panel (e).

During the early recovery phase (~12-19UT on 29 May), AE remained high but started to drop and the southward IMF $B_z$ started to decrease as well. Then in the late recovery phase, the AE index exhibited strong fluctuations with elevated activity during many intervals. The IMF $B_z$ also
fluctuated between southward and northward during the late recovery phase with the solar wind flow speed increasing gradually. In fact, the late recovery phase continued for a couple of days after 31 May until another storm occurred (not shown in Figure 2-1).

The 28-31 May 2010 storm event is chosen in this mode number estimation study for three reasons. First, we have exceptional GOES data coverage during this interval, as the magnetometer data from five GOES satellites are available, which is ideal for mode number calculations. Figure 2-2(a) depicts the locations of the five GOES satellites (GOES 11, 12, 13, 14, and 15) in the GSE coordinate system at 11UT of 28 May. The averaged west longitudes of these satellites during 28 May are listed in Figure 2-2(b) and the azimuthal spacings between the satellites are relatively fixed during the entire storm interval. Based on Figure 2-2, we see the azimuthal separation of every two adjacent GOES satellites is \(~15\) degrees, except for GOES 11 and GOES 14 which are separated by \(~31\) degrees. In the next section we will see that a close azimuthal separation of \(~15\) degrees can resolve azimuthal mode numbers ranging from \(-12\) to \(12\). In addition, the five satellites span a wide range of Magnetic Local Times (MLT) at each UT, which is useful to specify the local time extent of the mode structure. The second reason this event is selected is its distinct periods with either dominant solar wind dynamic pressure or AE activity. Based on Figure 2-1, we see that during the storm commencement the solar wind dynamic pressure increased to a high level while the AE index stayed low, except for two small enhancements. In contrast, during the storm main phase and early recovery phase, the solar wind dynamic pressure dropped to a very low level while the AE index became very active (up to around 1800 nT). Previous studies have shown that AE and solar wind \(P_{dyn}\) are correlated with different ULF mode structures (e.g., Tu et al. (2012)). For instance, by analyzing the mode structure of compressional \(B_z\) field from the LFM MHD simulation of a CIR storm in March 2008, Tu et al. (2012) showed that the power of \(B_z\) at \(m = 1\) is generally correlated with the solar wind \(P_{dyn}\), while the \(m > 1\) power is mostly enhanced when the AE index is high. The distinct periods of high \(P_{dyn}\) vs. high AE during our selected event provide a unique opportunity to examine how different patterns of AE and solar wind dynamic pressure affect the mode structure of ULF waves. Lastly, the event is chosen due to the concurrent enhancement of MeV electron flux. Panel (e) in Figure 2-1 plots the flux of \(>0.6\) MeV electrons measured by GOES 13, which shows a fast drop during the early main phase followed by a significant increase near the end of the main phase. To study the role of radial diffusion to the
observed enhancement of MeV electrons, accurate estimation of the ULF wave mode number is required which further motivates this work.

<table>
<thead>
<tr>
<th>Satellite</th>
<th>GOES 12</th>
<th>GOES 13</th>
<th>GOES 15</th>
<th>GOES 14</th>
<th>GOES 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>West longitude</td>
<td>59.9°</td>
<td>74.8°</td>
<td>89.8°</td>
<td>104.5°</td>
<td>135.5°</td>
</tr>
</tbody>
</table>

Figure 2-2. (a) Schematic of the locations of five GOES satellites in GSE coordinate at 11UT of 28 May 2010, with Δλ denoting the azimuthal separation between GOES 13 and GOES 12. The magnetic field perturbations on top of ambient magnetic field <\vec{B}> are decomposed into three components: \( B_{\parallel} \) compressional, \( B_{\nu} \) poloidal, and \( B_{\phi} \) toroidal. (b) Averaged west longitude (λ) of the five GOES satellites during 28 May 2010.

2.3. The Cross-Wavelet Transform (XWT) Method

To estimate the azimuthal mode structure of ULF waves, here we apply the Cross-Wavelet Transform (XWT) method developed in Sarris et al. (2013) and Sarris (2014). Wavelet analysis is a technique that is increasingly used due to its profound advantages over the conventional methods (such as Fourier analysis), e.g., it can better resolve the spectral analysis for nonstationary signals
and in low frequency regimes, and XWT provides a precise correlation between two signals while analyzing their phase-shifted, equi-wavelength values (Eriksson, 1998; Rioul & Vetterli, 1991). The Morlet wavelet transform is used in our spectral analysis (Eriksson, 1998; Grinsted et al., 2004; Torrence & Compo, 1998), and more details about the XWT technique can be found in in Sarris et al. (2013) and Sarris (2014). Here we briefly discuss how to use the XWT method to estimate the ULF wave mode number. If we have two magnetic field wave signals measured by two GOES satellites that are azimuthally separated by $\Delta \lambda$ (e.g., between GOES 12 and 13 in Figure 2-2), performing XWT on the pair of signals provides $XPhase$ values, which are the phase differences between the two signals, and $XPSD$ values, which are the cross power spectral density values. The results include one set of $(XPhase, XPSD)$ values for each time and frequency. Then the azimuthal wavenumber (mode number) can be calculated as:

$$m = \frac{XPhase}{\Delta \lambda}$$

2-1

One major assumption of using the XWT method to resolve the mode number is the good correlation between the two signals. Therefore, to ensure that the mode number results are reliable, for the selected time interval of the analysis we only include the XWT results with $XPSD$ values higher than the minimum value by at least $x\%$ multiplied by (maximum − minimum) (Sarris et al., 2013). This method works to select the correlated and high-amplitude signals from both satellites and to exclude the low-amplitude signals from both satellites that are below a certain low signal level. Although the less correlated signals with one at high amplitude and the other at low amplitude could still be included in the results, fortunately, this is not the case for our event for which the coherence between the two signals is very high (see detailed results in Section 4). Additionally, this method must be applied carefully if the $XPSD$ from both satellites is at a low level, which is not the case for our event as the storm-time ULF wave power is generally high. The high coherence grants the use of a certain $XPSD$ percentage threshold to select the relatively high-amplitude and correlated signals. The value of $x = 0.1$ was selected here based on visual inspection and testing of the correlations between the two observed signals. It is lower than the percentage value used in previous works (e.g., in Sarris et al. (2013) and Sarris (2014)) due to the higher correlation between the two observed signals in our event. We have also tested the results with different values of $x$ and there are no significant changes in our conclusions. Since the XWT method only resolves the trigonometric values of the phase difference between the two signals, the
resulting $XPhase$ has a so-called $2n\pi$ ambiguity, which leads to uncertainty in the mode number estimation. First, the $2n\pi$ ambiguity can affect the sign of the mode number. The $XPhase$ from the XWT analysis can be either constrained to be positive only, i.e., in the range of $[0^\circ, 360^\circ]$, or to include both positive and negative values, i.e., in the range of $[-180^\circ, 180^\circ]$. Many previous applications of the XWT and cross-spectrogram Fourier analyses (Sarris, 2014; Sarris & Li, 2017; Sarris et al., 2013) assumed only positive $XPhase$ (or mode number), which corresponds to eastward propagating ULF waves that can drift-resonate with radiation belt electrons. However, waves can propagate in both directions so here we also include negative mode numbers (i.e., $XPhase$ in the range of $[-180^\circ, 180^\circ]$ in Section 4) to incorporate both eastward and westward propagating waves. This is similar to the studies in Chisham and Mann (1999) and Murphy et al. (2018) using ground-based and in-situ magnetometer data and the SuperDARN studies in Yeoman et al. (2010). Second, the $2n\pi$ ambiguity suggests that the $XPhase$ cannot be uniquely determined since we don’t know how many full wave periods are in between the two measurements. However, this ambiguity can be largely reduced by validating the mode number results between two pairs of signals that overlap azimuthally, which will be carried out in Section 5.

2.4. The Mode Number Estimation Results

The XWT technique is applied to multiple pairs of GOES magnetometer data to estimate the azimuthal mode structure of thecompressional-mode ULF waves during the 28-31 May 2010 storm. Magnetic field measurements with 1-minute cadence from the four closely-spaced GOES magnetometers are used in the analysis, which are GOES 14, 15, 13, and 12, located as shown in Figure 2-2. The azimuthal separation between every two neighboring satellites are about $15^\circ$, which means that we can resolve the mode number in range of $[-12, 12]$ considering $XPhase$ range of $[-180^\circ, 180^\circ]$ from the XWT analysis and Equation 2-1 for mode estimation. To obtain the compressional component of the ULF waves, we first project the magnetic field measurements to a Mean Field-Aligned (MFA) coordinate system as in Sarris et al. (2013) (which is based on the averaged field over a 30-min time window as the ambient field). Then, after subtracting the 30-min window-averaged baseline fields, the remaining fluctuating magnetic fields are decomposed into compressional, poloidal, and toroidal components, denoted as $B_\parallel$, $B_\nu$, and $B_\phi$ respectively in Figure 2-2(a).
The \( B_\parallel \) components of the magnetic field pulsations measured by GOES 12 and 13 during the first two days of May 2010 storm are plotted in Figures 2-3(a) and 3(b) respectively. Their corresponding power spectral densities (PSD), calculated using the wavelet transform, are shown in Figures 2-3(c) and (d). The similarity between the two PSDs suggest strong correlation between the two signals, which support the use of the XWT method. After performing the XWT analysis on the GOES13&12 pair of signals, Figure 2-3(e) illustrates the XPSD vs. frequency and time and Figure 2-3(f) illustrates the estimated \textit{XPhase} between the two signals in the range [\(-180^\circ, 180^\circ\)]. As described above, a threshold of \( \chi = 0.1 \) corresponding to 0.1\% of the minimum-to-maximum difference above the minimum power has been applied on the XPSD to exclude the frequency-time regimes with low common power (low coherence) between the two signals, which are the blank regions in Figures 2-3(e) and 2.3(f). The results show stronger power of Pc5 ULF waves covering a wider range of frequencies during 29 May (which includes the storm main phase and
recovery phase) than 28 May (which includes the storm commencement). In addition, the $XPhase$ values and distributions are very different between the two days, which suggests distinct features in the ULF wave mode structure.

![Diagram](image)

Figure 2-4. Integrated wave power versus time and mode number based on the XWT analysis of the GOES 13&12 pair, by assuming (a) the mode number can be both positive and negative, or (b) the mode number can only be positive.

To estimate the mode numbers, Eq. (1) is used to compare the $XPhase$ in Figure 2-3(f) with the azimuthal separation $\Delta\lambda$ between GOES 13 and GOES 12 (which is $\sim 15^\circ$ from Figure 2-2). The nearest integer is used in the $m$ estimation, which means in general the positive mode number of $+|m|$ corresponds to $XPhase$ in the range of $\left(\frac{2m-1}{2}\Delta\lambda, \frac{2m+1}{2}\Delta\lambda\right)$, and the negative mode $-|m|$ corresponds to $XPhase$ in the range of $\left(-\frac{2m+1}{2}\Delta\lambda, -\frac{2m-1}{2}\Delta\lambda\right)$. Note here we assume the uncertainty in $m$ estimation is $< 0.5$, which is consistent with previous works in Sarris et al. (2013)
and Sarris (2014). However, accurately quantifying the uncertainty in \( m \) using Equation 2-1 requires estimation of the uncertainty in \( XPhase \) from the XWT analysis and the uncertainty in \( \Delta \lambda \) (which is almost negligible in our case), which will be pursued in a future effort. Based on this method, the \( m \) number at each time and frequency is estimated using the \( XPhase \) in Figure 2-3(f) with Figure 2-3(c) showing the corresponding XPSD at each mode. Then to achieve the total wave power at different mode numbers, we sum the XPSD values corresponding to the same mode number for each time and plot the integrated-over-frequency wave power versus time and mode number in Figure 2-4. In this analysis, we define the \( XPhase \) to be in the range of \([-180^\circ, 180^\circ]\) thus the resolved \( m \) range is \([-12,12]\] as shown in Figure 2-4(a) with the power distribution for the negative modes plotted on the top and the distribution for the positive modes plotted underneath.

We see that during the first day, 28 May, the wave power is mostly dominant at low mode numbers (both negative and positive), while for the second day, 29 May, the power is almost evenly distributed across all different modes from low to high. In contrast, if we only allow positive mode numbers, similar to previous studies in \( D_{LL} \) calculations, the \( XPhase \) will be defined in the range of \([0^\circ, 360^\circ]\) and the resolved \( m \) range is \([0,24]\] with results shown in Figure 2-4(b). We notice that the wave power in the low negative modes in Figure 2-4(a) is basically mirrored to the power in high positive modes in Figure 2-4(b) due to the different assumptions of the \( XPhase \) range (adding \( 2\pi \) to \( XPhase \) or adding 24 to \( m \) based on the 15° azimuthal separation between satellites, thus \( m = -1 \) flipping to \( m = 23 \), \( m = -2 \) flipping to \( m = 22 \), etc.). There are two subtle exceptions due to the edging effect: \( m = 0 \) corresponds to \( XPhase \) in the range of \((-0.5\Delta \lambda ,0.5\Delta \lambda)\) for the “Positive and Negative” case in Figure 2-4(a). Converting to the “All Positive” case in Figure 2-4(b), the \( m = 0 \) \( XPhase \) would be in the range of \((23.5\Delta \lambda,24\Delta \lambda)\) for the negative half after adding \( 2\pi \) but stay unchanged in the range of \((0^\circ, 0.5\Delta \lambda)\) for the positive half. The former now corresponds to \( m = 24 \) in the “All Positive” case and the latter corresponds to \( m = 0 \). Therefore, the \( m = 0 \) XPSD in the “Positive and Negative” case splits into the \( m = 0 \) and \( m = 24 \) XPSD in the “All Positive” case. Similarly, the \( m = 12 \) XPSD in the “All Positive” case corresponds to the sum of the \( m = 12 \) and \( m = -12 \) XPSD in the “Positive and Negative” case. Comparing the results in Figure 2-4(a) and (b) and considering the nature of ULF pulsations, the wave power is not expected to be higher at higher modes than at lower modes, especially during the period of high solar wind dynamic pressure (e.g., 28 May) when the pulsations are at large scales. Therefore, the power versus mode number distribution during the first day in Figure 2-4(b) by assuming all
positive modes seems unrealistic, while the distribution becomes more realistic when allowing for both positive and negative modes, as shown in Figure 2-4(a). However, this reasoning cannot be applied to the second day, when the power is almost evenly distributed over all modes. Here we define the XPhase range to be [-180°, 180°] to allow for both positive and negative modes for both days but leaving the detailed work of reducing the $2n\pi$ ambiguity during the second day (high AE period) in Section 5.

