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Static and Dynamic Analysis of Composite Plates using the MONNA Finite Element

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Static and Dynamic Analysis of Composite Plates using the MONNA Finite Element

Mohamed Omar

Thesis submitted to the
Benjamin M. Statler College of Engineering and Mineral Resources at
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in partial fulfillment of the requirements for the degree of

Master of Science
in
Mechanical Engineering

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Abstract

Static and Dynamic Analysis of Composite Plates using the MONNA Finite Element

Mohamed Omar

Composite materials are replacing metals in many fields due to their favorable qualities such as strength-to-weight ratio, fatigue characteristics, and corrosion resistance. An efficient, versatile, and accurate composite plate element applicable to the dynamic analysis of composite plates would help the advancement of composites tremendously.

The objective of this research is to analyze composite plates using the classical laminated plate theory (CLPT), first-order deformation theory (FSDT), and higher-order deformation theory (HSDT) with the applicable finite element modeling (FEM) for each theory. This thesis investigates which theory is more efficient and accurate and evaluates the benefits of using an h - p -version conforming plate element.

A higher-order composite plate element, called MONNA, is formulated and introduced to aid with the analysis. This element has the capability to accommodate both the first-order and higher-order shear deformation theories. The performance of MONNA is compared with that of the current elements for static and dynamic analysis. There is a minor difference between the FSDT and HSDT results for many of the cases considered, although the HSDT is more accurate. Overall, the new element is concluded to be very accurate and requires much fewer elements than many of the existing popular elements. One limitation (as are the many existing plate elements) is that it is applicable only to rectangular domains.

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Contents

List of Tables	vi
-----------------------	-----------

List of Figures	viii
------------------------	-------------

Introduction	1
1.1 Problem Statement.....	1
1.2 Literature Review	2
1.3 Need for Present Research	3
1.4 Objectives.....	4
1.5 Thesis Outline	4
Structural Models	6
2.1 Introduction.....	6
2.2 Description of the Classical Plate Theory	6
2.2.1 Assumptions	6
2.2.2 Equations of Kinematics	7
2.2.3 Constitutive Equations for CLPT Composite Laminate	9
2.2.4 Hamilton's Principle Applied to the CLPT Composite Laminate	10
2.2.5 Virtual Strain Energy of CLPT Composite Laminate	10
2.2.6 Virtual Kinetic Energy of CLPT Composite Laminate.....	11
2.3 Finite Element Formulation Based on CLPT	12
2.3.1 Introduction.....	12
2.3.2 CLPT MONNA Plate Element.....	13
2.3.3 CLPT Coordinate Transformation	14
2.3.4 CLPT Shape Functions.....	18
2.3.5 MONNA CLPT Physical Domain Transformation.....	23
2.3.6 MONNA CLPT Stiffness Matrix Formulation	26
2.3.7 MONNA CLPT Inertia Matrix Formulation	27
2.4 Formulation Based on the FSDT.....	28
2.4.1 First Order Plate Theory	28
2.4.2 FSDT Equations of Kinematics.....	29
2.5 Formulation Based on the HSDT	30
2.5.1 Higher Order Shear Deformation Plate Theory	30
2.5.2 HSDT Equations of Kinematics	30
2.5.3 Constitutive Equations for HSDT Composite Laminate	33
2.5.4 Hamilton's Principle used for HSDT Composite Laminate.....	36
2.5.5 Virtual Strain Energy of HSDT Composite Laminate.....	36
2.5.6 Virtual Kinetic Energy of HSDT Composite Laminate	37
2.6 Finite Element Formulation of the HSDT Plate	39

2.6.1	HSDT MONNA Plate Element.....	39
2.6.2	HSDT Coordinate Transformation	40
2.6.3	HSDT Shape Functions.....	40
2.6.4	MONNA HSDT Physical Domain Transformation.....	41
2.6.5	MONNA HSDT Stiffness Matrix Formulation	41
2.6.6	MONNA HSDT Inertia Matrix Formulation	43
2.6.7	MONNA Stress Formulation	44
2.7	BFS Plate Element	46
Numerical Methods		48
3.1	Gauss Quadrature Integration	48
3.2	Newmark's Integration Method	48
Numerical Verification		51
4.1	Isotropic Plates	51
4.1.1	Deflections	51
4.1.2	Natural Frequencies.....	54
4.2	Composite Plates	56
4.2.1	Deflection.....	56
4.2.2	Natural Frequencies.....	57
4.2.3	Stresses	60
4.2.4	Sandwich Plate.....	64
4.2.5	Time response	65
Results and Discussion		73
5.1	Convergence Study	73
5.2	Effect of Laminate Thickness	75
5.3	Number of Plies	77
5.4	Moduli ratio	78
Conclusions and Recommendations		80
6.1	Conclusions.....	80
6.2	Recommendations for Future Research	80
References		82
Bibliography		85

List of Tables

Table 3.1: Gauss quadrature sampling points and weights	48
Table 4.1: Isotropic plate material properties and dimensions.....	51
Table 4.2: Maximum Deflection of a square Isotropic plate	52
Table 4.3: Comparison and convergence study of maximum central deflection of a square isotropic plate	53
Table 4.4: Isotropic plate material properties and dimensions	54
Table 4.5: Natural frequencies of a simply supported isotropic square plate (rad/s)	54
Table 4.6: Natural frequencies of a simply supported isotropic square plate	55
Table 4.7: Isotropic plate material properties and dimensions	55
Table 4.8: Natural frequencies of a simply supported isotropic rectangular plate with sinusoidal loading.	55
Table 4.9: Composite plate material properties	56
Table 4.10: Central deflection of a simply supported composite square plate with sinusoidal loading.	57
Table 4.11: Composite plate material properties	58
Table 4.12: Natural frequencies of a simply supported composite square plate under sinusoidal loading.	59
Table 4.13: Nondimensionalized axial stress $\bar{\sigma}_{xx}$ of a simply supported composite square plate under sinusoidal loading.	60
Table 4.14: Nondimensionalized axial stress $\bar{\sigma}_{yy}$ of a simply supported composite square plate under sinusoidal loading.	61
Table 4.15: Nondimensionalized axial stress $\bar{\sigma}_{yz}$ of a simply supported composite square plate under sinusoidal loading.	61
Table 4.16: Nondimensionalized axial stress $\bar{\sigma}_{xx}$ of a simply supported composite square plate under sinusoidal loading.	63
Table 4.17: Nondimensionalized stresses of a simply supported composite square plate under sinusoidal loading.	64
Table 4.18: Composite material properties.....	65
Table 4.19: Isotropic material properties	65
Table 4.20: Nondimensionalized CLPT stress results.....	65
Table 4.21: Composite plate material properties.....	66
Table 4.22: Composite plate material properties	71
Table 5.1: Composite plate material properties	73
Table 5.2: Convergence study for a $[0/90]_5$ laminate with $b/h = 5$	74
Table 5.3: Nondimensional deflections of a simply supported square composite plate under sinusoidal transverse loading	76
Table 5.4: Composite plate material properties	77
Table 5.5: Comparison and convergence study of plate central deflection for a simply supported square composite plate under sinusoidal transverse loading.....	78
Table 5.6: Composite plate material properties	78

Table 5.7: Nondimensional fundamental frequencies of antisymmetric square plates at various moduli ratio	79
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List of Figures

Figure 2.1: Plate coordinates	7
Figure 2.2: The CLPT deformed geometry [Haught (2020)]	8
Figure 2.3: The CLPT MONNNA plate element with DOF distribution	13
Figure 2.4: A quadrilateral in the physical domain.	14
Figure 2.5: FSDT deformed geometry as per Haught (2020)	30
Figure 2.6: Transverse normal deformations of CLPT, FSDT, and HSDT [Haught (2020)]	32
Figure 2.7: HSDT MONNNA plate element with DOF distribution as per Haught (2020)	40
Figure 2.8: BFS plate element with DOF distribution as per Haught (2020)	47
Figure 4.1: Simply supported boundary conditions on all edges for a rectangular plate.	51
Figure 4.2: Step loading as a function of time.	67
Figure 4.3: Sine loading as a function of time.	67
Figure 4.4: Triangular loading as a function of time.	68
Figure 4.5: Explosive blast loading as a function of time.	68
Figure 4.6: Dynamic response of laminated plate under step loading.	69
Figure 4.7: Dynamic response of laminated plate under sine loading.	69
Figure 4.8: Dynamic response of laminated plate under triangular loading.	70
Figure 4.9: Dynamic response of laminated plate under explosive blast loading.	70
Figure 4.10: Dynamic response of a simply supported square laminated plate under sinusoidal pulse.	71
Figure 5.1: Convergence study for a $[0/90]_5$ laminate with $b/h = 5$	74

Chapter 1

Introduction

1.1 Problem Statement

The search for highly durable, strong, yet light-weight materials has become critical in the design of complex structures with stringent design constraints. Based on the extensive theoretical and experimental research that have been conducted by numerous researchers, it has been found that composites can provide the needed strength-to-weight ratio, fatigue characteristics, and corrosion resistance.

The modeling and analysis of structures made of composite materials are primarily done using the finite element method (FEM), which could be traced back to the 1950s, where it was first explored by Turner et al. in 1956 as mentioned by Felippa (2004). Because of its practicality, a number of industries have adopted this method in testing their products as they are being modeled; these industries include the aerospace, the automotive, the biomedical, the chemical, the energy, etc. This method also proved efficient as the finite element solution approaches the exact solution using the h -version method or the p -version method. The h -version method is used to increase the accuracy of the results by increasing the number of elements, while the p -version method is used to increase the accuracy of the results by increasing the number of internal nodes. However, a more recent version was introduced that explores the combination of both the h -version and the p -version, called the h - p version. Babuska and Guo (1992) established that the h - p -version is capable of achieving an exponential rate of convergence, unlike the other two version alone. Normally, h -version elements would have four nodes such as the Bogner element, but as the number of nodes increase, even though it is still a constant number of nodes, it effectively becomes an h - p -version. Hence, this thesis discusses the application of an h - p -version version element and its capabilities.

Different theories have been developed and implemented under the finite element modeling approach to analyze laminate plates. The first one to be introduced was the classical laminated plate theory (CLPT). The most significant limitation of this theory is that it neglects the effect of transverse shear deformation, but the shear effects are important for composite plates, even if the plate can be considered thin. The first order shear deformation theory (FSDT) still assumes that plane sections remain plane however, they are not normal to the deflected midplane of the plate. The higher order shear deformation theory (HSDT) by Reddy (2004) approximates the in-plane normal strains as cubic functions of the thickness coordinate.

In many structural applications, static and dynamic analyses of composite plates is essential – especially with the aid of a suitable finite element. In this context, one needs

to explore the application of efficient, accurate, h - p -version higher-order finite elements with C' continuity.

1.2 Literature Review

There have been a lot of publications on different composite laminate analyses. These publications involve different techniques and methods used to investigate and analyze composite laminates stacked in square or rectangular plates. One of the goals of this thesis is to study the effects of the shear deformation theories, namely FSDT and HSDT, applied to the static and dynamic response of composite plates. Therefore, this literature review primarily focuses on different composite plate theories that have been introduced to analyze plates.

The classical plate theory (CLPT) was introduced by Kirchhoff in the nineteenth century. It has been introduced as a two-dimensional classical plate theory and further developed by Love (1944) and Timoshenko (1959). This theory employs multiple assumptions, which lead to a simple governing equation. Despite the fact that these assumptions lead to limitations, it has been proven by Kirchhoff, Timoshenko and other authors that have demonstrated that it is a good approach to obtain quick and simple case approximations for thin composite plates.

The first-order shear deformation theory (FSDT) was first proposed by Reissner (1945) and Mindlin (1951). This theory can accommodate transverse shear effects and thus may be used to analyze thick plates. This is done through the assumption that the in-plane displacements vary linearly across the thickness. Unfortunately, FSDT needs a shear correction factor because the theory violates the equilibrium condition on the top and bottom faces of the plate. The theory has been proven to be fairly accurate for thin to moderately thick plates by different authors.

The limitations of CLPT and FSDT has influenced researchers to create the higher-order shear deformation theory (HSDT). Several authors can be noted for developing this theory such as Pagano (1970), who studied the calculation of inter laminar normal stress, which other authors used to formulate the HSDT. Wang and Choi (1982) have developed an eigenfunction technique to satisfy continuity while using the HSDT. Reddy (2004) has proposed various functions that helped in the development and application of the HSDT as he also created the third-order shear deformation theory (TSDT), which is a type of HSDT. One benefit of this theory is that it is better at handling thin and thick laminates. Phan and Reddy have (1985) applied the TSDT to free vibration, bending, and buckling of composite plates. The boundary conditions on the top and bottom of the plate are satisfied and consequently this theory does not require shear correction factors. The HSDT yields much more accurate results than the FSDT when applied to thick laminates.

Further improvements to the HSDT have been carried out by Abbas, Negm, and Elshafei (2013) with the introduction of a modified version of the higher-order deformation theory (MHSDT). This theory has proved to be better over other theories

with respect to computational time. The theory uses the Rayleigh-Ritz model, which is composed of a simple polynomial set of equations. A limitation of that theory is that it is not capable of implementing higher order polynomials. This theory shows general agreement with the analytical and experimental results.

After these theories were introduced, several different elements were created to implement the aforementioned theories such as the Melosh element (1963), which is a four-node rectangular element. This element is an h -version non-conforming element. Thus, it may have difficulty to achieve convergence as it does not have C^1 continuity. The BFS element is another well-known h -version element, created by Bogner, Fox, and Schmidt (1965). This element is conforming four-node rectangular element that allows genuine C^1 continuity and avoids convergence issues.

There have been many different applications to many differently modified theories such as Lin, Lu and Tarn (1989) who tested cantilevered composite plates with an 18 degree-of-freedom triangular finite element based on classical laminate plate theory (CLPT). Koo and Lee (1994) also tested the effect of structural damping on the aeroelasticity of composite plate wings using their nine-node isoparametric rectangular plate element with first-order shear deformation theory (FSDT). Thai et al. (2012) also created the NS-DSG3 element. This element relies on a combination of node-based smoothing discrete shear gap method with the higher-order shear deformation plate theory (HSDT). The formulation of the element uses only linear approximation. This element proved to be quite efficient as it also improves the accuracy if the transverse shear stresses.

To this end, a new C^1 continuous HSDT element, called MONNA, has been theoretically invented by Sivaneri (2019), and developed Haught (2020). This element is considered a conforming h - p -version element and was developed to investigate the aeroelasticity of composite plate wings. The element showed promise and proved to be efficient in static testing. Yet, the element was only used to examine deflections and natural frequencies within the structural analysis realm.

1.3 Need for Present Research

The need for composite materials has increased because of its desirable strength-to-weight ratio, fatigue characteristics, and corrosion resistance. With this comes the need for efficient finite element analysis tools in order to properly apply, test and study composites. Since composites are sensitive to the effects of shear deformation, Plate theories accounting for shear deformation need to be developed and sufficiently investigated. As mentioned before, the classical plate theory (CPT) is only adequate for simple cases since it neglects the effect of shear deformation. Also, it is not suitable for thick laminates as stated by Phan and Reddy (1985). The easiest way to get fairly accurate results for both thin and moderately thick composite laminates, would be to use the first-order shear deformation theory (FSDT). This theory is simple to implement but requires

a shear correction factor. The higher-order shear deformation theory resolves the issues of the first-order shear deformation theory. HSDT is very accurate and a very powerful tool to analyze composite laminates. The only issue that accompanies this theory is that it requires C^1 continuity of lateral displacements, which can be complicated to implement.

The MONNA element attempts to solve the shortcomings of the already established HSDT elements. However, it has not been tested enough, nor has it been investigated in dynamic tests. Therefore, this thesis investigates the application of the MONNA plate element to different static tests, such as stresses, sandwich plates, and dynamic tests as it reinvestigates the efficiency of the MONNA element and compares between it and other established elements. The MONNA finite element is also modified to accommodate CLPT and used to properly test, compare and verify all aspects of the CLPT element.

1.4 Objectives

The main goals of the present thesis are as follows:

- 1 Modify the MONNA HSDT finite element for the case of CLPT.
- 2 Test the element with respect to stresses, sandwich plates, and dynamic tests.
- 3 Test and compare among the different plate theories (HSDT, FSDT and CLPT), as applied to static and dynamic problems.
- 4 Evaluate the benefits of using a new conforming higher-order plate element and compare its performance with traditional plate elements for static and dynamic analysis.
- 5 Investigate the effect of laminate thickness, ply number, and moduli ratios

1.5 Thesis Outline

The Organization of this thesis is as follows:

- Chapter 2 presents the theoretical and finite element formulation of the composite plate structure. Composite plate theories are first explored, followed by a detailed theoretical formulation of the classical laminate plate theory, then the first-order shear deformation theory and then the higher-order shear deformation theory applied in this analysis. In this chapter, the new higher order plate element, called MONNA, is formulized and discussed.

- Chapter 3 discusses numerical methods used to aid with the solution.
- Chapter 4 includes the numerical results of the tested cases that are verified and compared to existing, established results.
- Chapter 5 provides a discussion on the behavior and performance of the element when investigating the effect of laminate thickness, ply orientation, material properties, different material properties ratios and a convergence study, while comparing between CLPT, FSDT, and HSDT.
- Chapter 6 offers the conclusions of the present analysis and recommendations for future research.

Chapter 2

Structural Models

2.1 Introduction

The basic mechanics of materials approach yields exact solutions (in the form of stresses, strains, and deflections) of simple structures such as bars, shafts, and beams. The finite element method has been the method of choice for tackling complex structures due to its versatility and convergence characteristics. The composite materials have found an exponential growth in many structural applications such as aerospace, automobile, bridges, home construction, and sporting goods. This is due to their advantage over conventional materials in the forms of strength to weight ratio, ability to tailor the material in the design process, and corrosion resistance.

The present thesis focuses on composite plates since composite laminates have usually large planar dimensions relative to their thickness. In this case composite laminates are treated as plates and hence composite finite elements are utilized in the present thesis.

This chapter provides descriptions of the common composite plate theories, namely the classical laminate plate theory (CLPT), first order shear deformation theory (FSDT), and higher order shear deformation theory (HSDT). Sivaneri (2019 & 2020) conceptually formulated a new higher-order composite plate element called MONNA (multi order nine-node advanced) element. The incorporation of this new element into a MATLAB code, verification pertaining to common applications has been carried out by Haught (2020) and the present author in this thesis.

2.2 Description of the Classical Plate Theory

2.2.1 Assumptions

The classical plate theory is one of the most used plate theories. Kirchhoff has formulated this simple theory for thin isotropic plates. Figure 2.1 shows a thin rectangular plate with cartesian coordinates (x , y , z) attached to it. The assumptions behind the Kirchhoff formulation as extended to composite laminates are listed here.

Figure 2.1: Plate coordinates

- 1 Straight lines perpendicular to the mid surface remain straight after deformation.
- 2 Transverse normals do not elongate ($\varepsilon_z = 0$).
- 3 Transverse normals rotate such that they are always perpendicular to the mid surface even after deformation ($\gamma_{xz} = 0$ and $\gamma_{yz} = 0$).
- 4 The transverse shear stresses are zero at the top and bottom of the plate.
- 5 Layers are perfectly bonded.
- 6 Each layer has a uniform thickness.
- 7 Each layer has a material that is linearly elastic and orthotropic.
- 8 The displacements are small compared to the thickness of the plate
- 9 The strains are small.