Figure 2-5. Mode number distribution based on the XWT analysis of the GOES13-GOES12 pair: (a) Integrated wave power for low mode (1 ≤ |m| ≤ 3, blue curve), high mode (|m| ≥ 4 up to 12, red curve), zero mode (m = 0, blue dotted curve), and all mode numbers (|m| ≤ 12 including m = 0, black curve); (b) Fraction of the power embedded in each mode number from -12 to 12; (c) Solar wind dynamic pressure $P_{dyn}$ and AE index; and (d) SYM/H index.

To further investigate the mode distribution of ULF waves, we calculate the integrated wave power for zero (m = 0), low (1 ≤ |m| ≤ 3), high (4 ≤ |m| ≤ 12), and all (|m| ≤ 12 including m = 0) mode numbers and plot the results in Figure 2-5(a). We see that the power in all mode numbers (black curve) is generally correlated with solar wind dynamic pressure, $P_{dyn}$, during high $P_{dyn}$
times and the AE index during high AE times (plotted in Figure 2-5(c)), and is higher during the active AE period. Note there is an interesting outlier during the $P_{dyn}$ peak at 9 UT of 28 May which will be discussed later. Again, panel (a) shows that the power at low mode (blue curve) is generally dominant over the high mode (red curve) during 28 May, the high $P_{dyn}$ period. It is interesting to note that the high mode power (red curve) becomes comparable to or even higher than the low mode power during 3–8 UT on 28 May when the two GOES satellites are near midnight (see the MLT labels for both satellites in the x-axis below UT). The power in higher mode numbers also shows a temporary increase during 13-15UT when the AE level is elevated during the high $P_{dyn}$ day (28 May). Then during the high AE and low $P_{dyn}$ period (29 May 0-17UT), the power at high mode stays comparable or even higher than low mode. We also calculated the fraction of total power embedded in each mode number and plot its variation in Figure 2-5(b). For the high $P_{dyn}$ period during 28 May, we see that the mode numbers at $m = 1$ (solid black curve) and $m = -1$ (dashed black curve) generally dominate over other modes. One interesting feature we find is that the sign of $m$ starts as generally negative from 5.5UT of 28 May and then changes to positive at 15.5 UT or 10.5h MLT (if using the MLT of the trailing satellite of the pair, GOES 13).

Subsequently, we investigate whether the change of sign in $m$, as marked by a red arrow in Figure 2-5(b), is a temporal or spatial effect. First, based on the solar wind conditions during the high $P_{dyn}$ period of 28 May, we cannot identify any temporal variations of the solar wind drivers that correspond to the pattern and change of the mode number, which indicates this sign change might not be temporal. To illustrate the spatial dependence, we convert the temporal variation of the mode fraction in Figure 2-5(b) to a polar plot in Figure 2-6(a) to display the MLT distribution of mode fraction during the high solar wind $P_{dyn}$ period (the day of 28 May). The color and line styles are of the same format as of Figure 2-5(b) and the MLT of the trailing satellite, GOES 13, is used for the plot. There is a clear change of sign in $m$ at around 10.5h MLT as denoted by the thick arrows in Figure 2-6(a). To further investigate if the sign change is spatial, we performed the same XWT analysis for another two pairs of GOES satellites and plot the resolved spatial distributions of the mode fraction in Figures 2-6(b) and 6(c) for the same high $P_{dyn}$ period.
Figure 2-6. MLT distribution of the power fractions at different mode numbers from the pairs of (a) GOES13&12; (b) GOES15&13, and (c) GOES14&15 during the day of high solar wind dynamic pressure (28 May). The wave propagation direction (or sign of $m$) is denoted by the red arrows.
According to the satellite locations in Figure 2-2, the GOES 15&13 pair passes through the same MLT region an hour later than the GOES 13&12 pair, with GOES 14&15 pair delayed for one more hour. However, Figure 2-6 shows that the three satellite pairs observe a sign change in the mode number at around the same MLT region (9.5-10.5h MLT), which suggests that the change of sign is a spatial effect. As discussed earlier, the sign of mode number indicates the propagation direction of the wave: $m > 0$ corresponds to east-propagating waves as denoted by the curved arrows in Figure 2-6, and $m < 0$ means west-propagating waves. The change of the wave propagation direction slightly before noon is consistent with the picture of compressional ULF waves driven by solar wind dynamic pressure variations (Hughes, 1994; Olson & Rostoker, 1978). The high-pressure solar wind buffets the Earth’s magnetosphere near noon and generates ULF waves propagating around the Earth in anti-sunward directions.

Figure 2-6 also shows that the MLT distributions of the mode fraction are not identical among all the pairs for the same period. This is due to the fact that the three pairs are located at different MLT sectors at the same time. The MLT coverage from the three pairs can be used to infer the local time extent of the resolved mode structure. For example, based on the x-axis in Figure 2-5, we see that at ~20UT on 28 May the GOES 13&12 pair is located at 15-16h MLT (considering that the two satellites are separated by 1h in MLT). At the same time, the GOES 15&13 pair is at 14-15h MLT and the GOES 14&15 pair is at 13-14h MLT. Figure 2-6 shows that all three pairs observe a dominant mode at $m = 1$ over all covered MLT regions (13-16h), suggesting that the low-mode structure covers at least 3h in MLT near the dayside. In contrast, at 9UT of 28 May when the GOES 13&12 pair is located at 04-05h MLT (based on Figure 2-5) and the GOES 15&13 pair is at 03-04h MLT, Figures 2-6(a) and (b) show that both pairs observe dominant low-mode structure at $m = -1$. However, the GOES 14&15 pair located at 02-03h MLT concurrently finds the wave power to be evenly distributed among all modes (up to ±12). Therefore, at this time the low-mode waves are only limited to MLT regions away from the midnight (> 3h MLT), which suggests that the low-mode ULF wave structure may not be as global as previously assumed in $D_{LL}$ quantification. This also explains the outlier in Figure 2-5(a). Even though the solar wind $P_{dyn}$ shows a peak at 9 UT of 28 May, the total power which is dominated by low modes does not show an increase since the GOES 13&12 pair is located near midnight at that time.
Figure 2-7. The same format as Figure 6 but for the period of high AE (29 May 0-17UT). The grey sectors in (b) and (c) are due to gaps in the GOES 15 magnetometer data.
The analyses above are for the high solar wind $P_{dyn}$ period. To analyze the high-AE period (29 May 0-17UT), the mode fraction results in Figure 2-5(b) shows that the power spreads over all mode numbers from low to high. Similar polar plots illustrating the MLT-distribution of the mode structure during the high AE period are included in Figure 2-7. Again, panel (a) corresponds to the same GOES13&12 pair as in Figure 2-5, while panels (b) and (c) correspond to the other two pairs, GOES15&13 and GOES14&15, which follow the GOES13&12 pair in MLT. The grey sectors in panels (b) and (c) are due to data gaps in the GOES 15 magnetometer data during this period. Figure 2-7 shows that the evenly distributed power across all modes is consistently observed by all three pairs, which further confirms the resolved mode structure. Furthermore, the results from three pairs show that the MLT coverage of the evenly distributed mode structure appears to be more global than generally assumed for high AE periods. Significant contribution from high modes is not limited to regions close to the midnight, and it can also cover the MLT regions all the way over to the dayside.

2.5. Reducing the $2n\pi$ Ambiguities in Mode Number Estimation

In Section 3 we discussed the $2n\pi$ ambiguity in the resolved $XPhase$ since the XWT method can only calculate the trigonometric values of the phase differences. The $2n\pi$ ambiguity not only affects the sign of the mode number but also leads to uncertainties in the calculated $XPhase$ or mode number, since we don’t know how many full wave periods are included in between the two measurement points. For low mode structure during high solar wind $P_{dyn}$ period, we have already demonstrated in Figure 2-4 that allowing for both positive and negative modes can lead to more realistic mode structure. To further reduce the $2n\pi$ ambiguity in the resolved $XPhase$, especially for the high AE period, here we perform a cross-pair analysis to compare and reconcile the mode number results from two pairs of signals that overlap azimuthally.

Specifically, the two overlapping pairs of GOES 13&12 and GOES 15&12 are used in this analysis. As shown in Figure 2-2, both pairs cover the MLT region between GOES 13 and GOES 12, with the GOES 13&12 pair separated by 1h in MLT and the GOES 15&12 pair separated by 2h in MLT. For the pair of GOES 13&12, Figures 2-8(a1) and (b1) show the power spectral densities measured by GOES 13 and GOES 12 respectively, and Figures 2-8(c1) and (d1) illustrate the resolved XPSD and $XPhase$ using the XWT analysis. Figures 2-8(a2) through 2.8(d2) show
the same results for the GOES 15&12 pair. The time period from 17UT of 28 May to 7UT of 29 May is selected for this analysis since it includes a high $P_{\text{dyn}}$ - low AE interval (17 to 22UT of 28 May) followed by a low $P_{\text{dyn}}$ - high AE interval (22UT of 28 May to 7UT of 29 May) as shown in Figure 2-9(c).

Figure 2-8. (a1-b1) Power Spectral densities (PSD) vs. frequency and time measured by GOES 13 and GOES 12 respectively; (c1-d1) Calculated cross power spectral density (X-PSD) and cross phase difference (X-Phase) for the pair of GOES 13&12 using XWT analysis; (e1) Integrated wave power at different mode number ranges plotted in different colors specified in the right legend; (f1) The same as (e1) but after the cross-pair analysis to reduce the $2\pi$ ambiguity. (a2-f2) The same format as (a1-f1) but for the GOES 15&12 pair.

The similarity of XPSD between the two pairs, as shown in panels (c1) and (c2) of Figure 2-8, demonstrates that the two pairs are likely measuring the same ULF pulsations during that time. Based on the XPSD and XPhase values, Panels (e1) and (e2) show the integrated wave power at different mode number ranges for the two pairs. Since the azimuthal separation between the GOES 13&12 pair is ~15° and the XPhase is defined to be in the range of [-180°, 180°], the resolved $m$ range is [-12, 12] in Panel (e1), with different colors and line styles showing power in different mode ranges, as indicated in the legend on the right side of the figure. Similarly, since the
azimuthal separation between the GOES 15&12 pair is about 30°, the resolved \( m \) range is only \([-6, 6]\) in Panel (e2). Even though the GOES 15&12 pair cannot cover as high \( m \) values as the GOES 13&12 pair, the mode results in the overlapping \( m \) range between the two pairs should be generally consistent. This is true for the high solar wind \( P_{\text{dyn}} \) interval (17-22UT of 28 May), showing dominant integrated wave power at low modes summed over \( m = 1 \) to 3 (blue curve), which is consistent between two pairs. However, for the subsequent high AE interval, at many times the mode distributions become inconsistent between the two pairs, especially at high mode numbers (the green curves). For example, at 5UT of 29 May, the GOES 15&12 pair in Panel (e2) shows a high power in the \( m \) range of \( 4 \leq |m| \leq 6 \) (solid and dashed green curves), while the GOES 13&12 pair in Panel (e1) shows a peak of power in the \( m \) range of \([-12, -10]\) (dashed magenta curve). To further illustrate the different results from the two pairs, in Figure 2-9(a) we plot the difference in the resolved \( m \) number (\( \Delta m \)) between the two pairs at each time and frequency. Green means good agreement between two pairs while red and blue indicate large disagreement. Consistent with the results of Figures 2-8(e1) and (e2), we see persistent good agreement between two pairs during the high \( P_{\text{dyn}} \) interval over all frequencies, but large disagreement up to \( |\Delta m| \approx 18 \) appears frequently during the high AE interval.

The mismatch between the two pairs during the high AE interval could be due to the \( 2\pi \) ambiguity in the resolved \( XPhase \). In the XWT analysis, the \( XPhase \) range is defined to be \([-180^\circ, 180^\circ]\), which is sufficient to resolve the low mode structures given the \( \sim 15^\circ \) azimuthal separation between the GOES 13&12 pair and even the \( \sim 30^\circ \) separation between the GOES 15&12 pair. This explains why during the high \( P_{\text{dyn}} \) interval when the low modes are dominant, the two pairs show consistent mode distributions from XWT analysis. However, the \([-180^\circ, 180^\circ]\) range may not be sufficient to cover the high mode numbers during the high AE interval. Yet due to the \( 2\pi \) ambiguity in the XWT analysis, the \( XPhase \) cannot be uniquely determined. Here we reduce the ambiguity in the phase difference and thus the mode number by comparing and reconciling the mode numbers between the two overlapping pairs. Specifically, for each pair and at each time and frequency (e.g., Figure 2-8(d1)), we allow the phase difference to be three possible values, \( XPhase - 360^\circ, XPhase, \) and \( XPhase + 360^\circ \), where \( XPhase \) is the phase difference directly from the XWT analysis in the range of \([-180^\circ, 180^\circ]\). Correspondingly, three possible values of mode number \( m1 \) are calculated using Eq. (1) for pair 1 (GOES 13&12) at each frequency-time
bin, and the same for \( m2 \) of pair 2 (GOES 15&12). Then for each time and frequency, we cross-compare the possible mode values from each pair to find the \( m1 \) from pair 1 that is the closest to the \( m2 \) from pair 2, i.e., \( |\Delta m| = |m1 - m2| \) reaching a minimum. As a result, the mode numbers, \( m1 \) and \( m2 \), that produce the minimum difference are used as the final and more realistic mode numbers for the two pairs.

![Diagram](image)

Figure 2-9. (a-b) The difference in the resolved mode number between two pairs of GOES 13&12 and GOES 15&12 before and after the cross-pair analysis; (c) Solar wind dynamic pressure \( P_{\text{dyn}} \) and AE index for the interval.

After performing this cross-pair analysis, Figure 2-9(b) shows the new results of the mode number difference \( \Delta m \) between two pairs. We see that the mode difference is mostly within the range of \( |\Delta m| < 6 \) (green or yellow colors), illustrating a significant improvement over Figure 2-9(a) before the analysis. A new comparison of the integrated wave power at different mode ranges is also shown in Figures 2-8(f1) and (f2) after the cross-pair analysis. First, we see that with the new
analysis, both pairs can resolve waves at higher mode numbers (e.g., the new red curves with $|m| \geq 13$), the power of which comes from the waves previously at lower mode numbers before applying the cross-pair analysis. For example, for the GOES 15&12 pair we find that from Figures 2-8(e2) before the cross-pair analysis to 2-8(f2) after the new analysis, the power in the range of $4 \leq |m| \leq 6$ (solid and dashed green curves) generally decreases, which is actually converted to the power at $|m| \geq 13$ (red curves). More importantly, by comparing the new power distributions after the cross-pair analysis between the two pairs (Figures 2-8(f1) and (f2)), we find that the distributions of wave power at different mode number ranges (different colors) reach a better agreement between the two pairs. Furthermore, we compare the histograms of the mode number before and after the cross-pair analysis for each pair in Figure 2-10. The mode numbers resolved at all the times and frequencies during the entire period of Figure 2-9 are included in the histogram analysis. X axes are the mode number $m$ values and the y axes show the counts of the time-frequency bins possessing the corresponding $m$ values. Panel (a1) of Figure 2-10 is for the GOES 13&12 pair before the cross-pair analysis, and Panel (a2) is after; similar for Panels (a2) and (b2) which are before and after the analysis for the GOES 15&12 pair. Again, before the cross-pair analysis the resolved mode numbers for the GOES 13&12 pair is confined in the range of $[-12,12]$ (Panel (a1)) and for the GOES 15&12 pair is in the range of $[-6, 6]$ (Panel (a2)). Both distributions show sharp cuts at the maximum and minimum mode numbers that can be resolved by the XWT analysis. The distributions are also inconsistent between the two pairs before performing the cross-pair analysis. However, after the cross-pair analysis, in Panels (b1) and (b2), we find that the mode numbers can reach higher ($\pm 24$ for GOES 13&12 pair and $\pm 18$ for GOES 15&12 pair) values and histograms show a more gradual drop as $|m|$ increases which is consistent with the physical nature of ULF pulsations. In addition, the histograms after the analysis show a better agreement between the two pairs (Panels (b1) versus (b2)). Therefore, the improved agreement of mode number distributions between the two pairs (Figures 2-8, 2-9, 2-10) and the more realistic distributions of mode number (the histograms) demonstrate that the cross-pair analysis is efficient in reducing the $2\pi n$ ambiguity in the phase difference and generating more reliable mode numbers during the high AE interval when the contribution from high mode numbers is significant.
2.6. Discussion and Conclusions

To understand and quantify the radial diffusion process in Earth’s radiation belts, a reliable estimation of the azimuthal mode structure of ULF waves is required. In this paper, we quantify the azimuthal mode numbers of the compressional Pc5 ULF waves based on multiple pairs of GOES satellite observations. The 28-31 May 2010 geomagnetic storm is selected for the study since we have five GOES satellites available during this time, which provide a wide range of local time coverage at the same time. In addition, the storm includes distinct intervals with either high solar wind dynamic pressure (\(P_{\text{dyn}}\))-low AE activity or low \(P_{\text{dyn}}\)-high AE, which may correlate with different ULF wave structures. A cross-spectral technique, XWT, is applied to each pair, and by combining the results from multiple pairs, temporal and spatial variations of the ULF wave mode structure during the storm are resolved. The work also makes improvements over many previous works (Li et al., 2016; Sarris, 2014; Sarris & Li, 2017; Sarris et al., 2013) by including both positive and negative mode numbers in the analysis (i.e., waves propagating in both eastward and westward directions) and by reducing the \(2n\pi\) ambiguity in the cross-spectral analysis, both of which contribute to more reliable mode number estimations. The specific results and conclusions are summarized below.

During the storm commencement, when the solar wind \(P_{\text{dyn}}\) is high while AE is low, the XWT results show that the ULF wave power is mostly dominated by low mode numbers (\(m = 1\) or \(-1\)), which is consistent with previous studies (Sarris & Li, 2017; Tu et al., 2012). Our results suggest that these low-mode ULF waves are probably driven by large-scale solar wind dynamic pressure variations on the dayside (Hughes, 1994; Olson & Rostoker, 1978), based on the strong correlation between the appearance of the low-mode wave structure and the enhanced solar wind \(P_{\text{dyn}}\) in Figure 2-5. Another possible driver of low-mode ULF wave is Kelvin-Helmholtz instability as a result of high-speed solar wind (Mann et al., 1999). However, the solar wind conditions in Figure 2-1 show that the solar wind speed remains elevated when the \(P_{\text{dyn}}\) drops, thus not well-correlated with the low-mode wave structure in Figure 2-5. This indicates that Kelvin-Helmholtz instability is less likely to be the main driver for the low-mode ULF waves during this event. Furthermore, we find that if only positive mode numbers are allowed, similar to some previous studies (Li et al., 2016; Sarris, 2014; Sarris & Li, 2017; Sarris et al., 2013) the mode structure becomes unrealistic.
during the high (Li et al., 2016; Sarris, 2014; Sarris & Li, 2017; Sarris et al., 2013) interval, which suggests that allowing for both positive and negative mode numbers is critical in the analysis. Accurately resolving the sign of mode number is important to precisely quantify the radial diffusion coefficient, $D_{LL}$. During the high $P_{dyn}$ period, an interesting change of sign in mode number, $m$, occurs in the slightly pre-noon region (around 9.5-10.5h MLT). After comparing the MLT distributions of the mode structure from multiple pairs of GOES satellites, we have confirmed that this sign change of $m$ is spatial rather than temporal, with eastward propagation in the noon to dusk sector and westward propagation in the noon to dawn sector. This is consistent with compressional ULF waves externally driven by high-pressure solar wind buffeting around noon, creating anti-sunward propagation of waves on both sides (Hughes, 1994; Olson & Rostoker, 1978). Furthermore, using the wide MLT coverage of the multiple pairs of GOES satellites at the same time, we find that the low-mode ULF waves may not be as global as previously assumed, limited to regions away from midnight. This limited local time extent of the low-mode structure needs to be properly considered in the calculation of $D_{LL}$ for energetic particles.

In contrast, during the storm main phase and early recovery phase when the solar wind $P_{dyn}$ is low and AE is high, the XWT results show that the wave power spreads almost evenly over all different mode numbers from low to high. The strong correlation between the appearance of the high-mode ULF waves and high AE in Figure 2-5 suggests that these waves may be driven by small-scale substorm injections from the nightside (James et al., 2013; Yeoman et al., 2010). By investigating the MLT-distribution of the mode structure during the high AE period from multiple pairs of GOES satellites, we find that the evenly distributed mode structure or the significant contribution from high modes is not limited to localized regions close to midnight, as generally assumed, and can spread all the way to the MLT regions on the dayside. Possible mechanisms for high-mode ULF waves near the dayside region include a wave-particle source mechanism involving drift-mirror instabilities (Walker et al., 1982; Yeoman et al., 1992) and a dispersive waveguide model with the high-m waves propagating both sunward and anti-sunward from the sources at flank regions (Fenrich et al., 1995). More detailed wave and particle analysis is needed in the future to resolve the driving mechanism during this event. This more global local time coverage of the evenly distributed mode structure up to high modes also needs to be considered realistically in $D_{LL}$ calculations.
Furthermore, to reduce the $2\pi n$ ambiguity in the resolved phase difference from the XWT analysis, a cross-pair analysis is performed on satellite field measurements here for the first time over two pairs of GOES satellites that overlap azimuthally. First, the XWT results from the two pairs suggest that a cross-pair analysis is necessary for the high AE intervals when the resolved mode numbers can be largely inconsistent. Then after comparing and reconciling the mode number results from the two pairs using the cross-pair analysis, the mode structures from the two pairs show a significantly improved agreement. These results demonstrate that the analysis is efficient in reducing the $2\pi n$ ambiguity and generating reliable mode structure during the high AE interval when the mode numbers can be high. In Section 5 the cross-pair analysis has only been applied as a proof of concept to the selected interval of the event. If applying to the entire event, we would expect that the low-mode dominated results in Figures 2-5 and 2.6 during high $P_{dyn}$ (and low AE) times remain mostly unchanged, while the detailed mode fraction results during high AE times in Figures 2-5 and 2.7 show changes similar to the high AE interval analyzed in Section 5. Applying
this cross-pair analysis to the entire event interval will be an important part of our future work. Additionally, in the cross-pair analysis, we only added one full phase period to both directions of the phase difference (i.e., \( \pm 2\pi \)) since our results show that this is sufficient to reach a good agreement between the pairs and to achieve accurate distributions of the mode numbers (e.g., the histograms in Figures 2-10(b1) and (b2)). This could be due to the fact that the contribution of even higher mode numbers \( m \geq 25 \) in the histograms is insignificant for this period. Even though a better agreement is reached, the mode structures are still not identical between the pairs. To further improve the results, the cross-pair analysis needs to be applied to more overlapping pairs (e.g., three pairs with small azimuthal separation rather than two) to cross-compare and reconcile the resolved mode structures.

Similar cross-pair analysis has been performed on ground-based ULF wave measurements in previous works (Baker, 2003; Chisham & Mann, 1999), but there are differences between our approaches. For instance, Chisham and Mann (1999) used three ground magnetometer pairs two of which are not overlapping, while here we only use the overlapping pairs (GOES13-12 and GOES 15-12) to better ensure they are measuring the same ULF wave. Additionally, in Chisham and Mann (1999) the sign of \( m \) is resolved by analyzing the arrival time of the waveform, while in our approach we let the sign of \( m \) be part of the comparison and choose \( m \) with the minimum difference between two pairs. Last but not least, our cross-pair analysis is performed on ULF waves measured in space while the analysis in Chisham and Mann (1999) is performed on ground magnetometers which makes estimating \( m > 50 \) nominally impossible due to the ionospheric screening.

The analysis presented in this work needs also to be applied to more case studies with different drivers of solar wind and geomagnetic conditions. In addition to multiple GOES satellites, the analysis can also be conducted for other space missions including NASA Van Allen Probes, THEMIS (Time History of Events and Macroscale Interactions during Substorms), and MMS (Magnetospheric Multiscale) missions to resolve the mode structure at a wide range of L shells and local times. With the electric field measurements from these other missions, similar analysis from this paper can be applied to study the mode structure of poloidal and toroidal ULF waves. In addition, the MMS mission is best suited due to the close separation between the satellites (as small as tens of kilometers), making it possible to resolve very high mode numbers (order of 100s).
(Le et al., 2017; Murphy et al., 2018). With a good database of the ULF mode numbers at different solar wind and geomagnetic conditions, we can construct statistical maps of mode structures under different driving conditions, which will be directly useful in quantifying the radial diffusion process in Earth’s inner magnetosphere and in resolving its contribution to the overall dynamics of energetic particles.
Chapter 3. High-fidelity Analysis of ULF Wave Mode Structure Following Interplanetary Shock Compression of the Dayside Magnetopause Using Multi-Point Observations from MMS

3.1. Overview

Among different mechanisms responsible for the transport of energetic charged particles in near Earth space, radial diffusion has long standing as an important mechanism. Radial diffusion is a phenomenon through which charged particles undergo a radial displacement with respect to their initial positions in the presence of a radial gradient in phase space density by interacting with fluctuating electric and magnetic fields which oscillate on a time scale comparable to their drift period (Fälthammar, 1965; Kellogg, 1959; Schulz & Lanzerotti, 1974). Despite its established importance (see review by Shprits et al. (2008)), many uncertainties and simplifying assumptions remain, some of which are addressed in Barani et al. (2019). These include a $2\pi n$ ambiguity in azimuthal mode number $m$ and the effect of both positive and negative $m$. As discussed in Chapter 1, radial diffusion is defined as a random walk on a radial gradient in phase space density $f$ at a rate determined by the coefficient $D_{LL}$. In radial diffusion the third adiabatic invariant is not conserved (Schulz & Lanzerotti, 1974).

It has been analytically shown that $m$, which can be interpreted as the rotational representation of the wave number, is one of the important inputs in the calculation of the radial diffusion coefficient (Fälthammar, 1965; Fei et al., 2006; Schulz & Lanzerotti, 1974). Determining $m$ based on the magnetometer measurements from the Magnetospheric Multiscale Mission is the focus of this paper.

In Chapter 1, it was discussed that the lack of a realistic estimation of azimuthal mode number in most models is a shortcoming (Barani et al. (2019); Sarris (2014); Sarris and Li (2017); Tu et al. (2012)) due to the difficulties and uncertainties embedded in all the different methods for quantifying $m$. For example, in Chapter 1 it was addressed that the maximum mode number that can be resolved ($m_{max}$) is either dependent upon the number of azimuthal measurements in the
case of equally-spaced virtual probes in the full coverage from MHD simulations (Li et al., 2020; Tu et al., 2012) or restricted by the separation of the probes using in-situ measurements (Barani et al., 2019). Furthermore, the physics of MHD simulations as well as the use of ground magnetometers inherently imposes limitations to the maximum resolvable $m$, which is quantitatively derived in the Appendix.

In this study, Cross-Spectral analysis will be applied to in-situ wave measurements from the Magnetospheric Multiscale Mission (MMS) that are very closely separated for mode number estimation of ULF waves. Previously, Sarris (2014) and Sarris and Li (2017) applied the Cross-Spectral analysis to time series of ULF wave magnetic fields measured by multiple GOES spacecraft to determine the azimuthal mode number and power distribution as a function of local time and their dependence on solar wind activity. Recently, Barani et al. (2019) developed new tools of the Cross Wavelet Spectral Analysis to look at multiple pairs of in-situ wave measurements from GOES magnetometers at geosynchronous orbit which was discussed in Chapter 2. In Chapter 2 we first applied this technique to one pair of GOES satellites to resolve power in Pc-5 ULF pulsations at both negative and positive mode numbers (corresponding to westward and eastward propagating pulsations, respectively) up to $|m| = 12$ following the minimum azimuthal spacing of $\sim 15^\circ$ between two GOES satellites. Then we introduced a cross-pair analysis to two pairs of GOES satellites to reduce the $2\pi\pi$ ambiguity in the cross-phase, based on which the accuracy of the mode number estimation was improved and maximum resolved mode number was increased to $|m| = 24$ for the same mentioned satellite separation. In this chapter, we will extend our previous analysis (Barani et al., 2019) which was limited to geosynchronous orbit to high-resolution closely separated multi-point ULF wave measurements from the MMS mission for high-fidelity mode number estimation.