2.2.2 Equations of Kinematics

Considering a thin plate (Figure 2.1) of thickness h with the (x, y) plane attached to the midplane of the plate and the z coordinate positive up. The CLPT has a displacement field (u, v, w) represented in the (x, y, z) directions as follows:

$$\begin{aligned}
 u(x, y, z, t) &= u_0(x, y, t) + z \phi_x(x, y, t) \\
 v(x, y, z, t) &= v_0(x, y, t) + z \phi_y(x, y, t) \\
 w(x, y, z, t) &= w_0(x, y, t)
 \end{aligned} \tag{2.1}$$

where u_0 and v_0 are the midplane displacements and w_0 is the lateral displacement, which is made up only of bending component, and hence the CLPT does not account for shear deformations. The ϕ_x and ϕ_y are rotations of the transverse normal about the y and x axes, respectively. These rotations are related to the bending deflection as

$$\begin{aligned}\phi_x &= \frac{\partial u_0}{\partial z} = -\frac{\partial w_0}{\partial x} \\ \phi_y &= \frac{\partial v_0}{\partial z} = -\frac{\partial w_0}{\partial y}\end{aligned}\quad (2.2)$$

Using these definitions in Eq. (2.1),

$$\begin{aligned}u &= u_0 - z \frac{\partial w_0}{\partial x} \\ v &= v_0 - z \frac{\partial w_0}{\partial y} \\ w &= w_0\end{aligned}\quad (2.3)$$

The CLPT model is illustrated below in Figure (2.2).

Figure 2.2: The CLPT deformed geometry [Haught (2020)]

The nonlinear kinematic equations for moderate rotations are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \{\varepsilon^{(0)}\} + z\{\varepsilon^{(1)}\} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \end{Bmatrix} + z \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}\quad (2.3)$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

2.2.3 Constitutive Equations for CLPT Composite Laminate

The constitutive equations for a CLPT composite laminate are

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \end{Bmatrix} \quad (2.4)$$

where $[N] = [N_x \ N_y \ N_{xy}]$ and it denotes the force resultants. The $\{M\}$ vector is similar to the $\{N\}$ vector, except it contains the moment resultants. The $[A]$, $[B]$, and $[D]$ matrices contain the extension stiffness, bending-extension coupling stiffness, and the bending stiffness, respectively. The stress resultants could be found through

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad (2.5)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad (2.6)$$

The stiffness coefficients are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \bar{Q}_{ij}^{(k)}(1, z, z^2) dz \quad i, j = 1, 2, 6 \quad (2.7)$$

The expanded form of the constitutive Eq. (2.4) becomes

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \\ \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (2.8)$$

2.2.4 Hamilton's Principle Applied to the CLPT Composite Laminate

The finite element equations of motions are derived from Hamilton's principle

$$\int_{t_1}^{t_2} (\delta U - \delta T - \delta W) dt = 0 \quad (2.9)$$

where δU is the virtual strain energy, δT is the virtual kinetic energy, and δW is the virtual work done by the applied load.

2.2.5 Virtual Strain Energy of CLPT Composite Laminate

The finite element stiffness matrix is found through the virtual strain energy expression. The expression for δU over a plate is given by

$$\delta U = \iiint_V (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy}) dV \quad (2.10)$$

where $\delta \varepsilon_x$, $\delta \varepsilon_y$, and $\delta \gamma_{xy}$, are the virtual strains and V is the volume of the plate. Next the volume integral is divided into an integral over the plate area A and another integral over the thickness h .

$$\delta U = \iint_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_x (\delta \varepsilon_x^{(0)} + z \delta \varepsilon_x^{(1)}) + \sigma_y (\delta \varepsilon_y^{(0)} + z \delta \varepsilon_y^{(1)}) + \tau_{xy} (\delta \gamma_{xy}^{(0)} + z \delta \gamma_{xy}^{(1)}) \right] dA dz \quad (2.11)$$

Substituting Eqs. (2.5 – 2.6) into Eq. (2.11) yields

$$\delta U = \iint_A \left[N_x \delta \varepsilon_x^{(0)} + M_x \delta \varepsilon_x^{(1)} + N_y \delta \varepsilon_y^{(0)} + M_y \delta \varepsilon_y^{(1)} + N_{xy} \delta \gamma_{xy}^{(0)} + M_{xy} \delta \gamma_{xy}^{(1)} \right] dA \quad (2.12)$$

The following abbreviated notations are used to describe the space and time partial derivatives:

$$\begin{aligned} ()^x &= \frac{\partial ()}{\partial x} \\ ()^y &= \frac{\partial ()}{\partial y} \end{aligned} \quad (2.13)$$

$$\dot{(\)} = \frac{\partial(\)}{\partial t}$$

Finally, the virtual strain energy is listed with the strain components fully written out using the shorthand notation. Eq. (2.12) becomes:

$$\begin{aligned} \delta U = \iint_A \{ & N_x(\delta u_0^x + w_b^x \delta w_b^x) - M_x \delta w_b^{xx} + N_y(\delta v_0^y + w_b^y \delta w_b^y) - M_y \delta w_b^{yy} \\ & + N_{xy}[\delta u_0^y + \delta v_0^x + (w_b^y)(\delta w_b^x) + (w_b^x)(\delta w_b^y)] \} dA \end{aligned} \quad (2.14)$$

2.2.6 Virtual Kinetic Energy of CLPT Composite Laminate

The finite element inertia matrix is found through the virtual kinetic energy expression. The expression for δT over a plate is given by

$$\delta T = \iiint_V \rho(\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) dV \quad (2.15)$$

where ρ is the material density. The expression of δT is further expanded using the displacement field Eq (2.1)

$$\delta T = \iiint_V \rho[(\dot{u}_0 - z\dot{w}_b^x)(\delta \dot{u}_0 - z\delta \dot{w}_b^x) + (\dot{v}_0 - z\dot{w}_b^y)(\delta \dot{v}_0 - z\delta \dot{w}_b^y) + (\dot{w}_b)(\delta \dot{w}_b)] dV \quad (2.16)$$

Rearranging the expanded expression,

$$\begin{aligned} \delta T = \iiint_V \rho[& (\dot{u}_0 - z\dot{w}_b^x)\delta \dot{u}_0 - (z\dot{u}_0 - z^2\dot{w}_b^x)\delta \dot{w}_b^x + (\dot{v}_0 - z\dot{w}_b^y)\delta \dot{v}_0 \\ & - (z\dot{v}_0 - z^2\dot{w}_b^y)\delta \dot{w}_b^y + (\dot{w}_b)\delta \dot{w}_b] dV \end{aligned} \quad (2.17)$$

Considering the time integral from Hamilton's principle from Eq. (2.9)

$$\begin{aligned} - \int_{t_1}^{t_2} \delta T dt = \iiint_V \left\{ \int_{t_1}^{t_2} -\rho[& (\dot{u}_0 - z\dot{w}_b^x)\delta \dot{u}_0 - (z\dot{u}_0 - z^2\dot{w}_b^x)\delta \dot{w}_b^x + (\dot{v}_0 - z\dot{w}_b^y)\delta \dot{v}_0 \right. \\ & \left. - (z\dot{v}_0 - z^2\dot{w}_b^y)\delta \dot{w}_b^y + (\dot{w}_b)\delta \dot{w}_b] dt \right\} dV \end{aligned} \quad (2.18)$$

Integrating by parts the time integral to eliminate terms containing time derivatives of the virtual displacements and grouping together all the boundary terms as

$$\iiint_V [B(t)]_{t_1}^{t_2} dV \quad (2.19)$$

This leads to

$$\begin{aligned} - \int_{t_1}^{t_2} \delta T dt = & \iiint_V \left\{ [B(t)]_{t_1}^{t_2} \right. \\ & + \int_{t_1}^{t_2} \rho [(\ddot{u}_0 - z\ddot{w}_b^x)\delta u_0 - (z\ddot{u}_0 - z^2\ddot{w}_b^x)\delta w_b^x + (\ddot{v}_0 - z\ddot{w}_b^y)\delta v_0 \\ & \left. - (z\ddot{v}_0 - z^2\ddot{w}_b^y)\delta w_b^y + (\ddot{w}_b)\delta w_b] dt \right\} dV \end{aligned} \quad (2.20)$$

The integrand is valid for all time and can be removed from the time integral. Furthermore, since the boundary terms, $B(t)$, do not contribute to the finite element inertia matrix, they may be neglected. The volume integral is split into a double integral, where one is integrated over the plate's area and the second is integrated over the plate's thickness. Introduce the mass moments of inertia as follows:

$$I_k = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho z^k dV \quad k = 0, 1, 2 \quad (2.21)$$

Then Eq. (2.20) becomes

$$\begin{aligned} -\delta T = & \iint_A [(I_0\ddot{u}_0 - I_1\ddot{w}_b^x)\delta u_0 - (I_1\ddot{u}_0 - I_2\ddot{w}_b^x)\delta w_b^x + (I_0\ddot{v}_0 - I_1\ddot{w}_b^y)\delta v_0 \\ & - (I_1\ddot{v}_0 - I_2\ddot{w}_b^y)\delta w_b^y + I_0(\ddot{w}_b)\delta w_b] dx dy \end{aligned} \quad (2.22)$$

where I_0 is the normal inertia, I_1 is the coupled normal-rotary inertia, and I_2 is the rotary inertia.

2.3 Finite Element Formulation Based on CLPT

2.3.1 Introduction

Interpolation Functions, and Continuity

The determination of the interpolation functions that are used to approximate the displacement distributions is a critical process when formulating the finite element. It is

important to know the different types of continuities and their conditions. There are two main types of continuities for this case of study, C^0 continuity and C^1 continuity. The C^0 continuity is required when the variable has first-order derivatives of displacements within the virtual strain and kinetic energy expressions. This means that the C^0 continuity is required when the interelement displacements are continuous, but slopes are not. The process to approximate these variables is done using the *Lagrange polynomials*. The C^1 continuity, on the other hand, is required when the variable has second-order derivatives of displacements within the virtual strain and kinetic energy expressions. This means that C^1 continuity is required when both the interelement displacements and slopes are continuous. The process to approximate these variables is done using the *Hermite polynomials*. It is important to understand and know the type of continuity that will be applied or needed for the plate theory and how to make it work properly, since it has major effects on its accuracy and capacity for transformation.

2.3.2 CLPT MONNA Plate Element

The goal of this section is to formulate a new conforming plate element based on CLPT theory. This element is a conforming Multi-Order Nine-Node Advanced (MONNA) plate element as shown in Figure (2.3). This element has been invented by Sivaneri (2019 and 2020) and developed and coded by Haught (2020) and the present author.

Figure 2.3: The CLPT MONNNA plate element with DOF distribution

This is a subparametric master element that has plate coordinate system of (ξ, η, z) at its origin. In the x and y directions, the in-plane dependent displacement variables are u and v while the transverse deflection variable is w_b . The element has four corner nodes (1-4), four mid-side nodes (5-8), and a ninth node at the center. That makes the number of degrees of freedom (DOF) equal to 43 as shown in Figure 2.3. This element is considered an h-p version. This means that the accuracy of the analyses increases by

increasing the number of elements and internal nodes. This also helps with the rate of convergence. The element has three different orders of polynomials. Bilinear polynomials are used as shape functions to transform the quadrilateral elements in physical domain to the standard domain. Bi-quadratic Lagrange polynomials are used to derive the shape functions for the in-plane displacements (u , v) and bi-quartic Hermite polynomials are used for interpolating the transverse deflection (w_b).

2.3.3 CLPT Coordinate Transformation

The goal is to be able to transform the geometry from the physical domain (x , y) to the standard domain (ξ , η) and vice versa. The bilinear shape functions that resemble the four-node isoparametric plane elements are utilized to achieve the transformation from one domain to the other. The bilinear shape functions will transform straight lines in the physical domain to a straight line in the standard domain. For a curved line scenario, the shape functions of an eight-node isoparametric will have to be utilized. Other than that, everything else will remain the same regarding the development of MONNA.

Consider a quadrilateral in the physical domain (x , y) as shown in Figure (2.4), where (x_i, y_i) signifies the coordinates of the i^{th} node as $i = 1 - 4$.

Figure 2.4: A quadrilateral in the physical domain.

The coordinate transformation between the standard and physical domains is shown in Eq. (2.23)

$$\begin{aligned} x(\xi, \eta) &= [N_i(\xi, \eta)]\{x_i\} \\ y(\xi, \eta) &= [N_i(\xi, \eta)]\{y_i\} \end{aligned} \quad (2.23)$$

where this expands to

$$\begin{aligned}
x(\xi, \eta) &= [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \\
y(\xi, \eta) &= [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}
\end{aligned} \tag{2.24}$$

where N_{1-4} are the shape functions and they are specified as

$$\begin{aligned}
N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta)
\end{aligned} \tag{2.25}$$

First Derivatives

The relationships between the first derivatives in the standard domain and the physical domain are:

$$\begin{aligned}
\frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \\
\frac{\partial}{\partial \eta} &= \frac{\partial}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \eta}
\end{aligned} \tag{2.26}$$

This is written in matrix notation as follows

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \tag{2.27}$$

The square matrix of Eq. (2.27) is called the *Jacobian matrix* and represented as $[J]$ as shown below:

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{2.28}$$

The determinant of the Jacobian matrix is known to be the *Jacobian* as shown bellow

$$|J| = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} \quad (2.29)$$

where

$$\begin{aligned} J_{11} &= \frac{\partial x}{\partial \xi} = \frac{1}{4} [-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4] \\ J_{21} &= \frac{\partial x}{\partial \eta} = \frac{1}{4} [-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4] \\ J_{12} &= \frac{\partial y}{\partial \xi} = \frac{1}{4} [-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4] \\ J_{22} &= \frac{\partial y}{\partial \eta} = \frac{1}{4} [-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4] \end{aligned} \quad (2.30)$$

Using the Jacobian, it becomes much easier to carry out the integrations (such as quadrature scheme) in the formulation as it relates the integration in the standard domain to those of the physical domain. That is

$$\iint_A F(x, y) dx dy = \int_{-1}^1 \int_{-1}^1 F[x(\xi, \eta), y(\xi, \eta)] |J| d\xi d\eta \quad (2.31)$$

Second Derivatives

The second derivatives are found by differentiating the first derivatives again. The second partial derivative with respect to ξ becomes

$$\frac{\partial^2}{\partial \xi^2} = \frac{\partial}{\partial \xi} \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) = \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \quad (2.32)$$

This comes out as

$$\frac{\partial^2}{\partial \xi^2} = \frac{\partial^2}{\partial x^2} \left(\frac{\partial x}{\partial \xi} \right)^2 + \frac{\partial^2}{\partial x \partial y} \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} + \frac{\partial}{\partial x} \frac{\partial x^2}{\partial \xi^2} + \frac{\partial^2}{\partial x \partial y} \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} + \frac{\partial^2}{\partial y^2} \left(\frac{\partial y}{\partial \xi} \right)^2 + \frac{\partial}{\partial y} \frac{\partial y^2}{\partial \xi^2} \quad (2.33)$$

That is

$$\frac{\partial^2}{\partial \xi^2} = \frac{\partial J_{11}}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial J_{12}}{\partial \xi} \frac{\partial}{\partial y} + J_{11}^2 \frac{\partial^2}{\partial x^2} + J_{12}^2 \frac{\partial^2}{\partial y^2} + 2J_{11}J_{12} \frac{\partial^2}{\partial x \partial y} \quad (2.34)$$

Similarly, the second partial derivative with respect to η is

$$\frac{\partial^2}{\partial \eta^2} = \frac{\partial J_{21}}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial J_{22}}{\partial \eta} \frac{\partial}{\partial y} + J_{21}^2 \frac{\partial^2}{\partial x^2} + J_{22}^2 \frac{\partial^2}{\partial y^2} + 2J_{21}J_{22} \frac{\partial^2}{\partial x \partial y} \quad (2.35)$$

The second partial derivative with respect to ξ and η is

$$\frac{\partial^2}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta} \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) = \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \eta} \right) \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \quad (2.36)$$

That is

$$\frac{\partial^2}{\partial \xi \partial \eta} = \frac{\partial J_{11}}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial J_{12}}{\partial \eta} \frac{\partial}{\partial y} + J_{11} J_{21} \frac{\partial^2}{\partial x^2} + J_{12} J_{22} \frac{\partial^2}{\partial y^2} + (J_{11} J_{22} + J_{12} J_{21}) \frac{\partial^2}{\partial x \partial y} \quad (2.37)$$

Equations (2.34), (2.35), and (2.37) are put in matrix form as

$$\begin{Bmatrix} \frac{\partial^2}{\partial \xi^2} \\ \frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial J_{11}}{\partial \xi} & \frac{\partial J_{12}}{\partial \xi} \\ \frac{\partial J_{21}}{\partial \xi} & \frac{\partial J_{22}}{\partial \xi} \\ \frac{\partial J_{11}}{\partial \eta} & \frac{\partial J_{12}}{\partial \eta} \\ \frac{\partial J_{21}}{\partial \eta} & \frac{\partial J_{22}}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} + \begin{bmatrix} J_{11}^2 & J_{12}^2 & 2J_{11}J_{12} \\ J_{21}^2 & J_{22}^2 & 2J_{21}J_{22} \\ J_{11}J_{21} & J_{12}J_{22} & J_{11}J_{22} + J_{12}J_{21} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x \partial y} \end{Bmatrix} \quad (2.38)$$

This is in compact form as

$$\{\partial_\xi^2\} = [C_A]\{\partial_x\} + [C_B]\{\partial_x^2\} \quad (2.39)$$

where $[C_A]$ is the 3×2 matrix and $[C_B]$ is the square matrix. However, in Eq. (2.38), the $J_{11}^\xi, J_{12}^\xi, J_{21}^\eta$ and J_{22}^η terms could be neglected because they are equal to zero for a quadrilateral.

Inverse Transformation

The transformation back from the standard domain to the physical domain is necessary to properly represent the physical variables. The inverse of the Jacobian matrix $[J^*]$ is

$$[J^*] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \quad (2.40)$$

The inverse transformation of the first derivative can be written as

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \quad (2.41)$$

This could be written in compact form as shown below

$$\{\partial_x\} = [J^*]\{\partial_\xi\} \quad (2.42)$$

Similarly inverse transformation of the second derivatives are written as

$$\{\partial_x^2\} = [T_A]\{\partial_\xi\} + [T_B]\{\partial_\xi^2\} \quad (2.43)$$

where $[T_A]$ is equal to $[C_A][J^*]$ and $[T_B]$ is the inverse of the $[C_B]$ matrix. The expanded version of the inverse of the second derivative is

$$\begin{Bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \begin{bmatrix} (T_A)_{11} & (T_A)_{12} \\ (T_A)_{21} & (T_A)_{22} \\ (T_A)_{31} & (T_A)_{32} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} + \begin{bmatrix} (J_{11}^*)^2 & (J_{12}^*)^2 & 2J_{11}^*J_{12}^* \\ (J_{21}^*)^2 & (J_{22}^*)^2 & 2J_{21}^*J_{22}^* \\ J_{11}^*J_{22}^* + J_{12}^*J_{21}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial^2}{\partial \xi^2} \\ \frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial \xi \partial \eta} \end{Bmatrix} \quad (2.44)$$

where the $[T_A]$, transformation matrix is

$$T_A = - \begin{bmatrix} 2J_{11}^*J_{12}^*J_{11}^\eta & 2J_{11}^*J_{12}^*J_{12}^\eta \\ 2J_{21}^*J_{22}^*J_{11}^\eta & 2J_{21}^*J_{22}^*J_{12}^\eta \\ (J_{11}^*J_{22}^* + J_{12}^*J_{21}^*)J_{11}^\eta & (J_{11}^*J_{22}^* + J_{12}^*J_{21}^*)J_{12}^\eta \end{bmatrix} \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \quad (2.45)$$

2.3.4 CLPT Shape Functions

The Shape functions of the finite element are derived in the standard domain. The in-plane displacements u and v have only first order derivatives thus Lagrange interpolation is needed, which means as discussed earlier that this is a case where C^0 continuity is needed. The lateral displacement on the other hand, w_b , have second-order derivatives and hence Hermite interpolation is needed for the solution, which implies that C^1 continuity is needed.