Our mode structure analysis in this work focuses on ULF wave signatures observed during two consecutive interplanetary shocks which compress the dayside magnetopause. The sudden compression of the magnetosphere due to an interplanetary shock, seen in solar wind dynamic pressure increase along with a rapid change in IMF $B_z$, could cause an intensification of the Chapman-Ferraro currents along the magnetopause. This dayside compression leads to an increase in the SYM-H index which is a global average of ground-based magnetometer measurements of the horizontal component of the geomagnetic field. Interplanetary shocks have been shown to be
important drivers of geomagnetic storms, which can excite strong ULF wave responses in the magnetosphere and affect the dynamics of ring current and radiation belt particles (see reviews by Hudson et al., 2020; Li and Hudson, 2019). For example, Hudson et al. (2004) show that large amplitude ULF waves were generated in the dayside magnetosphere after the arrival of interplanetary shocks. Zong et al. [2009] reported ULF wave oscillations observed by the Cluster constellation in the plasmasphere boundary which were induced by an interplanetary shock on 7 November 2004. They also applied the cross-spectral analysis on the observed oscillations and obtained a high wave mode number of $m \sim 50$. The estimated mode number corresponds to a drift-resonance energy that is consistent with what was observed in the electron flux enhancement. Therefore, high-fidelity estimation of the ULF wave mode structure during interplanetary shocks is critical to understanding the shock effects on ULF wave generation as well as energetic particle dynamics. However, detailed evolution of ULF wave mode structure and frequency spectrum during interplanetary shocks is still not fully investigated, which motivates our work here. This Chapter is organized as follows: Section 2 introduces the 9 March 2018 event with double shocks, the detailed methodology for the high-fidelity estimation of $m$ in this work using MMS data is described in Section 3, the mode structure and frequency spectrum results from our analysis of the double-shock event are discussed in Section 4, with further discussion and conclusions included in Section 5.

3.2. 9 March 2018 Shock Event

The 9 March 2018 event has been selected for detailed analysis in this work, with the NASA OMNIWeb solar wind measurements (https://omniweb.gsfc.nasa.gov/) and geomagnetic indices depicted in Figure 3-1. The event is accompanied by a two-step increase in dynamic pressure characterizing two shocks with approximately an hour lag time in between at around 1805UT and 1910UT on 9 March 2018, respectively. The initial disturbance is identified as an ICME shock (Interplanetary shock produced by a Coronal Mass Ejection) at L1 in the Cane-Richardson compilation of ICME shocks based on ACE measurements (http://www.srl.caltech.edu/ACE/ASC/DATA/level3/icmetable2.htm) and is propagated to the bow shock in the OMNIWeb data plotted in Figure 3-1 assuming a constant solar wind radial velocity between the measured L1 value and the bow shock (https://omniweb.gsfc.nasa.gov/html/ow_data.html#time_shift).
The two consecutive shocks enhance the dynamic pressure level from ~1.5 nPa to ~9 nPa with an increase of solar wind flow speed from ~350 km/s to ~450 km/s. At around 1.5 hours after the first shock (~19:40 UT), the Interplanetary Magnetic Field (IMF) $B_z$ turned southward and remained southward for almost two hours, with an extremum of ~-7 nT at around 2030 UT. The vertical magenta lines in Figure 3-1 are the selected times of interest which will be described in Section 4, and the time shift depicted between SYM-H and solar wind data explained in the chapter.

The Auroral Electrojet (AE) index was very low, below 200 nT for the entire study period (not shown in the figure), indicating that substorm dipolarization events, which can be an important source of ULF wave activity on the nightside, were not important for this event (please see Chapter 2 and Barani et al. (2019) for discussion of the dayside vs. nightside effects on ULF waves).

Figure 3-1. Solar wind parameters and geomagnetic indices during the 9 March 2018 shock event based on the OMNI data set, including IMF-Bz, total bulk solar wind flow speed $V_{SW}$, solar wind dynamic pressure $P_{dyn}$, Alfvén Mach number, and SYM-H index. The magenta lines mark four times of interest with the time shift between SYM-H and solar wind data explained in the chapter.
During the time interval studied, MMS was on the dayside probing from pre-noon to post-noon regions of the magnetosphere as shown in Figure 3-2, which plots the trajectory of the MMS spacecraft during the time 16:00 UT to 20:00 UT in GSE coordinate in xy (panel (a)), yz (panel (b)), and xz (panel (c)) planes. The radial variation of the MMS orbit corresponds to L values from ~8 to ~4, covering a range of L values from outside the geosynchronous orbit, where the ULF wave activity has been extensively analyzed in earlier studies (see Barani et al. (2019) and references), down to an L location where the peak flux of outer zone electrons typically occurs (Li & Hudson, 2019).

3.3. Methodology for High-Fidelity Estimation of \( m \)

3.3.1. Cross-Spectral Analysis

In this work we apply the cross-spectral analysis to the ULF waves observed by multiple MMS probes during the shock event to analyze the ULF wave power as a function of wave mode number, frequency, and time. A Cross-Wavelet Transform (XWT) method following the Morlet basis functions (Eriksson, 1998; Grinsted et al., 2004; Torrence & Compo, 1998) is used similar to the one used in Barani et al. (2019). Wavelet analysis and its advantages were discussed in Chapter 1.
A brief discussion of the usage of the XWT method to estimate the ULF wave mode number is found in Barani et al. (2019).

Equation 1-1 is the main relation for calculating $m$ values at every time-frequency bin while the Cross Power Spectral Density $XPSD$ values, as another outcome of the XWT analysis, gives the power content in the same corresponding time-frequency bins. In summary, a set of $(XPSD, XPhase)$ values as the outcome of the XWT spectral analysis, combined with calculation of $m$ from equation 1-1 at each time and frequency, are what we need for the mode structure analysis.

In this work, we take advantage of the high-resolution magnetic field measurements from the multi-probe MMS mission (Russell et al., 2014) to perform azimuthal mode number analysis of ULF waves. MMS is a mission with four spacecraft in a tetrahedron shape configuration launched in 2015 (Tooley et al., 2015). The MMS data were chosen for our analysis due to the following reasons: (1) The very close separation of the MMS probes compared to previous missions significantly increases the maximum resolved mode number $m_{max}$ from the cross-spectral analysis, which helps eliminate the $2\pi$ ambiguity in the mode number estimation. For example, GOES satellites had a typical minimum separation of $\sim$15 degrees as shown in our prior analysis (Barani et al., 2019), leading to $|m_{max}| = 12$ based on equation 1-1. The $2\pi$ ambiguity in the $XPhase$ means that in reality $|m|$ could be higher than 12 following equation 1-10. However, for a pair of MMS probes the azimuthal separation could be less than 1 degree, leading to $|m_{max}| = 400$ as shown in our results in Section 4, which is high enough to include almost all physically existing modes of ULF waves (also demonstrated in Section 4), thus eliminating the $2\pi$ ambiguity giving confidence that $n$ is always zero in Equation 1-5 and 1-6. (2) Due to its design to capture kinetic scale physics, the data sampling rate of MMS could be 1000 times higher than those in previous missions such as GOES. This unprecedented high time resolution is ideal to resolve the fast temporal variations of ULF waves during shock events with results demonstrated in Section 4. (3) Since MMS contains four probes which are very close to each other, we can perform cross-spectral analyses over multiple pairs of measurements at almost the same location in space, providing an unique opportunity to check the reliability of the resolved ULF wave mode structure (with details discussed in Appendix A1). (4) Since the multi-probe MMS mission passes through a wide range of radial distances from Earth during the studied double shock event, it
provides a great opportunity to study the mode structure at different radial distances or L shells which is important for radial diffusion quantifications (Fälthammar, 1965; Fei et al., 2006).

3.3.2. Threshold Setting

Strong coherence between a pair of wave signals is the underlying assumption of cross-spectral analysis. To increase the reliability of the estimated mode structure in this work, thresholds in the coherence of both $XPSD$ and $XPhase$ from the XWT analysis will be implemented for the first time. (1) Power coherence threshold: Low-common powers between two signals, or low values of $XPSD$ should be discarded since they could be due to signals that are not strongly correlated or background noise in the MMS magnetometer measurements (e.g., residual fields of the magnetometer). The $XPSD$ threshold used in this work is further discussed in Section 4. (2) Phase coherence threshold: Strong coherence between two signals also requires the $XPhase$ values to be stable in time. Here we use the method defined in Torrence and Compo (1998) to calculate the value of Wavelet coherence [e.g., Grinsted et al. (2004) and Eriksson (1998)] as the measure of the stability in $XPhase$ over time and frequency. The calculated Wavelet coherence is a value between 0 and 1, and in this work the phase coherence threshold is set to be 0.98 which is very close to perfect coherence. Please note that low values of Wavelet coherence (i.e., unstable $XPhase$) do not always correspond to low-common wave powers (or low $XPSD$) between two signals (like the case of random noise). Therefore, applying mere power coherence threshold is not always sufficient to ensure strong coherence between two signals.

3.3.3. Frequency Considerations in the Spectral Analysis

Generally, the range of frequency $f$ that can be resolved from a spectral analysis is:

$$\frac{1}{w_{\text{detrend}}} < f < \frac{f_s}{2}$$  \hspace{1cm} 3-1

where $f_s$ is the sampling frequency (with the factor 2 on the right-hand side of this inequality from the Nyquist sampling theorem) and $w_{\text{detrend}}$ is the time window used to detrend the signals. The MMS magnetic field data used in this study are at a cadence of 0.125 seconds which corresponds to $f_s = 8$ Hz. Since the width of the detrending time window determines the lower limit of the
resolved frequency, \(w_{\text{detrend}}\) needs to be large enough for the frequency to cover the Pc4-5 range ULF waves. On the other hand, it cannot be too large since MMS moves across L shells over time and we need to make sure MMS is located around the same L region over the time window \(w_{\text{detrend}}\) for the cross-spectral analysis to be valid at a constant L shell. Taking these two factors into consideration, the detrending time window is set to be 700 seconds, over which the L transit of MMS stays within \(\Delta L = 1\) during the studied event interval which corresponds to a lower frequency limit of \(1.43 \, mHz\). The frequency range investigated in this work is in the range of \(1.6 - 22.2 \, mHz\) (Pc4-5 range) which falls well inside the defined range from inequality 3-1.

### 3.4. Mode Structure and Frequency Spectrum Results

#### 3.4.1. Results from XWT Analysis

The time series signals and the corresponding wavelet spectra of the compressional magnetic field pulsations measured by MMS 4 and MMS 3 are shown in panels (a) to (d) of Figure 3-3 during the 2 hour and 20 minute time interval from 17:40 UT to 20:00 UT on 9 March 2018. The Magnetic Local Time (MLT) and L values of each probe are shown at the bottom of the figure at every 10 minutes. Both probes observed a large wave pulse with wave power peaked around 18:06:54UT (marked by the first vertical magenta line as a1), which is potentially induced by the first shock. The wave power dropped down to pre-shock level around 18:17:42UT (marked by the second magenta line as a2). About 1 hour later, the two probes observed another enhancement of wave power in panels (c) and (d), which peaked around 18:53:54UT (marked by magenta line b1) and dropped down around 19:02:42UT (marked by magenta line b2), potentially caused by the impact of the second shock. The compressional magnetic field values (and wave power) at the position of MMS during the second shock are lower than during the first shock since the probes were at \(L \approx 6.3\) during the impact of the second shock compared to \(L \approx 7.48\) during the first shock as denoted by the L values along the horizontal axis. It is not surprising that the magnetic perturbations are larger where the background magnetic field is weaker when excited by interplanetary shock compression of the dayside magnetopause, as is the case for the MMS field observations during the first shock. To confirm the correlation between the observed ULF wave responses and the two shocks, the same four time cuts are marked in the SYM-H plot in Figure 3-1(e), with time a1 corresponding to the start time of a large SYM-H increase, and time b1 marking the start of a moderate SYM-H decrease. Comparing the time variation of SYM-H to that of solar wind
conditions in the top four panels of Figure 3-1, the strong increase of SYM-H (time a1) should be caused by the sharp increase of solar wind dynamic pressure or the arrival of the first shock, which is also marked as a1 in the solar wind plots but with a time shift of ~10 min. Similarly, the decrease of SYM-H at time b1 should be related to the decrease of IMF Bz marked also as b1 in the solar wind plots but with a time shift slightly more than 10 min, which corresponds to the arrival of the second shock. The suggested time shift between the geomagnetic responses and solar wind conditions from the OMNI data set is typical for the passage of ICME shocks with time-varying solar wind velocity (as shown in Figure 3-1(b) for this event). Even though the OMNI solar wind parameters have already been propagated to the bow shock, constant solar wind radial velocity $V_x$ is assumed in calculating the propagation time from the L1 solar wind measurements to the bow shock, which could be inaccurate when $V_x$ rapidly changes with time. After shifting the times of a1 and b1 in the solar wind plots, the times a2 and b2 are also shifted by keeping the same time duration from a1 and b1 respectively. By marking the four time cuts in the solar wind plots after the shift, we have confirmed that the selected times of interest marked in Figure 3-3 correspond to a1 and b1 at the arrival of two shocks and a2 and b2 being shortly after the shock impact. The wavelet cross power spectral density ($XPSD$) values are shown in panel (e) of Figure 3-3 with the values below $40 \frac{nT^2}{Hz}$ filtered out by the power coherence threshold described in Section 3.3.2 (corresponding to the missing data in the panel). The total residual power in MMS magnetometers during this event was below $2.5 \times 10^{-3} \frac{nT^2}{Hz}$ which corresponds to the residual $XPSD$ in the range of $(0.114, 1.25) \frac{nT^2}{Hz}$ for the resolved frequency band of $(1.6, 22.2) mHz$. The compressional power density threshold for the entire frequency band of interest is set to be larger than the residual power measured by MMS (or the detection capability of magnetometers) by more than an order of magnitude. This threshold setting assures the reliability and coherence of the $XPSD$ in the analysis. The mode structure results from the XWT method, however, are not very sensitive to this specific threshold since it is the higher power that determines the resulting mode values rather than the lower ones. Panel (f) of Figure 3-3 shows the corresponding wavelet coherence of the signals, which is generally high (or close to 1), demonstrating high stability in $XPhase$ owing to the very close separation between the pair of MMS probes (the azimuthal separation between MMS 4 and MMS 3 was in the range of $(0.03, 0.150)$ degrees during the studied interval).
The phase coherence threshold is set to be 0.98 in this study and panel (g) shows the calculated wavelet cross phase values $XPhase$ after applying both the power and phase coherence thresholds. The filtered-out results after applying the thresholds are not shown in the panel or included in the mode structure estimates.
3.4.2. Mode Structure Results

The set of \((XPSD, XPhase)\) values from the XWT analysis in Figure 3-3 are then used to estimate the azimuthal mode structure of ULF waves at each time and frequency based on equation (1-1). In panel (c) of Figure 3-4, the integrated compressional wave power over the frequency range of 1.6 to 22.2 mHz and over different ranges of mode numbers are plotted vs. time during the interval. The integrated wave power at low mode \((|m| = 1, 2, 3)\) is in blue, with medium mode \((|m| = 4, 5, 6)\) in magenta, and high mode \((|m| = 7, 8, 9, ..., 400)\) in red. The positive modes corresponding to eastward propagating pulsations are shown in solid curves while the negative modes corresponding to westward propagating pulsations are shown in dashed curves.