Lagrange Interpolation

The quadratic Lagrange shape functions used to approximate the in-plane displacements for the nine-node element are listed below as theoretically formulated by Sivaneri (2019):

$$H_{L1}(\xi, \eta) = \frac{\xi\eta}{4} (1 - \xi)(1 - \eta) \quad (2.46)$$

$$\begin{aligned}
H_{L2}(\xi, \eta) &= -\frac{\xi\eta}{4}(1+\xi)(1-\eta) \\
H_{L3}(\xi, \eta) &= \frac{\xi\eta}{4}(1+\xi)(1+\eta) \\
H_{L4}(\xi, \eta) &= -\frac{\xi\eta}{4}(1-\xi)(1+\eta) \\
H_{L5}(\xi, \eta) &= -\frac{\eta}{2}(1-\xi^2)(1-\eta) \\
H_{L6}(\xi, \eta) &= \frac{\xi}{2}(1+\xi)(1-\eta^2) \\
H_{L7}(\xi, \eta) &= \frac{\eta}{2}(1-\xi^2)(1+\eta) \\
H_{L8}(\xi, \eta) &= -\frac{\xi}{2}(1-\xi)(1-\eta^2) \\
H_{L9}(\xi, \eta) &= (1-\xi^2)(1-\eta^2)
\end{aligned}$$

The shorthand notation that is used to represent the set of Lagrange shape functions is

$$[H_L(\xi, \eta)] = [H_{L1} \quad H_{L2} \quad \dots \quad H_{L9}] \quad (2.47)$$

Then, the in-plane displacements are given by

$$\begin{aligned}
u(\xi, \eta) &= [H_L(\xi, \eta)]\{q_u\} \\
v(\xi, \eta) &= [H_L(\xi, \eta)]\{q_v\}
\end{aligned} \quad (2.48)$$

where $\{q_u\}$ and $\{q_v\}$ are the axial displacement elements degree of freedom vector.

Hermite Interpolation

The Hermite Interpolation, also formulated by Sivaneri (2019), is needed to accommodate the lateral displacement DOF, w_b . The MONNA CLPT plate element has the following lateral displacement distribution

$$\begin{aligned}
\text{Corner nodes 1, 2, 3, 4: } &w_b, w_b^\xi, w_b^\eta, w_b^{\xi\eta} \\
\text{Midside nodes 5, 7: } &w_b, w_b^\xi \\
\text{Midside nodes 6, 8: } &w_b, w_b^\eta, \\
\text{Center nodes 9: } &w_b
\end{aligned} \quad (2.49)$$

Thus, for the transverse displacement w_b , there are 25 degrees of freedom. The form that the lateral displacement distribution take is as follows

$$w_b(\xi, \eta) = \sum_{i=0}^4 \sum_{j=0}^4 a_{ij} \xi^i \eta^j \quad (2.50)$$

Expanding Eq. (2.50) with a single-subscript coefficient yields

$$\begin{aligned} w_b(\xi, \eta) = & a_0 + a_1\xi + a_2\eta + a_3\xi^2 + a_4\xi\eta + a_5\eta^2 + a_6\xi^3 + a_7\xi^2\eta + a_8\xi\eta^2 \\ & + a_9\eta^3 + a_{10}\xi^4 + a_{11}\xi^3\eta + a_{12}\xi^2\eta^2 + a_{13}\xi\eta^3 + a_{14}\eta^4 + a_{15}\xi^4\eta \\ & + a_{16}\xi^3\eta^2 + a_{17}\xi^2\eta^3 + a_{18}\xi\eta^4 + a_{19}\xi^4\eta^2 + a_{20}\xi^3\eta^3 + a_{21}\xi^2\eta^4 \\ & + a_{22}\xi^4\eta^3 + a_{23}\xi^3\eta^4 + a_{24}\xi^4\eta^4 \end{aligned} \quad (2.51)$$

The procedure to obtain the unknown coefficients a_i of Eq. (2.51), which correspond to obtaining the shape functions, is done by plugging in the nodal coordinates into each D.O.F. in Eq. (2.51). The shape functions that will be yielded are the tensor products of the quartic Hermite polynomials in the standard domain (ξ , and η). The 25 shape functions are:

$$\begin{aligned} \bar{H}_1 = & \frac{9}{16}\xi\eta - \frac{3}{4}\xi\eta^2 - \frac{3}{16}\xi\eta^3 + \frac{3}{8}\xi\eta^4 - \frac{3}{4}\xi^2\eta + \xi^2\eta^2 + \frac{1}{4}\xi^2\eta^3 - \frac{1}{2}\xi^2\eta^4 \\ & - \frac{3}{16}\xi^3\eta + \frac{1}{4}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 - \frac{1}{8}\xi^3\eta^4 + \frac{3}{8}\xi^4\eta - \frac{1}{2}\xi^4\eta^2 \\ & - \frac{1}{8}\xi^4\eta^3 + \frac{1}{4}\xi^4\eta^4 \\ \bar{H}_2 = & \frac{3}{16}\xi\eta - \frac{1}{4}\xi\eta^2 - \frac{1}{16}\xi\eta^3 + \frac{1}{8}\xi\eta^4 - \frac{3}{16}\xi^2\eta + \frac{1}{4}\xi^2\eta^2 + \frac{1}{16}\xi^2\eta^3 \\ & - \frac{1}{8}\xi^2\eta^4 - \frac{3}{16}\xi^3\eta + \frac{1}{4}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 - \frac{1}{8}\xi^3\eta^4 + \frac{3}{16}\xi^4\eta \\ & - \frac{1}{4}\xi^4\eta^2 - \frac{1}{16}\xi^4\eta^3 + \frac{1}{8}\xi^4\eta^4 \\ \bar{H}_3 = & \frac{3}{16}\xi\eta - \frac{3}{16}\xi\eta^2 - \frac{3}{16}\xi\eta^3 + \frac{3}{16}\xi\eta^4 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta^3 \\ & - \frac{1}{4}\xi^2\eta^4 - \frac{1}{16}\xi^3\eta + \frac{1}{16}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 - \frac{1}{16}\xi^3\eta^4 + \frac{1}{8}\xi^4\eta \\ & - \frac{1}{8}\xi^4\eta^2 - \frac{1}{8}\xi^4\eta^3 + \frac{1}{8}\xi^4\eta^4 \\ \bar{H}_4 = & \frac{1}{16}\xi\eta - \frac{1}{16}\xi\eta^2 - \frac{1}{16}\xi\eta^3 + \frac{1}{16}\xi\eta^4 - \frac{1}{16}\xi^2\eta + \frac{1}{16}\xi^2\eta^2 + \frac{1}{16}\xi^2\eta^3 \\ & - \frac{1}{16}\xi^2\eta^4 - \frac{1}{16}\xi^3\eta + \frac{1}{16}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 - \frac{1}{16}\xi^3\eta^4 \\ & + \frac{1}{16}\xi^4\eta - \frac{1}{16}\xi^4\eta^2 - \frac{1}{16}\xi^4\eta^3 + \frac{1}{16}\xi^4\eta^4 \\ \bar{H}_5 = & -\frac{9}{16}\xi\eta + \frac{3}{4}\xi\eta^2 + \frac{3}{16}\xi\eta^3 - \frac{3}{8}\xi\eta^4 - \frac{3}{4}\xi^2\eta + \xi^2\eta^2 + \frac{1}{4}\xi^2\eta^3 - \frac{1}{2}\xi^2\eta^4 \\ & + \frac{3}{16}\xi^3\eta - \frac{1}{4}\xi^3\eta^2 - \frac{1}{16}\xi^3\eta^3 + \frac{1}{8}\xi^3\eta^4 + \frac{3}{8}\xi^4\eta - \frac{1}{2}\xi^4\eta^2 \\ & - \frac{1}{8}\xi^4\eta^3 + \frac{1}{4}\xi^4\eta^4 \end{aligned} \quad (2.52)$$

$$\begin{aligned}
\bar{H}_6 &= \frac{3}{16}\xi\eta - \frac{1}{4}\xi\eta^2 - \frac{1}{16}\xi\eta^3 + \frac{1}{8}\xi\eta^4 + \frac{3}{16}\xi^2\eta - \frac{1}{4}\xi^2\eta^2 - \frac{1}{16}\xi^2\eta^3 \\
&\quad + \frac{1}{8}\xi^2\eta^4 - \frac{3}{16}\xi^3\eta + \frac{1}{4}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 - \frac{1}{8}\xi^3\eta^4 - \frac{3}{16}\xi^4\eta \\
&\quad + \frac{1}{4}\xi^4\eta^2 + \frac{1}{16}\xi^4\eta^3 - \frac{1}{8}\xi^4\eta^4 \\
\bar{H}_7 &= -\frac{3}{16}\xi\eta + \frac{3}{16}\xi\eta^2 + \frac{3}{16}\xi\eta^3 - \frac{3}{16}\xi\eta^4 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta^3 \\
&\quad - \frac{1}{4}\xi^2\eta^4 + \frac{1}{16}\xi^3\eta - \frac{1}{16}\xi^3\eta^2 - \frac{1}{16}\xi^3\eta^3 + \frac{1}{16}\xi^3\eta^4 + \frac{1}{8}\xi^4\eta \\
&\quad - \frac{1}{8}\xi^4\eta^2 - \frac{1}{8}\xi^4\eta^3 + \frac{1}{8}\xi^4\eta^4 \\
\bar{H}_8 &= \frac{1}{16}\xi\eta - \frac{1}{16}\xi\eta^2 - \frac{1}{16}\xi\eta^3 + \frac{1}{16}\xi\eta^4 + \frac{1}{16}\xi^2\eta - \frac{1}{16}\xi^2\eta^2 - \frac{1}{16}\xi^2\eta^3 \\
&\quad + \frac{1}{16}\xi^2\eta^4 - \frac{1}{16}\xi^3\eta + \frac{1}{16}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 - \frac{1}{16}\xi^3\eta^4 \\
&\quad - \frac{1}{16}\xi^4\eta + \frac{1}{16}\xi^4\eta^2 + \frac{1}{16}\xi^4\eta^3 - \frac{1}{16}\xi^4\eta^4 \\
\bar{H}_9 &= \frac{9}{16}\xi\eta + \frac{3}{4}\xi\eta^2 - \frac{3}{16}\xi\eta^3 - \frac{3}{8}\xi\eta^4 + \frac{3}{4}\xi^2\eta + \xi^2\eta^2 - \frac{1}{4}\xi^2\eta^3 - \frac{1}{2}\xi^2\eta^4 \\
&\quad - \frac{3}{16}\xi^3\eta - \frac{1}{4}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 + \frac{1}{8}\xi^3\eta^4 - \frac{3}{8}\xi^4\eta - \frac{1}{2}\xi^4\eta^2 \\
&\quad + \frac{1}{8}\xi^4\eta^3 + \frac{1}{4}\xi^4\eta^4 \\
\bar{H}_{10} &= -\frac{3}{16}\xi\eta - \frac{1}{4}\xi\eta^2 + \frac{1}{16}\xi\eta^3 + \frac{1}{8}\xi\eta^4 - \frac{3}{16}\xi^2\eta - \frac{1}{4}\xi^2\eta^2 + \frac{1}{16}\xi^2\eta^3 \\
&\quad + \frac{1}{8}\xi^2\eta^4 + \frac{3}{16}\xi^3\eta + \frac{1}{4}\xi^3\eta^2 - \frac{1}{16}\xi^3\eta^3 - \frac{1}{8}\xi^3\eta^4 + \frac{3}{16}\xi^4\eta \\
&\quad + \frac{1}{4}\xi^4\eta^2 - \frac{1}{16}\xi^4\eta^3 - \frac{1}{8}\xi^4\eta^4 \\
\bar{H}_{11} &= -\frac{3}{16}\xi\eta - \frac{3}{16}\xi\eta^2 + \frac{3}{16}\xi\eta^3 + \frac{3}{16}\xi\eta^4 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta^3 \\
&\quad + \frac{1}{4}\xi^2\eta^4 + \frac{1}{16}\xi^3\eta + \frac{1}{16}\xi^3\eta^2 - \frac{1}{16}\xi^3\eta^3 - \frac{1}{16}\xi^3\eta^4 + \frac{1}{8}\xi^4\eta \\
&\quad + \frac{1}{8}\xi^4\eta^2 - \frac{1}{8}\xi^4\eta^3 - \frac{1}{8}\xi^4\eta^4 \\
\bar{H}_{12} &= \frac{1}{16}\xi\eta + \frac{1}{16}\xi\eta^2 - \frac{1}{16}\xi\eta^3 - \frac{1}{16}\xi\eta^4 + \frac{1}{16}\xi^2\eta + \frac{1}{16}\xi^2\eta^2 - \frac{1}{16}\xi^2\eta^3 \\
&\quad - \frac{1}{16}\xi^2\eta^4 - \frac{1}{16}\xi^3\eta - \frac{1}{16}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 + \frac{1}{16}\xi^3\eta^4 \\
&\quad - \frac{1}{16}\xi^4\eta - \frac{1}{16}\xi^4\eta^2 + \frac{1}{16}\xi^4\eta^3 + \frac{1}{16}\xi^4\eta^4
\end{aligned}$$

$$\begin{aligned}
\bar{H}_{13} &= -\frac{9}{16}\xi\eta - \frac{3}{4}\xi\eta^2 + \frac{3}{16}\xi\eta^3 + \frac{3}{8}\xi\eta^4 + \frac{3}{4}\xi^2\eta + \xi^2\eta^2 - \frac{1}{4}\xi^2\eta^3 - \frac{1}{2}\xi^2\eta^4 \\
&\quad + \frac{3}{16}\xi^3\eta + \frac{1}{4}\xi^3\eta^2 - \frac{1}{16}\xi^3\eta^3 - \frac{1}{8}\xi^3\eta^4 - \frac{3}{8}\xi^4\eta - \frac{1}{2}\xi^4\eta^2 \\
&\quad + \frac{1}{8}\xi^4\eta^3 + \frac{1}{4}\xi^4\eta^4 \\
\bar{H}_{14} &= -\frac{3}{16}\xi\eta - \frac{1}{4}\xi\eta^2 + \frac{1}{16}\xi\eta^3 + \frac{1}{8}\xi\eta^4 + \frac{3}{16}\xi^2\eta + \frac{1}{4}\xi^2\eta^2 - \frac{1}{16}\xi^2\eta^3 \\
&\quad - \frac{1}{8}\xi^2\eta^4 + \frac{3}{16}\xi^3\eta + \frac{1}{4}\xi^3\eta^2 - \frac{1}{16}\xi^3\eta^3 - \frac{1}{8}\xi^3\eta^4 - \frac{3}{16}\xi^4\eta \\
&\quad - \frac{1}{4}\xi^4\eta^2 + \frac{1}{16}\xi^4\eta^3 + \frac{1}{8}\xi^4\eta^4 \\
\bar{H}_{15} &= \frac{3}{16}\xi\eta + \frac{3}{16}\xi\eta^2 - \frac{3}{16}\xi\eta^3 - \frac{3}{16}\xi\eta^4 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta^3 \\
&\quad + \frac{1}{4}\xi^2\eta^4 - \frac{1}{16}\xi^3\eta - \frac{1}{16}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 + \frac{1}{16}\xi^3\eta^4 + \frac{1}{8}\xi^4\eta \\
&\quad + \frac{1}{8}\xi^4\eta^2 - \frac{1}{8}\xi^4\eta^3 - \frac{1}{8}\xi^4\eta^4 \\
\bar{H}_{16} &= \frac{1}{16}\xi\eta + \frac{1}{16}\xi\eta^2 - \frac{1}{16}\xi\eta^3 - \frac{1}{16}\xi\eta^4 - \frac{1}{16}\xi^2\eta - \frac{1}{16}\xi^2\eta^2 + \frac{1}{16}\xi^2\eta^3 \\
&\quad + \frac{1}{16}\xi^2\eta^4 - \frac{1}{16}\xi^3\eta - \frac{1}{16}\xi^3\eta^2 + \frac{1}{16}\xi^3\eta^3 + \frac{1}{16}\xi^3\eta^4 \\
&\quad + \frac{1}{16}\xi^4\eta + \frac{1}{16}\xi^4\eta^2 - \frac{1}{16}\xi^4\eta^3 - \frac{1}{16}\xi^4\eta^4 \\
\bar{H}_{17} &= -\frac{3}{4}\eta + \eta^2 + \frac{1}{4}\eta^3 - \frac{1}{2}\eta^4 + \frac{3}{2}\xi^2\eta - 2\xi^2\eta^2 - \frac{1}{2}\xi^2\eta^3 + \xi^2\eta^4 - \frac{3}{4}\xi^4\eta \\
&\quad + \xi^4\eta^2 + \frac{1}{4}\xi^4\eta^3 - \frac{1}{2}\xi^4\eta^4 \\
\bar{H}_{18} &= -\frac{1}{4}\eta + \frac{1}{4}\eta^2 + \frac{1}{4}\eta^3 - \frac{1}{4}\eta^4 + \frac{1}{2}\xi^2\eta - \frac{1}{2}\xi^2\eta^2 - \frac{1}{2}\xi^2\eta^3 + \frac{1}{2}\xi^2\eta^4 \\
&\quad - \frac{1}{4}\xi^4\eta + \frac{1}{4}\xi^4\eta^2 + \frac{1}{4}\xi^4\eta^3 - \frac{1}{4}\xi^4\eta^4 \\
\bar{H}_{19} &= \frac{3}{4}\xi - \frac{3}{2}\xi\eta^2 + \frac{3}{4}\xi\eta^4 + \xi^2 - 2\xi^2\eta^2 + \xi^2\eta^4 - \frac{1}{4}\xi^3 + \frac{1}{2}\xi^3\eta^2 - \frac{1}{4}\xi^3\eta^4 \\
&\quad - \frac{1}{2}\xi^4 + \xi^4\eta^2 - \frac{1}{2}\xi^4\eta^4 \\
\bar{H}_{20} &= -\frac{1}{4}\xi + \frac{1}{2}\xi\eta^2 - \frac{1}{4}\xi\eta^4 - \frac{1}{4}\xi^2 + \frac{1}{2}\xi^2\eta^2 - \frac{1}{4}\xi^2\eta^4 + \frac{1}{4}\xi^3 - \frac{1}{2}\xi^3\eta^2 \\
&\quad + \frac{1}{4}\xi^3\eta^4 + \frac{1}{4}\xi^4 - \frac{1}{2}\xi^4\eta^2 + \frac{1}{4}\xi^4\eta^4 \\
\bar{H}_{21} &= \frac{3}{4}\eta + \eta^2 - \frac{1}{4}\eta^3 - \frac{1}{2}\eta^4 - \frac{3}{2}\xi^2\eta - 2\xi^2\eta^2 + \frac{1}{2}\xi^2\eta^3 + \xi^2\eta^4 + \frac{3}{4}\xi^4\eta \\
&\quad + \xi^4\eta^2 - \frac{1}{4}\xi^4\eta^3 - \frac{1}{2}\xi^4\eta^4
\end{aligned}$$

$$\begin{aligned}
\bar{H}_{22} &= -\frac{1}{4}\eta - \frac{1}{4}\eta^2 + \frac{1}{4}\eta^3 + \frac{1}{4}\eta^4 + \frac{1}{2}\xi^2\eta + \frac{1}{2}\xi^2\eta^2 - \frac{1}{2}\xi^2\eta^3 - \frac{1}{2}\xi^2\eta^4 \\
&\quad - \frac{1}{4}\xi^4\eta - \frac{1}{4}\xi^4\eta^2 + \frac{1}{4}\xi^4\eta^3 + \frac{1}{4}\xi^4\eta^4 \\
\bar{H}_{23} &= -\frac{3}{4}\xi + \frac{3}{2}\xi\eta^2 - \frac{3}{4}\xi\eta^4 + \xi^2 - 2\xi^2\eta^2 + \xi^2\eta^4 + \frac{1}{4}\xi^3 - \frac{1}{2}\xi^3\eta^2 \\
&\quad + \frac{1}{4}\xi^3\eta^4 - \frac{1}{2}\xi^4 + \xi^4\eta^2 - \frac{1}{2}\xi^4\eta^4 \\
\bar{H}_{24} &= -\frac{1}{4}\xi + \frac{1}{2}\xi\eta^2 - \frac{1}{4}\xi\eta^4 + \frac{1}{4}\xi^2 - \frac{1}{2}\xi^2\eta^2 + \frac{1}{4}\xi^2\eta^4 + \frac{1}{4}\xi^3 - \frac{1}{2}\xi^3\eta^2 \\
&\quad + \frac{1}{4}\xi^3\eta^4 - \frac{1}{4}\xi^4 + \frac{1}{2}\xi^4\eta^2 - \frac{1}{4}\xi^4\eta^4 \\
\bar{H}_{25} &= 1 - 2\eta^2 + \eta^4 - 2\xi^2 + 4\xi^2\eta^2 - 2\xi^2\eta^4 + \xi^4 - 2\xi^4\eta^2 + \xi^4\eta^4
\end{aligned}$$

The shorthand notation that is used to represent the set of Hermite shape functions is

$$[\bar{H}(\xi, \eta)] = [\bar{H}_1 \quad \bar{H}_2 \quad \dots \quad \bar{H}_{25}] \quad (2.53)$$

In summary, the MONNA CLPT plate element has 9 nodes and 43 degrees of freedom. The lateral displacements are given by

$$\begin{aligned}
w_b(\xi, \eta) &= [\bar{H}(\xi, \eta)]\{\bar{q}_{w_b}\} \\
\frac{\partial w_b}{\partial \xi}(\xi, \eta) &= \left[\frac{\partial \bar{H}}{\partial \xi}(\xi, \eta) \right] \{\bar{q}_{w_b}\} \\
\frac{\partial w_b}{\partial \eta}(\xi, \eta) &= \left[\frac{\partial \bar{H}}{\partial \eta}(\xi, \eta) \right] \{\bar{q}_{w_b}\} \\
\frac{\partial w_b^2}{\partial \xi \partial \eta}(\xi, \eta) &= \left[\frac{\partial \bar{H}}{\partial \xi \partial \eta}(\xi, \eta) \right] \{\bar{q}_{w_b}\}
\end{aligned} \quad (2.54)$$

where $\{\bar{q}_{w_b}\}$ is the lateral displacement element degree of freedom vector in the standard domain. The bar accent represents the standard domain.

2.3.5 MONNA CLPT Physical Domain Transformation

The previous shape functions are worked out in the standard domain, and it is necessary to transform them back into the physical domain. There are two transformations that must occur to successfully achieve the desired outcome. These two transformations are the transformation of the degree of freedom vector and the transformation of the shape function differentiation.

Conformity, and Transformation

It could be seen so far from the formulation that the MONNA plate element is a C^1 continuous plate. There are two types of C^1 continuous plate elements. A *conforming element* and *non-conforming element*. The conforming one is where the lateral displacement w and the normal slope $\frac{\partial w}{\partial n}$ are continuous between elements along the edges of the elements. This is conformity is done by ensuring that both w and $\frac{\partial w}{\partial n}$ are defined in each element. So, for example, if $\frac{\partial w}{\partial n}$ is of cubic variation, then there must be three DOF available along that side to uniquely define its cubic variation and have a successful conforming plate element. A non-conforming element on the other hand is where the continuity of the slopes $\left(\frac{\partial w}{\partial n}\right)$ are not satisfied between the elements. This case is due to the fact that a non-conforming element has inadequate number of DOF between different elements. However, it is still capable of uniquely defining the $\frac{\partial w}{\partial n}$ along it varies along sides or elements. This also yields some difficulty with convergence as per Reddy (2004, 2006).

The transformation of the lateral degree of freedom, w , results in two strict requirements pertaining to the explicit degrees of freedom needed at a node if the physical domain is not a rectangle. First, a slope d.o.f. in the standard domain would become a function of both physical domain slopes. Second, the cross-derivative transformation requires second derivative d.o.f. that are not present in the MONNA formulation. Tests made by Haught (2020) has indicated that the presence of both slope d.o.f. at node results in singularities in the derivation process. The MONNA element can model only rectangular domains in the physical space but not quadrilateral regions.

Nevertheless, the MONNA element is a conforming element since it has a normal slope that varies quartically along the boundary, allowing at least four degrees of freedom to be present to define each normal slope variation.

Rectangular Transformation

In the case of rectangular transformation, the quantities $J_{12}, J_{21}, J_{11}^\eta$, and J_{12}^η are equal to zero. This helps simplify the transformation process. Using this simplification, the transformed shape functions associated with physical domain become:

$$\begin{aligned} H_1 &= \bar{H}_1 \\ H_2 &= J_{11} \bar{H}_2 \\ H_3 &= J_{22} \bar{H}_3 \\ H_4 &= J_{11} J_{22} \bar{H}_2 \\ &\vdots \\ H_{25} &= \bar{H}_{25} \end{aligned} \tag{2.55}$$

The shorthand notation that is used to represent the set of Hermite shape functions in the physical domain is

$$[H(\xi, \eta)] = [H_1 \quad H_2 \quad \dots \quad H_{25}] \quad (2.56)$$

Then, the displacement could be defined in physical domain as

$$w_b = [\bar{H}(\xi, \eta)]\{\bar{q}_{w_b}\} = [H(\xi, \eta)]\{q_{w_b}\} \quad (2.57)$$

Finally, for the shape function differentiation transformation, the inverse transformation expressions in Eqs. (2.41) and (2.44) were defined earlier to help with that. For the case of a rectangular shape profiles, J_{12}^*, J_{21}^* and T_A are zero. The first derivative transformations that the in-plane displacements which are dealt with using the Lagrange shape functions are given by

$$\begin{aligned} \frac{\partial H_L}{\partial x} &= J_{11}^* \frac{\partial H_L}{\partial \xi} \\ \frac{\partial H_L}{\partial y} &= J_{22}^* \frac{\partial H_L}{\partial \eta} \end{aligned} \quad (2.58)$$

The first and second derivatives are used for the lateral displacement. The Hermite shape functions are transformed in this case as follows

$$\begin{aligned} \frac{\partial H}{\partial x} &= J_{11}^* \frac{\partial H}{\partial \xi} \\ \frac{\partial H}{\partial y} &= J_{22}^* \frac{\partial H}{\partial \eta} \\ \frac{\partial^2 H}{\partial x^2} &= (J_{11}^*)^2 \frac{\partial^2 H}{\partial \xi^2} \\ \frac{\partial^2 H}{\partial y^2} &= (J_{22}^*)^2 \frac{\partial^2 H}{\partial \eta^2} \\ \frac{\partial^2 H}{\partial x \partial y} &= J_{11}^* J_{22}^* \frac{\partial^2 H}{\partial \xi \partial \eta} \end{aligned} \quad (2.59)$$

This could be simplified for a case of a rectangle in the physical domain. Considering a rectangle of size $2a$ in the x direction and $2b$ in the y direction. Nothing that, from Figure (2.4)

$$\begin{aligned} x_4 &= x_1 \\ x_3 &= x_2 \\ y_2 &= y_1 \\ y_4 &= y_3 \end{aligned} \quad (2.60)$$

This yields

$$J_{11} = a \quad (2.61)$$

$$\begin{aligned}
J_{12} &= 0 \\
J_{21} &= 0 \\
J_{22} &= b \\
|J| &= ab
\end{aligned} \tag{2.62}$$

Thus

$$\begin{aligned}
J_{11}^* &= 1/a \\
J_{12}^* &= 0 \\
J_{21}^* &= 0 \\
J_{22}^* &= 1/b
\end{aligned} \tag{2.63}$$

Therefore,

$$\begin{aligned}
\frac{\partial H}{\partial x} &= \frac{1}{a} \frac{\partial H}{\partial \xi} \\
\frac{\partial H}{\partial y} &= \frac{1}{b} \frac{\partial H}{\partial \eta} \\
\frac{\partial^2 H}{\partial x^2} &= \left(\frac{1}{a}\right)^2 \frac{\partial^2 H}{\partial \xi^2} \\
\frac{\partial^2 H}{\partial y^2} &= \left(\frac{1}{b}\right)^2 \frac{\partial^2 H}{\partial \eta^2} \\
\frac{\partial^2 H}{\partial x \partial y} &= \frac{1}{ab} \frac{\partial^2 H}{\partial \xi \partial \eta}
\end{aligned} \tag{2.64}$$

2.3.6 MONNA CLPT Stiffness Matrix Formulation

The virtual strain energy expression Eq. (2.14), δU , is used to derive the finite element stiffness. The relationship between the virtual strain energy and the element stiffness matrix is

$$\delta U_e = [\delta q_e][k_e]\{q_e\} \tag{2.65}$$

where $\{q_e\}$ is the element degree of freedom vector and $[k_e]$ is the element stiffness matrix. The shape functions are also used to express the real and virtual displacements as shown below for the in-plane displacements, the expressions for u are shown below.

$$\begin{aligned}
u &= [H_L]\{q_e\} \\
\delta u &= [\delta q_u]\{H_L\}
\end{aligned} \tag{2.66}$$

Similarly, this is the case for all other real and virtual variables, such as v , and w_b . The stiffness submatrices are derived below:

$$\begin{aligned}
[k_{uu}] &= \iint_A (A_{11}\{H_L^x\}[H_L^x] + A_{16}\{H_L^x\}[H_L^y] + A_{16}\{H_L^y\}[H_L^x] \\
&\quad + A_{66}\{H_L^y\}[H_L^y])dx \, dy \\
[k_{uv}] &= \iint_A (A_{12}\{H_L^x\}[H_L^y] + A_{16}\{H_L^x\}[H_L^x] + A_{26}\{H_L^y\}[H_L^y] \\
&\quad + A_{66}\{H_L^y\}[H_L^x])dx \, dy \\
[k_{uw_b}] &= \iint_A (-B_{11}\{H_L^x\}[H^{xx}] - B_{12}\{H_L^x\}[H^{yy}] - 2B_{16}\{H_L^x\}[H^{xy}] \\
&\quad - B_{16}\{H_L^y\}[H^{xx}] - B_{26}\{H_L^y\}[H^{yy}] - 2B_{66}\{H_L^y\}[H^{xy}])dx \, dy \\
[k_{vv}] &= \iint_A (A_{22}\{H_L^y\}[H_L^y] + A_{26}\{H_L^y\}[H_L^x] + A_{26}\{H_L^x\}[H_L^y] \\
&\quad + A_{66}\{H_L^x\}[H_L^x])dx \, dy \tag{2.67} \\
[k_{vw_b}] &= \iint_A (-B_{12}\{H_L^y\}[H^{xx}] - B_{22}\{H_L^y\}[H^{yy}] - 2B_{26}\{H_L^y\}[H^{xy}] \\
&\quad - B_{16}\{H_L^x\}[H^{xx}] - B_{26}\{H_L^x\}[H^{yy}] - 2B_{66}\{H_L^x\}[H^{xy}])dx \, dy \\
[k_{w_bw_b}] &= \iint_A (N_x\{H^x\}[H^x] + D_{11}\{H^{xx}\}[H^{xx}] + D_{12}\{H^{xx}\}[H^{yy}] \\
&\quad + 2D_{16}\{H^{xx}\}[H^{xy}] + N_y\{H^y\}[H^y] + D_{12}\{H^{yy}\}[H^{xx}] \\
&\quad + D_{22}\{H^{yy}\}[H^{yy}] + 2D_{26}\{H^{yy}\}[H^{xy}] + 2D_{16}\{H^{xy}\}[H^{xx}] \\
&\quad + 2D_{26}\{H^{xy}\}[H^{yy}] + 4D_{66}\{H^{xy}\}[H^{xy}])dx \, dy
\end{aligned}$$

The various submatrices could be organized together into a symmetric matrix. The stiffness matrix is symmetric about the diagonal as shown below.

$$[k_e] = \begin{bmatrix} [k_{uu}] & [k_{uv}] & [k_{uw_b}] \\ [k_{uv}]^T & [k_{vv}] & [k_{vw_b}] \\ [k_{uw_b}]^T & [k_{vw_b}]^T & [k_{w_bw_b}] \end{bmatrix} \tag{2.68}$$

2.3.7 MONNA CLPT Inertia Matrix Formulation

The virtual kinetic energy expression Eq. (2.22), δT , is used to derive the finite element inertia matrix. The relationship between the virtual kinetic energy and the element inertia matrix is

$$-\delta T_e = [\delta q_e][M_e]\{q_e\} \quad (2.69)$$

where $[M_e]$ is the element inertia matrix. The inertia submatrices are derived below:

$$\begin{aligned} [M_{uu}] &= \iint_A I_0\{H_L\}[H_L] dx dy \\ [M_{uv}] &= [0] \\ [M_{uw_b}] &= \iint_A -I_1\{H_L\}[H^x] dx dy \\ [M_{vv}] &= \iint_A I_0\{H_L\}[H_L] dx dy \\ [M_{vw_b}] &= \iint_A -I_1\{H_L\}[H^y] dx dy \\ [M_{w_bw_b}] &= \iint_A (I_2\{H^x\}[H^x] + I_2\{H^y\}[H^y] + I_0\{H\}[H]) dx dy \end{aligned} \quad (2.70)$$

Similar to the stiffness matrix, the various submatrices of the inertia could be organized together into a symmetric matrix. The inertia matrix is symmetric about the diagonal as shown below.

$$[M_e] = \begin{bmatrix} [M_{uu}] & [M_{uv}] & [M_{uw_b}] \\ [M_{uv}]^T & [M_{vv}] & [M_{vw_b}] \\ [M_{uw_b}]^T & [M_{vw_b}]^T & [M_{w_bw_b}] \end{bmatrix} \quad (2.71)$$

2.4 Formulation Based on the FSDT

2.4.1 First Order Plate Theory

The major difference between the classical laminated plate theory and the first-order plate theory is that Kirchhoff's third assumption is no longer assumed. This means that the transverse normals no longer remain perpendicular to the mid surface after deformation. This causes transverse shear strains and their effects to be introduced in the theory. This simple yet a major correction between the two theories that can affect the results of the analysis tremendously, as not including the transverse shear effects can lead

to huge errors in composite plates because of their low shear modulus. The deformed geometry of an FSDT plate is shown in Figure (2.5).

2.4.2 FSDT Equations of Kinematics

Similar to the CLPT element, the FSDT plate element has a displacement field shown below in Eq. (2.72)

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + z \phi_x(x, y, z) \\ v(x, y, z, t) &= v_0(x, y, t) + z \phi_y(x, y, z) \\ w(x, y, z, t) &= w_0(x, y, t) \end{aligned} \quad (2.72)$$

The lateral displacement is split into bending and shear components as

$$w(x, y, z, t) = w_b(x, y, t) + w_s(x, y, t) \quad (2.73)$$

where w_b is the lateral displacement due to bending and w_s is the lateral displacements due to shear. The rotations, ϕ_x and ϕ_y , of the FSDT element are defined as

$$\begin{aligned} \phi_x &= \gamma_{xz} - \frac{\partial w_0}{\partial x} \\ \phi_y &= \gamma_{yz} - \frac{\partial w_0}{\partial y} \end{aligned} \quad (2.74)$$

where γ_{xz} and γ_{yz} are the transverse shear strains. This is transformed into components due to bending and shear as

$$\begin{aligned} \phi_x &= \frac{\partial w_s}{\partial x} - \frac{\partial w}{\partial x} = -\frac{\partial w_b}{\partial x} \\ \phi_y &= \frac{\partial w_s}{\partial y} - \frac{\partial w}{\partial y} = -\frac{\partial w_b}{\partial y} \end{aligned} \quad (2.75)$$

The deformed Geometry is shown in Figure (2.5).

Figure 2.5: FSDT deformed geometry as per Haught (2020)

The formulation of the FSDT and the HSDT plate element are quite similar, but with some major differences. Hence differences are discussed as the HSDT formulation is introduced and discussed in the following sections.

2.5 Formulation Based on the HSDT

2.5.1 Higher Order Shear Deformation Plate Theory

In the higher-order shear deformation theory, not only is the third assumptions of Kirchhoff's hypothesis is no longer assumed, but also the first one. Consequently, the inplane displacements vary cubically across the thickness. This allows the transverse shear to vary quadratically through the thickness of each ply. This improves the accuracy of the transverse shear distribution.

2.5.2 HSDT Equations of Kinematics

The displacement fields used to formulate the HSDT element are the same ones used by Reddy (2004). Also known as Reddy's third-order shear deformation are shown below

$$u(x, y, z, t) = u_o(x, y, t) + z \phi_x(x, y, t) - z^3 c_1 \left(\phi_x + \frac{\partial w_0}{\partial x} \right) \quad (2.76)$$

$$v(x, y, z, t) = v_o(x, y, t) + z \phi_y(x, y, t) - z^3 c_1 \left(\phi_y + \frac{\partial w_o}{\partial y} \right)$$

$$w(x, y, z, t) = w_o(x, y, t) = w_b(x, y, t) + w_s(x, y, t)$$

where c_1 is called a tracer. The displacement field corresponding to FSDT can be recovered by setting $c_1 = 0$. The tracer value for HSDT is found by applying the condition that the transverse shear stress, τ_{yz} and τ_{xz} , must be zero at the top and bottom surfaces of the laminate at ($z = \pm h/2$). This makes the tracer to be

$$c_1 = \frac{4}{3h^3} \quad (2.77)$$

Substituting Eq. (2.75) into Eq. (2.76), the displacement field becomes

$$u(x, y, z, t) = u_o(x, y, t) - z \frac{\partial w_b}{\partial x} - z^3 c_1 \frac{\partial w_s}{\partial x}$$

$$v(x, y, z, t) = v_o(x, y, t) + z \frac{\partial w_b}{\partial y} - z^3 c_1 \frac{\partial w_s}{\partial y} \quad (2.78)$$

$$w(x, y, z, t) = w_o(x, y, t) = w_b(x, y, t) + w_s(x, y, t)$$

A comparison between the transverse normal deformation of the three main theories discussed in the present thesis is shown in Figure (2.6).

Figure 2.6: Transverse normal deformations of CLPT, FSDT, and HSDT [Haught (2020)]

The nonlinear kinematic strains for moderate rotations are given by

$$\begin{aligned} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} &= \begin{Bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_x^{(3)} \\ \varepsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \\ \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} &= \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \end{aligned} \quad (2.79)$$

where the various parts of the strains are defined as

$$\{\varepsilon^{(0)}\} = \begin{Bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_o}{\partial x} + \frac{1}{2} \left(\frac{\partial w_b}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w_s}{\partial x} \right)^2 \\ \frac{\partial v_o}{\partial y} + \frac{1}{2} \left(\frac{\partial w_b}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial w_s}{\partial y} \right)^2 \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{Bmatrix} \quad (2.80)$$

$$\{\varepsilon^{(1)}\} = \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix} \quad (2.81)$$

$$\{\varepsilon^{(3)}\} = \begin{Bmatrix} \varepsilon_x^{(3)} \\ \varepsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = C_1 \begin{Bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix} \quad (2.82)$$

$$\{\gamma^{(0)}\} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix} \quad (2.83)$$

$$\{\gamma^{(2)}\} = \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = -3c_1 \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix} \quad (2.84)$$

The third term in Eq. (2.80), $\gamma_{xy}^{(0)}$, is neglected to avoid nonlinearity in the governing equations.