Figure 3-4. (a)-(b) Solar wind dynamic pressure and SYM-H index during the 9 March 2018 shock event; (c) Integrated compressional wave power over the frequency range of 1.6 mHz to 22.2 mHz over different ranges of mode numbers, with \(|m| < 400\) in a black solid curve, \(m = 0\) in a black dotted curve, low modes of \(|m| \leq 3\) in blue, medium modes of \(3 < |m| \leq 6\) in magenta, and high modes \(6 < |m| \leq 400\) in red. The positive modes corresponding to eastward propagating pulsations are shown in solid curves while the negative modes corresponding to westward propagating pulsations are in dashed curves; (d) Fraction of the power in % over different ranges of mode numbers as described in (c).
modes corresponding to westward propagating pulsations are in dashed curves. The power of in-phase pulsations \( m = 0 \) is shown in the dashed black curves and total power over all the modes up to \( |m_{\text{max}}| = 400 \) is depicted in the solid black curve on top. Then for each time, we further calculate the fraction of ULF wave power at each mode number range by normalizing the respective integrated power in color by the total power up to \( |m_{\text{max}}| = 400 \) in black, with \% results plotted in panel (d). Again, the solid black curve on top is the total \% of all the power up to \( |m_{\text{max}}| = 400 \), which stays very close to 100\% at all times, demonstrating that the wave power at mode number above \( |m_{\text{max}}| = 400 \) is almost negligible. Therefore, even though the azimuthal separation between MMS 4 and MMS 3 increases from 0.03 degrees to 0.15 degrees during the interval, suggesting maximum resolvable mode number based on equation 1-1 well above 400, it is sufficient to investigate up to \( |m_{\text{max}}| = 400 \) in our analysis here. The solar wind dynamic pressure and SYM-H are depicted in panels (a) and (b), respectively, with the four times of interest from Figures 3-1 and 3-2 marked as vertical lines in this figure as well. The total ULF wave power (solid black in panel (c)) shows two peaks at the arrival of the two shocks marked by times a1 and b1, which is consistent with Figure 3-3. The power fraction results in panel (d) show distinct contributions of the low versus high \( m \) modes in the power of ULF waves during and after the shock impact. Lower modes (dashed blue curve at \(|m| \leq 3\)) are contributing dominantly to the wave power at the arrival of the first shock, whose contribution decreases gradually as the total wave power drops after the shock arrival. Shortly after the impact of the first shock (after time a2), higher modes (red curve at \(|m| \geq 7\)) constitute the dominant power contribution. To better illustrate the development of high mode structure after the impact of the first shock, detailed mode distribution of ULF waves over six different time intervals from 18:18UT (right after time a2) to 18:33UT is plotted in Figure 3-5, with each time interval about 3 minutes apart in time. The mode distribution in each panel of Figure 3-5 is plotted up to \(|m| = 200 \) and averaged over a time window of 12 seconds. The capability of resolving detailed mode structure of ULF waves over a short interval of 12 seconds is made possible by the high-time-resolution MMS field data. Figure 3-5 demonstrates that after the impact of the first shock, a clear bump-on-tail distribution at mode
number around 20 started to evolve around 18:18 UT (panel (a)), shifting to higher modes for around ten minutes (panels (b) to (e)), then reaching a peak of the bump at \( m \approx 50 \).

We also see that as the power at high modes grows, and the power at lower modes becomes less and less dominant, leaving a single peak at \( m \approx 50 \) in panel (f).

Looking back into the mode fraction results in panel (d) of Figure 3-4, a vivid switch of sign in \( m \) from negative to positive is observed around time a2 as the MMS spacecraft cross 10.97 MLT.
towards noon. This interesting pre-noon sign change in \( m \) is consistent with the observed change of sign in the mode analysis by Barani et al. (2019) using multiple pairs of GOES magnetometers, as well as Olson and Rostoker (1978) using ground magnetometers, suggesting that the observed ULF waves are launched by the dayside magnetopause compression and propagate anti-sunward. The contribution of the high-mode ULF waves (solid red curve in Figure 3-4 (d)) stays dominant after the first shock, which, interestingly, shows a clear drop as the wave power increases during the second shock (around time b1), with a concurrent increase of the fraction at low modes (solid blue curve) from times b1 to b2. Then the high-mode ULF waves become dominant again after the impact of the second shock. These distinct ULF wave mode structures during and after the shock revealed in our results will be further discussed in Section 5.

3.4.3. Frequency Spectrum Results

It is also of interest to investigate the frequency spectrum of ULF waves during the shock event since earlier studies have shown that in general lower-frequency wave power is concentrated in lower mode numbers while power at higher frequencies is distributed over a wider range of mode numbers (Olson & Rostoker, 1978). In Figure 3-6 the integrated wave power over the entire event from 17:40 UT to 20:00 UT is plotted in color versus mode number and frequency. The results

Figure 3-6. Integrated wave power over the entire event from 17:40 UT to 20:00 UT vs. mode number and frequency.
indeed show that the ULF wave power is more prominent at lower frequencies and the mode number spreads over a wider range at higher frequencies than at lower frequencies.

To better illustrate the change of ULF wave frequency spectrum during the two shocks, we look at the frequency spectrum of wave power integrated over 12 seconds, a much shorter interval than that of Figure 3-6 results. Analysis at this high time resolution, owing to the high-time-resolution field measurements by MMS, is critical to studying the abrupt variations in the ULF wave power and mode structure during shock events. Six 12-second time windows are selected to investigate the distinct patterns of the ULF wave frequency spectrum before, during, and after the shock impact for both shocks during the event, with a0 centered at 17:40:18 UT before the first shock, b0 centered at 18:45:06 UT before the second shock, and a1, a2, b1, b2 centered at the same times as marked in Figures 3-3 and 3-4. Results in Figure 3-7 demonstrate that, at both shock impacts (panels (a1) and (b1)), the total measured wave power is higher compared with the times right before (panels (a0) and (b0)) and after (panels (a2) and (b2)) the shocks. The contribution of higher frequency waves to the power (e.g., at > 7 mHz, corresponding to Pc-4 compressional pulsations as depicted in Figure 3-7), which is negligible before the shocks (panels (a0) and (b0)), becomes significant during the shocks (panels (a1) and (b1)). Right after the shocks impacted (panels (a2) and (b2)), the high-frequency power quickly drops down again and the wave power is mainly distributed over lower frequencies (< 7 mHz corresponding to Pc-5 compressional pulsations).
Figure 3-7. Patterns of ULF wave frequency spectrum over six selected 12-second time windows before (a0 and b0), during (a1 and b1), and after (a2 and b2) the impact of two shocks during the event. A subplot of solar wind dynamic pressure $P_{\text{dyn}}$ vs. time is embedded in each panel with a magenta line denoting the time around which the pattern was observed.
3.5. Discussion and Conclusions

A cross-spectral analysis using the XWT method has been performed on the high-resolution magnetic field measurements from the multi-probe MMS mission to estimate the azimuthal mode structure and frequency spectrum of ULF waves during a double-shock event. Our results based on the MMS 4 and MMS 3 pair of measurements show clear peaks in the compressional ULF wave power at the arrival of both shocks, with stronger wave power during the first shock than the second shock probably due to the fact that MMS probes are at lower L regions with stronger background magnetic field during the second shock. Strong power and phase coherence are found between the field measurements from the two MMS probes, owing to their close separation. The mode structure results from the XWT analysis show distinct contributions of low vs. high azimuthal modes in the power of ULF waves during and after the shock impact. For both shocks, the lower modes ($|m| \leq 3$) are shown to contribute dominantly to the wave power at the shock impacts, while the higher mode waves ($|m| \geq 7$) become dominant after the shock impact. Detailed mode distribution of ULF waves reveals the development of high mode structure after the impact of the first shock, first showing a bump-on-tail distribution at $m \approx 20$, then evolving to a dominant peak at $m \approx 50$ in about 10 minutes, with the power at lower modes becoming less and less prominent. In addition, an interesting change of sign in $m$ from negative to positive is observed as MMS crosses ~11 MLT pre-noon, which is consistent with the picture of wave generation by dayside magnetopause compression and then propagating anti-sunward. Finally, the observed frequency spectrum of ULF waves during the shock event shows that generally lower-frequency wave power is concentrated in lower mode numbers while power at higher frequencies is distributed over a wider range of mode numbers. Detailed analysis of the wave frequency spectrum before, during, and after the shock impact shows that the contribution of higher frequency waves (Pc-4 range compared with Pc-5) is negligible before and after the shock impact, but it becomes more significant during the shock impact for both shocks.

Generally, the effects of interplanetary shocks on the ULF wave mode structure and frequency spectrum can be understood as follows. Upon shock arrival, large-scale fluctuations in the compressional ULF waves are generated which correspond to lower-mode ULF waves over both Pc-5 and Pc-4 frequency ranges as seen in our results. This is consistent with previous studies using ground-based magnetometers by Araki et al. (1997), in-situ electric field measurements from
CRRES (Combined Release and Radiation Effects Satellite) (Brautigam et al., 2005), and more recently Van Allen Probes measurements (see review by Hudson et al. (2020)). Excitation of low-mode ULF waves and their relationship with high dynamic pressure have also been reported previously (see Barani et al. (2019) and references). Following the shock impact, smaller-scale fluctuations are generated which correspond to the dominant higher mode numbers seen in Figures 3-4 and 3-5 after the shocks. The high-$m$ ULF waves at $m\sim50$ were also reported by Eriksson et al. (2006) and Zong et al. (2009), but the mechanisms for the excitation of high-$m$ ULF waves after the impact of the shock are still not completely understood (Wang et al., 2015). One suggested mechanism for exciting $m > 40$ ULF waves is due to the drift bounce resonance with ring current ions (Chen & Hasegawa, 1988; Ozeke & Mann, 2001; Southwood & Kivelson, 1982); however this mechanism only accounts for westward propagating ($m < 0$) modes. It has also been reported that sudden changes in solar wind dynamic pressure accompanying interplanetary shocks can cause counter-rotating vortices in the magnetosphere (Shen et al., 2018; Shi et al., 2014) following the qualitative model introduced by Sibeck (1990). This mechanism could also potentially lead to small-scale and high-mode number ULF waves after the shock impact.

The high-resolution multi-probe MMS measurements of ULF waves are ideally suited to resolve the evolution of wave mode structure and frequency spectrum during interplanetary shocks. The first important advantage is its high time resolution, with a data cadence of 0.125 seconds, which has enabled the estimate of ULF wave mode numbers and frequency spectrum using the XWT method at a very high time resolution (i.e., 12 seconds as in the results in Figures 3-5 and 3-7). This is a significant advance compared to previous studies, e.g., with a time resolution of 3.5 minutes in Le et al. (2017) and of 30 minutes in Murphy et al. (2018) in resolved mode numbers. Being able to resolve the response of ULF waves at a high time resolution is of critical importance to events with sudden changes in its external drivers such as during interplanetary shocks in this study. This capability will enable better understanding of the potential mechanisms and drivers of the ULF wave pulsations at different mode numbers, low vs. high.

The second key advantage of using MMS data in the analysis is the very close separation between the MMS probes (much less than one degree during the event studied). The maximum resolved mode number is high enough ($|m_{max}| > 400$) for the XWT analysis to include almost all the existing modes of ULF waves, i.e., almost all of the integrated wave power over the frequency
range studied, eliminating the $2n\pi$ ambiguity in the mode number estimation which is an important uncertainty in previous work. The capability of resolving high mode number of ULF waves by applying the XWT method to closely separated MMS probes is also an advantage over other methods of estimating mode structure using MHD models and ground-based magnetometer data. In Appendix A1, we have derived the threshold mode value, $m_{th}$, above which ideal-MHD models are not capable of resolving. Our results in equation (A1-2) show that $m_{th}$ is proportional to $(L\mu)^{-\frac{1}{2}}$ where $\mu$ is the relativistic first adiabatic invariant of ions. The proportionality factor for dipole magnetic field approximation is ~1545 if $\mu$ is in $\text{MeV Gauss}^{-1}$ unit. For ions with a typical maximum energy value of 100 keV in the outer radiation belt (which corresponds to $\mu \sim 110.3$ and $\mu \sim 21 \text{MeV Gauss}^{-1}$ at $L = 7$ and $L = 4$, respectively), $m_{th}$ is around 56 and 170, respectively. The results in Figure 3-5 demonstrates that our analysis using the XWT method could resolve mode numbers well above the $m_{th}$ from ideal MHD simulations which are limited to scales greater than the largest ion gyro radius in the system corresponding to the highest typical energy of ions which is assumed to be 100 keV in this work. It is also well above the upper limit of $m$ that can be resolved by ground magnetometers since modes above 40 cannot typically reach Earth’s surface due to the ionospheric screening effect when the transvers scale length of a perturbation is comparable to the ionospheric scale height (Chisham & Mann, 1999).