2.5.3 Constitutive Equations for HSDT Composite Laminate

The constitutive equation for a HSDT composite laminate is

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [F] \\ [B] & [D] & [F] \\ [E] & [F] & [H] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \\ \{\varepsilon^{(3)}\} \end{Bmatrix} \quad (2.85)$$

$$\begin{Bmatrix} \{Q\} \\ \{R\} \end{Bmatrix} = \begin{bmatrix} [A] & [D] \\ [D] & [F] \end{bmatrix} \begin{Bmatrix} \{\gamma^{(0)}\} \\ \{\gamma^{(2)}\} \end{Bmatrix} \quad (2.86)$$

As discussed earlier, the $\{N\}$ vector denotes the force resultant and the $\{M\}$ denotes the moment resultant. The $[A]$, $[B]$, and $[D]$ matrices contain the extension stiffness, bending-extension coupling stiffness, and the bending stiffness, respectively. The matrices $[E]$, $[F]$, and $[H]$ have higher-order stiffness coefficient terms. The vector $\{Q\}$ represents the transverse force resultant and the vectors $\{P\}$ and $\{R\}$ are higher-order stress resultants. The vectors and matrices expressions are:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad (2.87)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad (2.88)$$

$$\begin{Bmatrix} P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^3 \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \quad (2.89)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz \quad (2.90)$$

$$\begin{Bmatrix} R \\ R_x \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz \quad (2.91)$$

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \bar{Q}_{ij}^{(k)}(1, z, z^2, z^3, z^4, z^6) dz \quad i, j = 1, 2, 6 \quad (2.92)$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \bar{Q}_{ij}^{(k)}(1, z, z^4) dz \quad i, j = 4, 5 \quad (2.93)$$

The expanded form of the constitutive equation Eq. (2.85) becomes

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \\ \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \\ \varepsilon_x^{(3)} \\ \varepsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \quad (2.94)$$

Also, the expanded form of the constitutive equation Eq. (2.86) becomes

$$\begin{Bmatrix} Q_x \\ Q_y \\ R_x \\ R_y \end{Bmatrix} = \begin{bmatrix} A_{55} & D_{55} & A_{45} & D_{45} \\ A_{45} & D_{45} & A_{44} & D_{44} \\ D_{55} & F_{55} & D_{45} & F_{45} \\ D_{45} & F_{45} & D_{44} & F_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} \quad (2.95)$$

Eq. (2.95) is rearranged such that the x terms are grouped together, and the y terms are grouped together as shown below

$$\begin{Bmatrix} Q_x \\ R_x \\ Q_y \\ R_y \end{Bmatrix} = \begin{bmatrix} A_{55} & D_{55} & A_{45} & D_{45} \\ D_{55} & F_{55} & D_{45} & F_{45} \\ A_{45} & D_{45} & A_{44} & D_{44} \\ D_{45} & F_{45} & D_{44} & F_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^{(0)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(0)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} \quad (2.96)$$

Then, introducing the following variables

$$D_{ij}^* = A_{ij} - 6c_1 D_{ij} + 9c_1^2 F_{ij} \quad i, j = 4,$$

$$Q_x^* = Q_x - 3c_1 R_x \quad (2.97)$$

$$Q_y^* = Q_y - 3c_1 R_y$$

Then, Eq. (2.96) is as

$$\begin{Bmatrix} Q_y^* \\ Q_x^* \end{Bmatrix} = K_s \begin{bmatrix} D_{44}^* & D_{45}^* \\ D_{45}^* & D_{55}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix} \quad (2.98)$$

where the variable K_s is introduced as the shear correction factor, discussed previously, which is needed for the FSDT formulation. for obtaining the FSDT results set $c_1 = 0$ and $K_s = 5/6$ and for HSDT, set $c_1 = 1$ and $K_s = 1$.

2.5.4 Hamilton's Principle used for HSDT Composite Laminate

The HSDT composite laminate uses that same Hamilton's principle as the CLPT case as shown below and discussed earlier.

$$\int_{t_1}^{t_2} (\delta U - \delta T - \delta W) dt = 0 \quad (2.99)$$

2.5.5 Virtual Strain Energy of HSDT Composite Laminate

For HSDT, the expression for δU over a plate is given by

$$\delta U = \iiint_V (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}) dV \quad (2.100)$$

The volume integral for HSDT is also divided into an integral over the plate area A and another integral over the thickness h .

$$\delta U = \iint_A \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}] dA dz \quad (2.101)$$

Expanding Eq. (2.101)

$$\begin{aligned}
\delta U = \iint_A \int_{-\frac{h}{2}}^{\frac{h}{2}} & \left[\sigma_x \left(\delta \varepsilon_x^{(0)} + z \delta \varepsilon_x^{(1)} + z^3 \delta \varepsilon_x^{(3)} \right) + \sigma_y \left(\delta \varepsilon_y^{(0)} + z \delta \varepsilon_y^{(1)} + z^3 \delta \varepsilon_y^{(3)} \right) \right. \\
& + \tau_{xy} \left(\delta \gamma_{xy}^{(0)} + z \delta \gamma_{xy}^{(1)} + z^3 \delta \gamma_{xy}^{(3)} \right) + \tau_{xz} \left(\delta \gamma_{xz}^{(0)} + z^2 \delta \gamma_{xz}^{(2)} \right) \\
& \left. + \tau_{yz} \left(\delta \gamma_{yz}^{(0)} + z^2 \delta \gamma_{yz}^{(2)} \right) \right] dA dz
\end{aligned} \tag{2.102}$$

Plugging Eqs. (2.87 – 2.93) into Eq. (2.102) yields

$$\begin{aligned}
\delta U = \iint_A & \left[N_x \delta \varepsilon_x^{(0)} + M_x \delta \varepsilon_x^{(1)} + P_x \delta \varepsilon_x^{(3)} + N_y \delta \varepsilon_y^{(0)} + M_y \delta \varepsilon_y^{(1)} + P_y \delta \varepsilon_y^{(3)} + N_{xy} \delta \gamma_{xy}^{(0)} \right. \\
& \left. + M_{xy} \delta \gamma_{xy}^{(1)} + P_{xy} \delta \gamma_{xy}^{(3)} + Q_x \delta \gamma_{xz}^{(0)} + R_x \delta \gamma_{xz}^{(2)} + Q_y \delta \gamma_{yz}^{(0)} + R_y \delta \gamma_{yz}^{(2)} \right] dA
\end{aligned} \tag{2.103}$$

Using the shorthand notation introduced in Eq. (2.13) to rewrite the virtual strain energy

$$\begin{aligned}
\delta U = \iint_A & \{ N_x (\delta u_0^x + w_b^x \delta w_b^x + w_s^x \delta w_s^x) - M_x \delta w_b^{xx} - P_x c_1 \delta w_s^{xx} \\
& + N_y (\delta v_0^y + w_b^y \delta w_b^y + w_s^y \delta w_s^y) - M_y \delta w_b^{yy} - P_y c_1 \delta w_s^{yy} \\
& + N_{xy} [\delta u_0^y + \delta v_0^x + (w_b^y + w_s^y) (\delta w_b^x + \delta w_s^x) \\
& + (w_b^x + w_s^x) (\delta w_b^y + \delta w_s^y)] - 2M_{xy} \delta w_b^{xy} - 2P_{xy} c_1 \delta w_s^{xy} + Q_x^* \delta w_s^x \\
& + Q_y^* \delta w_s^y \} dA
\end{aligned} \tag{2.104}$$

2.5.6 Virtual Kinetic Energy of HSDT Composite Laminate

The inertia matrix, similar to the CLPT plate element, is found for the HSDT plate element using the following δT expression

$$\delta T = \iiint_V \rho (\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) dV \tag{2.105}$$

This expression is further expanded using the displacement field equation (2.78)

$$\begin{aligned}
\delta T = \iiint_V & \rho [(\dot{u}_0 - z \dot{w}_b^x - c_1 z^3 \dot{w}_s^x) (\delta \dot{u}_0 - z \delta \dot{w}_b^x - c_1 z^3 \delta \dot{w}_s^x) \\
& + (\dot{v}_0 - z \dot{w}_b^y - c_1 z^3 \dot{w}_s^y) (\delta \dot{v}_0 - z \delta \dot{w}_b^y - c_1 z^3 \delta \dot{w}_s^y) \\
& + (\dot{w}_b + \dot{w}_s) (\delta \dot{w}_b + \delta \dot{w}_s)] dV
\end{aligned} \tag{2.106}$$

Rearranging Eq. (2.106)

$$\begin{aligned}
\delta T = \iiint_V \rho [& (\dot{u}_o - z\dot{w}_b^x - c_1 z^3 \dot{w}_s^x) \delta \dot{u}_o - (z\dot{u}_o - z^2 \dot{w}_b^x - c_1 z^4 \dot{w}_s^x) \delta \dot{w}_b^x \\
& - c_1 (z^3 \dot{u}_o - z^4 \dot{w}_b^x - c_1 z^6 \dot{w}_s^x) \delta \dot{w}_s^x + (\dot{v}_o - z\dot{w}_b^y - c_1 z^3 \dot{w}_s^y) \delta \dot{v}_o \\
& - (z\dot{v}_o - z^2 \dot{w}_b^y - c_1 z^4 \dot{w}_s^y) \delta \dot{w}_b^y - c_1 (z^3 \dot{v}_o - z^4 \dot{w}_b^y - c_1 z^6 \dot{w}_s^y) \delta \dot{w}_s^y \\
& + (\dot{w}_b + \dot{w}_s) \delta \dot{w}_b + (\dot{w}_b + \dot{w}_s) \delta \dot{w}_s] dV
\end{aligned} \tag{2.107}$$

Considering the time integral from Hamilton's principle from Eq. (2.99)

$$\begin{aligned}
- \int_{t_1}^{t_2} \delta T dt = \iiint_V \left\{ \int_{t_1}^{t_2} -\rho [& (\dot{u}_o - z\dot{w}_b^x - c_1 z^3 \dot{w}_s^x) \delta \dot{u}_o \right. \\
& - (z\dot{u}_o - z^2 \dot{w}_b^x - c_1 z^4 \dot{w}_s^x) \delta \dot{w}_b^x - c_1 (z^3 \dot{u}_o - z^4 \dot{w}_b^x - c_1 z^6 \dot{w}_s^x) \delta \dot{w}_s^x \\
& + (\dot{v}_o - z\dot{w}_b^y - c_1 z^3 \dot{w}_s^y) \delta \dot{v}_o - (z\dot{v}_o - z^2 \dot{w}_b^y - c_1 z^4 \dot{w}_s^y) \delta \dot{w}_b^y \\
& - c_1 (z^3 \dot{v}_o - z^4 \dot{w}_b^y - c_1 z^6 \dot{w}_s^y) \delta \dot{w}_s^y + (\dot{w}_b + \dot{w}_s) \delta \dot{w}_b \\
& \left. + (\dot{w}_b + \dot{w}_s) \delta \dot{w}_s] dt \right\} dV
\end{aligned} \tag{2.108}$$

Integrating the time integral by parts to eliminate terms containing time derivatives of virtual displacements and grouping together all the boundary terms as

$$\iiint_V [B(t)]_{t_1}^{t_2} dV \tag{2.109}$$

This yields

$$\begin{aligned}
- \int_{t_1}^{t_2} \delta T dt = \iiint_V \left\{ [B(t)]_{t_1}^{t_2} \right. \\
+ \int_{t_1}^{t_2} \rho [& (\ddot{u}_o - z\ddot{w}_b^x - c_1 z^3 \ddot{w}_s^x) \delta u_o \\
& - (z\ddot{u}_o - z^2 \ddot{w}_b^x - c_1 z^4 \ddot{w}_s^x) \delta w_b^x \\
& - c_1 (z^3 \ddot{u}_o - z^4 \ddot{w}_b^x - c_1 z^6 \ddot{w}_s^x) \delta w_s^x + (\ddot{v}_o - z\ddot{w}_b^y - c_1 z^3 \ddot{w}_s^y) \delta v_o \\
& - (z\ddot{v}_o - z^2 \ddot{w}_b^y - c_1 z^4 \ddot{w}_s^y) \delta w_b^y \\
& - c_1 (z^3 \ddot{v}_o - z^4 \ddot{w}_b^y - c_1 z^6 \ddot{w}_s^y) \delta w_s^y + (\ddot{w}_b + \ddot{w}_s) \delta w_b \\
& \left. + (\ddot{w}_b + \ddot{w}_s) \delta w_s] dt \right\} dV
\end{aligned} \tag{2.110}$$

Once again, through this process, the virtual displacements no longer include time derivatives. This makes the integrand valid for all time and can be removed from the time integral. Furthermore, since the boundary terms, $B(t)$, may be neglected, because they do

not contribute to the finite element inertia matrix. Next, the volume integral is split into a double integral where one is integrated over the plate's area and the second is integrated over the plate's thickness. Then the mass moments of inertia are introduced as

$$I_k = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho z^k dV \quad k = 0, 1, 2, \dots, 6 \quad (2.111)$$

The expression becomes

$$\begin{aligned} -\delta T = \iint_A & [(I_0 \ddot{u}_0 - I_1 \ddot{w}_b^x - c_1 I_3 \ddot{w}_s^x) \delta u_0 - (I_1 \ddot{u}_0 - I_2 \ddot{w}_b^x - c_1 I_4 \ddot{w}_s^x) \delta w_b^x \\ & - c_1 (I_3 \ddot{u}_0 - I_4 \ddot{w}_b^x - c_1 I_6 \ddot{w}_s^x) \delta w_s^x + (I_0 \ddot{v}_0 - I_1 \ddot{w}_b^y - c_1 I_3 \ddot{w}_s^y) \delta v_0 \\ & - (I_1 \ddot{v}_0 - I_2 \ddot{w}_b^y - c_1 I_4 \ddot{w}_s^y) \delta w_b^y - c_1 (I_3 \ddot{v}_0 - I_4 \ddot{w}_b^y - c_1 I_6 \ddot{w}_s^y) \delta w_s^y \\ & + I_0 (\ddot{w}_b + \ddot{w}_s) \delta w_b + I_0 (\ddot{w}_b + \ddot{w}_s) \delta w_s] dx dy \end{aligned} \quad (2.112)$$

where I_0 is the normal inertia, I_1 is the coupled normal-rotary inertia, and I_2 is the rotary inertia, and the remaining terms are higher-order terms.

2.6 Finite Element Formulation of the HSDT Plate

2.6.1 HSDT MONNA Plate Element

The goal of this section is to formulate a new conforming plate element based on HSDT theory. As introduced by Sivaneri (2019), the element d.o.f. for the conforming MONNA plate element are shown in Figure (2.7).

Figure 2.7: HSDT MONNNA plate element with DOF distribution as per Haught (2020)

The HSDT MONNNA plate element is also formulated to be a subparametric element that has the plate. It also has a plate coordinate system of (ξ, η, z) at its origin. It has an in-plane dependent displacements variables, u and v and transverse deflections variables, w_b and w_s . The element has four corner nodes (1-4), four midside nodes (5-8), and a ninth node in the center, which makes the number of degrees of freedom (DOF) equal to 68 degrees of freedom as they are shown in Figure (2.7). It is also an h-p version. It also uses the three different orders of polynomials, the bilinear polynomials, the bi-quadratic Lagrange polynomials, and the bi-quartic Hermite polynomials, that were discussed earlier in the CLPT formulation.

2.6.2 HSDT Coordinate Transformation

The main goal of this section is to be able to transform the geometry from the physical domain to the standard domain (ξ, η) and vice versa. The HSDT plate element uses the same coordinate transformation procedure that the CLPT plate element uses. This includes the first derivatives, second derivatives and the inverse transformations. Hence, this section references section (2.3.3) for more details.

2.6.3 HSDT Shape Functions

The main goal of this section is to explore the quadratic Lagrange shape functions used to approximate the in-plane displacements. The HSDT formulation, like the CLPT, uses the same Lagrange shape functions discussed earlier in section (2.3.4). It also explores the Hermite shape functions that are used to accommodate the lateral displacement DOF, w_b and w_s . The MONNNA HSDT formulation also uses Hermite interpolation equations that are used in the CLPT shape function, except for the HSDT

case, these equations are also used to accommodate the lateral displacement, w_s , which is first introduced in the HSDT formulation along with its derivatives. Adding w_s to the formulation increases the number of the degrees of freedom, for the HSDT case, to 68 DOF. However, the procedure is similar to that of the CLPT formulation discussed in section (2.3.4). Thus, for more details, section (2.3.4) is referenced.

2.6.4 MONNA HSDT Physical Domain Transformation

As previously discussed, the MONNA HSDT formulation uses the same transformation as the CLPT formulation as shown in section (2.6.2). Since that is the case, it also uses the same transformation used by the CLPT formulation to transform from the standard domain to the physical domain, discussed in section (2.3.5). It also follows the same conformity and transformation rules, which are also discussed earlier. This also means that the rectangular transformation applies for the HSDT case, as this was the case for the CLPT case. The only difference is that for the HSDT formulation, the w_s degree of freedom is accommodated along with its derivatives accordingly. The same procedure used in the CLPT formulation is also used for the HSDT formulation. Thus, this section references section (2.3.5) for more details.

2.6.5 MONNA HSDT Stiffness Matrix Formulation

The finite element stiffness is derived from the virtual strain energy expression Eq. (2.104), δU . The relationship between the virtual strain energy and the element stiffness matrix is shown below

$$\delta U_e = [\delta q_e][k_e]\{q_e\} \quad (2.113)$$

The shape functions are used to express the real and virtual displacements are shown below

$$\begin{aligned} u &= [H_L]\{q_e\} \\ \delta u &= [\delta q_u]\{H_L\} \end{aligned} \quad (2.114)$$

Similarly, all other real and virtual variables, such as v , w_b , and w_s , are also expressed the same way. The stiffness submatrices are derived below:

$$\begin{aligned} [k_{uu}] &= \iint_A (A_{11}\{H_L^x\}[H_L^x] + A_{16}\{H_L^x\}[H_L^y] + A_{16}\{H_L^y\}[H_L^x] \\ &\quad + A_{66}\{H_L^y\}[H_L^y])dx dy \end{aligned} \quad (2.115)$$

$$\begin{aligned}
[k_{uv}] &= \iint_A (A_{12}\{H_L^x\}[H_L^y] + A_{16}\{H_L^x\}[H_L^x] + A_{26}\{H_L^y\}[H_L^y] \\
&\quad + A_{66}\{H_L^y\}[H_L^x])dx \, dy \\
[k_{uw_b}] &= \iint_A (-B_{11}\{H_L^x\}[H^{xx}] - B_{12}\{H_L^x\}[H^{yy}] - 2B_{16}\{H_L^x\}[H^{xy}] \\
&\quad - B_{16}\{H_L^y\}[H^{xx}] - B_{26}\{H_L^y\}[H^{yy}] - 2B_{66}\{H_L^y\}[H^{xy}])dx \, dy \\
[k_{uw_s}] &= \iint_A C_1(-E_{11}\{H_L^x\}[H^{xx}] - E_{12}\{H_L^x\}[H^{yy}] - 2E_{16}\{H_L^x\}[H^{xy}] \\
&\quad - E_{16}\{H_L^y\}[H^{xx}] - E_{26}\{H_L^y\}[H^{yy}] - 2E_{66}\{H_L^y\}[H^{xy}])dx \, dy \\
[k_{vv}] &= \iint_A (A_{22}\{H_L^y\}[H_L^y] + A_{26}\{H_L^y\}[H_L^x] + A_{26}\{H_L^x\}[H_L^y] \\
&\quad + A_{66}\{H_L^x\}[H_L^x])dx \, dy \\
[k_{vw_b}] &= \iint_A (-B_{12}\{H_L^y\}[H^{xx}] - B_{22}\{H_L^y\}[H^{yy}] - 2B_{26}\{H_L^y\}[H^{xy}] \\
&\quad - B_{16}\{H_L^x\}[H^{xx}] - B_{26}\{H_L^x\}[H^{yy}] - 2B_{66}\{H_L^x\}[H^{xy}])dx \, dy \\
[k_{vw_s}] &= \iint_A C_1(-E_{12}\{H_L^y\}[H^{xx}] - E_{22}\{H_L^y\}[H^{yy}] - 2E_{26}\{H_L^y\}[H^{xy}] \\
&\quad - E_{16}\{H_L^x\}[H^{xx}] - E_{26}\{H_L^x\}[H^{yy}] - 2E_{66}\{H_L^x\}[H^{xy}]) \, dx \, dy \\
[k_{w_bw_b}] &= \iint_A (N_x\{H^x\}[H^x] + D_{11}\{H^{xx}\}[H^{xx}] + D_{12}\{H^{xx}\}[H^{yy}] \\
&\quad + 2D_{16}\{H^{xx}\}[H^{xy}] + N_y\{H^y\}[H^y] + D_{12}\{H^{yy}\}[H^{xx}] \\
&\quad + D_{22}\{H^{yy}\}[H^{yy}] + 2D_{26}\{H^{yy}\}[H^{xy}] + 2D_{16}\{H^{xy}\}[H^{xx}] \\
&\quad + 2D_{26}\{H^{xy}\}[H^{yy}] + 4D_{66}\{H^{xy}\}[H^{xy}])dx \, dy \\
[k_{w_bw_s}] &= \iint_A C_1(F_{11}\{H^{xx}\}[H^{xx}] + F_{12}\{H^{xx}\}[H^{yy}] + 2F_{16}\{H^{xx}\}[H^{xy}] \\
&\quad + F_{12}\{H^{yy}\}[H^{xx}] + F_{22}\{H^{yy}\}[H^{yy}] + 2F_{26}\{H^{yy}\}[H^{xy}] \\
&\quad + 2F_{16}\{H^{xy}\}[H^{xx}] + 2F_{26}\{H^{xy}\}[H^{yy}] \\
&\quad + 4F_{66}\{H^{xy}\}[H^{xy}])dx \, dy
\end{aligned}$$

$$\begin{aligned}
[k_{w_s w_s}] = \iint_A [& N_x \{H^x\} [H^x] + N_y \{H^y\} [H^y] + C_1^2 (H_{11} \{H^{xx}\} [H^{xx}] \\
& + H_{12} \{H^{xx}\} [H^{yy}] + 2H_{16} \{H^{xx}\} [H^{xy}] + H_{12} \{H^{yy}\} [H^{xx}] \\
& + H_{22} \{H^{yy}\} [H^{yy}] + 2H_{26} \{H^{yy}\} [H^{xy}] + 2H_{16} \{H^{xy}\} [H^{xx}] \\
& + 2H_{26} \{H^{xy}\} [H^{yy}] + 4H_{66} \{H^{xy}\} [H^{xy}]) + D_{45}^* \{H^x\} [H^y] \\
& + D_{55}^* \{H^x\} [H^x] + D_{44}^* \{H^y\} [H^y] + D_{45}^* \{H^y\} [H^x]] dx dy
\end{aligned}$$

The various submatrices are organized together into the symmetric matrix shown below.

$$[k_e] = \begin{bmatrix} [k_{uu}] & [k_{uv}] & [k_{uw_b}] & [k_{uw_s}] \\ [k_{uv}]^T & [k_{vv}] & [k_{vw_b}] & [k_{vw_s}] \\ [k_{uw_b}]^T & [k_{vw_b}]^T & [k_{w_b w_b}] & [k_{w_b w_s}] \\ [k_{uw_s}]^T & [k_{vw_s}]^T & [k_{w_b w_s}]^T & [k_{w_s w_s}] \end{bmatrix} \quad (2.116)$$

2.6.6 MONNA HSDT Inertia Matrix Formulation

The finite element Inertia is derived from the virtual kinetic energy expression Eq. (2.112), δT . The relationship between the virtual kinetic energy and the element inertia matrix is shown below

$$-\delta T_e = [\delta q_e] [M_e] \{q_e\} \quad (2.117)$$

Thus, the inertia submatrices are derived below as

$$\begin{aligned}
[M_{uu}] &= \iint_A I_0 \{H_L\} [H_L] dx dy \\
[M_{uv}] &= [0] \\
[M_{uw_b}] &= \iint_A -I_1 \{H_L\} [H^x] dx dy \\
[M_{uw_s}] &= \iint_A -C_1 I_3 \{H_L\} [H^x] dx dy \\
[M_{vv}] &= \iint_A I_0 \{H_L\} [H_L] dx dy \\
[M_{vw_b}] &= \iint_A -I_1 \{H_L\} [H^y] dx dy
\end{aligned} \quad (2.118)$$

$$\begin{aligned}
[M_{vw_s}] &= \iint_A -C_1 I_3 \{H_L\} [H^y] \, dx \, dy \\
[M_{w_b w_b}] &= \iint_A (I_2 \{H^x\} [H^x] + I_2 \{H^y\} [H^y] + I_0 \{H\} [H]) \, dx \, dy \\
[M_{w_b w_s}] &= \iint_A [C_1 (I_4 \{H^x\} [H^x] + I_4 \{H^y\} [H^y]) + I_0 \{H\} [H]] \, dx \, dy \\
[M_{w_s w_s}] &= \iint_A [C_1^2 (I_6 \{H^x\} [H^x] + I_6 \{H^y\} [H^y]) + I_0 \{H\} [H]] \, dx \, dy
\end{aligned}$$

Similar to the stiffness matrix, the various submatrices of the inertia could be organized together into a symmetric matrix. The inertia matrix is symmetric about the diagonal as shown below.

$$[M_e] = \begin{bmatrix} [M_{uu}] & [M_{uv}] & [M_{uw_b}] & [M_{uw_s}] \\ [M_{uv}]^T & [M_{vv}] & [M_{vw_b}] & [M_{vw_s}] \\ [M_{uw_b}]^T & [M_{vw_b}]^T & [M_{w_b w_b}] & [M_{w_b w_s}] \\ [M_{uw_s}]^T & [M_{vw_s}]^T & [M_{w_b w_s}]^T & [M_{w_s w_s}] \end{bmatrix} \quad (2.119)$$

2.6.7 MONNA Stress Formulation

In this section, ply stresses are determined at a point (x_1, y_1) in the global coordinates of a composite plate modelled with MONNA elements. The pseudo code that is enunciated in this section is due to Sivaneri (2020). The nonlinear kinematic equations for moderate rotations are given by

$$\begin{aligned}
\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} &= \begin{Bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \epsilon_x^{(3)} \\ \epsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \\
\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} &= \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}
\end{aligned} \quad (2.120)$$

where the various parts of the strain are

$$\{\varepsilon^{(0)}\} = \begin{Bmatrix} \varepsilon_x^{(0)} \\ \varepsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_o}{\partial x} + \frac{1}{2} \left(\frac{\partial w_b}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w_s}{\partial x} \right)^2 \\ \frac{\partial v_o}{\partial y} + \frac{1}{2} \left(\frac{\partial w_b}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial w_s}{\partial y} \right)^2 \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{Bmatrix} \quad (2.121)$$

$$\{\varepsilon^{(1)}\} = \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix} \quad (2.122)$$

$$\{\varepsilon^{(3)}\} = \begin{Bmatrix} \varepsilon_x^{(3)} \\ \varepsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = C_1 \begin{Bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix} \quad (2.123)$$

$$\{\gamma^{(0)}\} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix} \quad (2.124)$$

$$\{\gamma^{(2)}\} = \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = -3c_1 \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix} \quad (2.125)$$

The third term, $\gamma_{xy}^{(0)}$ in Eq. (2.121) is neglected to avoid any nonlinearity in the governing equations. In order to properly calculate the stresses, the first step is to locate the element corresponding to (x_1, y_1) and then find the following values for that element.

$$\begin{aligned}
u_0^x(x_1, y_1) &= [H_{L1}^x(x_1, y_1) \quad \dots \quad H_{L9}^x(x_1, y_1)] [\bar{u}_1 \quad \dots \quad \bar{u}_9]^T \\
u_0^y(x_1, y_1) &= [H_{L1}^y(x_1, y_1) \quad \dots \quad H_{L9}^y(x_1, y_1)] [\bar{u}_1 \quad \dots \quad \bar{u}_9]^T \\
v_0^x(x_1, y_1) &= [H_{L1}^x(x_1, y_1) \quad \dots \quad H_{L9}^x(x_1, y_1)] [\bar{v}_1 \quad \dots \quad \bar{v}_9]^T \\
v_0^y(x_1, y_1) &= [H_{L1}^y(x_1, y_1) \quad \dots \quad H_{L9}^y(x_1, y_1)] [\bar{v}_1 \quad \dots \quad \bar{v}_9]^T \\
w_b^x(x_1, y_1) &= [H_1^x(x_1, y_1) \quad \dots \quad H_{25}^x(x_1, y_1)] [\bar{w}_{b1} \quad \dots \quad \bar{w}_{b25}]^T \\
w_b^y(x_1, y_1) &= [H_1^y(x_1, y_1) \quad \dots \quad H_{25}^y(x_1, y_1)] [\bar{w}_{b1} \quad \dots \quad \bar{w}_{b25}]^T \\
w_b^{xx}(x_1, y_1) &= [H_1^{xx}(x_1, y_1) \quad \dots \quad H_{25}^{xx}(x_1, y_1)] [\bar{w}_{b1} \quad \dots \quad \bar{w}_{b25}]^T \\
w_b^{yy}(x_1, y_1) &= [H_1^{yy}(x_1, y_1) \quad \dots \quad H_{25}^{yy}(x_1, y_1)] [\bar{w}_{b1} \quad \dots \quad \bar{w}_{b25}]^T \\
w_b^{xy}(x_1, y_1) &= [H_1^{xy}(x_1, y_1) \quad \dots \quad H_{25}^{xy}(x_1, y_1)] [\bar{w}_{b1} \quad \dots \quad \bar{w}_{b25}]^T \\
w_s^x(x_1, y_1) &= [H_1^x(x_1, y_1) \quad \dots \quad H_{25}^x(x_1, y_1)] [\bar{w}_{s1} \quad \dots \quad \bar{w}_{s25}]^T \\
w_s^y(x_1, y_1) &= [H_1^y(x_1, y_1) \quad \dots \quad H_{25}^y(x_1, y_1)] [\bar{w}_{s1} \quad \dots \quad \bar{w}_{s25}]^T \\
w_s^{xx}(x_1, y_1) &= [H_1^{xx}(x_1, y_1) \quad \dots \quad H_{25}^{xx}(x_1, y_1)] [\bar{w}_{s1} \quad \dots \quad \bar{w}_{s25}]^T \\
w_s^{yy}(x_1, y_1) &= [H_1^{yy}(x_1, y_1) \quad \dots \quad H_{25}^{yy}(x_1, y_1)] [\bar{w}_{s1} \quad \dots \quad \bar{w}_{s25}]^T \\
w_s^{xy}(x_1, y_1) &= [H_1^{xy}(x_1, y_1) \quad \dots \quad H_{25}^{xy}(x_1, y_1)] [\bar{w}_{s1} \quad \dots \quad \bar{w}_{s25}]^T
\end{aligned} \tag{2.126}$$

Then after finding the element values, the strain quantities are calculated using Eqs. (2.121 - 2.125) and then plugged into the kinematic Eq. (2.120). Then the ply strains at the top and the bottom of the k^{th} ply, $[\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]_{kt}$, are found. From there, the on-axis stresses and the off-axis stress at the top and the bottom of the k^{th} ply are found as

$$[\sigma_x \quad \sigma_y \quad \tau_{xy}]_{kt}^T = [\bar{Q}]_k [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]_{kt}^T \tag{2.127}$$

$$[\sigma_1 \quad \sigma_2 \quad \sigma_6]_{kt}^T = [T_\sigma]_k [\sigma_x \quad \sigma_y \quad \tau_{xy}]_{kt}^T \tag{2.128}$$

similarly, the ply transverse shear stresses and strains, $[\tau_{yz} \quad \tau_{xz}]$, can be found for the top and bottom of the k^{th} ply.

2.7 BFS Plate Element

To determine the efficiency of the MONNA plate element, it is compared with the performance of different traditional and well-established plate elements. One of those plate elements is the conforming Bogner, Fox and Schmidt (BFS) element (1965).

The BFS element is a four-noded conforming rectangular C^1 plate finite element with 10 degrees of freedom at each node for the HSDT composite BFS element. Making a total of 40 degrees of freedom. In the present analysis, the BFS plate element shown in Figure (2.8) is used to validate and articulate the efficiency of the MONNA plate element. The BFS element is an h-version element with four types of lateral displacement at each

47

Chapter 3

Numerical Methods

This chapter describes the different numerical techniques used in the thesis that aid in the solution process. The Gauss Quadrature integration technique is used to integrate the finite element stiffness and the inertia matrices. The Newmark time integration is one of the most efficient techniques used to solve equations in the time domain. Thus, it solves for the response of the structure as a function of time.

3.1 Gauss Quadrature Integration

The Gauss quadrature integration could be defined as an approximation of the definite integral of a function. The integration works by using n sampling points that yield exact results for polynomials of the degree $(2n - 1)$ or less by a suitable number of nodes, x_i , and weights, w_i for $i = 1, \dots, n$. The two-dimensional integration scheme is shown in Eq. (3.1)

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \sum_{i=1}^n \sum_{j=1}^m w_i w_j f(a_i, b_j) \quad (3.1)$$

where a_i and b_j are the ξ and η coordinates at the sampling points, respectively while, w_i and w_j are the corresponding weights. The sampling points and their corresponding weights for a seven-point scheme are shown below in Table (3.1).

Table 3.1: Gauss quadrature sampling points and weights

Sampling Points	Weights
0.0000000000000000	0.4179591836734694
± 0.4058451513773972	0.3818300505051189
± 0.7415311855993945	0.2797053914892766
± 0.9491079123427585	0.1294849661688697

3.2 Newmark's Integration Method

The Newmark's time integration method is a frequently used technique in the structural dynamics area of research. It is coupled with the finite element method to solve

equations in the time domain. This method could be considered a direct extension of the linear acceleration method. There are different types of Newmark's integration method. In this thesis, the direct integration method, used by Polina (2014), is employed for its simplicity, accuracy, and stability. The assumptions behind this method are:

1. The equilibrium equation is satisfied at discrete time intervals.
2. The displacements, velocities, and accelerations vary within each time interval.

The equation of motion is

$$[M]\{\ddot{q}_{t+\Delta t}\} + [C]\{\dot{q}_{t+\Delta t}\} + [K]\{q_{t+\Delta t}\} = \{Q_{t+\Delta t}\} \quad (3.2)$$

where $[M]$ is the finite element inertia matrix and $[K]$ is the finite element stiffness matrix, $[C]$ is the damping matrix, and $\{Q\}$ is the global load vector or also could be called the external excitation vector at time $t + \Delta t$. The effective loading is defined later in chapter 4. The $\{\ddot{q}\}$, $\{\dot{q}\}$, and $\{q\}$ are the acceleration, velocity, and the displacement vectors. For simplicity, the damping matrix, $[C]$, is not considered. There for the governing equation becomes

$$[M]\{\ddot{q}_{t+\Delta t}\} + [K]\{q_{t+\Delta t}\} = \{Q_{t+\Delta t}\} \quad (3.3)$$

There are a number of constants in this scheme, and they are defined as

$$\begin{aligned} a_0 &= \frac{1}{\alpha \Delta t^2} & a_1 &= \frac{\delta}{\alpha \Delta t} & a_2 &= \frac{1}{\alpha \Delta t} & a_3 &= \frac{1}{2\alpha} - 1 \\ a_4 &= \frac{\delta}{\alpha} - 1 & a_5 &= \frac{\Delta t}{2} \left(\frac{\delta}{\alpha} - 2 \right) & a_6 &= \Delta t (1 - \delta) & a_7 &= \delta \Delta t \end{aligned} \quad (3.4)$$

where α and δ are the Newmark parameters. For the present thesis, we take $\alpha = 0.25$ and $\delta = 0.5$. Setting the two parameters to these two values reduces the Newmark method to the constant acceleration method which is unconditionally stable. Also, the time step, Δt , is to be set based on the type of analysis.

After finding the stiffness and inertia matrices, setting the parameters, the time step and calculating the constants, initialize the acceleration, velocity and displacement matrices. Then for each time step

1. Form the effective stiffness matrix, which is found through Eq. (3.5)

$$[\hat{K}] = [K] + a_0[M] + a_1[C] \quad (3.5)$$

2. Calculate the effective loads at time $t + \Delta t$ through Eq. (3.6)

$$\{\hat{Q}_{t+\Delta t}\} = \{Q_{t+\Delta t}\} + [M](a_0\{q_t\} + a_2\{\dot{q}_t\} + a_3\{\ddot{q}_t\}) + [C](a_0\{q_t\} + a_4\{\dot{q}_t\} + a_5\{\ddot{q}_t\}) \quad (3.6)$$

3. Calculate the displacements at time $t + \Delta t$ through Eq. (3.7)

$$\{q_{t+\Delta t}\} = [\hat{K}]^{-1}\{\hat{Q}_{t+\Delta t}\} \quad (3.7)$$

4. Then calculate the accelerations at time $t + \Delta t$ through Eq. (3.8)

$$\{\ddot{q}_{t+\Delta t}\} = a_0(\{q_{t+\Delta t}\} - \{q_t\}) - a_2\{\dot{q}_t\} - a_3\{\ddot{q}_t\} \quad (3.8)$$

5. Finally, calculate the velocities at time $t + \Delta t$ through Eq. (3.9)

$$\{\dot{q}_{t+\Delta t}\} = \{\dot{q}_t\} + a_6\{\ddot{q}_t\} + a_7\{\ddot{q}_{t+\Delta t}\} \quad (3.9)$$

This process is to be repeated for each time step in the chosen time interval.

Chapter 4

Numerical Verification

The structural models presented in previous chapters are incorporated into a MATLAB code to obtain the numerical solutions. It is important to verify the model before generating new results. The numerical verification process is done by comparing the present results with existing analytical and computational results that are available in literature. This chapter details the verification process.

4.1 Isotropic Plates

4.1.1 Deflections

The material and the geometric properties chosen for the static deflection analysis are indicated in Table (4.1). This square plate is simply supported on all four sides and subjected to a uniformly distributed load (UDL). The Navier solution for the central deflection is shown in Table (4.2) and compared with that of the BFS element and the MONNA element. The boundary conditions used for the HSDT plate that is simply supported on all sides are shown in Figure (4.1).

Figure 4.1: Simply supported boundary conditions on all edges for a rectangular plate.

Table 4.1: Isotropic plate material properties and dimensions

Material properties	Dimensions (m)
---------------------	----------------

$$\begin{array}{cc}
E = 70 \text{ GPA} & a \times b = 5 \times 5 \\
v = 0.25 & h = 0.01
\end{array}$$

Table 4.2: Maximum Deflection of a square Isotropic plate

Model	Type	Elements	Deflection (m)	%Error
Navier Solution	Analytical	-	0.004081	-
MONNA Element	FEM	4×4	0.004081	0%
BFS Element	FEM	8×8	0.004081	0%

The results based on the MONNA (HSDT) and BFS elements matched the analytical results, and it should be noted that the BFS element required 4 times as many elements as that of MONNA.

Singh, Raveendranath, and Venkateswara (2000) created a four-noded shear-flexible composite plate element and used it to test isotropic as well as orthotropic cases. They verified their solutions by comparing them to the analytical solutions of Saleino and Goldberg. Singh et al conducted a convergence study by increasing the number of elements for a square isotropic plate ($a \times a \times h$) that is simply supported on all four sides and subjected to a uniformly distributed load; they reported the central deflection in nondimensional form:

$$\bar{w} = w_c D \times 10^3 / q a^4 \quad (4.1)$$

where D is the bending rigidity of the plate. Various thickness ratios (h/a) have been considered. The Comparison of the present results with that provided in Singh et al is seen in Table (4.3). The results show that the MONNA HSDT formulation converges very fast, and the results are in excellent agreement with the analytical solution.

Table 4.3: Comparison and convergence study of maximum central deflection of a square isotropic plate

h/a	Model	type	Elements	$\bar{w}$	%Error
0.1	Saleino and Goldberg	Analytical	-	0.424	-
			2×2	0.425	0.24%
	Singh et al.	FEM	4×4	0.426	0.47%
			6×6	0.427	0.70%
	MONNA	FEM	2×2	0.427	0.70%
			4×4	0.427	0.70%
			6×6	0.427	0.70%
	BFS	FEM	2×2	0.432	1.85%
			4×4	0.426	0.47%
			6×6	0.426	0.47%
0.05	Saleino and Goldberg	Analytical	-	0.411	-
			2×2	0.411	0.00%
	Singh et al.	FEM	4×4	0.411	0.00%
			6×6	0.411	0.00%
	MONNA	FEM	2×2	0.411	0.00%
			4×4	0.411	0.00%
			6×6	0.411	0.00%
	BFS	FEM	2×2	0.417	1.44%
			4×4	0.411	0.00%
			6×6	0.411	0.00%
0.01	Saleino and Goldberg	Analytical	-	0.407	-
			2×2	0.407	0.00%
	Singh et al.	FEM	4×4	0.406	0.25%
			6×6	0.407	0.00%
	MONNA	FEM	2×2	0.406	0.25%
			4×4	0.406	0.25%
			6×6	0.406	0.25%
	BFS	FEM	2×2	0.412	1.21%
			4×4	0.407	0.00%
			6×6	0.406	0.25%

4.1.2 Natural Frequencies

The natural frequency is arguably one of the most important properties of a mechanical system. Hence, it is one of the most important analysis done within structural analysis field. It is important for any proper analysis tools used in the industry to have the capabilities to calculate the structural natural frequencies. Thus, MONNA is built to accommodate this aspect of the structural analysis field.