Furthermore, the four MMS probes that are closely separated have provided unprecedented opportunity to verify the resolved mode structure among different pairs of measurements. For example, in Figure 3-8 we show the mode fraction results using two different pairs of MMS, with the MMS 4 – MMS 3 pair in panel (a) which is identical to Figure 3-4(d) and the MMS 1 – MMS 3 pair in panel (b). Excellent consistency in the resolved mode structure is found across the two different pairs, including the low vs. high modes during and after the shocks and the change of sign in $m$ observed pre-noon, which demonstrates the reliability of the mode results. Good consistency is also confirmed with other MMS pairs even though only two pairs are shown in Figure 3-8. This is an advance over some previous work (Le et al., 2017; Murphy et al., 2018) in which estimates using different MMS pairs give different mode number results and an interpolation (or best fit) is needed to achieve a final estimate of $m$. The multiple MMS pairs have also enabled a quantitative error analysis of the resolved mode number, which is described in more
Finally, the XWT analysis performed in this work resolves the distribution of ULF wave power among different active modes at each time, while in many previous works (e.g., Le et al. (2017); Murphy et al. (2018); Loto'aniu et al. (2006); Chisham and Mann (1999)) only a single dominant mode is reported during just a few considered time windows of the studied events. This is important because in reality more than one mode of ULF waves could be generated at the same time.

Figure 3-8. Mode fraction results using the MMS 4 – MMS 3 pair (panel (a), identical to Figure 3-4(d)) and the MMS 1 – MMS 3 pair (panel (b)). The figure format and line styles are the same as in Figure 3-4(d).
Chapter 4. Summary, Conclusion, and Future Work

4.1. GOES Study

In this work, we quantified the azimuthal mode number of compressional ULF magnetic pulsations using in-situ magnetometer data. The magnetic field data from multiple probes of two missions was used. The results from multiple geostationary probes of GOES spacecraft were reported in Chapter 2. The GOES study was chosen to show the dependence of the mode distribution on a few main geomagnetic and solar wind indices such as AE and $P_{\text{dyn}}$. The 28-31 May 2010 geomagnetic storm was selected for the GOES study since five GOES satellites were available during this time, which provide a wide range of local time coverage at the same time. The opportunity to study the ULF wave mode behavior using multiple pairs paved the way for the cross-pair analysis introduced in Chapter 2. In addition, the storm includes distinct intervals with either high $P_{\text{dyn}}$ accompanied with low AE activity on 28 May or the opposite on the next day with low $P_{\text{dyn}}$ and high AE. These distinct intervals allowed us to examine the effects of either high $P_{\text{dyn}}$ (while AE-related effects are very low) on the mode number structures as well as the opposite scenario of high AE while $P_{\text{dyn}}$ is not effective to make any correlation with the ULF wave mode structures. The spread of the five GOES probes over local time also allowed the study of distinct mode structure over night versus noon MLTs. Another motivation for selecting this event for the GOES study was the detection of the concurrent enhancement of MeV electron flux. It was shown that the flux of $>0.6$ MeV electrons measured by GOES 13 has a fast drop during the early main phase of the studied storm followed by a significant increase near the end of the main phase.

The GOES study well resolved the low-versus-high mode number behaviors. The dominancy of ULF wave power over low mode numbers (e.g., $|m| = 1$ and $m = 0$) is well demonstrated during the storm sudden commencement, when the solar wind $P_{\text{dyn}}$ is high while AE is low, which is consistent with previous studies (Sarris & Li, 2017; Tu et al., 2012). That suggests a more global scale magnetic fluctuations is driven by dayside compression of the magnetosphere compared to the non-storm and the low $P_{\text{dyn}}$ times. The possibility of Kelvin-Helmholtz Instability (KHI) being the driver of these low-mode ULF fluctuations was also examined. As KHI is correlated with high-speed solar wind, it is expected to see that the power in low modes decreases when the solar wind speed decreases (Mann et al., 1999) while the opposite happened: solar wind speed remains
elevated when the $P_{dyn}$ drops, thus it is not well-correlated with the low-mode wave structure. This reduces the likelihood of KHI -which happens around flank regions- as the main mechanism for these low-mode ULF waves during this event. Therefore, during this event the low-mode ULF wave activity was most likely produced by mechanisms depicted in Figure 1-4(a) of Chapter 1 rather than Figure 1-4(b).

Another outcome of the GOES event study was resolving the sign in the estimated mode numbers. Many previous works did not take the sign of $m$ into account (Li et al., 2016; Li et al., 2017; Oliifer et al., 2019; Sarris, 2014; Sarris & Li, 2017; Sarris et al., 2013; Sarris et al., 2009). We showed that not considering the sign of $m$ does not give a realistic pattern of ULF waves due to solar wind interactions with Earth’s magnetosphere; while considering both signs resolves the anti-sunward pattern of the wave propagation which is in agreement with what we understand as the wave generation by the solar wind buffeting Earth’s dayside magnetosphere acting as a driver (Hughes, 1994; Olson & Rostoker, 1978). Specifically during the high $P_{dyn}$ period, an interesting change of sign in $m$ occurs in the slightly pre-noon region (around 9.5-10.5h MLT) and it was confirmed that this sign change of $m$ is spatial rather than temporal, with eastward propagation in the noon to dusk sector and westward propagation in the noon to dawn sector. One possibility for this pre-noon deflection is the shift of the solar wind flow to prenoon by the bow shock effect and the effect of aberration due to Earth’s orbital motion. The mechanism was suggested by Walters (1964) and observed by Olson and Rostoker (1978). Using plasma shock relations, Walters (1964) estimated this deflection angle to be between 10° and 25° westward from the subsolar point which corresponds to 11.33h and 10.33h MLT, respectively. However, the resolved deflection angle of 30° for the pair GOES13-12 (or even larger for other pairs at slightly different times) in this work is larger than the theoretically predicted values. Therefore, the estimated deflection cannot be fully explained by the described deflection mechanism and further investigations are required to understand the underlying mechanisms.

Furthermore, using the exceptionally wide MLT coverage of the multiple pairs of GOES satellites at the same time, it was found that low-mode ULF waves are limited to regions away from midnight and may not be as global as previously assumed (Sarris & Li, 2017; Tu et al., 2012). This
limited local time extent of the low-mode structure needs to be properly considered in the calculation of $D_{LL}$ for energetic particles.

During the storm main phase as well as early recovery phase when AE was high and $P_{dyn}$ was low (opposite to the commencement period of the storm), the GOES study showed that all mode numbers contribute to the power of detected ULF waves such that there is no distinct mode to carry the dominant power in magnetic fluctuations. It was demonstrated that the excitation of these small-scale ULF fluctuations can originate from nightside substorm injections of azimuthally drifting ions and electrons (James et al., 2013; Yeoman et al., 2010). However, it was demonstrated in this work that the detection of these small-scale structures is not limited to the nightside and these structures can spread all the way to the dayside, and therefore the suggested mechanism needs further investigation. Possible mechanisms for high-mode ULF waves near the dayside region include a wave-particle source mechanism involving drift-mirror instabilities (Walker et al., 1982; Yeoman et al., 1992) and a dispersive waveguide model with the high-$m$ waves propagating both sunward and anti-sunward from the sources at the flank regions (Fenrich et al., 1995). More detailed wave and particle analysis is needed in the future to resolve the driving mechanism during this event. This more global coverage of the evenly distributed mode structure up to high modes also needs to be considered realistically in $D_{LL}$ calculations.

The availability of magnetometer data from five GOES satellites for the mentioned event study for the 28-31 May 2010 geomagnetic storm allowed us to check consistency of the results among different pairs. During the high $P_{dyn}$ day, the mode results show very good agreement between different pairs: all pairs show dominant power at low mode numbers with a prenoon excitation region as discussed above. However, during the high AE times, the agreement was not found among the same pairs. We believe this is related to the high mode structure of the pulsations: higher $m$ during this time means smaller spatial scales (correspondingly larger XPhase values, following the definition). This means, over the course of azimuthal changes, the amplitude of these ULF waves fluctuates more in contrast to the low $m$ waves. Therefore, for a given pair of probes with fixed separation, the likelihood of having phase wrapping between the two spacecraft is higher. This means that the $2n\pi$ ambiguity is very likely to play a role during these high-$m$ events, implying that the separation of the probes was not small enough to capture the more-than-one-
period signatures of the waves that occurred in between the two spacecraft. To reduce the $2n\pi$ ambiguity in the resolved phase difference from the XWT analysis, a cross-pair analysis is introduced in this work and is performed here on field measurements for the first time over two pairs of GOES satellites that overlap azimuthally. It is suggested that this cross-pair analysis is necessary for the high AE intervals when the resolved mode numbers are largely inconsistent and when there is no other means to resolve the inconsistency, such as using other pairs (or other missions) with smaller azimuthal separation. It was shown that after comparing and reconciling the mode number results from the two pairs of spacecraft using the cross-pair analysis introduced in Chapter 2, the mode structures from the two pairs show significantly improved agreement. The results demonstrate that the analysis is efficient in reducing the $2n\pi$ ambiguity and generating reliable mode structure during the high AE interval when the mode numbers are high. The main points of this scheme are: (1) making sure that the two pairs involved in the comparing and reconciling process are overlapping pairs, which means both are looking at the same region, so the same physics is involved that makes the assumption of having the same mode structure within that region legitimate. This means that the effectiveness and reliability of this suggested cross-pair scheme will be reduced when analyzing pairs that have simultaneously less/no overlap in azimuth angles. (2) having two overlapping pairs looking at the same mode behavior allows estimating the level of ambiguity, corresponding to $n$ in equations (1-5) and (1-6), for each pair such that both pairs give the same or closest mode value $m_{\text{real}}$. In the mentioned event study, only a maximum of one full phase period was added to both directions of the phase difference (i.e., $0, \pm 2\pi$ for $XPhase$ values, corresponding to $n = 0, \pm 1$ in equation (1-5)) since our results showed that this is sufficient to reach a good agreement between the pairs and to achieve accurate distributions of the mode numbers (e.g., the histogram figures in Chapter 2). This could be due to the fact that the contribution of higher mode numbers ($m \geq 25$ in the histograms) was insignificant for this period. Even though better agreement is reached, the mode structures are still not identical between the pairs. The suggested scheme in general is applicable to more than two overlapping pairs.

Although similar cross-pair analysis has been performed on ground-based ULF wave measurements in previous work (Baker, 2003; Chisham & Mann, 1999), there are differences between our approaches. For instance, Chisham and Mann (1999) used three ground magnetometer pairs, two of which are not overlapping, while in this work we only used the overlapping pairs.
(GOES13-12 and GOES 15-12) to better ensure that they are measuring the same ULF waves. Additionally, in Chisham and Mann (1999) the sign of \( m \) is resolved by analyzing the arrival time of the waveform, while in our approach we let the sign of \( m \) be part of the comparison scheme and chose \( m \) with the minimum difference between two pairs to be the realistic one \( m_{\text{real}} \). Last but not least, our cross-pair analysis is performed on ULF waves measured in space while the analysis in Chisham and Mann (1999) is performed on ground magnetometers which makes estimating \( m > 40 \) nominally impossible due to ionospheric screening.

4.2. MMS Study

In Chapter 3, the 09 March 2018 double ICME shock event was chosen for a detailed mode number analysis during and after the shocks using high-fidelity MMS magnetometer data. This mission was chosen because of its unprecedented time resolution, ability for testing/checking the results by using magnetometer pairs from each of the four spacecraft (each pair gives a complete set of mode numbers and the MMS mission offers 6 pairs), confidence in not having the \( 2n\pi \) ambiguity (Barani et al., 2019) with access to measurements of sufficiently high mode numbers due to the close separations of each pair as described in Chapter 2, and the access to a broad range of \( L \). Two threshold setting criteria were addressed in Chapter 3: (1) the purpose of power thresholding is to discard the time-frequency bins with low-common powers that might have the origin of noise or other persistent instrument-related phenomena which are not the target of the study. (2) Phase coherent thresholding to keep only the time-frequency bins with stable \( XP \) values in time. Both thresholds are required and must be implemented to assure that the analysis is neither from unexpected phenomena such as noise nor from data with incoherent sources. Attention must be paid to the fact that it is not always the case that an unstable relative phase comes from the low-common power in the two signals. For example, if at some time domain the power in one signal is high while it is not high in the other signal of a pair, such that the common power or \( XPSD \) (which can be interpreted as the multiplication of the powers in Fourier space) would not lie below the set threshold, then the power thresholding would not be enough to exclude the interval with poor phase coherence.

It was discussed that different time windows involved in different spacecraft data processing such as smoothing, coordinate transformation, and detrending that would leave a frequency signature
in the spectrum of the signal should not interfere with the intended study frequency band. Therefore, we introduced inequality relations that must be satisfied when considering frequency content in the signals subject to spectral analysis. For example, it was demonstrated that if it takes around half an hour for MMS to pass through the geosynchronous region, \((L - 0.5, L + 0.5)\) with \(L = 6.6\), and if we intend to study and describe the behavior of \(m\) at a given radial distance from Earth, \(\frac{1}{t_{\text{pass}}}\) which gives 0.52 mHz should be the number used as the lower limit of the target frequency bands to study. That specific value, which was the case in our MMS event study, allows capturing the signatures in the investigation of PC-5 and PC-4 bands of ULF waves and their relations with different \(L\) values. On the contrary, if the detrending window used in a study is half an hour (corresponding to \(\frac{1}{w_{\text{detrend}}}\) of \(\sim 0.56\) mHz), while the spacecraft passage time through each specific \(L\) range is less than half an hour, investigation of resolving some of the physical behaviors that depend upon radial distance would not be allowable.