Naiver also calculated analytically the natural frequencies of an isotropic plate. The material properties and dimensions used in this example are given in Table (4.4). For this test, the plate uses the same simply supported boundary condition shown in Figure (4.1). The natural frequency results of the test are shown in Table (4.5). The results show that both the MONNA and BFS elements generate natural frequencies that are in excellent agreement with the analytical solution produced by Naiver.

Table 4.4: Isotropic plate material properties and dimensions

Material properties	Dimensions (m)
$E = 30 \text{ Msi}$	$a \times b = 10 \times 10$
$\nu = 0.3$	$h = 0.1$
$\rho = 0.001 \frac{\text{slugs}}{\text{in}^3}$	

Table 4.5: Natural frequencies of a simply supported isotropic square plate (rad/s)

Model	Type	Elements	ω_1	ω_2	ω_3	ω_4
Navier Solution	Analytical	-	1035	2587	4138	5173
MONNA Element	FEM	4×4	1034	2584	4133	5167
MONNA % Error			0.10%	0.12%	0.12%	0.12%
BFS Element	FEM	10×10	1034	2584	4132	5166
BFS %Error			0.10%	0.12%	0.15%	0.14%

Phan and Reddy (1985) tested isotropic plates as well. In their test, they compared their finite element results for natural frequencies with the exact solutions of Srinivas, Rao and Rao (1970). The results are for an isotropic square plate ($a \times a \times h$) that is simply supported on all four sides with a Poisson's ratio (ν) of 0.3 and a thickness ratio (h/a) of 0.1; The modulus of elasticity (E) is not specifically given since the natural frequencies are expressed in non-dimensional form as

$$\bar{\omega} = \omega h \sqrt{\rho/G} \quad (4.2)$$

where G is the shear modulus, h is the plate thickness and ρ is the density of the material. The first four nondimensional frequencies from the present HSDT and FSDT models are compared with the analytical results of Srinivas et al and the finite element results of Reddy and Phan in Table (4.6). The present results are in excellent agreement with the other two.

Table 4.6: Natural frequencies of a simply supported isotropic square plate

Model	Type	Elements	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
Srinivas, et al.	Analytical	-	0.0932	0.226	0.3421	0.4171
Reddy and Phan (HSDPT)	FEM	N/A	0.0931	0.2222	0.3411	0.4158
% Error			0.11%	1.68%	0.29%	0.31%
Reddy and Phan (FSDPT)	FEM	N/A	0.093	0.2219	0.3406	0.4149
% Error			0.21%	1.81%	0.44%	0.53%
MONNA Element (HSDT)	FEM	4×4	0.093	0.222	0.3406	0.4153
MONNA % Error			0.21%	1.77%	0.44%	0.43%
MONNA Element (FSDT)	FEM	4×4	0.093	0.2219	0.3406	0.4151
MONNA % Error			0.21%	1.81%	0.44%	0.48%

Reddy and Phan (1985) also have modelled rectangular plates ($a \times b \times h$) and compared their finite element results for natural frequencies with the exact solutions of Reismann and Lee (1969). The properties used in this modeling are listed in Table (4.7). The comparison of the present results in the form of nondimensional natural frequencies [Eq. (4.2)] with that of Reismann and Lee and Phan and Reddy is shown in Table (4.8). The present FEM results are in line with the FEM results of Reddy and Phan with very little deviation from the analytical results.

Table 4.7: Isotropic plate material properties and dimensions

Material properties	Dimensions (m)
$\nu = 0.3$	$a/h = 10$
	$a/b = \sqrt{2}$

Table 4.8: Natural frequencies of a simply supported isotropic rectangular plate with sinusoidal loading.

Model	Type	Elements	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
Reismann and Lee	Analytical	-	0.704	1.376	2.018	2.431
Reddy and Phan (HSDPT)	FEM	N/A	0.7038	1.3738	2.0141	2.4263
% Error			0.03%	0.16%	0.19%	0.19%
Reddy and Phan (FSDPT)	FEM	N/A	0.7036	1.3729	2.0123	2.4235
% Error			0.06%	0.23%	0.28%	0.31%

MONNA Element (HSDT)	FEM	4×4	0.7036	1.373	2.0125	2.4249
% Error			0.06%	0.22%	0.27%	0.25%
MONNA Element (FSDT)	FEM	4×4	0.7036	1.3729	2.0124	2.4246
% Error			0.06%	0.23%	0.28%	0.26%

The results from Tables (4.6) and (4.8) prove that the MONNA element is eminently capable of generating accurate natural frequencies based on the HSDT and FSDT theories. Also, the results show that MONNA gives very promising solutions for both square and rectangular plates.

4.2 Composite Plates

4.2.1 Deflection

Reddy (2004) has published extensive amount of data that compares the analytical to finite element solution of composite plates with the three discussed theories, CLPT, FSDT, and HSDT. In this section, deflection results based on MONNA are compared with Reddy's finite element and analytical results. The composite material properties (in the form of ratios) used in this exercise are given in Table (4.9). The plate is an unsymmetric cross ply laminate of the form $[0/90]_{N/2}$, where N is the total number of layers. Two values of N , namely 2 and 10 are considered. The geometry is a square plate of width $b = a$ and thickness h . Two b/h ratios are considered, namely 5 (thick plate) and 10. The plate is simply supported on all sides. The applied loading is of the form

$$q(x, y) = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (4.3)$$

Table 4.9: Composite plate material properties

Material Properties	
$E_1 = 25E_2$	$G_{23} = 0.2E_2$
$G_{12} = G_{13} = 0.5E_2$	$\nu_{12} = 0.25$

The central deflection is nondimensionalized as

$$\bar{w} = 100w_{max} \frac{E_2 t^3}{q_0 b^4} \quad (4.4)$$

Table 4.10: Central deflection of a simply supported composite square plate with sinusoidal loading.

N	b/h	Model	Type	Elements	HSDT	FSDT	CLPT
2	5	Reddy	Analytical	-	1.667	1.758	1.064
		Reddy	FEM	4×4	1.667	1.759	1.043
		%Error	-		0.00%	0.06%	1.97%
		MONNA	FEM	4×4	1.666	1.759	1.064
		%Error	-		0.05%	0.04%	0.00%
		BFS	FEM	4×4	1.646	1.724	-
		%Error	-		1.29%	1.95%	-
	10	Reddy	Analytical	-	1.216	1.237	1.064
		Reddy	FEM	4×4	1.214	1.238	1.043
		%Error	-		0.16%	0.08%	1.97%
		MONNA	FEM	4×4	1.216	1.238	1.064
		%Error	-		0.00%	0.05%	0.00%
		BFS	FEM	4×4	1.186	1.202	-
		%Error	-		2.43%	2.81%	-
	10	Reddy	Analytical	-	1.129	1.137	0.442
		Reddy	FEM	4×4	1.135	1.137	0.444
		%Error	-		0.53%	0.00%	0.45%
		MONNA	FEM	4×4	1.130	1.137	0.442
		%Error	-		0.04%	0.04%	0.05%
		BFS	FEM	4×4	1.129	1.137	-
		%Error	-		0.01%	0.02%	-
10	5	Reddy	Analytical	-	0.616	0.615	0.442
		Reddy	FEM	4×4	0.619	0.616	0.444
		%Error	-		0.49%	0.16%	0.45%
		MONNA	FEM	4×4	0.616	0.616	0.442
		%Error	-		0.00%	0.08%	0.05%
		BFS	FEM	4×4	0.616	0.615	-
		%Error	-		0.06%	0.03%	-
	10	Reddy	Analytical	-	0.616	0.615	0.442
		Reddy	FEM	4×4	0.619	0.616	0.444
		%Error	-		0.49%	0.16%	0.45%
		MONNA	FEM	4×4	0.616	0.616	0.442
		%Error	-		0.00%	0.08%	0.05%
		BFS	FEM	4×4	0.616	0.615	-
		%Error	-		0.06%	0.03%	-

The central deflection comparisons are shown in Table (4.10). The MONNA based results agree excellently with the analytical solution and are more accurate than the FEM solutions of Reddy. The MONNA results converged with a mesh size of 4×4 . As a benchmark, the results generated using the BFS element are also included in this table.

4.2.2 Natural Frequencies

Reddy (2004) also documented the natural frequency results from his analysis with a layup of $[0/90]_{N/2}$ and the material properties used are shown in Table (4.11)

Table 4.11: Composite plate material properties

Material Properties	
$E_1 = 40E_2$	$G_{23} = 0.5E_2$
$G_{12} = G_{13} = 0.6E_2$	$\nu_{12} = 0.25$

The square plate ($b \times b \times h$) is simply supported on all four edges. The natural frequency is nondimensionalized as

$$\bar{\omega} = \omega \left(\frac{b^2}{t} \right) \sqrt{\frac{\rho}{E_2}} \quad (4.5)$$

The nondimensionalized natural frequency results are shown in Table (4.12)

Table 4.12: Natural frequencies of a simply supported composite square plate under with sinusoidal loading.

N	b/h	Model	Type	Elements	HSDT	FSDT	CLPT
2	5	Reddy	Analytical	-	9.087	8.833	10.721
		Reddy	FEM	4×4	9.103	8.837	11.192
		%Error	-		0.18%	0.05%	4.39%
		MONNA	FEM	4×4	9.088	8.834	10.722
		%Error	-		0.01%	0.01%	0.00%
		BFS	FEM	4×4	9.173	8.937	-
		%Error	-		0.95%	1.18%	-
	10	Reddy	Analytical	-	10.568	10.473	11.154
		Reddy	FEM	4×4	10.594	10.48	11.383
		%Error	-		0.25%	0.07%	2.05%
		MONNA	FEM	4×4	10.5695	10.475	11.1556
		%Error	-		0.01%	0.02%	0.01%
		BFS	FEM	4×4	10.717	10.632	-
		%Error	-		1.41%	1.52%	-
	10	Reddy	Analytical	-	11.673	11.644	12.167
		Reddy	FEM	4×4	11.664	11.647	18.624
		%Error	-		0.08%	0.03%	53.07%
		MONNA	FEM	4×4	11.673	11.644	10.922
		%Error	-		0.00%	0.00%	10.24%
		BFS	FEM	4×4	11.679	11.649	-
		%Error	-		0.05%	0.04%	-
10	10	Reddy	Analytical	-	15.771	15.779	18.492
		Reddy	FEM	4×4	15.787	15.787	18.637
		%Error	-		0.10%	0.05%	0.78%
		MONNA	FEM	4×4	15.770	15.779	18.492
		%Error	-		0.01%	0.00%	0.00%
		BFS	FEM	4×4	15.782	15.790	-
		%Error	-		0.07%	0.07%	-
	10	Reddy	Analytical	-	15.771	15.779	18.492
		Reddy	FEM	4×4	15.787	15.787	18.637
		%Error	-		0.10%	0.05%	0.78%
		MONNA	FEM	4×4	15.770	15.779	18.492
		%Error	-		0.01%	0.00%	0.00%
		BFS	FEM	4×4	15.782	15.790	-
		%Error	-		0.07%	0.07%	-

It could be seen that the present element, MONNA, performs better than the established elements, as it has less overall error percentage than Reddy's element and the BFS element.

4.2.3 Stresses

Another major aspect composite plates structural analysis is the stresses. Reddy (2004) also documented his exact and FEM solutions for the stresses. This analysis has a sinusoidal loading applied on the composite square plate, Eq. (4.3). The material properties used for this analysis are the same ones shown in Table (4.9). The layup is also in the form of a layup of $[0/90]_{N/2}$. The nondimensionalizations used for σ_{xx} , σ_{yy} , and σ_{yz} are

$$\begin{aligned}\bar{\sigma}_{xx} &= -\sigma_{xx} \left(\frac{a}{2}, \frac{b}{2}, -\frac{h}{2} \right) \frac{h^2}{q_0 b^2} \times 10 \\ \bar{\sigma}_{yy} &= -\sigma_{yy} \left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2} \right) \frac{h^2}{q_0 b^2} \times 10 \\ \bar{\sigma}_{yz} &= -\sigma_{yz} \left(0, 0, \frac{h}{2} \right) \frac{h}{q_0 b} \times 10\end{aligned}\tag{4.6}$$

The nondimensionalized stress results are shown in Table (4.13 – 4.15)

Table 4.13: Nondimensionalized axial stress ($\bar{\sigma}_{xx}$) of a simply supported composite square plate under sinusoidal loading.

N	b/h	Model	Type	Elements	HSDT	FSDT	CLPT
2	5	Reddy	Analytical	-	8.385	7.157	7.157
		Reddy	FEM	4×4	7.669	6.948	6.659
		%Error	-		8.54%	2.92%	6.96%
		MONNA	FEM	4×4	8.376	7.277	7.277
		%Error	-		0.10%	1.68%	1.68%
	10	Reddy	Analytical	-	7.468	7.157	7.157
		Reddy	FEM	4×4	6.829	6.948	6.659
		%Error	-		8.56%	2.92%	6.96%
		MONNA	FEM	4×4	7.5288	7.277	7.2772
		%Error	-		0.81%	1.68%	1.68%
10	5	Reddy	Analytical	-	6.34	5.009	5.009
		Reddy	FEM	4×4	5.762	4.864	4.611
		%Error	-		9.12%	2.89%	7.95%
		MONNA	FEM	4×4	6.337	4.998	4.998
		%Error	-		0.06%	0.22%	0.22%
	10	Reddy	Analytical	-	5.346	5.009	5.009
		Reddy	FEM	4×4	4.842	4.863	4.611
		%Error	-		9.43%	2.91%	7.95%
		MONNA	FEM	4×4	5.3382	4.998	4.998
		%Error	-		0.15%	0.22%	0.22%

Table 4.14: Nondimensionalized axial stress ($\bar{\sigma}_{yy}$) of a simply supported composite square plate under sinusoidal loading.

N	b/h	Model	Type	Elements	HSDT	FSDT	CLPT
2	5	Reddy	Analytical	-	8.385	7.157	7.157
		Reddy	FEM	4×4	7.669	6.948	6.659
		%Error	-		8.54%	2.92%	6.96%
		MONNA	FEM	4×4	8.376	7.277	7.277
		%Error	-		0.10%	1.68%	1.68%
	10	Reddy	Analytical	-	7.468	7.157	7.157
		Reddy	FEM	4×4	6.829	6.948	6.659
		%Error	-		8.56%	2.92%	6.96%
		MONNA	FEM	4×4	7.5288	7.277	7.2772
		%Error	-		0.81%	1.68%	1.68%
10	5	Reddy	Analytical	-	6.34	5.009	5.009
		Reddy	FEM	4×4	5.762	4.864	4.611
		%Error	-		9.12%	2.89%	7.95%
		MONNA	FEM	4×4	6.337	4.998	4.998
		%Error	-		0.06%	0.22%	0.22%
	10	Reddy	Analytical	-	5.346	5.009	5.009
		Reddy	FEM	4×4	4.842	4.863	4.611
		%Error	-		9.43%	2.91%	7.95%
		MONNA	FEM	4×4	5.3382	4.998	4.998
		%Error	-		0.15%	0.22%	0.22%

Table 4.15: Nondimensionalized axial stress ($\bar{\sigma}_{yz}$) of a simply supported composite square plate under sinusoidal loading.

N	b/h	Model	Type	Elements	HSDT	FSDT
2	5	Reddy	Analytical	-	3.155	2.729
		MONNA	FEM	4×4	3.141	2.728
		%Error	-		0.46%	0.04%
	10	Reddy	Analytical	-	3.19	2.729
		MONNA	FEM	4×4	3.156	2.728
		%Error	-		1.07%	0.04%
10	5	Reddy	Analytical	-	3.362	2.729
		MONNA	FEM	4×4	3.363	2.728
		%Error	-		0.04%	0.04%
	10	Reddy	Analytical	-	3.408	2.729
		MONNA	FEM	4×4	3.410	2.728
		%Error	-		0.05%	0.04%

The MONNA stress analysis shows that the results are in excellent agreement with the analytical solution and better than the FEM results of Reddy.

Reddy (2004) also tested the stresses on a symmetric five-ply composite square plate with the ply layup of [0/90/0/90/0]. The material properties for this test are shown in Table (4.9). The applied sinusoidal loading is as shown in Eq. (4.3). The nondimensionalizations employed for the stresses are

$$\bar{\sigma}_{xx} = -\sigma_{xx} \left(\frac{a}{2}, \frac{b}{2}, -\frac{h}{2} \right) \frac{h^2}{q_0 b^2} = -\sigma_{xx} \left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2} \right) \frac{h^2}{q_0 b^2} \quad (4.7)$$

The results shown in Table (4.16) are compared to the 3-D elasticity solution (ELS) with the percentage error shown between Reddy's results and MONNA's results. It could be seen through the results produced, that the MONNA plate element shows a better agreement with the elasticity solution than Reddy's FEM solution. MONNA also shows that it converges much faster than Reddy's solution, as it is shown that MONNA FSDT solution does not change as a/h changes. It is seen that the elasticity solution converges around a/h of 20, while compared to MONNA converges at a/h of 4 and does not change throughout the variation of a/h . Also, MONNA's CLPT solution is in good agreement with the provided CLPT solution. Therefore, it can be concluded that the MONNA is much more efficient than Reddy's finite element.

Table 4.16: Nondimensionalized axial stress ($\bar{\sigma}_{xx}$) of a simply supported composite square plate under sinusoidal loading.

a/h	Model	Theory	Elements	$\bar{\sigma}_{xx}$
4	ELS	exact	-	0.685
	Reddy	FSDT	4×4	0.4339
	%Error	-	-	36.66%
	MONNA	FSDT	4×4	0.5394
	%Error	-	-	21.26%
10	ELS	exact	-	0.545
	Reddy	FSDT	4×4	0.4986
	%Error	-	-	8.51%
	MONNA	FSDT	4×4	0.5394
	%Error	-	-	1.03%
20	ELS	exact	-	0.539
	Reddy	FSDT	4×4	0.5239
	%Error	-	-	2.80%
	MONNA	FSDT	4×4	0.5394
	%Error	-	-	0.07%
100	ELS	exact	-	0.539
	Reddy	FSDT	4×4	0.5345
	%Error	-	-	0.83%
	MONNA	FSDT	4×4	0.5394
	%Error	-	-	0.07%
		CLPT	-	0.5387
	MONNA	CLPT	4×4	0.5394
	%Error	-	-	0.13%

Lee and Kim (2013) tested a case of $[45/-45]_4$ layup and compared their results to the HSDT results of Latheswary (2004) et al. and Kant and Pandya (1988). The material properties used for this test are shown in Table (4.9). For this test, a sinusoidal loading [Eq. (4.3)] is applied. The nondimensionalizations for the stresses are

$$\begin{aligned}
 \bar{\sigma}_{xx} &= \sigma_{xx} \left(\frac{a}{2}, \frac{b}{2}, -\frac{h}{2} \right) \frac{h^2}{q_0 b^2} \\
 \bar{\sigma}_{yy} &= \sigma_{yy} \left(\frac{a}{2}, \frac{b}{2}, \frac{h}{2} \right) \frac{h^2}{q_0 b^2} \\
 \bar{\tau}_{xy} &= \tau_{xy} \left(0, 0, \frac{h}{2} \right) \frac{h}{q_0 b}
 \end{aligned} \tag{4.8}$$

The results from this example are shown in Table (4.17) where the MONNA results are compared to the published results by Lee and Kim and their references. The MONNA results are in better agreement than that of Lee and Kim's with their references. After examining and verifying the MONNA element results with various established authors, it is very clear that MONNA stress results are much better than those of the previous authors.