As mentioned in the first chapter, our scheme of resolving mode number of the ULF waves allows us to resolve the power at mode numbers much higher than \(|m| = 1\). To address the question of how to assure that most or the entire power of the ULF pulsations is resolved through this scheme, the next task is to determine the maximum \(m\), or \(m_{\text{max}}\), in the analysis that is always above any active \(m\) in the system with the total power above the power level threshold at any time. To address this question the power relation \(\sum_{m=-m_{\text{max}}}^{m_{\text{max}}} p(t,m) \leq P_{\text{tot}}(t)\) was introduced. To resolve the entire or most of the ULF wave power, \(m_{\text{max}}\) must be chosen such that this relation holds a value close to 1 at every time during the entire analysis. \(P(t,m)\) is the cross power at time \(t\) and mode \(m\). \(P_{\text{tot}}(t)\) is the total cross power at time \(t\) regardless of the modes that were active during that time. It was demonstrated that for the mentioned event study, \(|m_{\text{max}}| = 400\) is an excellent choice for giving at least the suggested relation close to or larger than 0.99 for the entire time of the analysis. However, the choice of \(|m_{\text{max}}|\) is an event-based choice and depends on many factors of a specific geomagnetic event subject to study such as severity of the event as well as the power and coherence threshold values used and the frequency interval of the studied waves. Typically, for higher frequency intervals and for lower threshold setting, to get the same close-to-one value for the aforementioned power relation, \(|m_{\text{max}}|\) has to increase.
The high-fidelity of the magnetic field data analyzed from MMS has paved the way to investigate with detail the dynamics of the mode distribution. It was demonstrated that contribution of the low versus high modes in power of the pulsations is well resolved: Lower modes ($|m| \leq 6$) are contributing dominantly to the power of pulsations during the first shock, while immediately after that, higher modes ($|m| > 6$) constitute the dominant power contribution. This observation clearly shows that shock onset corresponds to more in-phase magnetic fluctuations in the Pc-5 and Pc-4 regimes than what follows. Lower-mode-number pulsations are produced upon shock arrival, which can be thought of as generating a large scale response seen in ground-based magnetometer studies (Araki et al., 1997) as well as in-situ electric field measurements dating back to Combined Release and Radiation Effects Satellite (CRRES) (Wygant et al., 1994) and more recently from Van Allen Probes (see review by Hudson et al. (2020)). Following the shock impacts, smaller spatial scale fluctuations are implied by the dominant high mode numbers observed after both shocks hit and passed the magnetosphere. A vivid switch of sign in $m$ from negative to positive was observed as the MMS spacecraft cross 10.97 MLT towards noon. This interesting pre-noon sign change is also consistent with the observed change of sign in the mode analysis by Barani et al. (2019) also discussed in Chapter 2 as well as earlier in this chapter. The evolving high mode signatures right after passage of the first shock was discussed. To resolve the dynamics of the mode structure and correlation with shock signatures in the solar wind $P_{dyn}$, a study of ULF fluctuating power over modes and frequencies was carried out during small intervals of 12 seconds. This time interval is adequately small to capture the significance of shock impacts and evolution of the power spectrum in frequency and mode over sequential times much smaller than the total time interval. Distinguishable patterns of frequency distribution during the shock impact versus right after the shock impact are observed for both reported shocks during this event: At both shock impacts, the total measured power is higher compared with the times right before and after the shocks, and the contribution of higher frequencies (e.g. $> 7$ mHz, corresponding to Pc-4 compressional pulsations) to the power is not negligible, while right after the shocks impacted, the power distributes significantly over lower frequencies (e.g. $< 7$ mHz, corresponding to Pc-5 compressional pulsations), making contributions of higher frequencies to the power spectral density negligible or significantly less prominent. Therefore, the slope of the power versus frequency is steeper after
the shocks have impacted the region than right upon impact when higher frequencies are excited, leaving a very low contribution for PC-4 magnetic pulsations of ~7 to ~22 mHz.

The mechanism by which shock compression of the dayside magnetopause can excite high-\(m\) ULF waves is not completely understood (Wang et al., 2015). Mechanisms which have been suggested for exciting \(m > 40\) ULF waves include drift bounce resonance with ring current ions (Chen & Hasegawa, 1988; Ozeke & Mann, 2001; Southwood & Kivelson, 1982), however this mechanism only accounts for westward propagating \(m < 0\) modes since ions drift westward contrary to the eastward drifting nature of electrons around Earth. Excitation of low-mode ULF waves and their relationship with high dynamic pressure have been reported previously (see Barani et al. (2019) and references), and high-\(m\) wave detection around \(m \sim 50\) were also reported by Eriksson et al. (2006) and Zong et al. (2009). The high-mode number ULF wave signatures described here focus on the local picture of the dynamics of the mode number rather than the global picture, therefore can offer a complementary view improving our previous understanding of the larger scale detection of ULF wave behaviors through analytical studies (such as the qualitative model introduced by Sibeck (1990)) or through measurements of the wave behavior through missions with pairs of larger separations such as the original five (now three) THEMIS spacecraft (Angelopoulos, 2008), studying the magnetospheric response to sudden changes in solar wind dynamic pressure accompanying interplanetary shocks impacting the magnetopause (Shen et al., 2018; Shi et al., 2014).

The key advantages of the mode analyses tools and computational packages developed for this study are: (1) Time resolution: In this work, due to the novel scheme used in the analysis, a high data cadence of 0.125 seconds, and the performed XWT analysis, the time resolution in the resolved mode structure allows revealing signatures of prompt transition in the mode behavior during sudden changes such as interplanetary shock impacts. The nature of XWT in limiting the time-versus-frequency resolution -as briefly described in the first chapter- still holds and limits the time resolution specifically for the lower edge of the frequency bands subject to study. The 12-second integration time performed for the integrated \(XPower\) versus mode number can be interpreted as the resolution time for this study. The best time resolution of the azimuthal mode number resolved using MMS in recent work is \(~3.5\) minutes in Le et al. (2017) and 30 minutes in Murphy et al. (2018) following the chosen time window in these work. (2) Reproducibility of the
results among different pair measurements: The level of agreement depends on the separation of the probes of a pair as addressed in the error analysis. In the previous mentioned works using MMS (Le et al., 2017; Murphy et al., 2018), one has to perform an interpolation (best fit) of the results from different pairs since consistent mode results were not achieved, while in this work it is demonstrated that the outcome of the analysis from different pairs is so close and consistent that it makes the use of the best-fit procedures quite unnecessary. This comparison of results from different probe pairs shows the effectiveness of our method in resolving accurate mode structure.

(3) Resolving multiple active modes simultaneously. At every instant of time, the distribution of power among all active modes is well resolved, while in much of the previous work, only a single \( m \) is reported during the considered time window. This work shows that assuming a single mode to be the mode in which the power is embedded is not a reliable estimation no matter the width of the considered averaging time window.

In this work we introduced the threshold mode \( m_{th} \), larger than which ideal MHD becomes invalid and in Appendix A.1 it is shown that this threshold mode is proportional to \( (L \mu)^{-\frac{1}{2}} \) where \( \mu \) is the relativistic first adiabatic invariant for the ions. The proportionality is given by \[ \frac{\pi R_E c m_{th}^{1/2}}{5 \times 10^6 \left[ \frac{m_{th} c^2}{2} \right]^{1/2}} \] where for simplicity the dimensions for \( m_{0,i} c^2 \) and \( \mu \) are \( MeV \) and \( \frac{MeV}{T} \) respectively while everything else is in SI units. Here the dipole magnetic field of \( \frac{B_E}{L} \) is considered in which \( B_E \) is the magnetic field at Earth’s surface at the equator. This threshold value gives the approximate range of the largest resolvable \( m \) for ULF estimations using ideal MHD models. For the first adiabatic invariant of around \( 110.29 \times 10^4 \frac{MeV}{T} \) (or \( 110.29 \frac{MeV}{Gauss} \) corresponding to 100 keV ions which is considered in this work to be a typical maximum value for ion energy in outer radiation belt), the threshold mode value at \( L = 7 \) is around 56. Therefore, to resolve the pulsation signatures with larger \( m \) than the mentioned threshold value, resorting to more sophisticated (such as kinetic, multi-fluid, or hybrid) models is inevitable. As \( m_{th} \) can be truly interpreted as the border between the kinetic and MHD models, it becomes obvious that ground-based magnetometers are not capable of fully resolving those features of the ULF waves which require kinetic description since it is well accepted that pulsations with mode values above \( m \approx 40 \) barely reach Earth’s surface due to the ionospheric screening effect (Chisham & Mann, 1999).
4.3. Significance of Mode Number in Radial Diffusion Coefficient (Analytical Estimation)

To address the key question in the long-lasting debate about how higher ULF wave azimuthal mode number (e.g., $|m| > 1$) would affect and contribute to the radial diffusion rates ($D_{\perp \perp}$) of energetic electrons, here for the first time a first-principles calculation is given:

In the most general realistic case of a system whose Power Spectral Densities (PSD) are distributed over different frequencies and azimuthal mode numbers (negative and zero values also allowable), the total PSD that contributes to the radial diffusion (equation 1-9) is the power spectral density at all different harmonics of the drift frequency, while the number of harmonic and mode number must be equal, which is a subgroup of the following PSD:

$$PSD (all\ modes, all\ harmonics) = \sum_{m=-m_{\text{max}}}^{m_{\text{max}}} \sum_{n=1}^{N} PSD (m, n\omega_d)$$  \hspace{1cm} 4-1$$

Here $n$ is the number of harmonics of drift frequency and $N$ is a large number such that for frequencies of $\omega > N\omega_d$ there is no remaining harmonic power left, i.e., $\lim_{n \to N} PSD(m, n\omega_d) = 0$.

In Appendix A.3, starting from equation 1-9, we show that in the most general case (with no restriction on mode number values) the magnetic field contribution to the radial diffusion coefficient is:

$$D_{\perp \perp}^B = \alpha (\mu, L) \left\{ \frac{PSD (all\ modes, all\ harmonics)}{\sum_{n=1}^{N} \frac{1}{n^k}} - \sum_{m=-m_{\text{max}}}^{0} PSD (m, \omega_d) \right. \left. - \sum_{m=\text{min}(2,m_{\text{max}})}^{m_{\text{max}}} \left(1 - \frac{1}{m^{k-2}}\right) PSD (m, \omega_d) \right\}$$  \hspace{1cm} 4-2$$

The second term on the RHS of the above expression contains the negative mode and in-phase ($m = 0$) contributions in ULF waves and the third term is the contribution of the modes beyond $m = +1$. In the simplified case of only $m = +1$, we have only the first term in the above expression:
\[
D_{LL}^{B,m=+1} = \alpha(\mu, L) \left\{ \frac{PSD \ (all \ modes, \ all \ harmonics)}{\sum_{n=1}^{N} \frac{1}{n^k}} \right\} 
\]

Therefore, to address the key question and calculate the difference between the realistic case versus the simplified case of only \( m = +1 \), all that is needed is to subtract the previous two equations,

\[
D_{LL}^{B,m=+1} - D_{LL}^{B} = \alpha(\mu, L) \left\{ \sum_{m=-m_{max}}^{0} PSD \ (m, \omega_d) + \sum_{m=\min (2, m_{max})}^{m_{max}} (1 - \frac{1}{m^{k-2}}) PSD \ (m, \omega_d) \right\} 
\]

It is interesting to note that since for all dimensionless \( k > 1 \) values, as we saw that the higher frequencies contribute less to the total power of ULF wave magnetic pulsations, the above expression is always positive: \( D_{LL}^{B,m=+1} - D_{LL}^{B} > 0 \). In our MMS event study discussed in detail in Chapter 3, \( k \) was a number between 3 and 4. It was 3.97 for frequencies around 2 to 3 mHz and lower for higher frequency ranges. It is important to note that, based on this expression and the realistic assumption of having typically less power at higher frequencies, the simplified case of only \( m = +1 \) would overestimate the radial diffusion of electrons in the radiation belt. Although this simplified case is unrealistic, lacking any information about the direction of pulsation propagation as well as the in-phase \( (m = 0) \) waves and higher \( m \) contribution, \( m = +1 \) is typically assumed (rather than a distribution of the \( m \) modes) in most radiation belt models to estimate the radial diffusion coefficient \( D_{LL} \) of electrons as discussed previously.

To show the significance and the importance of higher (than \( m = +1 \)) as well as the negative mode values, here two scenarios are compared following the above derived equation and the realistic dimensionless value of \( k = 3 \) based on the MMS event study: Scenario 1. The more realistic situation that allows ULF wave magnetic pulsations to have power in the modes up to \( |m_{max}| = 2 \), meaning that the active modes can be \(-2, -1, 0, +1, +2\). Scenario 2. All power in the ULF wave magnetic pulsations comes from only the \( m = +1 \) mode. The amount of overestimation following the above relation is the aforementioned \( D_{LL}^{B,m=+1} - D_{LL}^{B} \) relation while setting \( m_{max} = 2 \). Following this calculation, \( D_{LL}^{B,m=+1} = 4 D_{LL}^{B} \) which gives relative overestimation of 300% which is significant. Since all of the terms on the RHS of equation 4-4 are positive, the overestimation would be even larger if ULF wave magnetic pulsations have
considerable power in higher than $m_{max} = 2$. This calculation is specifically important as it was shown in Chapter 3 through the MMS study that, to capture the full power in all ULF wave modes, having $m_{max} = 200$ is typical (Figure 3-6) for a moderate shock event. Therefore, inclusion of the negative mode values as well as higher (than $m = +1$) modes is both important and should be considered in future radial diffusion calculations.

4.4. Future work

The current work is a step forward in making estimations of the electron transport in the radiation belt more realistic. However, there are still unresolved issues regarding the estimation of $D_{LL}$. The effect of limited local time evaluation of $m$ on the radial diffusion coefficient estimation is still an open question. It is expected and previously reported that the ULF waves of low mode numbers would not be global in local time (as discussed in the GOES study case of Chapter 2). In this case, electrons during their drift would experience a less persistent driving (resonant) force and therefore any transport phenomenon would be slower than the similar case of global ULF wave activity. This reduction in radial transport efficiency might be significant enough not to allow electron behavior to be diffusive, rather other types of transport might be expected specifically if the interaction time $\tau$, discussed in Chapter 1, approaches the drift period of electrons. Estimating this phenomenon requires development of models that do not currently exist since the radial diffusion formalism that has been developed, e.g., Fälthammar (1965) and Fei et al. (2006), assumes azimuthal symmetry of field fluctuations in a reduced 1-D Fokker-Planck equation of 1-8.

The focus of this study was only the magnetic field portion of the ULF wave power. To acquire a complete picture of the electron diffusion in the magnetosphere, especially the outer radiation belt, the electric field contribution to $D_{LL}$ should be examined. Although the main magnetic driver of the radial diffusion of electrons in Earth’s dipole field approximation is the compressional component, the role of the other components is not well examined either. With the electric field measurements from several missions including MMS, similar analysis to this thesis work can be applied to study the mode structure of poloidal and toroidal ULF waves. The contribution of azimuthal electric field ($E_\phi$) to $D_{LL}$ has been examined by Ali et al. (2016) using Van Allen Probes electric field measurements; Brautigam et al. (2005) using CRRES; and Liu et al. (2016) using THEMIS in situ electric field measurements. Others have modeled the electric field contribution
to $D_{L_L}$ using a standing Alfven wave model to map ground-based magnetometer measurements of magnetic field perturbations into the equatorial plane $E_\varphi$ (Ozeke et al., 2014). All of these studies have assumed $m = 1$. The availability of MMS electric field measurements (Torbert et al., 2014) for analysis similar to that performed here for magnetometer measurements is an outstanding future project.

The analysis presented in this thesis work also needs to be applied to more case studies with different drivers of solar wind and geomagnetic conditions. In addition to multiple GOES and MMS satellites, the analysis can also be conducted for other space missions including NASA Van Allen Probes, CLUSTER and THEMIS missions to resolve the mode structure at a wider range of $L$ shells and local times at higher and negative $m$ values than previous studies. With a good database of the ULF mode numbers at different solar wind and geomagnetic conditions including both magnetic and electric contributions, we can construct statistical maps of mode structures under different driving conditions, which will be directly useful in quantifying the radial diffusion process in Earth’s inner magnetosphere and in resolving its contribution to the overall dynamics of energetic particles.
A. Appendix

A.1. Estimation of the Threshold Mode $m_{th}$ in Ideal-MHD Simulations

To estimate the threshold mode value $m_{th}$ over which ideal-MHD models are not capable of resolving, first we calculate the linear wavelength $\lambda$ associated with the mode $m$. As the circumference of drift shell at $L$ is $2\pi LR_E$ which equals $m\lambda$, the linear wavelength is $\lambda = \frac{2\pi LR_E}{m}$.

In ideal MHD models, this value should be considerably greater than the gyro radius of the magnetized fluid through which the pulsations are happening. Therefore, we set the condition for validity of ideal MHD to be $\lambda > 10 \rho_i$ in which $\rho_i$ is the ion Larmor radius and can be calculated as:

$$\rho_i = \left[ \frac{2 \mu m_{0,i} c^2}{q_i B^{1/2} c} \right]^{1/2} \tag{A.1-1}$$

where $m_{0,i}$ and $q_i$ are mass and charge of the ions, respectively, $B$ is the magnetic field and $c$ is the speed of light.

Setting the condition $\frac{\lambda}{\rho_i} > 10$ and simplifying the inequality give:

$$m < \frac{\pi LR_E c B^{1/2}}{5 \times 10^6 \left[ 2 \mu m_{0,i} c^2 \right]^{1/2}} = m_{th} \tag{A.1-2}$$

where for simplicity the dimensions for $m_{0,i} c^2$ and $\mu$ are $MeV$ and $\frac{MeV}{T}$ respectively while everything else is in SI units. If we consider the dipole magnetic field of $B = B_E L^3$ in which $B_E$ is the magnetic field on the Earth surface at equator and assume $q_i = +e$, the threshold mode can be further simplified to $m_{th} = 154.49 \times 10^3 \frac{1}{\sqrt{\mu L}}$. 


A.2. Error Analysis

Estimating the error in the resolved $m$ is important but not trivial. Based on equation (1-1), the relative error in $m$ is defined as:

$$\frac{\delta m}{m} = \frac{\delta(XPhase)}{XPhase} + \frac{\delta(\Delta\lambda)}{\Delta\lambda}$$  \hspace{1cm} \text{(A.2-1)}

which includes two terms: the first term is the relative uncertainty in $XPhase$ and the second term is relative uncertainty in longitudinal separation. Quantifying the second term is relatively straightforward. For the 9 March 2018 event studied in this work, the separation of the MMS 4 – MMS 3 pair monotonically increased from $0.03^\circ$ to $0.14^\circ$ from 17:40 UT to 20:00 UT. Our estimates show that the relative error due to the second term of equation (A.2-1) is below 1%. As the spacecraft longitudinal separation $\Delta\lambda$ is in the denominator of the second term, it is crucial to exclude times when the two probes switch order in space, i.e., when the trailing probe becomes the leading one. When that happens, $\Delta\lambda$ approaches zero which significantly increases the mode uncertainty due to the second term. This switch did not occur during this event.

There are three different methods to either quantify the error in the $XPhase$ for the first term in equation A.2-1 or making its effect significantly small: (1) Directly calculating the error in $XPhase$ at each step in the XWT analysis, which can be challenging since XWT requires a combination of different mathematical calculations including sums, FFTs, and trigonometric computations. (2) Cross spectral error estimation as described by Green (1976), following the confidence limits for the phase estimation (Bendat & Piersol, 2010). (3) Consistency check using multi-pairs by increasing the phase coherence threshold value which is implemented in this work.

While gradually increasing the coherence threshold, we find that after a specific value the consistency in the resolved mode structure between different pairs (such as MMS 4 – MMS 3 pair and MMS 1 – MMS 3 shown in Figure 3-8) does not significantly improve. Then that specific coherence threshold value (0.98 specifically for this work) is selected for the least error. Since the consistency between multiple MMS pairs is very high as shown in our results, it is expected that the error in the calculated $m$ due to the first term in equation (A.2-1) is very low or negligible compared to that from the second term (which is below 1% as discussed above).
A.3. Contribution of High Modes in $D_{LL}$ (Analytical Estimation)

To address the key question in the long-lasting debate about how higher ULF wave azimuthal mode numbers (e.g., $|m| > 1$) affect and contribute to the radial diffusion rates $D_{LL}$ of energetic electrons, here for the first time a first-principles calculation is given:

In a most general realistic case of a system whose Power Spectral Densities (PSD) are distributed over different frequencies and different azimuthal mode numbers (negative values also allowable), the total PSD that contributes to the radial diffusion (equation 1-9) is the total power spectral density at all different harmonics of drift frequency which is a sub-group of the following PSD at all modes and all harmonics:

$$
PSD (all \ modes, all \ harmonics) = 
\sum_{n=1}^{N} PSD (-|m_{max}|, n \omega_d) + \sum_{n=1}^{N} PSD (-|m_{max}| + 1, n \omega_d) + \sum_{n=1}^{N} PSD (-|m_{max}| + 2, n \omega_d) + \cdots + \sum_{n=1}^{N} PSD (-1, n \omega_d) + \sum_{n=1}^{N} PSD (0, n \omega_d) + \sum_{n=1}^{N} PSD (1, n \omega_d) + \cdots + \sum_{n=1}^{N} PSD (m_{max}, n \omega_d)
$$

The first argument in PSD is mode number (spatial content) and the second argument is frequency harmonic (time content) of the ULF wave fluctuations. In the above expression, $n$ is the number of harmonics for drift frequency and $N$ is a big number such that for $\omega > N \omega_d$ there is no remaining harmonic power left, i.e., $\lim_{n \to N} PSD(m, n \omega_d) = 0$.

To relate the PSD values at any frequency harmonic to the PSD at the first harmonic ($n = 1$), due to the nature of the pulsations, we assume that PSD has a power-law behavior versus frequency. For example, PSD at mode $m = -3$ and harmonic $n \omega_d$ is $PSD(-3, \omega_d) = c_{-3} \frac{1}{\omega^k}$ where $k > 1$.

Following this assumption, corresponding to $m = -3$ we would have $PSD(-3,2 \omega_d) = c_{-3} \frac{1}{2^k \omega^k} = \frac{1}{2^k} PSD(-3, \omega_d)$. Therefore, the general case for the power at mode $m$ gives
\[ \text{PSD}(m, n\omega_d) = \text{PSD}(m, \omega_d) \frac{1}{n^k} \]  
A.3-2

This result allows expressing all the terms in equation A.3-1 in terms of PSD at the first frequency harmonic. For example, in the case with \(|m_{\text{max}}| = 3\), the expression 4-1 gives:

\[
\begin{align*}
\text{PSD (all modes up to 3, all harmonics)} &= [\text{PSD}(-3, \omega_d) + \text{PSD}(-2, \omega_d) + \text{PSD}(-1, \omega_d) + \text{PSD}(0, \omega_d) \\
&\quad + \text{PSD}(1, \omega_d) + \text{PSD}(2, \omega_d) + \text{PSD}(3, \omega_d)] \sum_{n=1}^{N} \frac{1}{n^k}
\end{align*}
\]  
A.3-3

Following the same logic, in the most general case, the 4-1 expression gives

\[
\text{PSD (all modes, all harmonics)} = \left( \sum_{m=-m_{\text{max}}}^{m_{\text{max}}} \text{PSD}(m, \omega_d) \right) \left( \sum_{n=1}^{N} \frac{1}{n^k} \right)
\]  
A.3-4

Having that, equation 1-9 that gives the magnetic field contribution of the radial diffusion coefficient becomes:

\[
\begin{align*}
D_{\text{LL}}^B &= \alpha(\mu, L) \sum_{m=1}^{m_{\text{max}}N} m^2 \text{PSD}(m, m\omega_d) \\
&= \alpha \left( \sum_{m=1}^{N} m^2 \text{PSD} (m, m\omega_d) + \sum_{m=N+1}^{m_{\text{max}}} m^2 \text{PSD} (m, m\omega_d) \right)
\end{align*}
\]  
A.3-5

In the general case, we set \(m_{\text{max}} \geq N\), therefore the second term on the RHS of the above relation is negligible since for \(\omega > N\omega_d\) there is no remained power above the threshold power, so

\[
D_{\text{LL}}^B = \alpha(\mu, L) \sum_{m=1}^{N} m^2 \text{PSD} (m, m\omega_d)
\]  
A.3-6

Therefore, as an example, for the case of \(|m_{\text{max}}| = 3\), we have
\[ D_{LL}^{B,m_{\text{max}}=3} = \alpha(\mu, L) \left[ \frac{I}{\text{PSD}(1, \omega_d)} + 2^2 \frac{II}{\text{PSD}(2, 2\omega_d)} + 3^2 \frac{III}{\text{PSD}(3, 3\omega_d)} \right] \]  

Each of the three terms in the above expression are

\[ I: \text{PSD}(1, \omega_d) = \frac{\text{PSD (all modes up to 3, all harmonics)}}{\sum_{n=1}^{N} \frac{1}{n^k}} \]
\[ - \left[ \text{PSD}(-3, \omega_d) + \text{PSD}(-2, \omega_d) + \text{PSD}(-1, \omega_d) + \text{PSD}(0, \omega_d) \\
+ \text{PSD}(1, \omega_d) + \text{PSD}(2, \omega_d) + \text{PSD}(3, \omega_d) \right] \]

\[ II: \frac{1}{2^k} \text{PSD}(2, \omega_d) \]

\[ III: \frac{1}{2^k} \text{PSD}(3, \omega_d) \]

Substituting \( I, II, \) and \( III \) into the diffusion relation of Equation 4-7, for the case of \( |m_{\text{max}}| = 3 \), gives:

\[ D_{LL}^{B,m_{\text{max}}=3} = \alpha(\mu, L) \left\{ \frac{\text{PSD (all modes up to 3, all harmonics)}}{\sum_{n=1}^{N} \frac{1}{n^k}} \right. \]
\[ - \sum_{m=-m_{\text{max}}}^{0} \text{PSD} (m, \omega_d) \]
\[ - \sum_{m=\text{min (2, } m_{\text{max}})}^{m_{\text{max}}} \left(1 - \frac{1}{m^{k-2}}\right) \text{PSD} (m, \omega_d) \]

Now going back to the most general case for radial diffusion with no restriction on mode number values, using Equation 4-4, we have:

\[ D_{LL}^{B} = \alpha(\mu, L) \left\{ \frac{\text{PSD (all modes, all harmonics)}}{\sum_{n=1}^{N} \frac{1}{n^k}} \right. \]
\[ - \sum_{m=-m_{\text{max}}}^{0} \text{PSD} (m, \omega_d) \]
\[ - \sum_{m=\text{min (2, } m_{\text{max}})}^{m_{\text{max}}} \left(1 - \frac{1}{m^{k-2}}\right) \text{PSD} (m, \omega_d) \]
The second term on the RHS of the above expression contains the negative mode and in-phase contributions in ULF waves and the third term is the contribution of the modes beyond \( m = +1 \) while in the simplified case of only \( m = +1 \), we have only the first term in the above expression:

\[
D_{LL}^{B,m=+1} = \alpha(\mu,L) \left\{ \frac{\text{PSD (all modes, all harmonics)}}{\sum_{n=1}^{N} \frac{1}{n^n}} \right\}
\]  

A.3-13

Therefore, to address the key question and calculate the difference between the realistic case versus the simplified case of only \( m = +1 \), all is needed is to subtract the previous two equations:

\[
D_{LL}^{B,m=+1} - D_{LL}^{B} = \alpha(\mu,L) \left\{ \sum_{m=-m_{max}}^{0} \text{PSD} (m, \omega_d) \right. \\
+ \left. \sum_{m=\min(2,m_{max})}^{m_{max}} \left(1 - \frac{1}{m^{k-2}}\right) \text{PSD} (m, \omega_d) \right\}
\]  

A.3-14

which gives a significant overestimation.

In the case study of \( m_{max} = 2 \), the calculated overestimation following the above relation is 300\%

In this estimation, a symmetric propagation of westward versus eastward is assumed which means \( \text{PSD} (m, \omega_d) = \text{PSD} (-m, \omega_d) \) for each \( m \).
References