Table 4.17: Nondimensionalized stresses of a simply supported composite square plate under sinusoidal loading.

a/h	Model	Type	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\tau}_{xy}$
10	Kant and Pandya	HSDT	0.1633	0.1633	0.1601
	Latheswary et al.	HSDT	0.1627	0.1627	0.1547
	Lee and Kim	HSDT	0.1612	0.1612	0.1545
	%Error	-	1.29%	1.29%	3.50%
	MONNA	HSDT	0.1616	0.1616	0.1549
	%Error	-	1.04%	1.04%	3.25%
100	Kant and Pandya	HSDT	0.1462	0.1462	0.143
	Latheswary et al.	HSDT	0.1456	0.1456	0.1377
	Lee and Kim	HSDT	0.1439	0.1439	0.1379
	%Error	-	1.57%	1.57%	3.57%
	MONNA	HSDT	0.144	0.144	0.138
	%Error	-	1.50%	1.50%	3.50%

4.2.4 Sandwich Plate

The next example is a sandwich plate from Pagano (1970), where the material properties of the outer layers of the plate are different than those of the mid-layer of the plate. The ply orientation for this test is [0/90/0]. The MONNA CLPT code is edited to be able to accommodate such case. The material properties are shown in Tables (4.18 – 4.19)

Table 4.18: Composite material properties

Composite Material Properties	
E_1 (psi)	2.50E+07
E_2 (psi)	1.00E+07
G_{12} (psi)	5.00E+05
G_{23} (psi)	2.00E+05
ν_{12}	0.25
ν_{23}	0.25

Table 4.19: Isotropic material properties

Isotropic core Material Properties	
E_{xx} (psi)	4.00E+04
E_{yy} (psi)	4.00E+04
E_{zz} (psi)	5.00E+05
G_{xz} (psi)	6.00E+04
G_{xy} (psi)	1.60E+04
ν_{xy}	0.25

The stresses were nondimensionalized as

$$(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}) = \frac{1}{\sigma S^2} (\sigma_x, \sigma_y, \tau_{xy}) \quad (4.9)$$

$$S = a/h$$

The nondimensionalized CLPT stress results are shown in Table (4.20). The CLPT MONNA results are in excellent agreement with Pagano's analytical solution.

Table 4.20: Nondimensionalized CLPT stress results

$\bar{\sigma}_x$			$\bar{\sigma}_y$			$\bar{\tau}_{xy}$		
Pagano	MONNA	%Error	Pagano	MONNA	%Error	Pagano	MONNA	%Error
1.097	1.0985	0.137%	0.0543	0.0544	0.184%	0.0433	0.0433	0%

4.2.5 Time response

Lee and Kim (2013) tested a case of $[45/-45]_4$ layup and compared their results to the HSDT results of Latheswary (2004) et al. and Kant and Pandya (1988).

Another major aspect to be analyzed is the dynamic response. Gupta, and Ghosh (2017) tested a composite symmetric cross-ply laminate and compared their FSDT results to Valizadeh (2012). The laminate tested is made up of three plies, $[0/90/0]$, that is subjected to a transverse load which is sinusoidally distributed and varies with time as

$$q(x, y, t) = q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) F(t) \quad (4.10)$$

The material properties shown in Table (4.21) were used for this analysis.

Table 4.21: Composite plate material properties

Material Properties	
$E_1 = 172.369 \text{ GPa}$	$\rho = 1603.3 \text{ kg/m}^3$
$E_2 = 6.895 \text{ GPa}$	$a = b = 20h$
$G_{12} = 3.448 \text{ GPa}$	$h = 0.0381 \text{ m}$
$G_{13} = 3.448 \text{ GPa}$	$t_1 = 0.006 \text{ s}$
$G_{23} = 2.758 \text{ GPa}$	
$\nu_{12} = 0.25$	$\Delta t = 0.01e^{-3} \text{ s}$
$P_0 = 3.448 \text{ MPa}$	$\gamma = 330 \text{ s}^{-1}$

Where t_1 is the time that the response is carried out for and Δt is the time step. The time function of the transient load, $F(t)$, is

$$F(t) = \begin{cases} \begin{cases} 1, & 0 \leq t \leq t_1 \\ 0, & t \geq t_1 \end{cases} & \text{step loading} \\ \begin{cases} 1 - \left(\frac{t}{t_1}\right), & 0 \leq t \leq t_1 \\ 0, & t \geq t_1 \end{cases} & \text{triangular loading} \\ \begin{cases} \sin\left(\frac{\pi t}{t_1}\right), & 0 \leq t \leq t_1 \\ 0, & t \geq t_1 \end{cases} & \text{sine loading} \\ \begin{cases} e^{-\gamma t}, & 0 \leq t \leq t_1 \\ 0, & t \geq t_1 \end{cases} & \text{explosive blast loading} \end{cases} \quad (4.11)$$

As it can be seen, the transient load is applied at $t = 0$ and it is applied until the end of the response time, t_1 . A graphical representation of these is shown in Figures (4.2 - 4.5).

Figure 4.2: Step loading as a function of time.

Figure 4.3: Sine loading as a function of time.

Figure 4.4: Triangular loading as a function of time.

Figure 4.5: Explosive blast loading as a function of time.

Figures (4.6-4.9) show the results of the dynamic response.

Figure 4.6: Dynamic response of laminated plate under step loading.

Figure 4.7: Dynamic response of laminated plate under sine loading.

Figure 4.8: Dynamic response of laminated plate under triangular loading.

Figure 4.9: Dynamic response of laminated plate under explosive blast loading.

The results from the Dynamic response test shows that MONNA FSDT element, is in good agreement with the established data. However, the difference between the established FSDT element and the presented FSDT element, is that the degrees of freedom used in the formulation of the presented element are more than those presented in the established element. Thus, MONNA is more accurate with 68 degrees of freedom and for it being an h - p -version. Also, the difference between the CLPT and the FSDT is caused due to the fact that CLPT does not account for shear deformation.

Jing and Liao (1990), who introduced a partial hybrid stress element, have conducted the dynamic response of a composite square plate. They have compared their results to a higher-order plate element solution (HOPE) and Mindlin's solution, which is a first order plate element. They laminate is an antisymmetric cross ply made up of two plies, [0/90], that is subjected to a Heaviside step function as the response varies with time as shown in Eq. (4.10). The material properties shown in Table (4.22) are used for this analysis.

Table 4.22: Composite plate material properties

Material Properties	
$E_1 = 25 E_2$	$a/h = 25 \text{ cm}$
$E_2 = 2.1 \times 10^6 \text{ N/cm}^2$	$h = 5 \text{ cm}$
$G_{12} = G_{13} = G_{23} = 0.5 E_2$	$t_1 = 200 \mu\text{s}$
$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$	$\Delta t = 5 \mu\text{s}$
$\rho = 8 \times 10^{-6} \text{ Nsec}^2/\text{cm}^4$	$\alpha = -0.3$

The nondimensionalization of the deflection is

$$\bar{w} = \frac{1000wE_2}{Q_0hS^4} \quad (4.12)$$

$$S = a/h$$

Figure 4.10: Dynamic response of a simply supported square laminated plate under sinusoidal pulse.

The comparison between MONNA's HSDT and FSDT, Jing and Liao's partial hybrid stress element, Mindlin's first-order element, and the higher-order plate element is seen in Figure (4.10). The MONNA HSDT shows that it underpredicts the solution with an acceptable margin of error. Also, the MONNA FSDT solution gave almost the exact same results as Jing and Liao's partial hybrid stress element. This means that the results of MONNA are in good agreement with the authors' results and that of their references.

Chapter 5

Results and Discussion

After establishing the capabilities of the MONNA plate element, it's important to analyze how efficient it is. In this chapter, the performance and efficiency of the MONNA plate element are analyzed. A convergence study of the MONNA element is investigated, then the effects of laminate length-to-thickness ratios, increasing number of plies, and material properties ratios.

5.1 Convergence Study

A convergence study is performed to gauge the efficiency and accuracy of the MONNA element.

In this section both conforming elements, MONNA and BFS, are compared with each other and with the Melosh element, a nonconforming element. This is to understand how efficient the conforming elements are in structural analysis. The material properties are found in Table (5.1). The plate is square with the layup $[0/90]_5$. The loading is

$$q(x, y) = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (5.1)$$

Table 5.1: Composite plate material properties

Material Properties	
$E_1 = 25E_2$	$G_{23} = 0.2E_2$
$G_{12} = G_{13} = 0.5E_2$	$\nu_{12} = 0.25$

The nondimensionalization of the central deflection is

$$\bar{w} = 100w_{max} \frac{E_2 t^3}{q_0 b^4} \quad (5.2)$$

Table 5.2: Convergence study for a $[0/90]_5$ laminate with $b/h = 5$

Elements	Melosh (non-conforming)	BFS (conforming)	MONNA (conforming)
-	Maximum deflection, HSDT (Exact: $w = 1.129$)		
2×2	1.3098	1.1328	1.1305
4×4	1.1734	1.1286	1.1295
6×6	1.1484	1.1288	1.1294
8×8	1.14	1.129	1.1294
10×10	1.1361	1.1291	1.1294
14×14	1.1328	1.1292	1.1294
20×20	1.131	1.1293	1.1294
24×24	1.1305	1.1293	1.1294
30×30	1.1301	1.1293	1.1294
34×34	1.13	1.1294	1.1294
40×40	1.1298	1.1294	1.1294
50×50	1.1296	1.1294	1.1294

Figure 5.1: Convergence study for a $[0/90]_5$ laminate with $b/h = 5$

It is seen from Table (5.2) and Figure (5.1) that the MONNA element is the fastest to converge, with a mesh size of only 4×4 required to achieve convergence to the exact value of the maximum deflection. Next the BFS element converges at a mesh size of 8×8 . Then finally, the Melosh element converges at a higher number of elements, which is 50×50 . This indicates that the conforming elements are more efficient.

5.2 Effect of Laminate Thickness

Adim, Daouadji, and Rabahi (2016) have tested the effects of varying the laminate thickness and compared their results to Reddy's HSDT element and Pagano's elasticity solution. They developed and tested two models, a hyperbolic shear deformation theory, also abbreviated HSDT, and exponential shear deformation theory abbreviated ESDT. The material properties are shown in Table (5.1).

Table 5.3: Nondimensional deflections of a simply supported square composite plate under sinusoidal transverse loading

a/h	Model	Theory	$\bar{w}$	%Error
2	Pagano	Elasticity	4.936	-
	Reddy	HSDT	4.562	7.58%
	A.D.R. Model 1	HSDT	4.573	7.36%
	A.D.R. Model 2	ESDT	4.562	7.58%
	MONNA	HSDT	4.657	5.66%
	MONNA	CLPT	1.064	78.44%
	Pagano	Elasticity	1.728	-
5	Reddy	HSDT	1.667	3.54%
	A.D.R. Model 1	HSDT	1.668	3.48%
	A.D.R. Model 2	ESDT	1.667	3.54%
	MONNA	HSDT	1.667	3.56%
	MONNA	CLPT	1.064	38.43%
	Pagano	Elasticity	1.232	-
	Reddy	HSDT	1.216	1.27%
10	A.D.R. Model 1	HSDT	1.216	1.25%
	A.D.R. Model 2	ESDT	1.216	1.27%
	MONNA	HSDT	1.216	1.28%
	MONNA	CLPT	1.064	13.62%
	Pagano	Elasticity	1.106	-
	Reddy	HSDT	1.102	0.38%
	A.D.R. Model 1	HSDT	1.102	0.37%
20	A.D.R. Model 2	ESDT	1.102	0.38%
	MONNA	HSDT	1.102	0.38%
	MONNA	CLPT	1.064	3.80%
	Pagano	Elasticity	1.074	-
	Reddy	HSDT	1.065	0.85%
	A.D.R. Model 1	HSDT	1.065	0.85%
	A.D.R. Model 2	ESDT	1.065	0.85%
100	MONNA	HSDT	1.065	0.84%
	MONNA	CLPT	1.064	0.95%

All elements produced the expected results, and they are all in good agreement among themselves. It is obvious that for the case of a thick plate ($a/h = 2$, and 5), the accuracy of the results is not perfect. However, all of the plates are behaving the same way, hence the results obtained by MONNA are valid. Nevertheless, when compared to Reddy's HSDT solution, the results are very accurate. The conclusion that as the plate becomes thinner, the ratio of length-to-thickness increases, the results accuracy improves. This is the case for all the examined elements in Table (5.3). The CLPT plate element is compared with the HSDT to show that it is not affected by changing the plate thickness. The FSDT plate element is not shown however, from previous tests, it could be seen that it behaves similarly to the HSDT.

5.3 Number of Plies

Another important aspect of understanding well the MONNA element is to investigate how increasing the number of plies affect its efficiency and performance. The material properties of this test are shown in Table (5.4)

Table 5.4: Composite plate material properties

Material Properties	
$E_1 = 25E_2$	$G_{23} = 0.2E_2$
$G_{12} = G_{13} = 0.5E_2$	$\nu_{12} = 0.25$

The nondimensionalization used for this analysis is shown earlier in Eq. (5.2). The results are shown in Table (5.5). The results produced, prove again that MONNA, behaves similar to other plate elements and does not yield the most accurate results in the case of thick plates ($h/a = 0.25$); however, these results are acceptable since this is one of the disadvantages of the HSDT element. Other authors tested this and produced similar results as seen from Table (5.5). On the other hand, once more plies are added into the laminate, the accuracy is improved drastically, even for thick plates. The test also shows that the MONNA element does converge relatively quickly. Also, it adds to the fact that the accuracy improves as the length-to-thickness ratio of the plate increases. In conclusion, increasing the number of plies will drastically improve the results and increasing the length-to-thickness ratio of the plate will also contribute to a better accuracy. The FSDT plate element is not shown however, from previous tests, it could be seen that it behaves similarly to the HSDT.

Table 5.5: Comparison and convergence study of plate central deflection for a simply supported square composite plate under sinusoidal transverse loading

Plies	h/a	Exact	MONNA HSDT			
			2x2	4x4	6x6	8x8
[0/90] ₁	0.25	2.1492	1.9955 7.15%	1.9975 7.06%	1.9983 7.02%	1.9984 7.02%
	0.1	1.2372	1.219 1.47%	1.216 1.71%	1.216 1.71%	1.2161 1.71%
	0.01	1.0653	1.0702 0.46%	1.0655 0.02%	1.0652 0.01%	1.0651 0.02%
[0/90] ₂	0.25	1.5921	1.6146 1.41%	1.61 1.12%	1.6095 1.09%	1.6094 1.09%
	0.1	0.6802	0.6881 1.16%	0.6868 0.97%	0.6866 0.94%	0.6866 0.94%
	0.01	0.5083	0.509 0.14%	0.5084 0.02%	0.5084 0.02%	0.5083 0.00%
[0/90] ₄	0.25	1.5335	1.5189 0.95%	1.517 1.08%	1.5169 1.08%	1.5168 1.09%
	0.1	0.6216	0.6234 0.29%	0.6229 0.21%	0.6229 0.21%	0.6229 0.21%
	0.01	0.4496	0.4499 0.07%	0.4497 0.02%	0.4496 0.00%	0.4496 0.00%

5.4 Moduli ratio

It is important to understand how the material properties ratio can affect the efficiency of the present element. The test is done by calculating the nondimensional natural frequency while varying the modulus ratio E_1/E_2 to examine the efficiency of the MONNA element in comparison with other established results such as Adim, Daouadji, and Rabahi (2016). They compare their results with that of Noor and Reddy's polynomial shear deformation theory (PSDT). The material properties and dimensions are shown below in Table (5.6).

Table 5.6: Composite plate material properties

Material and dimensions	Properties and
$a/h = 5$	$G_{23} = 0.5E_2$
$G_{12} = G_{13} = 0.6E_2$	$\nu_{12} = 0.25$

The natural frequency is nondimensionalized as

$$\bar{\omega} = \omega \left(\frac{b^2}{t} \right) \sqrt{\frac{\rho}{E_2}} \quad (5.3)$$

Testing an antisymmetric cross ply square plate under sinusoidal loading and simply supported boundary conditions, the results are shown in Table (5.7)

Table 5.7: Nondimensional fundamental frequencies of antisymmetric square plates at various moduli ratio

# Layers	Model	Theory	E_1/E_2							
			3	%Error	10	%Error	20	%Error	40	%Error
[0/90] ₁	NOOR	Exact	6.258	-	6.985	-	7.675	-	8.563	-
	Reddy	PSDT	6.217	0.65%	6.989	0.06%	7.821	1.91%	9.087	6.13%
	A.D.R.M 1	HSDT	6.217	0.66%	6.988	0.05%	7.820	1.89%	9.084	6.09%
	A.D.R.M.2	ESDT	6.217	0.65%	6.989	0.06%	7.821	1.91%	9.087	6.13%
	MONNA	HSDT	6.217	0.65%	6.989	0.07%	7.822	1.92%	9.088	6.14%
[0/90] ₂	NOOR	Exact	6.546	-	8.145	-	9.406	-	10.679	-
	Reddy	PSDT	6.501	0.68%	8.195	0.62%	9.627	2.35%	11.172	4.61%
	A.D.R.M 1	HSDT	6.501	0.68%	8.196	0.63%	9.627	2.36%	11.173	4.62%
	A.D.R.M.2	ESDT	6.501	0.68%	8.195	0.62%	9.627	2.35%	11.172	4.61%
	MONNA	HSDT	6.501	0.68%	8.196	0.63%	9.627	2.35%	11.172	4.61%
[0/90] ₃	NOOR	Exact	6.610	-	8.414	-	9.840	-	11.272	-
	Reddy	PSDT	6.556	0.16%	8.405	3.20%	9.918	5.45%	11.501	7.70%
	A.D.R.M 1	HSDT	6.556	0.16%	8.405	3.20%	9.918	5.45%	11.501	7.70%
	A.D.R.M.2	ESDT	6.556	0.16%	8.405	3.20%	9.918	5.45%	11.501	7.70%
	MONNA	HSDT	6.556	0.16%	8.405	3.20%	9.918	5.45%	11.501	7.70%
[0/90] ₅	NOOR	Exact	6.646	-	8.563	-	10.084	-	11.625	-
	Reddy	PSDT	6.584	0.93%	8.513	0.58%	10.067	0.17%	11.673	0.42%
	A.D.R.M 1	HSDT	6.584	0.93%	8.513	0.58%	10.067	0.17%	11.672	0.41%
	A.D.R.M.2	ESDT	6.584	0.93%	8.513	0.58%	10.067	0.17%	11.673	0.42%
	MONNA	HSDT	6.584	0.93%	8.513	0.58%	10.067	0.17%	11.673	0.42%

It is seen that MONNA is in total agreement with all the other elements that it is being compared to. It is seen from Table (5.7) that as E_1/E_2 ratio increases, the error increases for all the elements.

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

After modeling with and testing the MONNA element, the following conclusions are made:

- A new conforming h - p -version plate element, called MONNA, capable of implementing the higher-order shear deformation theory (HSDT), the first-order shear deformation theory (FSDT), and the classical laminated plate theory (CLPT), has been successfully formulated, developed, and implemented in a MATLAB code to structurally analyze composite plates.
- Similar to other C^1 continuous rectangular plate elements, the MONNA plate element is a conforming element.
- As presently formulated, the MONNA element is restricted to modelling rectangular plates.
- The MONNA plate element converges relatively faster than other existing element, making it very efficient and less time consuming to use with a very high accuracy level.
- The accuracy of the element is slightly affected by a couple of factors such as laminate length-to-thickness ratios, the number of plies, and material properties ratios. Nevertheless, the performance, efficiency and accuracy of the element does not change significantly as the element still produces highly accurate results.
- The MONNA higher-order element produces stress results that are more accurate than most of the existing plate elements.

6.2 Recommendations for Future Research

- A more thorough investigation of thick and thin symmetric plate results should be looked into, where deflections, natural frequencies, stresses, and time responses of MONNA are tested and compared to other established results.

- Investigate, using MONNA, the buckling load tests and if the element is capable of handling this aspect of structural analysis, then compare the results to other established results.
- Make MONNA amenable for non-rectangular domains.

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